The Local-Ratio Technique: LP Formulation

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The General Problem)

Given: $w \in (\mathbb{R}^+)^n$, I
Find:

$$\begin{array}{ll} \min[\max] & w \cdot x \\ s.t. & Constraint(x, I) \end{array}$$

r-Approximation

x is an r-approximation if
$Constraint(x, I)$ and $w \cdot x \leq [\geq] r \cdot w^*$

An algorithm is an r-approximation if
for any w, I it returns an r-approximation.

Weight Decomposition

Local-Ratio Theorem:

$$\left.\begin{array}{l} x \text{ is an } r\text{-approximation w.r.t. } \delta \\ x \text{ is an } r\text{-approximation w.r.t. } w-\delta \end{array}\right\}$$

$$\Longrightarrow$$

x is an r-approximation w.r.t. w

Proof:

$$\left.\begin{array}{l} \delta \cdot x \leq r * \delta^* \\ (w-\delta) \cdot x \leq r * (w-\delta)^* \end{array}\right\}$$

$$\Longrightarrow$$

$$\begin{aligned} w \cdot x &\leq r * (\delta^* + (w-\delta)^*) \\ &\leq r * w^* \end{aligned}$$

For max problems use "$\geq$". □

Weight Decomposition - Cont.

Special Case: optimization (r=1):

$$\left.\begin{array}{l} x \text{ is an optimum w.r.t. } \delta \\ x \text{ is an optimum w.r.t. } w - \delta \end{array}\right\}$$

$$\Longrightarrow$$

x is an optimum w.r.t. w

Minimum Spanning Tree

Given: $G = (V, E)$, $w \in (\mathbb{R}^+)^{|E|}$

Find:

$$\begin{array}{ll} \min & w \cdot x \\ s.t. & x \in \{0,1\}^{|E|} \\ & G(V, \{e : x_e = 1\}) \text{ is connected} \end{array}$$

How to get 1-Approximation

Add constraint: $|\{e : x_e = 1\}| = |V| - 1$

For all edges e select $\delta_e = \epsilon$

Shrink edges e s.t. $w(e) = 0$

Local-Ratio=1: $\delta \cdot x = \epsilon(|V| - 1)$

...Kruskal

Shortest Path

Given: $G = (V, E)$, $s, t \in V$, $w \in (\mathbb{R}^+)^{|E|}$
Find:

$$\begin{array}{ll} \min & w \cdot x \\ s.t. & x \in \{0,1\}^{|E|} \\ & \text{There is a path from s to t in } G(V, \{e : x_e = 1\}) \end{array}$$

How to get 1-Approximation

Add constraint: $|\{e : x_e = 1\}|$ is a simple path from s to t.

Select $\delta_{(s,v)} = \epsilon$ and 0 for all other edges.

Shrink edges e s.t. $w(e) = 0$

Local-Ratio=1: $\delta \cdot x = \epsilon$

...Dijkstra

Vertex Cover

Given: $G = (V, E)$, $w \in (I\!R^+)^{|V|}$
Find:

$$\begin{array}{lll} \min & w \cdot x & \\ s.t. & x \in \{0,1\}^{|V|} & \\ & x_i + x_j \geq 1 & \forall_{\{i,j\} \in E} \end{array}$$

How to get 2-Approximation

Option 1: For $\{i, j\} \in E$, $\delta_i = \delta_j = \epsilon$ and 0 for all others.

Local-Ratio=2: $\epsilon \leq \delta \cdot x \leq 2\epsilon$

Option 2: For $i \in V$, $\delta_i = \epsilon d(i)$ $\delta_j = \epsilon$ for $j \in D(i)$ and 0 for all others.

Local-Ratio=2: $\epsilon d(v) \leq \delta \cdot x \leq 2\epsilon d(v)$

Homogeneous: $\delta_i = \epsilon d(i)$

Local-Ratio=2: $\epsilon |E| \leq \delta \cdot x \leq 2\epsilon |E|$

Set Cover

Given: $H = (V, E)$, $w \in (I\!R^+)^{|V|}$
Find:

$$\begin{array}{ll} \min & w \cdot x \\ s.t. & x \in \{0,1\}^{|V|} \\ & \sum_{i \in e} x_i \geq 1 \quad \forall_{e \in E} \end{array}$$

How to get Δ_E-Approximation

Option 1: For $e \in E$, $\forall_{i \in e} \delta_i = \epsilon$ and 0 for all others.

Local-Ratio=Δ_E: $\epsilon \leq \delta \cdot x \leq \epsilon \Delta_E$

Homogeneous: $\delta_i = \epsilon d(i)$

Local-Ratio=Δ_E: $\epsilon|E| \leq \delta \cdot x \leq \epsilon|E|\Delta_E$

Generalized Vertex Cover

Given: $G = (V, E)$, $w \in (I\!R^+)^{|V|+|E|}$
Find:

$$\begin{array}{ll} \min & w \cdot x \\ s.t. & x \in \{0,1\}^{|V|+|E|} \\ & x_i + x_j + x_{(i,j)} \geq 1 \quad \forall_{\{i,j\} \in E} \end{array}$$

How to get 2-Approximation

Add costreint: $x_i + x_j + x_{(i,j)} \leq 2 \forall_{\{i,j\} \in E}$

For $\{i, j\} \in E$, $\delta_i = \delta_j = \delta_{(i,j)} = \epsilon$ and 0 for all others.

