Local Ratio: A Unified Framework for Approximation Algorithms

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The *local ratio* technique is a methodology for the design and analysis of algorithms for a broad range of optimization problems. The technique is remarkably simple and elegant, and yet can be applied to several classical and fundamental problems (including covering problems, packing problems, and scheduling problems). The local ratio technique uses elementary math and requires combinatorial insight into the structure and properties of the problem at hand. Typically, when using the technique, one has to invent a weight function for a problem instance under which every "reasonable" solution is "good." The local ratio technique is closely related to the *primal-dual* schema, though it is not based on weak LP duality (which is the basis of the primal-dual approach) since it is not based on linear programming.

In this survey we introduce the local ratio technique and demonstrate its use in the design and analysis of algorithms for various problems. We trace the evolution path of the technique since its inception in the 1980's, culminating with the most recent development, namely, *fractional local ratio*, which can be viewed as a new LP rounding technique.

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1. INTRODUCTION

The *local ratio* technique is an approximation paradigm for NP-hard optimization problems which has been enjoying a great measure of success since its inception in the early 1980's. Its main feature of attraction is its simplicity and elegance; it is very easy to understand, and has surprisingly broad applicability. The technique is based on exploiting the combinatorial structure of the problem at hand—often by making just one or two straightforward observations—and is therefore accessible to the non-specialist. In this sense the technique is not unlike *dynamic programming*; it is easily mastered, and can be used effectively in a wide variety of problems. Although it is often the case that employing heavier machinery can produce better results, the local ratio technique usually provides a good head start on the problem.

In this survey we introduce the local ratio technique and demonstrate its use in the design and analysis of algorithms for various problems. Among the multitude of algorithms we describe some are algorithms that were developed initially using the technique, and others are local ratio interpretations of pre-existing algorithms. Several of them appear here for the first time. Whenever possible we present a unified analysis for a number of similar algorithms. We cover minimization and maximization problems, approximation and exact optimization algorithms, classical results and more recent developments.

1.1 A First Glimpse

Before actually beginning, let us briefly introduce the basic notions of optimization and approximation, as the intended audience of this survey is the general Computer Science community. (We give more complete and precise definitions in Section 2.) An *optimization problem* is one where we seek a solution of minimum or maximum cost among a set of candidate solutions. For example, if the problem is to find the longest (simple) path between two given vertices in a graph, the candidate solutions are all simple paths connecting the two vertices, and the cost of each path is its length. Since this problem, and many others, are NP-hard, we are often interested in efficiently obtaining *approximate* solutions, i.e., solutions that, although suboptimal, are not "far" from optimal. A solution whose cost is within a factor of r of the optimum is said to be r*-approximate*. Thus, in the *longest path* example, a path whose length is at least one half the length of the longest path is 2-approximate. An r*-approximation* algorithm for a given problem is an algorithm that finds r-approximate solutions.

Say we want to develop an r-approximation algorithm for a minimization problem. A key step in the design of such an algorithm is to establish a good lower bound b on the cost of the optimal solution. This bound can later be used in the analysis to prove that the solution found by the algorithm is r-approximate by showing that its cost is no more than $r \cdot b$. At a high level of abstraction, the local ratio technique is based on a "local" variation of this scheme. In essence, the idea is to break down the cost c of the solution found by the algorithm into a sum of "partial costs" $c = c_1 + c_2 + \ldots + c_k$, and similarly break down the lower bound b into $b = b_1 + b_2 + \ldots + b_k$, and to show that $c_i \leq r \cdot b_i$ for all i. (For maximization problems, b is an upper bound and we show that $c_i \geq b_i/r$ for all i.) This high-level idea of decomposing a cost and charging each component to some corresponding object is fairly standard, and is ubiquitous, for example, in time complexity analyses of algorithms. The local ratio technique is all about implementing it in a certain systematic manner. In the local ratio methodology, the decomposition of c and b is not a mere artifact of the analysis—existing in hindsight—but is built directly into the algorithm. Roughly speaking, the algorithm constructs a solution by steps designed to explicitly decompose c and b in such a way as to create the desired effect. The actual ideas involved in the technique are, of course, more elaborate; we get down to the specifics in subsequent sections.

To give a first taste of the local ratio technique, we demonstrate it on the well known *vertex cover* problem. In this problem we are given a graph $G = (V, E)$ and a non-negative weight function w on its vertices, and the objective is to find a minimum weight *vertex cover*, i.e., a minimum weight set of vertices $C \subseteq V$ such that every edge in the graph has at least one endpoint in C. Let us develop a 2-approximation algorithm for this problem in intuitive terms. Imagine that we are shopping for a vertex cover of the graph at the local branch of *Graphs 'Я' Us*. The direct approach would be to somehow select a vertex cover and then pay for it, but rather than doing so, we agree with the shopkeeper on the following method of payment. We repeatedly select vertices and make certain down payments on them. (The down payment we make each time may be less than the actual cost of the vertex, and we may make several payments on the same vertex.) The shopkeeper, in turn, does not want to be bothered with keeping track of all of our down payments, so he adopts the following strategy. Whenever we make a payment of ϵ on a vertex, he responds by simply decreasing the price of that vertex by ϵ.

More specifically, we conduct the business transaction in rounds. In each round we select an edge (u, v) whose two endpoints have non-zero cost, and make a down payment of $\epsilon = \min\{\text{price of } u, \text{price of } v\}$ on each of its endpoints. In response, the shopkeeper lowers the prices of u and v by ϵ each. With these new prices in effect, at least one of the edge's endpoints has zero cost. Thus after $O(|V|)$ rounds, prices will have dropped sufficiently for each edge to have at least one zero-cost endpoint. When that happens, the set of all zero-cost vertices is a vertex cover, and it is free. We take this set as our solution.

We formalize the above process as the following algorithm.

Algorithm VC

1. While there exists an edge (u, v) such that $\min\{w(u), w(v)\} > 0$:
2. Let $\epsilon = \min\{w(u), w(v)\}$.
3. $w(u) \leftarrow w(u) - \epsilon$.
4. $w(v) \leftarrow w(v) - \epsilon$.
5. Return the set $C = \{v \mid w(v) = 0\}$.

What is the cost of the solution in terms of the original weights, and how far is it from the optimal cost? Consider the ith round. Let (u_i, v_i) be the edge selected in this round, and let ϵ_i be the down payment made on each of its endpoints. Since every vertex cover must contain at least one of u_i and v_i (in order to cover the edge connecting them), decreasing both their prices by ϵ_i has the effect of lowering the optimal cost, denoted OPT, by at least ϵ_i. Thus, in the ith round we pay $2\epsilon_i$ and effect a drop of at least ϵ_i in OPT. Hence the *local ratio* between our payment and the drop in OPT is at most 2 in every round. It follows that the ratio between our total payment and the total drop in OPT, summed over all rounds, is at most 2. Clearly, the cost of the solution we find is fully covered by the sum of all down payments we make, and the total drop in the optimal cost is the original value of OPT (as evidenced by the fact that we find a vertex cover of zero cost with respect to the final weights). Thus the cost of our solution is at most $2OPT$.

Note that our approximation ratio analysis makes no reference to the actual value of ϵ in any given iteration. The choice of $\epsilon = \min\{w(u), w(v)\}$ is not compulsory; any $0 \le \epsilon \le \min\{w(u), w(v)\}$ would do just fine and would result in a factor 2 approximation. We use $\epsilon = \min\{w(u), w(v)\}$ for reasons of efficiency. This choice ensures that the number of vertices with positive cost decreases with each iteration, thus limiting the number of iterations to less than $|V|$. (Note that although a naive implementation of the algorithm runs in $\Theta(|V||E|)$ time, the algorithm can be made to run in linear time by performing a single sweep through the edges in arbitrary order.)

Another point worth noting is that our analysis of algorithm **VC** contains some slack in that it bounds the cost of the solution by the sum of *all* down payments, rather than by the sum of down payments made for vertices that are actually taken to the solution. It might seem that a more refined analysis could yield a better approximation ratio, but unfortunately this is not true for the *vertex cover* problem; one can easily construct examples in which all vertices for which down payments are made are eventually taken to the solution. Another source of slack is that in the final step, all zero-cost vertices are taken to the solution, and no attempt is made to prune it by removing unnecessary vertices. One obvious idea is to return a *minimal* (with respect to set inclusion) subset of C that is still a vertex cover. This idea fails too, as it is easy to construct worst-case examples in which C is minimal to begin with. As we shall see, though, these ideas are useful (indeed, necessary) in the context of other optimization problems.

1.2 Historical Highlights

The idea of bounding (or computing exactly) a numerical quantity by decomposing it into parts and bounding each by a more easily computable quantity is age-old, and has appeared early on in the history of approximation algorithms. Some examples include Garey et al. [1972] and Garey et al. [1976], who analyzed *bin packing* algorithms by decomposing the solution cost and comparing each component with a corresponding component of a suitably decomposed lower bound; Gavril (see Garey and Johnson [1979, pp. 134]), who obtained an approximation ratio of 2 for *unweighted vertex cover* by charging the two endpoints of an edge to the one endpoint that must be contained in the optimal solution; and Rosenkrantz et al. [1977], who used a similar charging idea in the context of the *traveling salesman* problem.

The origins of the local ratio technique can be traced back to a paper by Bar-Yehuda and Even [1981] on *vertex cover* and *set cover*. In this paper, the authors presented a generalization of Algorithm **VC** suited for *set cover*, and gave a primal-dual analysis of it. This linear time algorithm was motivated by a previous algorithm of Hochbaum [1982] which was based on LP duality[1] and required the solution of a linear program. Although Bar-Yehuda and Even's primal-dual analysis contains an implicit local ratio argument, the debut of the local ratio technique only occurred in a followup paper [Bar-Yehuda and Even 1985] several years later, where the authors presented a local ratio analysis of the same algorithm. They also developed a specialized $(2 - \frac{\log_2 \log_2 n}{2 \log_2 n})$-approximation algorithm for *vertex cover* based on local-ratio principles. More than a decade later, Bafna et al. [1999] devised a local ratio 2-approximation algorithm for the *feedback vertex set* problem. In this landmark paper, they incorporated the idea of *minimal solutions* into the local ratio technique. Subsequently, Fujito [1998] presented a unified local ratio approximation algorithm for node-deletion problems, and later still, Bar-Yehuda [2000] developed a generic local ratio algorithm that explained most local ratio and primal-dual approximation algorithms known at the time. At this point the local ratio technique had reached a certain level of maturity, but only in the context of minimization problems. No local ratio (or primal-dual) algorithms were known for any maximization problem. This changed in a paper of Bar-Noy et al. [2001], who presented the first local ratio algorithms for maximization problems. The key idea facilitating these algorithms was the use of *maximal* solutions rather than minimal ones. The most recent development, due to Bar-Yehuda et al. [2002], is a novel extension of the local ratio technique called *fractional local ratio*.

1.3 Connection to the Primal-Dual Schema

The *primal-dual* schema for approximation algorithms is a widely used LP based method for the design and analysis of approximation algorithms. (See Williamson [2002] for a good recent survey or Goemans and Williamson [1997] for an older one.) While it does not require the actual solution of linear programs, it does rely on LP duality and related concepts. It is frequently regarded as an outgrowth of the primal-dual method for the (exact) solution of linear programs.

It has often been observed that primal-dual algorithms have local ratio interpretations, and vice versa. Some notable examples include Bar-Yehuda and Even's primal-dual algorithm for *vertex cover* [Bar-Yehuda and Even 1981], which was later formulated in local ratio terms [Bar-Yehuda and Even 1985]; Bafna et al.'s local ratio algorithm for *feedback vertex set* [Bafna et al. 1999], which was later recast as a primal-dual algorithm [Chudak et al. 1998]; and Bar-Noy et al.'s approximation framework for several maximization scheduling problems [Bar-Noy et al. 2001], which was developed initially using the local-ratio approach, and then explained (in the same paper) in primal-dual terms. Thus, over the years there was a growing sense that the two seemingly distinct approaches share a common ground, but the exact nature of the connection between them remained unclear (see, e.g., [Williamson 2002], where this was posed as an open question). The issue was resolved only recently by Bar-Yehuda and Rawitz [2001]. They defined two abstract frameworks, one encompassing the primal-dual schema, and the other encompassing the local ratio technique, and showed that these two frameworks are

[1]LP stands for *linear programming.* The use of linear programming in the context of approximation algorithms is beyond the scope of this survey, as are the details of the *primal-dual* schema, which is an approximation paradigm based on linear programming concepts.

equivalent[2]. The equivalence is constructive, meaning that an algorithm formulated within one paradigm can be translated quite mechanically to the other paradigm.

1.4 Overview of this Survey

The survey is organized as follows. In Section 2 we define the basic notions that are used in the field of approximation algorithms. We also establish some terminology and notation to be used later in the survey. In Section 3 we begin the survey proper by presenting two introductory examples in the spirit of Algorithm **VC** above. In Section 4 we state and prove the Local Ratio Theorem (for minimization problems) and formulate the local ratio technique as a design and analysis framework based on it. In Section 5 we introduce the idea of minimal solutions into the framework, making it powerful enough to encompass many known approximation algorithms for covering problems (e.g., *feedback vertex set* and *network design*). In Section 6 we develop a local ratio framework for maximization problems, which is, in a sense, a mirror image of its minimization counterpart developed in Sections 4 and 5. We apply the framework to a few (interrelated) scheduling problems. In Section 7 we present several algorithms that deviate from the standard local ratio approach. In particular we show local ratio analyses of exact optimization algorithms (for polynomial-time solvable problems). In the final section of the survey, Section 8, we describe a recent development, namely, the fractional local ratio technique.

Since this write up is intended as a primer as well as a survey, it is written somewhat textbook style. We have removed nearly all citations and references from the running text, and instead have included at the end of each major section a subsection titled *Background*, in which we cite sources for the material covered in the section and discuss related work. The first such subsection, discussing *vertex cover*, appears next.

1.5 Background

The *vertex cover* problem is known to be NP-complete even for planar cubic graphs with unit weights [Garey et al. 1976]. Håstad [1997] proves a lower bound on the approximability of the problem. He shows, using PCP arguments, that *vertex cover* cannot be approximated within a factor of $\frac{7}{6}$ unless P=NP. Dinur and Safra [2002] improve this bound to $10\sqrt{5} - 21 \approx 1.36067$. The first 2-approximation algorithm for the weighted version of *vertex cover* is due to Nemhauser and Trotter [1975]. Hochbaum [1983] uses this algorithm to obtain an approximation algorithm with performance ratio $2 - \frac{2}{d_{\max}}$, where $d_{\max}$ is the maximum degree of a vertex. Gavril (see [Garey and Johnson 1979]) gives a linear time 2-approximation algorithm for the non-weighted case. (Algorithm **VC** reduces to this algorithm on non-weighted instances.) Hochbaum [1982] presents two approximation algorithms for *set cover*, both requiring the solution of a linear program. The first constructs a cover based on the optimal dual solution, whereas the second is a simple LP rounding algo-

[2] There are a small number of algorithms that are usually considered to be primal-dual algorithms, but which deviate in important respects from the mainstream primal-dual schema. Although these algorithms do not fit neatly in the abstract framework of Bar-Yehuda and Rawitz [2001], and hence have no clean local ratio counterparts, they do have non-LP interpretations in the local ratio spirit (see, e.g., Freund and Rawitz [2003]).

rithm. Her algorithms achieve an approximation guarantee of 2 on instances that are graphs (i.e., on *vertex cover* instances). Bar-Yehuda and Even [1981] present an LP based approximation algorithm for weighted *set cover* that does not solve a linear program directly. Rather, it constructs simultaneously a primal integral solution and a dual feasible solution without solving either the primal or dual programs. It is the first algorithm to operate in this method, a method which later became known as the *primal-dual schema*. Their algorithm reduces to Algorithm **VC** on instances that are graphs. In a subsequent paper, Bar-Yehuda and Even [1985] provide an alternative local ratio analysis for this algorithm, making it the first local ratio algorithm as well. They also present a specialized $(2 - \frac{\log_2 \log_2 n}{2 \log_2 n})$-approximation algorithm for *vertex cover*. Independently, Monien and Speckenmeyer [1985] achieved the same ratio for the unweighted case. Recently, Halperin [2000] improved this result to $2 - (1 - \mathrm{o}(1))\frac{2 \ln \ln d_{\max}}{\ln d_{\max}}$ using *semidefinite programming*.

2. DEFINITIONS AND NOTATION

2.1 Optimization Problems

An optimization problem is a family of problem *instances*, i.e., possible inputs. With each instance is associated a collection of *solutions*, each of which is either *feasible* or *infeasible*, and a *cost* function assigning a cost to each solution. Each optimization problem is classified as either a *maximization* problem or a *minimization* problem. For a given problem instance, an *optimal* solution is a feasible solution whose cost is optimal, which is to say that it is either minimal or maximal (depending, respectively, on whether the problem is one of minimization or maximization) among all feasible solutions. The cost of an optimal solution is referred to as the *optimum value*, or simply the *optimum*. For example, the familiar *minimum spanning tree* problem is a minimization problem in which the instances are edge-weighted graphs, solutions are subgraphs, feasible solutions are spanning trees, the cost of a given solution is the total weight of its edges, and optimal solutions are spanning trees of minimum total edge weight. Throughout this survey we use the terms *cost* and *weight* interchangeably.

Many optimization problems can be formulated as problems of selecting a subset (satisfying certain constraints) of a given set of weighted objects. For example, *minimum spanning tree* can be viewed as the problem of selecting a subset of the edges forming a spanning tree. In such problems we consider the cost function to be defined on the objects, and extend it to subsets in the natural manner. Nearly all of the problems we shall encounter in this survey are problems of this type.

2.2 Approximation Algorithms

An approximation algorithm for an optimization problem is an algorithm that takes an input instance and finds a feasible solution for it. The word *approximation* in the term *approximation algorithm* expresses the idea that our goal in the design of such an algorithm is to make it so that the algorithm will always return a feasible solution whose value approximates the optimum in some sense. Approximation algorithms are of interest chiefly in the context of NP-hard problems, where finding optimal solutions efficiently is not possible (unless P=NP), but finding approximate solutions efficiently quite often is.

The most popular measure of closeness to the optimum is *approximation ratio*, defined as follows. For $r \geq 1$, a feasible solution is said to be *r-approximate* if its cost is within a factor of r of the optimum. More specifically, let $w(X)$ be the cost of a given feasible solution X and let w^* be the optimum value. Then, in the case of minimization, X is said to be r-approximate if $w(X) \leq r \cdot w^*$, and in the case of maximization, it is said to be r-approximate if $w(X) \geq w^*/r$. (Note that for both types of problems the smaller r is, the closer X is to being optimal, and if $r = 1$, X is optimal.) In the context of approximation algorithms we only consider problems in which the optimum is non-negative for all input instances. An r-approximate solution is also called an *r-approximation*. An algorithm that always returns r-approximate solutions is said to achieve an *approximation factor* of r, and it is called an *r-approximation* algorithm. Also, r is said to be a *performance guarantee*, or an *approximation guarantee*, for it. The *approximation ratio* of a given algorithm is $\inf \{r \mid r$ is a performance guarantee for the algorithm$\}$. Nevertheless, the term *approximation ratio* is sometimes used as a synonym for *performance guarantee*.

2.3 Conventions and Notation

In the sequel we assume the following conventions, except where specified otherwise.

—All weights are non-negative and denoted by w. We denote by $w(x)$ the weight of element x, and by $w(X)$ the total weight of set X, i.e., $w(X) = \sum_{x \in X} w(x)$.

—We denote the optimum value for a given problem instance by OPT.

—Graphs are simple and undirected. A graph is denoted $G = (V, E)$. Whenever a digraph is specified, it is denoted $G = (N, A)$. We denote the number of vertices/nodes by n and the number of edges/arcs by m. We denote the degree of vertex v by $\deg(v)$.

—We denote by H_n the nth harmonic number, that is, $H_n = \sum_{i=1}^{n} \frac{1}{i}$.

3. TWO INTRODUCTORY EXAMPLES

In this section we demonstrate two easy applications of the local ratio technique. We describe approximation algorithms for the *prize collecting* version of *vertex cover* and for a certain *graph editing* problem. Our treatment of these problems is similar to the treatment of *vertex cover* in the introduction.

3.1 Prize Collecting Vertex Cover

The *prize collecting vertex cover* problem is a generalization of *vertex cover* in which we are not obligated to cover all edges, but must pay a penalty for those left uncovered. More specifically, both vertices and edges have non-negative weights; every set of vertices is a feasible solution; and the cost of a feasible solution is the total weight of its vertices plus the total weight of the edges it does not cover. Our algorithm for this problem is very similar to Algorithm **VC**.

Algorithm PCVC

1. While there exists an edge $e = (u, v)$ such that $\min\{w(u), w(v), w(e)\} > 0$:
2. Let $\epsilon = \min\{w(u), w(v), w(e)\}$.
3. $w(u) \leftarrow w(u) - \epsilon$.
4. $w(v) \leftarrow w(v) - \epsilon$.
5. $w(e) \leftarrow w(e) - \epsilon$.
6. Return the feasible solution $C = \{v \mid w(v) = 0\}$.

The analysis here is similar to that of *vertex cover*. First, observe that when the algorithm halts, every edge not covered by C has zero cost, and thus the cost of C is zero. Next, consider the ith iteration. Let $e_i = (u_i, v_i)$ be the edge involved, and let ϵ_i be the amount by which the weights are reduced. We pay $3\epsilon_i$ and effect a drop of at least ϵ_i in the optimal cost (since every feasible solution must either contain one (or both) of u_i and v_i, or else pay for the edge e_i). Thus, a factor of 3 is immediate. However, we can tighten the analysis by taking into account only payments that actually contribute towards covering the cost of C. Of the $3\epsilon_i$ we seem to pay, only $2\epsilon_i$ are actually "consumed," since we must either pay for e_i or for one or both of u_i and v_i, but never for all three. Thus the approximation ratio is 2.

We remark that our analysis still holds if we set $\epsilon = \min\{w(u), w(v), \frac{1}{2}w(e)\}$ in Line 2, and decrease $w(e)$ by 2ϵ in Line 5. The ratio of 2 remains intact because OPT still drops by at least ϵ and we still pay in actuality at most 2ϵ. In fact, we can set $\epsilon = \min\{w(u), w(v), \frac{1}{\alpha}w(e)\}$, and decrease $w(e)$ by $\alpha \cdot \epsilon$, for any $1 \leq \alpha \leq 2$.

3.2 Graph Editing

A *graph editing* problem asks to transform, by means of *editing operations*, a given graph into a new graph possessing some desired property. The editing operations we consider are:

Vertex deletion. Delete a vertex (and all incident edges).

Edge deletion. Delete an edge.

Non-edge addition. Connect two non-adjacent vertices by an edge. We refer to a pair of non-adjacent vertices as a *non-edge*, and to the act of connecting them by an edge as *adding* the corresponding non-edge to the graph.

The input consists of a graph with weights associated with its vertices, edges, and non-edges, collectively known as the *graph elements*. The cost of a transformation is defined as the total cost of the graph elements involved, that is, the total cost of edges removed (including edges removed as part of vertex deletion operations), vertices removed, and non-edges added. Different graph editing problems may impose restrictions on the graph-element weights or on the types of permitted editing operations. For example, the *vertex cover* problem is a graph editing problem in which all edge weights are zero, only vertex deletion operations are allowed, and the desired property of the target graph is *independent set*, i.e., it must contain no edges.

As another example, consider the *prize collecting vertex cover* problem. One might be tempted to think that it is a graph editing problem (for the *independent set* property) in which the permitted editing operations are vertex deletion and edge deletion. This is not the case, however, because in the graph editing framework, the cost of deleting a vertex is its weight plus the total weight of its incident edges, whereas in the context of *prize collecting vertex cover*, the cost of a vertex is only its own weight. Nevertheless, the *prize collecting vertex cover* problem can be viewed as a graph editing problem by means of the following reduction. Given a vertex-and-edge weighted graph G, construct the *complement graph* G', that is, the graph obtained from G by turning every edge into a non-edge and every non-edge into an edge. Define weights on the graph element of G' as follows. The weight of a vertex in G' is the same as its weight in G; the weight of a non-edge in G' is the same as the weight of the corresponding edge in G; the weights of all edges in G' are zero. It can be easily seen that the problem of finding a prize collecting vertex cover in G is equivalent to the graph editing problem on G' in which the desired property is *clique* (i.e., the target graph must be complete).

We consider the following graph editing problem. We are given a t-partite graph $G = (V, E)$ together with a partitioning of G into $t \geq 2$ (non-empty) sides $V = \bigcup_{i=1}^{t} V_i$, and a weight function w on G's elements. The objective is to find a minimum cost transformation yielding a complete t'-partite graph, for some $t' \leq t$, whose sides are subsets of the sides of G. We stress that t' is not part of the input, but rather is defined by the target graph. We also consider a special case of this problem in which non-edge additions are not allowed. (Or, in other words, $w(e) = \infty$ for any non-edge e.) In the remainder of this section we develop an approximation algorithm for the above mentioned editing problem and its special case with approximation guarantees of 3 and 2, respectively.

To analyze the problem, let us first introduce some terminology. We define the *extended weight* of a vertex u, denoted $\overline{w}(u)$, as the sum of $w(u)$ and the total weight of edges incident on u. We say that a non-edge $e = (u, v)$ is *costly* if $\min\{\overline{w}(u), \overline{w}(v), w(e)\} > 0$. Finally, we use the term *non-edge* as an abbreviation of *non-edge between two vertices belonging to different sides of the partition.*

Consider a non-edge (u, v). If we delete u or v from the graph, we will not be able to add (u, v). On the other hand, if we leave them both in the graph, we will be forced to add (u, v) (since the target graph must be complete t'-partite). Similarly, if (u, v) is an edge, it will be deleted if and only if at least one of its endpoints is deleted. Thus, every feasible solution is completely defined by the set of vertices which it removes from the graph, and, conversely, every set of vertices defines a feasible solution in a similar manner. We can therefore identify feasible solutions with sets of vertices.

OBSERVATION 1. *The optimal cost is* 0 *if and only if the graph contains no costly non-edges, and if that is the case, the set* $\{v \,|\, \overline{w}(v) = 0\}$ *is an optimal solution.*

The following algorithm uses a similar idea to the one behind Algorithm **PCVC**.

Algorithm (t,t)-Graph Editing

1. While there exists a costly non-edge $e = (u, v)$:
2. Let $\epsilon = \min\{\overline{w}(u), \overline{w}(v), w(e)\}$.
3. $\overline{w}(u) \leftarrow \overline{w}(u) - \epsilon$.
4. $\overline{w}(v) \leftarrow \overline{w}(v) - \epsilon$.
5. $w(e) \leftarrow w(e) - \epsilon$.
6. Return the feasible solution $\{v \mid \overline{w}(v) = 0\}$.

There is a subtle point regarding the implementation of Lines 3 and 4. Recall that the extend weight $\overline{w}(x)$ of vertex x is defined as the sum of $w(x)$ and the total weight of edges incident on x. When we say "$\overline{w}(x) \leftarrow \overline{w}(x) - \epsilon$" we actually mean "decrease $w(x)$ and the weights of the edges incident on x by some non-negative (but not necessarily equal) amounts, such that $\overline{w}(x)$ drops by ϵ as a result." We do not care exactly how much is subtracted from the weight of each of the graph elements involved, only that the total decrease amounts to ϵ and the weights of the graph elements remain non-negative. Note that the effect of this is not only to decrease $\overline{w}(x)$ by ϵ, but possibly to also decrease the extended weight of some of x's neighbors (as a result of decreasing the weights of the edges connecting them to x). The algorithm must keep track of these changes. Regarding this point, note that u and v (in Lines 3 and 4) are non-adjacent, so decreasing the extended weight of one of them has no effect on the extended weight of the other.

The analysis is similar to the analysis of Algorithm **PCVC**. In the ith iteration we pay $3\epsilon_i$ and manage to lower the optimal cost by at least ϵ_i (because every feasible solution must either delete at least one of the two vertices, or else add the non-edge). Thus, the solution returned is 3-approximate. Note that it is possible to be forced to actually pay nearly $3\epsilon_i$ if the weights of u_i and v_i are zero and each is adjacent to an edge contributing only negligibly to the decrease in its extended weight. In such a case it might happen that these two edges will remain in the graph, all other edges incident on u_i or v_i will be deleted by the removal of neighboring vertices, and the non-edge (u_i, v_i) will be added. As a result, only the negligible down payments made on the two surviving edges will be recoverable.

In the special case where non-edge additions are prohibited we only pay $2\epsilon_i$, yielding an approximation guarantee of 2.

3.3 Background

Prize collecting vertex cover. The *prize collecting vertex cover* problem (also known as *generalized vertex cover*) was introduced and studied by Hochbaum [2002], who presented an $O(nm \log \frac{n^2}{m})$-time 2-approximation algorithm. Algorithm **PCVC** shown here is due to Bar-Yehuda and Rawitz [2001]. The lower bounds of Håstad [1997] and Dinur and Safra [2002] for *vertex cover* apply to the prize collecting version because the plain *vertex cover* problem is a special case of the prize collecting version (where edge costs are infinite).