Local-Ratio=2: $\epsilon \leq \delta \cdot x \leq 2\epsilon$

Partial Vertex Cover

Given: $G = (V, E)$, $w \in (I\!R^+)^{|V|}$, $t \leq |E|$
Find:

$$\begin{array}{ll} \min & w \cdot x \\ s.t. & x \in \{0,1\}^{|V|} \\ & |\{\{i,j\} \in E : x_i + x_j \geq 1\}| \geq t \end{array}$$

How to get 2-Approximation

Add constraint:
$\forall_{x_k=1} |\{\{i,j\} \in E - E(k) : x_i + x_j \geq 1\}| < t$

Homogeneous: $\forall_{i \in V} \delta_i = \epsilon \min(t, d(i))$

Local-Ratio=2: $\epsilon t \leq \delta \cdot x \leq 2\epsilon t$

Vertex Feedback Set

Given: $G = (V, E)$, $w \in (\mathbb{R}^+)^{|V|}$
Find:

$$\begin{array}{ll} \min & w \cdot x \\ s.t. & x \in \{0,1\}^{|V|} \\ & G(\{i : x_i = 0\}) \text{ is a forest} \end{array}$$

How to get 2-Approximation

Add constraint: $\{i : x_i = 0\} is maximal.$

Homogeneous: Assume $\forall_{i \in V} d(i) > 1$ (Why?)
Define $\forall_{i \in V} \delta_i = \epsilon d(i)$

Becker Geiger: $\delta^* \leq \delta \cdot x \leq 2\delta^*$

Maximum Set Packing

Comment : Also called Maximum Hypergraph Matching.

Given: $H = (V, E)$, $w \in (\mathbb{R}^+)^{|E|}$
Find:

$$\begin{array}{ll} \max & w \cdot x \\ s.t. & x \in \{0,1\}^{|E|} \\ & \sum_{e \in D(i)} x_e \leq 1 \end{array}$$

How to get Δ_E-Approximation

Add constraint: $\forall_{x_e=0} \exists_{i \in V, e' \in D(i)} x'_e = 1$

Homogeneous: Define $\forall_{e \in E} \delta_e = \epsilon$

Local-Ratio=Δ_E: $\frac{\delta_e^*}{\Delta_E} \leq \delta \cdot x \leq \delta_e^*$

Arkin Hasin 97: $\Delta_E - 1 + \varepsilon$

Min 2VIP

Given:

- n variables, m constraints, $w \in (\mathbb{R}^+)^n$
- $a, b, c \in \mathbb{Z}^m$.
- $\ell, u \in \mathbb{N}^n$. $U = \max_i \{u_i - \ell_i\}$.

Find:

$$\begin{array}{lll} (2VIP) & \min & w \cdot x \\ & \text{s.t.} & a_k x_{i_k} + b_k x_{j_k} \geq c_k \\ & & \ell_i \leq x_i \leq u_i, \;\; x_i \in \mathbb{N} \end{array}$$

How to get 2-Approximation

Hint: If $x_i < i_1 \implies x_j \geq j_1$ Define $\delta_i(i_1) = \delta_j(j_1) = \epsilon$ and 0 for all others.

Generalized Steiner Forest

Given: $G = (V, E)$, $w \in (\mathbb{R}^+)^{|E|}$,
$T_1, T_2, ..., T_t \subseteq V$ $t > 0$, $T \in (\{0,1\}^V)^t$
Find:

$$\begin{array}{ll} \min & w \cdot x \\ s.t. & x \in \{0,1\}^n \\ & \forall_{0 \le i \le t} \forall_{v,u \in T_i} \\ & \exists_{P \in (v,u) path} \forall_{e \in P} x_e = 1 \end{array}$$

How to get 2-Approximation

Add constraint: $\{e : x_e = 1\}$ is minimal.

Homogeneous: $\delta_e = \epsilon TerminalDegree(e)$

Shrink edges e s.t. $w(e) = 0$

Williamson, Goemans, Mihail, Vazirani:
$t \le \delta \cdot x \le 2t - 2^*$

Min s-t Cut

Given: $G = (V, E)$, $w \in (I\!R^{+})^{|E|}$, $s, t \in V$

$$\textbf{Find}: \min \quad w \cdot x$$
$$s.t. \qquad x \in \{0,1\}^{|E|}$$
$$\forall_{P \in (s,t) path} \sum_{e \in P} x_e \geq 1$$

How to get 1-Approximation

Assume each non-zero weight edge e has anti-parallel e^{-} with zero weight

Add constraint:

$\forall_{P \in (s,t) path} \sum_{e \in P} x_e - x_{e^-} = 1$

Select δ: For a non-zero weighted path (s, t) path P: $\forall_{e \in P} \delta_e = \epsilon$, $\forall_{e \in P^-} \delta_e = -\epsilon$, and 0 for all other edges.

Local-Ratio=1: $\delta \cdot x = \epsilon$

Summary

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r-Approximation:

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Vertex Cover:

Generalized Vertex Cover:

Partial Vertex Cover:

Vertex Feedback Set:

Maximum Set Packing:

Min 2VIP:

Generalized Steiner Forest:

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