Graph editing. The *graph editing* problem discussed in this section has its roots in a number of node and edge deletion problems considered by Hochbaum [1998]. She approximates these problems by reducing them to max-flow problems. Bar-

Yehuda and Rawitz [2002] generalize the problems and use the local ratio technique to approximate them. They consider several variants of the following optimization problem: given a graph and a non-negative weight function on the vertices and edges, find a minimum weight set of vertices and edges whose removal from the graph leaves a complete k-partite graph. The editing problem discussed in this section generalizes one of these variants.

4. THE LOCAL RATIO THEOREM

As we have seen, the local ratio technique is based on the idea of paying in each iteration at most $r \cdot \epsilon$, for some r, in order to achieve a reduction of at least ϵ in OPT. If the same r is used in all iterations, the solution returned is r-approximate. This idea has a very intuitive appeal, and works well in the examples we have encountered in the previous section. However, a closer look at these examples reveals that the idea worked mainly because we were able to make a very localized payment and argue that OPT must drop by a proportional amount because every solution must involve some of the items we paid for. For example, in *vertex cover* the payment was localized to a single edge; we paid ϵ for each of its two endpoints and argued that OPT must drop by at least ϵ because every vertex cover must include at least one of them. This localization of the payments is at the root of the simplicity and elegance of the analysis, but it is also a source of weakness: how can we deal with problems in which no single set of items is necessarily involved in every optimal solution? Consider for example the *feedback vertex set* problem, in which we are given a vertex-weighted graph and are asked to remove a minimum weight set of vertices such that the remaining graph will contain no cycles. As we shall see, there is a 2-approximation local ratio algorithm for this problem, yet surely it is not always possible to find two vertices such that at least one of them is part of every optimal solution! The Local Ratio Theorem, which we present below, allows us to go beyond localized payments by shifting the focus of attention from the weight function itself to the *changes* in the weight function, and treating these changes as weight functions in their own right. A consequence of this is that algorithms based on the theorem are more readily formulated as recursive algorithms, rather than iterative ones. This, in turn, leads to the idea of concentrating on minimal solutions in a very natural manner. By *minimal* we mean minimal with respect to set inclusion. Such solutions are an essential ingredient in many applications of the technique. We explain and demonstrate all this in this section and subsequent ones.

The Local Ratio Theorem is deceptively simple[3]. It applies to optimization problems that can be formulated as follows.

> Given a *weight vector* $w \in \mathbb{R}^n$ and a set of *feasibility constraints* $\mathcal{C}$, find a *solution vector* $x \in \mathbb{R}^n$ satisfying the constraints in $\mathcal{C}$ that minimizes the scalar product $w \cdot x$.

(For convenience we deal in this section with minimization problems only. In Section 6 we discuss maximization problems.)

[3]Indeed, some authors refer to it as the Local Ratio *Lemma*. We prefer *Theorem* in light of its fundamental role.

The most common type of optimization problem that can be formulated as above consists of instances in which the input contains a set I of n weighted elements and a specification of feasibility constraints on subsets of I. Feasible solutions are subsets of I satisfying the feasibility constraints. The cost of a feasible solution is the total weight of the elements it comprises. In terms of the above formulation, the weight function on the elements is expressed by the weight vector (where the ith component of the vector is the weight of the ith element), and subsets are expressed as 0–1 vectors (where the ith component of the vector is 1 if the ith element is in the subset, and is 0 otherwise). The scalar product $w \cdot x$ is then the weight of the subset corresponding to x.

Theorem 2 Local Ratio—Minimization Problems. *Let $\mathcal{C}$ be a set of feasibility constraints on vectors in $\mathbb{R}^n$. Let $w, w_1, w_2 \in \mathbb{R}^n$ be such that $w = w_1 + w_2$. Let $x \in \mathbb{R}^n$ be a feasible solution (with respect to $\mathcal{C}$) that is r-approximate with respect to w_1 and with respect to w_2. Then, x is r-approximate with respect to w as well.*

Proof. Let x^*, x_1^*, and x_2^* be optimal solutions with respect to w, w_1, and w_2, respectively. Clearly, $w_1 \cdot x_1^* \leq w_1 \cdot x^*$ and $w_2 \cdot x_2^* \leq w_2 \cdot x^*$. Thus, $w \cdot x = w_1 \cdot x + w_2 \cdot x \leq r(w_1 \cdot x_1^*) + r(w_2 \cdot x_2^*) \leq r(w_1 \cdot x^*) + r(w_2 \cdot x^*) = r(w \cdot x^*)$. $\square$

(Note that the theorem holds even when negative weights are allowed.)

The Local Ratio Theorem leads naturally to the formulation of recursive algorithms with the following general structure.

(1) If a zero-cost solution can be found, return one.

(2) Otherwise, find a suitable decomposition of w into two weight functions w_1 and $w_2 = w - w_1$, and solve the problem recursively using w_2 as the weight function in the recursive call.

We demonstrate this on *vertex cover*.

Algorithm RecursiveVC(G,w)

1. If $\min\{w(u), w(v)\} = 0$ for all edges (u, v):
2. Return the set $\{v \mid w(v) = 0\}$.
3. Else:
4. Let (u, v) be an edge such that $\epsilon \triangleq \min\{w(u), w(v)\} > 0$.
5. Define $w_1(x) = \begin{cases} \epsilon & x = u \text{ or } x = v, \\ 0 & \text{otherwise,} \end{cases}$
   and define $w_2 = w - w_1$.
6. Return **RecursiveVC(G,w_2)**.

Algorithm **RecursiveVC** is clearly just a recursive formulation of Algorithm **VC**. However, the recursive formulation is amenable to direct analysis based on the Local Ratio Theorem. We prove that the solution returned by the algorithm is 2-approximate by induction on the recursion. In the base case, the algorithm returns a vertex cover of zero cost, which is optimal. For the inductive step, consider the solution returned by the recursive call. By the inductive hypothesis it is 2-approximate with respect to w_2. We claim that it is also 2-approximate with respect to w_1. In fact, we claim that *every* feasible solution is 2-approximate with respect to w_1. To see this, observe that the cost (with respect to w_1) of every vertex cover is at most 2ϵ, since only u and v have non-zero cost and they each cost ϵ. On the other hand, the minimum cost of a cover is at least ϵ, since every vertex cover must include at least one of u and v in order to cover the edge (u, v). Thus, by the Local Ratio Theorem the solution returned is 2-approximate with respect to w.

Note that different algorithms (for different problems) conforming to the general structure outlined above differ from one another only in the decomposition of w, and this decomposition is determined completely by the choice of w_1. (Actually, they might also differ in the way they search for a zero-cost solution, but most often the only candidate that needs to be examined is the set of all zero-cost elements.) Accordingly, such algorithms also share most of their analyses. Specifically, the proof that a given algorithm is an r-approximation algorithm is by induction on the recursion. In the base case the solution is optimal (and therefore r-approximate) because it has zero cost, and in the inductive step, the solution returned by the recursive call is r-approximate with respect to w_2 by the inductive hypothesis. Thus, different algorithms differ from one another only in the choice of w_1 and in the proof that every feasible solution is r-approximate with respect to w_1. (Obviously, they may also differ in the value of the approximation ratio r.) The following definition formalizes this notion.

DEFINITION 1. *Given a set of constraints $\mathcal{C}$ on vectors in $\mathbb{R}^n$ and a number $r \geq 1$, a weight vector $w \in \mathbb{R}^n$ is said to be* fully r-effective *if there exists a number b such that $b \leq w \cdot x \leq r \cdot b$ for all feasible solutions x.*

The analysis of algorithms in our framework boils down to proving that w_1 is r-effective. Proving this amounts to proving that b is a lower bound on the optimum value and $r \cdot b$ is an upper bound on the cost of every feasible solution, and thus every feasible solution is r-approximate (all with respect to w_1). For this reason we will henceforth focus solely on w_1, and neglect to mention the remaining details explicitly.

In the remainder of this section we demonstrate the above framework on two problems: *hitting set* and *feedback vertex set in tournaments*. We remark that these algorithms can be formulated just as easily in terms of localized payments; the true power of the Local Ratio Theorem will become apparent in the next section.

4.1 Hitting Set

The input in a *hitting set* instance is a collection of non-empty sets $\mathcal{C} = \{S_1, \ldots, S_m\}$ and a weight function w on the sets' elements $U = \bigcup_{i=1}^{m} S_i$. An element $x \in U$ is said to *hit* a given set $S_i \in \mathcal{C}$ if $x \in S_i$. A subset $H \subseteq U$ is said to *hit* a given set $S_i \in \mathcal{C}$ if $H \cap S_i \neq \emptyset$. The objective is to find a minimum-cost subset of U that hits all sets $S_i \in \mathcal{C}$.

Let $s_{\max} = \max_{1 \leq i \leq m} |S_i|$. The following decomposition of w achieves an approximation factor of $s_{\max}$. Let i be any index such that S_i contains no zero-weight elements, and let $\epsilon = \min\{w(x) \mid x \in S_i\}$. Define:

$$w_1(x) = \begin{cases} \epsilon & x \in S_i, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that w_1 is $s_{\max}$-effective. Clearly, the cost of every feasible solution is bounded by $\epsilon \cdot |S_i| \leq \epsilon \cdot s_{\max}$. On the other hand, every feasible solution must hit S_i, i.e., must contain at least one element of S_i. Thus every feasible solution costs at least ϵ.

Remarks. The *hitting set* problem generalizes many covering problems. For example, *vertex cover* can be seen as a *hitting set* problem in which the sets are the edges and the elements are the vertices. Indeed, Algorithm **RecursiveVC** is just a special case of the algorithm for *hitting set* we have just described. (Note that $s_{\max} = 2$ for the formulation of *vertex cover* in terms of *hitting set.*)

We also wish to comment on the relation between *hitting set* and *set cover*. In the *set cover* problem we are given a collection of sets $\mathcal{C} = \{S_1, \ldots, S_m\}$, as in *hitting set*, but this time the weight function is on the sets, not on the elements. The objective is to find a minimum-cost collection of sets that "covers" all elements. In other words, the union of the sets in the solution must be equal to $\bigcup_{i=1}^{m} S_i$. It is well known that *set cover* and *hitting set* are equivalent problems in the sense that each is obtained from the other by switching the roles of sets and elements. In *set cover* terms, the approximation ratio obtained by the above algorithm is the maximum *degree* of an element, where the degree of an element is the number of sets containing it as a member.

4.2 Feedback Vertex Set in Tournaments

A *tournament* is an orientation of a complete (undirected) graph, i.e., it is a directed graph with the property that for every unordered pair of distinct nodes $\{u, v\}$ it either contains the arc (u, v) or the arc (v, u), but not both. The *feedback vertex set in tournaments* problem is the following. Given a tournament and a weight function w on its nodes, find a minimum-weight set of nodes whose removal leaves a graph containing no directed cycles.

An immediate observation, expressed in the following lemma, is that we may restrict our attention to cycles of length 3. (Note that the smallest possible cycle length in a tournament is 3.) We refer to the set of (three) nodes on a directed cycle of length 3 as a *triangle*. We say that a triangle is *positive* if all of its nodes have strictly positive weights.

LEMMA 3. *A tournament contains a directed cycle if and only if it contains a triangle.*

PROOF. The existence of a triangle implies the existence of a directed cycle by definition. The existence of a directed cycle implies the existence of a triangle by the following argument. Suppose the minimum length of a directed cycle in the tournament is $k \geq 3$. If $k = 3$, we are done. Otherwise, the tournament contains some directed cycle $v_1, v_2, v_3, \ldots, v_k, v_1$. If the tournament contains the arc (v_3, v_1), then it contains the directed cycle v_1, v_2, v_3, v_1 of length 3. Otherwise, it contains the arc (v_1, v_3) and therefore it contains the directed cycle $v_3, \ldots, v_k, v_1, v_3$ of length $k-1$. In either case we reach a contradiction with the minimality of k. $\square$

Lemma 3 implies that the set of all zero-cost nodes is an optimal solution (of zero cost) if and only if the tournament contains no positive triangles. Thus we obtain a 3-approximation algorithm by means of the following fully 3-effective weight function. Let $\{v_1, v_2, v_3\}$ be a positive triangle, and let $\epsilon = \min\{w(v_1), w(v_2), w(v_3)\}$. Define:

$$w_1(v) = \begin{cases} \epsilon & v \in \{v_1, v_2, v_3\}, \\ 0 & \text{otherwise.} \end{cases}$$

The maximum cost, with respect to w_1, of a feasible solution is clearly at most 3ϵ, while the minimum cost is at least ϵ, since every feasible solution must contain at least on of v_1, v_2, v_3.

Cai et al. [2001] describe a 2.5-approximation algorithm. They present an algorithm that finds an *optimal* solution in any tournament T that does not contain *forbidden* sub-tournaments, i.e. sub-tournaments of the forms shown in Figure 1 (where the two arcs not shown in T_1 may take any direction). They use this algorithm to obtain a 2.5-approximation algorithm for the general case, employing the following fully 2.5-effective weight function. Let F be a subset of five positive-weight vertices inducing a forbidden sub-tournament and let $\epsilon = \min\{w(v) \mid v \in F\}$. Define:

$$w_1(v) = \begin{cases} \epsilon & v \in F, \\ 0 & \text{otherwise.} \end{cases}$$

This function is fully 2.5-effective since the cost of every feasible solution is clearly at most 5ϵ, whereas the minimum cost is at least 2ϵ (as every four vertices in F contain a triangle). At the basis of the recursion—the case where every forbidden sub-tournament in T contains at least one zero-weight vertex—an optimal solution is found by removing the set Z of all zero-weight vertices (thereby eliminating all forbidden sub-tournaments), solving the problem optimally on the remaining graph, and adding Z to the solution found.

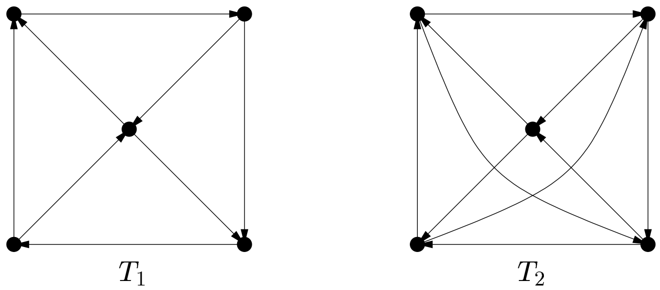

Fig. 1. Forbidden sub-tournaments.

4.3 Background

Hitting set. For the unweighted *hitting set* (or equivalently *set cover*) problem, Johnson [1974] and Lovász [1975] show that the greedy algorithm is an $H_{d_{\max}}$-approximation algorithm, where $d_{\max}$ is the maximum degree of an element. Chvátal [1979] generalizes this result to the weighted case. His analysis has a *dual fitting* interpretation (see Jain et al. [2002]). Hochbaum [1982] gives two $s_{\max}$-approximation algorithms, both of which are based on solving a linear program. Bar-Yehuda and Even [1981] suggest a linear time primal-dual $s_{\max}$-approximation algorithm. In subsequent work [Bar-Yehuda and Even 1985], they present the Local Ratio Theorem and provide a local ratio analysis of the same algorithm. (Their analysis is the one given in this section.) Feige [1996] proves a lower bound of $(1 - \mathrm{o}(1)) \ln m$ (unless NP$\subseteq$DTIME$(n^{\mathrm{O}(\log \log n)})$).
Feedback vertex set in tournaments. The 2.5-approximation algorithm for *feedback vertex set in tournaments* discussed in this section is due to Cai et al. [2001].

5. MINIMAL SOLUTIONS

In the previous section we introduced the idea of weight decomposition as a means of going beyond localized payments, but the examples we gave of its use did not in fact depart from that paradigm. All we did was translate the "payment" arguments into weight decomposition language. In each case we identified a small subset of elements such that every feasible solution had to contain at least one of them, and by associating a weight of ϵ with each of these elements and a weight of zero with all others, we were able to obtain an approximation ratio bounded by the size of the subset. There are many problems, though, where identifying such a small subset is impossible, simply because no such subset necessarily exists.

A case in point is the *partial hitting set* problem. This problem is similar to *hitting set* except that not all sets need to be hit. More specifically, the input consists of a collection of sets, a weight function on the sets' elements, and a number k. The objective is to find a minimum-cost subset of elements that hits at least k of the sets. The crucial difference between *hitting set* and *partial hitting set* is that in the latter, there is no single set that must be hit by all feasible solutions. Recall that the algorithm for *hitting set* picked some set S_i and associated a weight of ϵ with each of its elements. The analysis was based on the fact that $\epsilon \cdot |S_i|$ is an upper bound on the cost of every feasible solution, while ϵ is a lower bound since S_i must be hit. This approach fails for *partial hitting set* because ϵ is no longer a lower bound—an optimal solution need not necessarily hit S_i. Thus if we use the above weight function, we will end up with a solution whose weight with respect to w_1 is positive, while the optimum value may be equal to 0.

Our method of handling such situations is a logical extension of the same upper-bound/lower-bound idea. Roughly speaking, we do not know of any single subset that must contribute to all solutions, but the set of *all* elements must surely do so (otherwise, the empty set is feasible and optimal). Thus, to prevent OPT from being equal to 0, we can assign a positive weight to every element. This takes care of the lower bound but opens up the question of how to obtain a non-trivial upper bound—clearly, it is almost never the case that the cost of every feasible solution is within some reasonable factor of the cost of a single element. It turns out that the simple idea of considering only *minimal* solutions works well in many situations. By *minimal solution* we mean a feasible solution that is minimal with respect to set inclusion, i.e., a feasible solution whose proper subsets are all infeasible. Minimal solutions are meaningful mainly in the context of *covering* problems. (In our context, *covering* problems are problems for which feasible solutions are monotone inclusion-wise, i.e., if a set X is a feasible solution, then so is every superset of X. For example, adding a vertex to a vertex cover yields a vertex cover, so *vertex cover* is a covering problem. In contrast, adding an edge to a spanning tree does not yield a tree, so *minimum spanning tree* is not a covering problem.) The idea of focusing on minimal solutions leads to the following definition, which is an adaptation of Definition 1 to minimal solutions.

DEFINITION 2. *Given a set of constraints $\mathcal{C}$ on vectors in $\mathbb{R}^n$ and a number $r \geq 1$, a weight vector $w \in \mathbb{R}^n$ is said to be* r-effective *if there exists a number b such that $b \leq w \cdot x \leq r \cdot b$ for all minimal feasible solutions x.*

If we can show that our algorithm uses an r-effective w_1 and returns minimal solutions, we will have essentially proved that it is an r-approximation algorithm. Designing an algorithm to return minimal solutions is quite easy. Most of the creative effort is therefore expended in finding an r-effective weight function (for a small r).

In this section we present local ratio algorithms for the problems of *partial hitting set*, *network design*, and *feedback vertex set*, algorithms that depend on obtaining minimal solutions. We describe and analyze the first of these algorithms in full detail. We then outline the general framework of local ratio algorithms obtaining minimal solutions, and discuss the subsequent algorithms informally with reference to this framework.

5.1 Partial Hitting Set

In the *partial hitting set* problem, the input consists of a collection of non-empty sets $\mathcal{C} = \{S_1, \ldots, S_m\}$, a weight function w on the sets' elements $U = \bigcup_{i=1}^{m} S_i$, and a number k. The objective is to find a minimum-cost subset of U that hits at least k sets in $\mathcal{C}$. We assume that a feasible solution exists, i.e., $k \leq m$.

The *partial hitting set* problem generalizes *hitting set*, and as we have remarked in Section 4.1, *hitting set* and *set cover* are equivalent problems. For this reason, the *partial hitting set* problem is often described in the literature in *set cover* terms and is referred to as *partial covering*. We prefer the *partial hitting set* formulation here in order to highlight the similarities and differences between our treatment of this problem and the *hitting set* problem.

Given an instance of *partial hitting set*, let $\mathcal{S}(x) = \{S_i \in \mathcal{C} \mid x \in S_i\}$ for $x \in U$. Define the *degree* of an element $x \in U$ as $d(x) = |\mathcal{S}(x)|$. We present a $\max\{s_{\max}, 2\}$-approximation algorithm. (Recall the definition $s_{\max} = \max_{1 \leq i \leq m} |S_i|$.)

Algorithm PHS$(\mathcal{C}, w, k)$

1. If $k \leq 0$, return $\emptyset$.
2. Else, if there exists an element $x \in U$ such that $w(x) = 0$ do:
3. $H' \leftarrow$ **PHS**$(\mathcal{C} \setminus \mathcal{S}(x), w, k - d(x))$.
4.   If H' hits at least k sets in $\mathcal{C}$:
5.   Return the solution $H = H'$.
6.   Else:
7.   Return the solution $H = H' \cup \{x\}$.
8. Else:
9.   Let ϵ be maximal such that $\epsilon \cdot \min\{d(x), k\} \leq w(x)$ for all $x \in U$.
10.   Define the weight functions $w_1(x) = \epsilon \cdot \min\{d(x), k\}$ and $w_2 = w - w_1$.
11.   Return **PHS**$(\mathcal{C}, w_2, k)$.

Note the slight abuse of notation in Line 3. The weight function in the recursive call is not w itself, but rather the restriction of w to $\bigcup(\mathcal{C} \setminus \mathcal{S}(x))$. We will continue to silently abuse notation in this manner.

Let us analyze the algorithm. We claim that the algorithm finds a minimal solution that is $\max\{s_{\max}, 2\}$-approximate. Intuitively, Lines 2–7 ensure that the solution returned is minimal, while the weight decomposition in Lines 8–11 ensures that every minimal solution is $\max\{s_{\max}, 2\}$-approximate. This is done by associating with each element x a weight that is proportional to its "covering power," which is the number of sets it hits, but not more than k, since hitting more than k sets is no better than hitting k sets.

Proposition 4. *Algorithm **PHS** returns a feasible minimal solution.*

Proof. The proof is by induction on the recursion. At the recursion basis the solution returned is the empty set, which is both feasible (since $k \leq 0$) and minimal. For the inductive step, $k > 0$ and there are two cases to consider. If Lines 9–11 are executed, then the solution returned is feasible and minimal by the inductive hypothesis. Otherwise, Lines 3–7 are executed. By the inductive hypothesis H' is minimal and feasible with respect to $(\mathcal{C} \setminus \mathcal{S}(x), k - d(x))$. If $H' = \emptyset$ then $d(x) \geq k$ and $H = H' \cup \{x\}$ is clearly feasible and minimal. Otherwise, H' hits at least $k - d(x)$ sets in $\mathcal{C}$ that do not contain x, and by minimality, for all $y \in H'$, the set $H' \setminus \{y\}$ hits less than $k - d(x)$ sets that do not contain x. (Note that $H' \neq \emptyset$ implies $k > d(x)$.) Consider the solution H. Either $H = H'$, which is the case if H' hits at least k or more sets in $\mathcal{C}$, or else $H = H' \cup \{x\}$, in which case H hits at least $k - d(x)$ sets that do not contain x and additional $d(x)$ sets that do contain x. In either case H hits at least k sets and is therefore feasible. It is also minimal (in either case), since for all $y \in H$, if $y \neq x$, then $H \setminus \{y\}$ hits less than $k - d(x)$ sets that do not contain x and at most $d(x)$ sets that do contain x, for a total of less than k sets, and if $y = x$, then $x \in H$, which implies $H = H' \cup \{x\}$, which is only possible if $H \setminus \{y\} = H'$ hits less than k sets. $\square$

Proposition 5. *The weight function* w_1 *used in Algorithm* ***PHS*** *is* $\max\{s_{max}, 2\}$*-effective.*

Proof. In terms of w_1, every feasible solution costs at least $\epsilon \cdot k$, since it either contains an element whose cost is $\epsilon \cdot k$, or else consists solely of elements with degree less than k, in which case the cost of the solution equals ϵ times the total degree of elements in the solution, which must be at least k in order to hit k sets. To prove that every minimal solution costs at most $\epsilon \cdot k \cdot \max\{s_{\max}, 2\}$, consider a non-empty minimal feasible solution H. If H is a singleton, its cost is at most $\epsilon \cdot k$, and the claim follows. Otherwise, we prove the claim by showing that $\sum_{x \in H} d(x) \leq k \cdot \max\{s_{\max}, 2\}$. We say that set S_i is *hit* r *times* by H if $|S_i \cap H| = r$. We bound the total number of times sets are hit by H, since this number is equal to $\sum_{x \in H} d(x)$. Clearly every set S_i may be hit at most $|S_i| \leq s_{\max}$ times. Thus, if t is the number of sets that are hit by H twice or more, these sets contribute at most $t \cdot s_{\max}$ to the count. The sets that are hit only once contribute exactly $\sum_{x \in H} d'(x)$, where $d'(x)$ is the number of sets hit by x but not by any other member of H. Let $x^* = \arg\min\{d'(x) \,|\, x \in H\}$. Then (by the choice of x^* and the fact that H is not a singleton) $d'(x^*) \leq \sum_{x \in H \setminus \{x^*\}} d'(x)$. In addition, $t + \sum_{x \in H \setminus \{x^*\}} d'(x) < k$ by the minimality of H. Thus the sets that are hit only once contribute $d'(x^*) + \sum_{x \in H \setminus \{x^*\}} d'(x) \leq 2 \sum_{x \in H \setminus \{x^*\}} d'(x) < 2(k - t)$, and the total is less than $t \cdot s_{\max} + 2(k - t) \leq k \cdot \max\{s_{\max}, 2\}$. □

Theorem 6. *Algorithm* ***PHS*** *returns* $\max\{s_{max}, 2\}$*-approximate solutions.*

Proof. The proof is by induction on the recursion. In the base case the solution returned is the empty set, which is optimal. For the inductive step, if Lines 3–7 are executed, then H' is $\max\{s_{\max}, 2\}$-approximate with respect to $(\mathcal{C} \setminus \mathcal{S}(x), w, k - d(x))$ by the inductive hypothesis. Since $w(x) = 0$, the cost of H equals that of H' and the optimum value for $(\mathcal{C}, w, k)$ cannot be smaller than the optimum value for $(\mathcal{C} \setminus \mathcal{S}(x), w, k - d(x))$ because if H^* is an optimal solution for $(\mathcal{C}, w, k)$, then $H^* \setminus \{x\}$ is a feasible solution of the same cost for $(\mathcal{C} \setminus \mathcal{S}(x), w, k - d(x))$. If, on the other hand, Lines 9–11 are executed, then by the inductive hypothesis the solution returned is $\max\{s_{\max}, 2\}$-approximate with respect to w_2, and by Proposition 5 it is also $\max\{s_{\max}, 2\}$-approximate with respect to w_1. Thus by the Local Ratio Theorem it is $\max\{s_{\max}, 2\}$-approximate with respect to w as well. □

5.2 A Framework

The general structure of a local ratio algorithm obtaining minimal solutions consists of a three-way *if* condition that directs execution to one of the following three options: *optimal solution*, *problem size reduction*, and *weight decomposition*. The top-level description of the algorithm is:

(1) If a zero-cost minimal solution can be found, do: *optimal solution*.
(2) Otherwise, if the problem contains a zero-cost element, do: *problem size reduction*.
(3) Otherwise, do: *weight decomposition*.

The three types of steps are:

Optimal solution. Find a zero-cost minimal solution and return it. (Typically, the solution will simply be the empty set.) This is the recursion basis.

Problem size reduction. This option consists of three steps.

(1) Reduce the problem size by deleting a zero-cost element. This changes the problem instance, and may entail further changes (to achieve consistency with the idea that the element has potentially been taken to the solution). For example, in the *partial hitting set* problem we select a zero-cost element and delete all sets containing it and reduce the hitting requirement parameter k by the number of these sets. Sometimes the problem instance must be adjusted more radically to reflect the deletion of the element. For example, a graph edge might be eliminated by *contracting* it. As a result, two vertices get "fused" together and the edges incident on them undergo changes and possibly fusions. In other words, the modification consists of eliminating some existing elements and introducing new ones. This, in turn, requires that the weight function be extended to cover the new elements. It is important to realize, though, that the adjustment of the weight function amounts to a re-interpretation of the existing weights in terms of the new instance and not to an actual change of weights.
(2) Solve the problem recursively on the new instance.
(3) If the solution returned (when re-interpreted in terms of the original instance) is feasible (for the original instance), return it. Otherwise, extend it by adding the deleted zero-cost element, and return this solution.

Weight decomposition. Find an r-effective weight function w_1 such that $w - w_1$ is non-negative, and solve the problem recursively using $w - w_1$ as the weight function in the recursive call. Return the solution obtained.

The above formulation of the generic algorithm should not be taken too literally. Each branch of the three-way *if* statement may actually consist of several sub-cases, only one of which is to be executed. For example, Line 2 might hypothetically represent something like: *If there is a vertex of degree less than 2, take appropriate action. Otherwise, if there is a zero-weight vertex, take appropriate action.* In both cases, *appropriate action* would fall in the category of *problem size reduction*. We shall see examples of this in the sequel.

The analysis of an algorithm in the above framework follows the pattern of our analysis for *partial hitting set*. It consists of proving the following three claims.

(1) The algorithm returns a minimal feasible solution.
(2) The weight function w_1 is r-effective.
(3) The algorithm returns an r-approximate solution.

The proof of the first claim is by induction on the recursion. In the base case the solution is feasible and minimal by design. For the inductive step, if the algorithm performs *weight decomposition*, then the solution is feasible and minimal by the inductive hypothesis. If the algorithm performs *problem size reduction*, the claim follows intuitively from the fact that the solution returned by the recursive call is feasible and minimal with respect to the modified instance (by the inductive hypothesis) and it is extended only if it is infeasible with respect to the original instance. Although the argument is straightforward, the details of a rigorous proof tend to be slightly messy, as they depend heavily on the precise manner in which the instance is modified.

The proof of the second claim follows from the combinatorial structure of the problem. There is no fixed pattern here. Indeed, the key to the design of a local ratio algorithm is understanding the combinatorial properties of the problem under study and finding the right r and an r-effective weight function.

The proof of the third claim is also by induction on the recursion, based on the first two claims. In the *optimal solution* case it is true by design. In the *problem size reduction* case, the solution found recursively for the modified instance is r-approximate by the inductive hypothesis, and it has the same cost as the solution generated for the original instance since the two solutions may only differ by a zero-weight element. This, combined with the fact that OPT can only decrease as a result of the instance modification, yields the claim. (Again, the details of a rigorous proof tend to be somewhat messy.) In the *weight decomposition* case, the claim follows by the inductive hypothesis and the Local Ratio Theorem, based on the fact that the solution is feasible and minimal, and that w_1 is r-effective.

In the next two sections we describe algorithms for the *network design* and *feedback vertex set* problems in the context of this framework.

5.3 Network Design

The *network design* problem is defined as follows. Given an edge weighted graph $G = (V, E)$ and a *demand* function $f : 2^V \to \mathbb{N}$, find a minimum weight set of edges F that for all $S \subseteq V$, contains at least $f(S)$ edges from $\delta(S)$, where $\delta(S)$ is the set of edges in the cut defined by S (i.e., edges with exactly one endpoint in S). We assume $f(V) = f(\emptyset) = 0$, for otherwise no solution exists. Furthermore, we assume the existence of a polynomial-time oracle that provides $f(S)$ when given a subset S. We make this assumption because the number of subsets $S \subseteq V$ is exponential, and enumerating $f(S)$ for all of them in the input would be unreasonable. Finally, to simplify the exposition, we generalize the problem and allow G to be a multi-graph, i.e., G may contain multiple edges between pairs of vertices.

Many familiar problems can be formulated as network design problems with *0-1 demand functions*, i.e., demand functions with range $\{0, 1\}$. For example, in the *shortest path* problem $f(S) = 1$ if $|S \cap \{s, t\}| = 1$ and $f(S) = 0$ otherwise; in the *minimum spanning tree* problem, $f(S) = 1$ if $\emptyset \neq S \subsetneq V$ and $f(S) = 0$ otherwise; and in the *Steiner tree* problem, $f(S) = 1$ if $\emptyset \neq S \subsetneq T$ and $f(S) = 0$ otherwise. The *Steiner tree* problem is: given a graph $G = (V, E)$ and a set of *terminals* $T \subseteq V$, find a minimum weight set of edges that induces a connected subgraph containing all terminals (and possibly other vertices as well). The key property of 0-1 demand functions is that every minimal solution induces a forest, for if a feasible solution contains a cycle, removing a single edge from this cycle cannot destroy the solution's feasibility. (Of course, multiple edges are redundant too.)

In our treatment of the problem we restrict our attention to 0-1 demand functions. We further restrict our attention to two families of 0-1 demand functions for which deciding whether a given set of edges constitutes a feasible solution is easy. Specifically, we consider 0-1 *proper* functions and 0-1 *downwards monotone* functions.

0-1 proper functions.. A 0-1 demand function f is *proper* if it satisfies two conditions.

Symmetry. $f(S) = f(V \setminus S)$ for all $S \subseteq V$.

Maximality. For all pairs of disjoint sets A and B, if $f(A) = f(B) = 0$, then $f(A \cup B) = 0$.

Note that the examples given above for applications of 0-1 functions are actually applications of 0-1 proper functions.

0-1 downwards monotone functions.. A 0-1 demand function f is *downwards monotone* if $f(S) = 1$ implies $f(S') = 1$ for all nonempty $S' \subseteq S$. Notice that a downwards monotone function satisfies maximality, but not necessarily symmetry.

Downwards monotone functions can be used to model several familiar problems. For example, the *edge covering* problem is that of selecting a minimum weight set of edges spanning the entire vertex set. This problem (which is polynomial time solvable [Grötschel et al. 1988]) can be modeled by: $f(S) = 1$ if and only if $|S| = 1$, which is downwards monotone. Another example is the *lower capacitated tree partitioning* problem. Here we must find a minimum weight set of edges F such that (V, F) is a forest in which each tree contains at least k vertices, for some parameter k. This problem can be modeled by: $f(S) = 1$ if and only if $0 < |S| < k$, which again is downwards monotone.

Notice that if a 0-1 demand function f is proper or downwards monotone, we can check efficiently whether an edge set F is feasible by checking if $f(C) = 0$ for every connected component C of (V, F). If $f(C) = 1$ for some component C, then F is clearly infeasible. Otherwise, F is feasible since every $S \subseteq V$ that is a union of components obeys $f(S) = 0$ by maximality, and every S that is not a union of components is satisfied by some edge in F.

5.3.1 *The Algorithm.* We now present a 2-approximation algorithm. We begin with some definitions and an observation, and then describe and analyze the algorithm.

Definitions.

(1) The vertices v such that $f(\{v\}) = 1$ are called *terminals*.

(2) We denote by $\tau(e)$ the number of endpoints of edge e that are terminals. (Thus, $\tau(e) \in \{0, 1, 2\}$ for all e.)

(3) In the course of its execution, the algorithm modifies the graph by performing *edge contractions*. Contracting an edge consists in "fusing" its two endpoints u and v into a single (new) vertex z. The edge (or multiple edges) connecting u and v are deleted and every other edge incident on u or v becomes incident on z instead. In addition, the demand function f is replaced with a new demand function f', defined by

$$f'(S) = \begin{cases} f((S \setminus z) \cup \{u, v\}) & z \in S, \\ f(S) & z \notin S. \end{cases}$$

Clearly, if f is 0-1 proper or downwards monotone then so is f'.

Observation 7. *Let I be an instance of the problem and let I' be the problem instance obtained from it by contracting a single edge e. Then:*

(1) *The optimum value for I is not less than the optimum value for I'.*

(2) *If F' is a minimal feasible solution for I', then either F' is minimal feasible for I, or else it is infeasible for I but $F' \cup \{e\}$ is minimal feasible for I.*

The algorithm follows.

Algorithm ND(G, w, f)

1. If G contains no terminals, return $\emptyset$.
2. Else, if there exists an edge e such that $w(e) = 0$ do:
3.   Let (G', w', f') be the instance obtained by contracting e.
4. $F' \leftarrow \mathbf{ND}(G', w', f')$.
5.   If F' is a feasible solution with respect to G:
6.   Return the solution $F = F'$.
7.   Else:
8.   Return the solution $F = F' \cup \{e\}$.
9. Else:
10.   Let $\epsilon = \min\{w(e)/\tau(e) \mid \tau(e) \geq 1\}$.
11.   Define the weight functions $w_1(e) = \epsilon \cdot \tau(e)$ and $w_2 = w - w_1$.
12.   Return $\mathbf{ND}(G, w_2, f)$.

In terms of our framework, Line 1 implements the *optimal solution* case (correctness follows from the maximality property of f), Lines 2–8 implement the *problem size reduction* case (correctness follows from Observation 7), and Lines 9–12 implement the *weight decomposition* case.

We claim that the solution returned by the algorithm is 2-approximate, and to prove this we only need to show that w_1 is 2-effective.

Lemma 8. *Let F be a minimal feasible solution. Then every tree in the forest induced by F contains at most one non-terminal leaf (and all other leaves are terminals).*

Proof. We first consider the case that f is 0-1 proper. Suppose there exists some leaf v that is not a terminal. Let T_v be the tree containing v. Let F' be the set of edges obtained from F by removing the edge incident on v. Since F is minimal, F' is infeasible. Thus there exists $\emptyset \neq S \subsetneq V$ such that $f(S) = 1$ and $\delta(S) \cap F' = \emptyset$. By symmetry, we can assume $v \in S$ without loss of generality. The fact that $\delta(S) \cap F' = \emptyset$ implies that S is a union of trees in the forest induced by F' (we consider trees to be sets of vertices). The trees in this forest are exactly the trees in the forest induced by F, except that T_v is split into two trees $\{v\}$ and $T_v \setminus \{v\}$. Since F is feasible, every tree T in the forest induced by it satisfies $f(T) = 0$, and by maximality, so does the union of any number of them. Thus, $f(S \cap T_v) = 1$, for otherwise maximality would imply $f(S) = 0$. We know that S is a union of trees in the forest induced by F', and $\emptyset \neq S \cap T_v \neq T_v$ since $f(S \cap T_v) = 1$.

Thus, either $S \cap T_v = T_v \setminus \{v\}$ or $S \cap T_v = \{v\}$. The latter is impossible since v is not a terminal, so $S \cap T_v = T_v \setminus \{v\}$. But this contradicts our assumption that $v \in S$. Thus in the case of 0-1 proper functions, *all* leaves are terminals.

Now consider the case that f is 0-1 downwards monotone. Again, suppose some leaf v is a non-terminal. The proof proceeds as before, except that this time we cannot assume $v \in S$ (since symmetry does not necessarily hold). Thus in the final step of the proof, we conclude that $S \cap T_v = T_v \setminus \{v\}$, and there is no contradiction. Hence, $f(T_v \setminus \{v\}) = 1$, which implies (by downwards monotonicity) that all vertices in T_v except for v are terminals. We remark that the proof can be extended quite easily to show that in fact if a tree contains *any* non-terminal (not necessarily a leaf) then all other vertices in the tree are terminals. □

The proof that w_1 is 2-effective follows from Lemma 8. Consider a minimal feasible solution F. We claim that $\epsilon|V_t| \le w_1(F) \le 2\epsilon|V_t|$, where V_t denotes the set of terminals. To show this, we prove the equivalent claim $|V_t| \le \sum_{e \in F} \tau(e) \le 2|V_t|$. For the first inequality, observe that the feasibility of F implies that every terminal must be adjacent to at least one of the edges in F, and thus $\sum_{e \in F} \tau(e) \ge |V_t|$. Turning to the second inequality, let V_F be the set of vertices in the forest induced by F, let $\deg_F(v)$ denote the degree of vertex v in the forest, and let k be the number of trees in the forest. Then,

$$\begin{aligned}
\sum_{e \in F} \tau(e) &= \sum_{v \in V_t} \deg_F(v) \\
&= \sum_{v \in V_F} \deg_F(v) - \sum_{v \in V_F \setminus V_t} \deg_F(v) \\
&= 2(|V_F| - k) - \sum_{v \in V_F \setminus V_t} \deg_F(v).
\end{aligned}$$

By Lemma 8, all but one of the non-terminals in a given tree are not leaves, so each has degree at least 2 in the forest. Thus,

$$\sum_{v \in V_F \setminus V_t} \deg_F(v) \ge 2|V_F \setminus V_t| - k = 2|V_F| - 2|V_t| - k,$$

and thus $\sum_{e \in F} \tau(e) \le 2|V_t| - k \le 2|V_t|$.

In a slightly more refined analysis we can take into account the $-k$ term in the last inequality and see that w_1 is actually $(2 - 1/|V_t|)$-effective (since $k \ge 1$). Furthermore, if f is 0-1 proper, then *all* leaves are terminals, and we can replace the $-k$ by $-2k$, so in this case w_1 is even $(2 - 2/|V_t|)$-effective. (Note that V_t decreases as the recursion proceeds to ever deeper levels, so in the above expressions we must understand $|V_t|$ to represent the number of terminals in the *initial* graph.)

In particular, the ratio is 1 for *shortest path*, since this problem is modeled by a 0-1 proper function and $|V_t| = 2$. In fact, Algorithm **ND** can be viewed as a recursive implementation of a variant of Dijkstra's algorithm using bi-directional search (see Nicholson [1966]). Also, in the case of *minimum spanning tree*, w_1 is really 1-effective, since every minimal solution costs exactly $2\epsilon(n - 1)$. For this problem Algorithm **ND** can be seen as a recursive implementation of Kruskal's minimum spanning tree algorithm [Kruskal 1956].

5.4 Feedback Vertex Set

A set of vertices in an undirected graph is called a *feedback vertex set* (FVS for short) if its removal leaves an acyclic graph (i.e., a forest). Another way of saying this is that the set intersects all cycles in the graph. The *feedback vertex set* problem is: given a vertex-weighted graph, find a minimum-weight FVS. In this section we describe and analyze a 2-approximation algorithm for the problem following our minimal-solutions framework.

The algorithm is as follows.

Algorithm $\mathbf{FVS}(G, w)$

If G is empty, return $\emptyset$.
If there exists a vertex $v \in V$ such that $\deg(v) \leq 1$ do:
    return $\mathbf{FVS}(G \setminus \{v\}, w)$.
Else, if there exists a vertex $v \in V$ such that $w(v) = 0$ do:
    $F' \leftarrow \mathbf{FVS}(G \setminus \{v\}, w)$.
    If F' is an FVS with respect to G:
        Return F'.
    Else:
        Return $F = F' \cup \{v\}$.
Else:
    Let $\epsilon = \min_{v \in V} \frac{w(v)}{\deg(v)}$.
    Define the weight functions $w_1(v) = \epsilon \cdot \deg(v)$ and $w_2 = w - w_1$.
    Return $\mathbf{FVS}(G, w_2)$

The analysis follows the pattern outlined above—the only interesting part is showing that w_1 is 2-effective. Since $w_1(F) = \epsilon \cdot \sum_{v \in F} \deg(v)$ for any FVS F, it is sufficient to demonstrate the existence of a number b such that for all minimal solutions F, $b \leq \sum_{v \in F} \deg(v) \leq 2b$. Note that the weight decomposition is only applied to graphs in which all vertices have degree at least 2. We shall henceforth assume that our graph $G = (V, E)$ is such a graph.

Consider a feasible solution F. There is a clear connection between $\sum_{v \in F} \deg(v)$ and the number of edges deleted by removing F from the graph, so it makes sense to reason about these edges. Consider the graph obtained by removing F from G. This graph is a forest on $|V| - |F|$ nodes, and it therefore contains precisely $|V| - |F| - k$ edges, where k is the number of its connected components (trees). The number of edges deleted by the removal of F is thus $|E| - |V| + |F| + k > |E| - |V| + |F|$, and since each of these edges contributes to the degree of some vertex in F, we get the following lower bound: $|E| - |V| + |F| < \sum_{v \in F} \deg(v)$.

Let us fix some feasible solution F^* that minimizes $|E| - |V| + |F^*|$, i.e., let F^* be a minimum cardinality FVS. We claim that all minimal solutions F satisfy $|E| - |V| + |F^*| < \sum_{v \in F} \deg(v) \leq 2(|E| - |V| + |F^*|)$, which proves that w_1 is 2-effective. Let F be a minimal solution. The first inequality follows immediately from the discussion above and our choice of F^*. The second inequality, however, is not immediate, and to prove it we must first explore the structure of G.

Consider a vertex $v \in F$. The minimality of F implies that there exists at least one cycle in G such that v is the only vertex in F contained in this cycle. Denote this cycle by C_v (if more than one such cycle exist, choose one arbitrarily) and let P_v be the path obtained by removing v from C_v. Let $\mathcal{P} = \{P_v \mid v \in F\}$. Now consider the connected components of the graph G' obtained by removing F from G. Some of these connected components contain vertices from the paths $\mathcal{P}$. We refer to them as *vpath-components*. Let V_1 be the set of vertices in the vpath-components and let V_2 be the remaining vertices in G'. The original graph G is thus partitioned into three parts, as illustrated in Figure 2. (Although F and V_2 are each drawn in the figure as a single oval, the subgraphs they induce are not necessarily connected.) Observe that there are no edges between V_1 and V_2 since such edges would appear in G', but V_1 is a union of connected components in G'. Let $n_1 = |V_1|$ and $n_2 = |V_2|$. Let m_1 be the number of edges in G that are incident on vertices in V_1 (i.e., edges between vertices in V_1 and edges connecting vertices in V_1 with vertices in F). Similarly, let m_2 be the number of edges in G that are incident on vertices in V_2. Let m'_1 be the number of edges connecting vertices in F with vertices in V_1 and let m'_2 be the number of edges connecting vertices in F with vertices in V_2. Let m_F be the number of edges in the subgraph induced by F. Finally, let κ be the number of vpath-components.

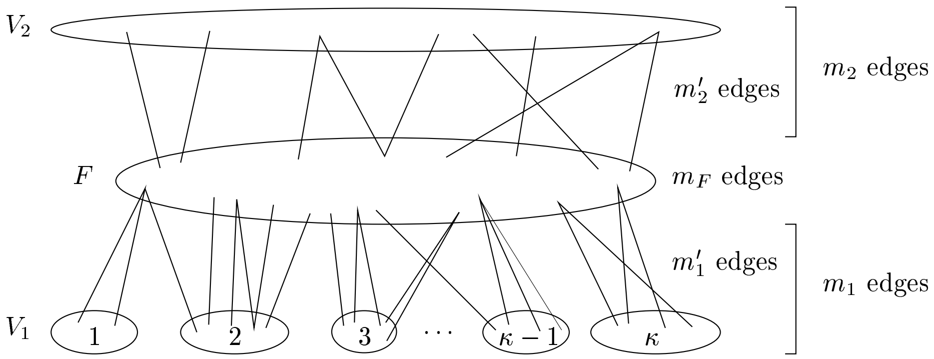

Fig. 2. Partition of G.

With the above notation in place, we are ready to bound $\sum_{v \in F} \deg(v)$. First note that $|V| = n_1 + n_2 + |F|$ and $|E| = m_1 + m_2 + m_F$. Also,

$$\sum_{v \in F} \deg(v) = m'_1 + m'_2 + 2m_F.$$

To achieve the desired bound on $\sum_{v \in F} \deg(v)$, we bound m'_1 and m'_2, and sum these bounds together with $2m_F$. We will also need a bound on $2|F|$.

As we have already pointed out, the removal of F leaves a forest and thus each of the vpath-components is a tree. Therefore, the number of edges in the subgraph induced by V_1 is $n_1 - \kappa$, and thus $m'_1 = m_1 - n_1 + \kappa$. Let us bound κ. Consider a path $P_v \in \mathcal{P}$. This path is obtained by deleting v from the cycle C_v. By definition,

C_v contains exactly one vertex in F, namely v, so P_v contains no vertices from F, and therefore exists in the graph G'. It follows that P_v is fully contained in exactly one vpath-component. To bound the number of vpath-components, observe that for all $v \in F$, the cycle C_v contains at least one vertex of F^* (since F^* is an FVS), so either $v \in F^*$, in which case $v \in F^* \cap F$, or else P_v contains a vertex of F^*, in which case this vertex belongs to $F^* \setminus F$. Thus each vpath-component either contains a path P_v such that $v \in F^* \cap F$ or else contains a vertex of $F^* \setminus F$. Thus $\kappa \leq |F^* \cap F| + |F^* \setminus F| = |F^*|$, and hence

$$m'_1 \leq m_1 - n_1 + |F^*|.$$

The last expression also bounds $2|F|$ since every vertex $v \in F$ is connected by at least two edges to vertices in V_1 (the two edges connecting it to its neighbors in the cycle C_v), and thus $2|F| \leq m'_1$, which implies

$$m_1 - n_1 + |F^*| \geq 2|F|.$$

To bound m'_2, recall that the degree of every vertex in the graph is at least 2. Thus $2n_2 \leq \sum_{u \in V_2} \deg(u) = 2(m_2 - m'_2) + m'_2 = 2m_2 - m'_2$, which implies

$$m'_2 \leq 2m_2 - 2n_2.$$

Putting the pieces together, we get

$$\begin{aligned}\sum_{v \in F} \deg(v) &= m'_1 + m'_2 + 2m_F \\ &\leq m_1 - n_1 + |F^*| + 2m_2 - 2n_2 + 2m_F \\ &= 2(m_1 + m_2 + m_F) - 2n_1 - 2n_2 - (m_1 - n_1 + |F^*|) + 2|F^*| \\ &\leq 2|E| - 2n_1 - 2n_2 - 2|F| + 2|F^*| \\ &= 2(|E| - |V| + |F^*|).\end{aligned}$$

Remark. In addition to the above choice of w_1, the following two alternative weight decompositions have been proposed in the literature, and both have been shown to be 2-effective. The first is obtained by replacing Lines 11 and 12 of Algorithm **FVS** with the following:

If G contains a *semi-disjoint* cycle[4] C:
  Let $\epsilon = \min_{v \in C} w(v)$.
  Define the weight functions $w_1(v) = \begin{cases} \epsilon & v \in C, \\ 0 & \text{otherwise,} \end{cases}$ and $w_2 = w - w_1$.
Else:
  Let $\epsilon = \min_{v \in V} w(v)/(\deg(v) - 1)$.
  Define the weight functions $w_1(v) = \epsilon \cdot (\deg(v) - 1)$
  and $w_2 = w - w_1$.

The second is obtained by:

[4] A cycle C is *semidisjoint* if there exists a vertex $x \in C$ such that $\deg(u) = 2$ for every vertex $u \in C \setminus \{x\}$.

Choose an *endblock*[5] B, and let $\epsilon = \min_{v \in B} w(v)/(\deg(v) - 1)$.

Define the weight functions $w_1(v) = \begin{cases} \epsilon \cdot (\deg(v) - 1) & v \in B, \\ 0 & \text{otherwise,} \end{cases}$

and $w_2 = w - w_1$.

5.5 Background

Minimal solutions and feedback vertex set. The use of minimal solutions in the local ratio setting appeared for the first time in the context of *feedback vertex set* [Bafna et al. 1999]. This problem is NP-hard [Karp 1972] and MAX SNP-hard [Lund and Yannakakis 1993], and at least as hard to approximate as vertex cover [Lewis and Yannakakis 1980]. An approximation algorithm for the unweighted case that achieves a performance ratio of $2 \log_2 n$ is essentially contained in a lemma due to Erdös and Pósa [1962]. Monien and Shultz [1981] improve the ratio to $\sqrt{\log n}$. Bar-Yehuda et al. [1998] give a 4-approximation algorithm for the unweighted case, and an $O(\log n)$-approximation algorithm for the weighted case. Bafna et al. [1999] present a local ratio 2-approximation algorithm for the weighted case (using the first of the two alternative weight decompositions presented above). Their algorithm is the first local ratio algorithm to make use of the concept of minimal solutions (although this concept was used earlier in primal-dual algorithms [Ravi and Klein 1993; Agrawal et al. 1995; Goemans and Williamson 1995]). At about the same time, Becker and Geiger [1996] also obtained a 2-approximation algorithm for the problem. Algorithm **FVS** is a recursive formulation of their algorithm. Chudak et al. [1998] interpret the above two algorithms in primal-dual terms, and suggest a third algorithm, whose local ratio interpretation is described by the second alternative weight decomposition given above. Fujito [1998] proposes a generic local ratio algorithm for node-deletion problems with nontrivial and hereditary graph properties[6]. Algorithm **RecursiveVC**, Algorithm **FVS**, and the algorithm of Bafna et al. [1999] can be seen as instances of Fujito's generic algorithm. Bar-Yehuda [2000] presents a unified local ratio approach for covering problems. His presentation contains a short generic approximation algorithm which can explain many known exact optimization and approximation algorithms for covering problems. Bar-Yehuda and Rawitz [2001] devise a framework that extends the generic algorithm from Bar-Yehuda [2000]. The notion of effectiveness of a weight function first appeared in Bar-Yehuda [2000]. A similar idea appeared earlier in the primal-dual setting [Bertsimas and Teo 1998].

Partial hitting set. The *partial hitting set* problem generalizes *hitting set*, and as we have remarked in Section 4.1, *hitting set* and *set cover* are equivalent problems. For this reason, the *partial hitting set* problem is often described in the literature in *set cover* terms and is referred to as *partial covering*. The problem was first studied by Kearns [1990] in relation to learning. He proves that the performance ratio of the greedy algorithm is at most $2H_m + 3$. (Recall that H_m is the mth harmonic number.) Slavík [1997] shows that it is actually bounded by H_k.

[5]An *endblock* is a biconnected component containing at most one articulation point.

[6]A graph property π is *nontrivial* if it is true for infinitely many graphs and false for infinitely many graphs; it is *hereditary* if every subgraph of a graph satisfying π also satisfies π.

The special case in which the cardinality of every set is exactly 2 is called the *partial vertex cover* problem. This problem was studied by Bshouty and Burroughs [1998], who obtained the first polynomial time 2-approximation algorithm for it. The $\max\{s_{\max}, 2\}$-approximation algorithm for *partial hitting set* given in this section (Algorithm **PHS**) is due to Bar-Yehuda [2001]. In fact, his approach can be used to approximate an extension of the partial hitting set problem in which there is a *length* l_i associated with each set S_i, and the goal is to hit sets of total length at least k. (The plain *hitting set* problem is the special case where $l_i = 1$ for all i.) Gandhi et al. [2001] present a multi-phase primal-dual algorithm for *partial hitting set* achieving a performance ratio of $\max\{s_{\max}, 2\}$.

Network design. As we have seen, Algorithm **ND** can be applied to problems that can be modeled by 0-1 downwards monotone functions. These include *lower capacitated partitioning* problems [Goemans and Williamson 1995; Imielińska et al. 1993] and *location design* problems [Laporte 1988; Laporte et al. 1983]. (For further discussion see Goemans and Williamson [1997].) Algorithm **ND** has evolved from work on the *generalized Steiner network* problem (also called the *survival network design* problem), in which we are given a graph $G = (V, E)$, edge costs, and a nonnegative integer r_{ij} for each pair of vertices i and j. Our goal is to find a minimum weight subset of edges $E' \subseteq E$ such that there are at least r_{ij} edge-disjoint paths between each pair of vertices $i, j \in V$ in the graph (V, E'). This problem can be modeled as a *network design* problem with the proper function[7] $f(S) = \max_{i \in S, j \notin S} r_{ij}$. Note that the *Steiner tree* problem is a special case of this problem where $r_{ij} = 1$ if i and j are terminals, and $r_{ij} = 0$ otherwise. Agrawal et al. [1995] describe an approximation algorithm for a variant of the *generalized Steiner network* problem in which an edge can be selected (and paid for) multiple times. The performance ratio of their algorithm is $2\lceil \log_2(r_{\max} + 1)\rceil$, where $r_{\max} = \max_{ij} r_{ij}$. Goemans and Williamson [1995] present a primal-dual approximation framework for network design problems in which the demand function is 0-1 proper. The approximation ratio of their generic algorithm is $2 - \frac{2}{|V_t|}$. (In the case of *Steiner tree* their algorithm simulates the algorithm of Agrawal et al. [1995].) Algorithm **ND** is a local ratio version of this algorithm. Williamson et al. [1995] present the first approximation algorithm for general proper functions. Their algorithm finds a solution within a factor of $\max\{2f_{\max} - 1, 2\}$ of the optimal, where $f_{\max} = \max_S f(S)$. In addition to the primal-dual approach, their algorithm uses the idea of satisfying f in "phases," which was introduced by Ravi and Klein [1993] (in the context of a 3-approximation algorithm for proper functions with range $\{0, 2\}$). Gabow et al. [1993] improve the efficiency of this algorithm. Goemans et al. [1994] improve the ratio for the general case to $2H_{f_{\max}}$. Jain [1998] considers *weakly super-modular* functions, that is, functions f satisfying (1) $f(V) = 0$, and (2) $f(A) + f(B) \leq \max\{f(A \setminus B) + f(B \setminus A), f(A \cap B) + f(A \cup B)\}$ for all $A, B \subseteq V$. (Note that proper functions are weakly super-modular.) He describes a 2-approximation algorithm based on LP rounding.

[7] A function that satisfies symmetry and maximality is called proper.

6. MAXIMIZATION PROBLEMS AND SCHEDULING APPLICATIONS

In this section we describe applications of the local ratio technique in the context of maximization problems, focusing on problems of resource allocation and scheduling. Our goal in this section is twofold. First, we develop a local ratio theory for maximization problems in general, and second, we demonstrate the technique's applicability in an important branch of optimization, namely, resource allocation and scheduling. Resource allocation and scheduling problems are immensely popular objects of study in the field of approximation algorithms and combinatorial optimization, owing to their direct applicability to many real-life scenarios and their richness in terms of mathematical structure. Historically, they were among the first to be analyzed in terms of worst-case approximation ratio, and research into these problems continues actively to this day.

We begin by presenting a local ratio theorem for maximization problems and sketching a general framework based on it in Section 6.1. We apply this framework to a collection of scheduling maximization problems in Section 6.2. We briefly discuss other kinds of scheduling problems in Section 6.3.

6.1 Local Ratio for Maximization Problems

The Local Ratio Theorem for maximization problems is very similar to its minimization counterpart. It applies to optimization problems that can be formulated as follows.

> Given a *weight vector* $w \in \mathbb{R}^n$ and a set of *feasibility constraints* $\mathcal{C}$, find a *solution vector* $x \in \mathbb{R}^n$ satisfying the constraints in $\mathcal{C}$ that maximizes the scalar product $w \cdot x$.

Before stating the Local Ratio Theorem for maximization problems, we remind the reader that we call a feasible solution to a maximization problem *r-approximate* if its weight is at least $1/r$ times the optimum weight (hence approximation factors are greater than or equal to 1).

THEOREM 9 LOCAL RATIO—MAXIMIZATION PROBLEMS. *Let $\mathcal{C}$ be a set of feasibility constraints on vectors in $\mathbb{R}^n$. Let $w, w_1, w_2 \in \mathbb{R}^n$ be such that $w = w_1 + w_2$. Let $x \in \mathbb{R}^n$ be a feasible solution (with respect to $\mathcal{C}$) that is r-approximate with respect to w_1 and with respect to w_2. Then, x is r-approximate with respect to w as well.*

PROOF. Let x^*, x_1^*, and x_2^* be optimal solutions with respect to w, w_1, and w_2, respectively. Clearly, $w_1 \cdot x_1^* \geq w_1 \cdot x^*$ and $w_2 \cdot x_2^* \geq w_2 \cdot x^*$. Thus, $w \cdot x = w_1 \cdot x + w_2 \cdot x \geq \frac{1}{r}(w_1 \cdot x_1^*) + \frac{1}{r}(w_2 \cdot x_2^*) \geq \frac{1}{r}(w_1 \cdot x^*) + \frac{1}{r}(w_2 \cdot x^*) = \frac{1}{r}(w \cdot x^*)$. □

The general structure of a local ratio approximation algorithm for a maximization problem is similar to the one described for the minimization case in Section 5.2. It too consists of a three-way *if* condition that directs execution to one of three main options: *optimal solution*, *problem size reduction*, and *weight decomposition*. There are several differences though. In contrast to the minimization case, we make no effort to keep the weight function non-negative, i.e., in *weight decomposition* steps we allow w_2 to take on negative values. Also, in *problem size reduction* steps we usually remove an element whose weight is either zero *or negative*. Finally and most importantly, we strive to construct *maximal* solutions rather than minimal ones. This affects our choice of w_1 in *weight decomposition* steps. The weight function w_1 is chosen such that every *maximal* solution (a feasible solution that cannot be extended) is r-approximate with respect to it.[8] Accordingly, when the recursive call returns in *problem size reduction* steps, we extend the solution if possible (rather than if necessary), but we attempt to do so only for zero-weight elements (and not for negative weight ones).

As in the minimization case, we use the notion of effectiveness.

DEFINITION 3. *In the context of maximization problems, a weight function w is said to be r-*effective *if there exists a number b such that $b \leq w \cdot x \leq r \cdot b$ for all maximal feasible solutions x.*

6.2 Throughput Maximization Problems

Consider the following general problem. The input consists of a set of *activities*, each requiring the utilization of a given limited *resource*. The amount of resource available is fixed over time; we normalize it to unit size for convenience. The activities are specified as a collection of sets $\mathcal{A}_1, \ldots, \mathcal{A}_n$. Each set represents a single activity: it consists of all possible *instances* of that activity. An instance $I \in \mathcal{A}_i$ is defined by the following parameters.

(1) A half-open time interval $[s(I), e(I))$ during which the activity will be executed. We call $s(I)$ and $e(I)$ the *start-time* and the *end-time* of the instance.
(2) The amount of resource required for the activity. We refer to this amount as the *width* of the instance and denote it $d(I)$. Naturally, $0 < d(I) \leq 1$.
(3) The *weight* $w(I) \geq 0$ of the instance. The weight represents the profit to be gained by scheduling this instance of the activity.

Different instances of the same activity may have different parameters of duration, width, or weight. A *schedule* is a collection of instances. It is *feasible* if (1) it contains at most one instance of every activity, and (2) for all time instants t, the total width of the instances in the schedule whose time interval contains t does not exceed 1 (the amount of resource available). The goal is to find a feasible schedule that maximizes the total weight of instances in the schedule.

In the following sections we describe local ratio algorithms for several special cases of the general problem. We use the following notation. For a given activity instance I, $\mathcal{A}(I)$ denotes the activity to which I belongs and $\mathcal{I}(I)$ denotes the set of all activity instances that intersect I but belong to activities other than $\mathcal{A}(I)$.

6.2.1 *Interval Scheduling.* In the *interval scheduling* problem we must schedule jobs on a single processor with no preemption. Each job consists of a finite collection of time intervals during which it may be scheduled. The problem is to select a maximum weight subset of non-conflicting intervals, at most one interval for each job. In terms of our general problem, this is simply the special case where every activity consists of a finite number of instances and the width of every instance is 1.

To design the weight decomposition for this problem, we examine the properties of maximal schedules. Let J be the activity instance with minimum end-time among

[8] We actually impose a somewhat weaker condition, as described in Section 6.2.

all activity instances of all activities (breaking ties arbitrarily). Note that the choice of J ensures that all of the intervals intersecting it intersect each other at J's right endpoint (see Figure 3). Consider a maximal schedule $\mathcal{S}$. Clearly $\mathcal{S}$ cannot contain more than one instance from $\mathcal{A}(J)$, nor can it contain more than one instance from $\mathcal{I}(J)$, since all of these instances intersect each other. Thus $\mathcal{S}$ contains at most two intervals from $\mathcal{A}(J) \cup \mathcal{I}(J)$. On the other hand, $\mathcal{S}$ must contain at least one instance from $\mathcal{A}(J) \cup \mathcal{I}(J)$, for otherwise it would not be maximal (since J could be added to it). This implies that the weight function

$$w_1(I) = \epsilon \cdot \begin{cases} 1 & I \in \mathcal{A}(J) \cup \mathcal{I}(J), \\ 0 & \text{otherwise}, \end{cases}$$

is 2-effective for any choice of $\epsilon > 0$, and we can expect to obtain a 2-approximation algorithm based on it.

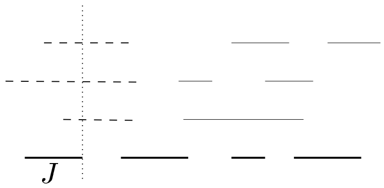

Fig. 3. J, $\mathcal{A}(J)$, and $\mathcal{I}(J)$: heavy lines represent $\mathcal{A}(J)$; dashed lines represent $\mathcal{I}(J)$.

A logical course of action is to fix $\epsilon = \min\{w(I) \mid I \in \mathcal{A}(J) \cup \mathcal{I}(J)\}$ and to solve the problem recursively on $w - w_1$, relying on two things: (1) w_1 is 2-effective; and (2) the solution returned is maximal. However, we prefer a slightly different approach. We show that w_1 actually satisfies a stronger property than 2-effectiveness. For a given activity instance I, we say that a feasible schedule is *I-maximal* if either it contains I, or it does not contain I but adding I to it will render it infeasible. Clearly, every maximal schedule is also I-maximal for any given I, but the converse is not necessarily true. The stronger property satisfied by the above w_1 is that every J-maximal schedule is 2-approximate with respect to w_1 (for all $\epsilon > 0$). To see this, observe that no optimal schedule may contain more than two activity instances from $\mathcal{A}(J) \cup \mathcal{I}(J)$, whereas every J-maximal schedule must contain at least one (if it contains none, it cannot be J-maximal since J can be added). The most natural choice of ϵ is $\epsilon = w(J)$.

Our algorithm for *interval scheduling* is based on the above observations. The initial call is $\mathbf{IS}(\mathcal{A}, w)$, where $\mathcal{A}$ is the set of jobs, which we also view as the set of all instances, i.e., $\cup_{i=1}^{m} \mathcal{A}_i$.

Algorithm $\mathbf{IS}(\mathcal{A}, w)$

1. If $\mathcal{A} = \emptyset$, return $\emptyset$.
2. If there exists an instance I such that $w(I) \leq 0$ do:
3.   Return $\mathbf{IS}(\mathcal{A} \setminus \{I\}, w)$.
4. Else:
5.   Let J be the instance with minimum end-time in $\mathcal{A}$.
6.   Define the weight functions
$$w_1(I) = w(J) \cdot \begin{cases} 1 & I \in \mathcal{A}(J) \cup \mathcal{I}(J), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } w_2 = w - w_1.$$
7. $\mathcal{S}' \leftarrow \mathbf{IS}(\mathcal{A}, w_2)$.
8.   If $\mathcal{S}' \cup \{J\}$ is feasible:
9.   Return $\mathcal{S} = \mathcal{S}' \cup \{J\}$.
10.   Else:
11.   Return $\mathcal{S} = \mathcal{S}'$.

As with similar previous claims, the proof that Algorithm **IS** is 2-approximation is by induction on the recursion. At the basis of the recursion (Line 1) the schedule returned is optimal and hence 2-approximate. For the inductive step there are two possibilities. If the recursive call is made in Line 3, then by the inductive hypothesis the schedule returned is 2-approximate with respect to $(\mathcal{A} \setminus \{I\}, w)$, and since the weight of I is non-positive, the optimum for $(\mathcal{A}, w)$ cannot be greater than the optimum for $(\mathcal{A} \setminus \{I\}, w)$. Thus the schedule returned is 2-approximate with respect to $(\mathcal{A}, w)$ as well. If the recursive call is made in Line 7, then by the inductive hypothesis $\mathcal{S}'$ is 2-approximate with respect to w_2, and since $w_2(J) = 0$ and $S \subseteq S' \cup \{J\}$, it follows that $\mathcal{S}$ too is 2-approximate with respect to w_2. Since $\mathcal{S}$ is J-maximal by construction (Lines 8–11), it is also 2-approximate with respect to w_1. Thus, by the Local Ratio Theorem, it is 2-approximate with respect to w as well.

6.2.2 *Scheduling on Parallel Identical Machines.* In this problem the resource consists of k parallel identical machines. Each activity instance may be assigned to any of the k machines. Thus $d(I) = 1/k$ for all I.

In order to approximate this problem we use Algorithm **IS**, but with a different choice of w_1, namely,

$$w_1(I) = w(J) \cdot \begin{cases} 1 & I \in \mathcal{A}(J), \\ 1/k & I \in \mathcal{I}(J), \\ 0 & \text{otherwise.} \end{cases}$$

The analysis of the algorithm is similar to the one used for the case $k = 1$ (i.e., interval scheduling). It suffices to show that every J-maximal schedule is 2-approximate with respect to w_1. This is so because every J-maximal schedule either contains an instance from $\mathcal{A}(J)$ or a set of instances intersecting J that prevent J from being added to the schedule. In the former case, the weight of the schedule with respect to w_1 is at least $w(J)$. In the latter case, since k machines are available but J cannot be added, the schedule must already contain k activity instances from

$\mathcal{I}(J)$, and its weight (with respect to w_1) is therefore at least $k \cdot \frac{1}{k} \cdot w(J) = w(J)$. Thus the weight of every J-maximal schedule is at least $w(J)$. On the other hand, an optimal schedule may contains at most one instance from $\mathcal{A}(J)$ and at most k instances from $\mathcal{I}(J)$ (as they all intersect each other), and thus its weight cannot exceed $w(J) + k \cdot \frac{1}{k} \cdot w(J) = 2w(J)$.

Remark. Our algorithm only finds a set of activity instances that can be scheduled, but does not construct an actual assignment of instances to machines. This can be done easily by scanning the instances (in the solution found by the algorithm) in increasing order of end-time, and assigning each to an arbitrary available machine. It is easy to see that such a machine must always exist.

6.2.3 *Bandwidth Allocation of Sessions in Communication Networks.* Consider a scenario in which the bandwidth of a communication channel must be allocated to sessions. Here the resource is the channel's bandwidth, and the activities are sessions to be routed through the channel. A session is specified as a list of intervals in which it can be scheduled, together with a bandwidth requirement and a weight for each interval. The goal is to find the most profitable set of sessions that can utilize the available bandwidth.

To approximate this problem we first consider the following two special cases.

Special Case 1. All instances are *wide*, i.e., $d(I) > 1/2$ for all I.

Special Case 2. All activity instances are *narrow*, i.e., $d(I) \leq 1/2$ for all I.

In the case of wide instances the problem reduces to interval scheduling since no pair of intersecting instances may be scheduled together. Thus, we can use Algorithm **IS** to find a 2-approximate schedule.

In the case of narrow instances we find a 3-approximate schedule by a variant of Algorithm **IS** in which w_1 is defined as follows:

$$w_1(I) = w(J) \cdot \begin{cases} 1 & I \in \mathcal{A}(J), \\ 2 \cdot d(I) & I \in \mathcal{I}(J), \\ 0 & \text{otherwise.} \end{cases}$$

To prove that the algorithm is a 3-approximation it suffices to show that every J-maximal schedule is 3-approximate with respect to w_1. (All other details are essentially the same as for interval scheduling.) A J-maximal schedule either contains an instance of $\mathcal{A}(J)$ or contains a set of instances intersecting J that prevent J from being added to the schedule. In the former case the weight of the schedule is at least $w(J)$. In the latter case, since J cannot be added, the combined width of activity instances from $\mathcal{I}(J)$ in the schedule must be greater than $1 - d(J) \geq 1/2$, and thus their total weight (with respect to w_1) must be greater than $w(J) \cdot 2 \cdot \frac{1}{2} = w(J)$. Thus, the weight of every J-maximal schedule is at least $w(J)$. On the other hand, an optimal schedule may contain at most one instance from $\mathcal{A}(J)$ and at most a set of instances from $\mathcal{I}(J)$ with total width 1 and hence total weight $2w(J)$. Thus, the optimum weight is at most $3w(J)$, and therefore every J-maximal schedule is 3-approximate with respect to w_1.

In order to approximate the problem in the general case where both narrow and wide activity instances are present, we solve it separately for the narrow instances and for the wide instances, and return the solution of greater weight. Let OPT

be the optimum weight for all activity instances, and let OPT_n and OPT_w be the optimum weight for the narrow instance and for the wide instances, respectively. Then, the weight of the schedule found is at least $\max\left\{\frac{1}{3}OPT_n, \frac{1}{2}OPT_w\right\}$. Now, either $OPT_n \geq \frac{3}{5}OPT$, or else $OPT_w \geq \frac{2}{5}OPT$. In either case the schedule returned is 5-approximate.

6.2.4 *Variations and Extensions.* In addition to the three problems we have just discussed, there are several other scheduling problems that can be approximated by various extensions and variations of Algorithm **IS**. We briefly mention these problems here without showing their solutions. Full details can be found in the papers describing them.

Independent set in interval graphs [Bar-Noy et al. 2001]. It is easy to see that the special case of *interval scheduling* in which each activity consists of a single instance is just a reformulation of the problem of finding a maximum weight independent set in an interval graph. It turns out that for this special case, Algorithm **IS** achieves an approximation ratio of 1, i.e., it solves the problem optimally. Algorithm **IS** is not the first to do so, however. See, e.g., Golumbic [1980].

Scheduling on parallel unrelated machines [Bar-Noy et al. 2001]. Scheduling on unrelated machines differs from scheduling on identical machines in that the profit derived from assigning an activity instance to a machine may be different for different machines, and furthermore, each activity instance may specify a list of *forbidden* machines, i.e., machines on which it is not willing to be scheduled. A variant of Algorithm **IS** approximates this problem with approximation guarantee 2.

Continuous input [Bar-Noy et al. 2001]. In our treatment of the general scheduling problem we have implicitly assumed that each activity is specified as a finite list of instances. We can generalize the problem and allow each activity to consist of infinitely many instances by specifying the activity as a finite collection of *time windows*. A time window is an interval of time $\mathcal{T}$, which together with a *length* parameter $l(\mathcal{T})$ and a *weight* parameter $w(\mathcal{T})$, represent the (infinite) set of instances $[s,t)$ such that $s \in \mathcal{T}$ and $t = s + l(\mathcal{T})$. Each of these instances has profit $w(\mathcal{T})$. The various variants of Algorithm **IS** can be modified to handle such "continuous" input efficiently, but a sacrifice must be made in the approximation ratio. Specifically, for arbitrarily small $\epsilon > 0$, the algorithm can be made to run in $O(n^2/\epsilon)$ time and achieve an approximation factor of $r + \epsilon$, where r is the factor it would achieve on "discrete" input.

Batching [Bar-Noy et al. 2002]. While most scheduling problems forbid the scheduling of two simultaneous jobs on the same machine, it is sometimes desirable to relax this constraint and allow multiple jobs (with similar characteristics) to be *batched* together and scheduled simultaneously. (For example, it is possible to broadcast a movie to several clients who have asked for it.) The introduction of batching into our general scheduling problem complicates things considerably, yet the basic idea of Algorithm **IS** is still valid. Bar-Noy et al. [2002] developed 2-approximation and 4–approximation algorithms for two variants of the problem of scheduling on identical machines with batching. Their algorithms are based on the same idea as Algorithm **IS**, though in a far more complex manner.

Combinatorial auctions [Akcoglu et al. 2002]. A *combinatorial auction* is an auction in which each bidder places a bid on some subset of the available goods (rather than on individual items), and the auctioneer accepts a collection of mutually disjoint bids, with the aim of maximizing revenue. Akcoglu et al. [2002] describe such auctioning as finding a maximum weight independent set in a *bid graph* whose vertices are the bids and whose edges connect pairs of conflicting bids. Since *maximum weight independent set* is hard to approximate in general [Håstad 1996], they focus on special cases. They present an approximation algorithm based on Algorithm **IS** for the problem on ordered graphs. The approximation guarantee they achieve is the maximum over vertices v of the maximum independent set size in the subgraph induced by the neighbors of v that appear after v in the ordering. They show that several types of combinatorial auctions can be modeled as bid graphs in which this value is a constant.

6.3 Additional Scheduling Applications

Generally speaking, a scheduling problem is a problem in which jobs compete for the use of some limited resource, and the goal is to resolve all conflicts. Conflicts can be resolved either by scheduling only a subset of the jobs or by enlarging the amount of available resource to accommodate all jobs. In the previous section we considered the first option. We assumed that the amount of resource was fixed and that we could reject jobs. We measured the quality of a schedule by its *throughput* (total weight of the accepted jobs), which we wished to maximize. In this section we mention two additional applications of the local ratio technique; one in the fixed-resource setting—but from a minimization viewpoint, and another in the enlargeable-resource setting.

In the fixed-resource setting we can adopt a minimization point of view by measuring the quality of a schedule by its *loss* (total weight of rejected jobs) instead of its throughput. Using this measure turns throughput maximization problems into loss minimization ones. While the throughput and loss measures are equivalent in terms of optimal solutions, they are completely distinct when one considers approximate solutions.

Consider the loss minimization version of the bandwidth allocation problem of Section 6.2.3. The loss associated with a schedule is the total weight of activity instances that are not scheduled. This measure is unsatisfactory if multiple instances per activity are allowed, since even in the best case at most one instance per activity can be scheduled (and all others contribute to the loss),[9] but the loss measure does make sense in the special case where each activity consists of a single instance. Bar-Noy et al. [2001] provided a 4-approximation variant of Algorithm **IS** for the generalization of this special case in which the amount of available resource may vary in time. Using a suitable reduction, their algorithm can be used to approximate (with approximation factor 4) the *general caching* problem, in which pages of

[9]A more reasonable definition of loss is to associate with each activity $\mathcal{A}$ a loss which is the difference between the maximum weight of an instance of $\mathcal{A}$ and the weight of the instance actually selected (if any). However, this changes the nature of the problem completely. We now associate a nonnegative penalty with each activity instance, and we must schedule exactly one instance of each activity. (To represent the possibility of rejecting the activity entirely, we add to it a virtual instance that intersects no other activity instances.)

varying sizes and penalties must be swapped in and out of a fixed size cache. (The motivation for this problem is caching World Wide Web pages; see Albers et al. [1999] and Bar-Noy et al. [2001] for more details.)

The second setting we consider is the enlargeable-resource setting. Here we must satisfy all jobs, and can achieve this by increasing the amount of resource. The objective is to minimize the cost of the resource used, hence such problems are called *resource minimization* problems. An example of a resource minimization problem is the *bandwidth trading* problem. In this problem we are given a set of machine *types* $\mathcal{T} = \{T_1, \ldots, T_m\}$ and a set of *jobs* $J = \{1, \ldots, n\}$. Each machine type T_i is defined by two parameters: a time interval $I(T_i)$ during which it is *available*, and a weight $w(T_i)$, which represents the cost of allocating a machine of this type. Each job j is defined by a single time interval $I(j)$ during which it must be processed. A given job j may be *scheduled feasibly* on a machine of type T if type T is available throughout the job's interval, i.e., if $I(j) \subseteq I(T)$. A *schedule* is a set of machines together with an assignment of each job to one of them. It is *feasible* if every job is assigned feasibly and no two jobs with intersecting intervals are assigned to the same machine. The cost of a feasible schedule is the total cost of the machines it uses, where the cost of a machine is defined as the weight associated with its type. The goal is to find a minimum-cost feasible schedule. Bhatia et al. [2003] present a 3-approximation local ratio algorithm for the *bandwidth trading* problem. Their algorithm is somewhat unusual in that it does not use an r-effective (or fully r-effective) weight function in the weight decomposition steps, nor does it return minimal solutions. Instead, it uses a weight function for which "good" solutions satisfy a certain property different from minimality. Accordingly, the algorithm modifies the solution returned from each recursive call in a rather elaborate manner, in order to transform it into a "good" solution.

6.4 Background

As mentioned earlier, single machine scheduling with one instance per activity is equivalent to *maximum weight independent set in interval graphs* and hence polynomial-time solvable [Golumbic 1980]. Arkin and Silverberg [1987] solve the problem efficiently even for unrelated multiple machines. The problem becomes NP-hard (even in the single machine case) if multiple instances per activity are allowed [Spieksma 1999] or if instances may require arbitrary amounts of the resource. (In the latter case the problem is NP-hard as it contains *knapsack* [Garey and Johnson 1979] as a special case in which all time intervals intersect.) Spieksma [1999] studies the unweighted interval scheduling problem. He proves that it is Max-SNP-hard, and presents a simple greedy 2-approximation algorithm. Bar-Noy et al. [2001] consider *real-time scheduling*, in which each job is associated with a release time, a deadline, a weight, and a processing time on each of the machines. They give several constant factor approximation algorithms for various variants of the throughput maximization problem. They also show that the problem of scheduling unweighted jobs on unrelated machine is Max-SNP-hard.

Bar-Noy et al. [2001] present a general framework for solving resource allocation and scheduling problems that is based on the local ratio technique. Given a resource of fixed size, they present algorithms that approximate the maximum throughput or the minimum loss by a constant factor. The algorithms apply to many problems, among which are: real-time scheduling of jobs on parallel machines; bandwidth allocation for sessions between two endpoints; general caching; dynamic storage allocation; and bandwidth allocation on optical line and ring topologies. In particular, they improve most of the results from Bar-Noy et al. [2001] either in the approximation factor or in the running time complexity. Their algorithms can also be interpreted within the primal-dual schema (see also Bar-Yehuda and Rawitz [2001]) and are the first local ratio (or primal-dual) algorithms for maximization problems. Section 6.2 is based on Bar-Noy et al. [2001]. Independently, Berman and DasGupta [2000] also improve upon the algorithms given in Bar-Noy et al. [2001]. They develop an algorithm for interval scheduling that is nearly identical to the one from Bar-Noy et al. [2001]. Furthermore, they employ the same rounding idea used in Bar-Noy et al. [2001] in order to contend with time windows. In addition to single machine scheduling, they also consider scheduling on parallel machines, both identical (obtaining an approximation guarantee of $(k+1)^k/((k+1)^k - k^k))$ and unrelated (obtaining an approximation guarantee of 2).

Chuzhoy et al. [2001] consider the unweighted *real-time scheduling* problem and present an $(e/(e-1)+\epsilon)$-approximation algorithm. They generalize this algorithm to achieve a ratio of $(1 + e/(e-1) + \epsilon)$ for unweighted *bandwidth allocation.*

7. ODDS AND ENDS

In this section we present three local ratio algorithms that do not fit properly in the frameworks introduced previously. Each algorithm deviates from these frameworks in a different manner, but they are all based on the Local Ratio Theorem and related ideas. The first two algorithms are interpretations of known algorithms for *longest path in a DAG* and *minimum s-t cut.* These problems are polynomial-time solvable, and the algorithms presented here demonstrate the use of the local ratio technique in the design of 1-approximation algorithms and the utilization of negative weights in this context. The third problem we consider is *capacitated vertex cover*, for which we describe a 2-approximation algorithm. The interesting feature of this NP-hard problem is that feasible solutions are not necessarily 0-1 vectors.

7.1 Longest Path in a DAG

The *longest path* problem is, given an edge-weighted graph and two distinguished vertices s and t, find a simple path from s to t of maximum *length*, where the *length* of a path is defined as the sum of weights of its edges. For general graphs (either directed or undirected) the problem is NP-hard, but for directed acyclic graphs (DAGs) it is solvable in linear time by a Dijkstra-like algorithm that processes the nodes in topological order. In this section we present a local ratio interpretation of this algorithm. We allow negative arc weights, and we assume without loss of generality that s is a root, i.e., that every node is reachable from s. (Otherwise, simply delete all unreachable nodes.)

Let us first introduce some terminology. We say that node u is a *proper successor* of node v if u is entered by exactly one arc, and this arc is (v, u).

OBSERVATION 10. *Let s be root of a DAG on two or more nodes. Then s has at least one proper successor in the graph. (For example, the second node in a topological sort of the graph is a proper successor of s.)*

We also define the act of *contracting* an arc (u,v) as the following three-step process:

(1) Delete the arc (u,v).
(2) Redirect every arc leaving (or entering) u to leave (or enter) v instead. This modification may create pairs of parallel arcs. For each such pair, delete the arc of lesser weight (breaking ties arbitrarily).
(3) Delete u.

Let G' be the graph obtained from G by contracting an arc (u,v), and let $P' = v, v_1, \ldots, v_k$ be a path leaving v in G'. The arc (v,v_1) in G' may have originated from (u,v_1) or from (v,v_1) in G. Depending on which the case is, we define the path P in G corresponding to P' to be either the path $u, v_1, \ldots, v_k$ or the path $u, v, v_1, \ldots, v_k$.

OBSERVATION 11. *Let G be an arc-weighted DAG with root s. Let v be a proper successor of s, and suppose $w(s,v) = 0$. Let G' be the graph obtained from G by contracting (s,v). Then:*

(1) G' is a DAG with root v.
(2) The maximum length of a path from s to t in G is equal to the maximum length of a path from v to t in G'.
(3) If P' is a longest path from v to t in G', then the corresponding path P in G is a longest path from s to t.

The algorithm is based on Observations 10 and 11. It repeatedly contracts the arc connecting the DAG's root to one of its proper successors.

Algorithm LPDAG(G, s, t, w)

1. If $s = t$ return the path consisting of the single node s.
2. Else, if s has a proper successor v such that $w(s,v) = 0$ do:
3. Let G' be the graph obtained by contracting (s,v).
4. $P' \leftarrow$ **LPDAG**(G', v, t, w).
5. Return the path in G corresponding to P'.
6. Else:
7. Let x be a proper successor of s and let $\epsilon = w(s,x)$.
8. Define the weight functions $w_1(u,v) = \begin{cases} \epsilon & u = s, \\ 0 & \text{otherwise,} \end{cases}$
   and $w_2 = w - w_1$.
9. Return **LPDAG**(G, s, t, w_2).

It is easy to see that w_1 is fully 1-effective for all ϵ (even negative!) Hence, Algorithm **LPDAG** solves the problem optimally.

7.2 Minimum s-t Cut

The *minimum s-t cut* problem is defined as follows. Given an arc-weighted directed graph $G = (N, A)$ and a pair of distinct distinguished nodes s and t, a *directed s-t* *cut* (or simply *cut* for short) is a set of arcs whose removal disconnects all directed paths from s to t. The goal is to find a *minimum* cut, i.e., a cut of minimum weight. Note that there always exists a minimum cut which is also minimal (in the sense of set inclusion), since arc weights are non-negative. Thus we restrict our attention to minimal cuts, and therefore have the following equivalent definition (which is more common in the literature). A cut is a partition of the graph nodes into two sets S and $\bar{S}$ such that $s \in S$ and $t \in \bar{S}$, and the *cut arcs* are the arcs leaving S and entering $\bar{S}$. We use the two definitions of *cut* interchangeably. We remark that the undirected version of the problem can be reduced to the directed case by replacing each edge by two arcs (in opposite directions) with the same weight as the original edge.

It is well known that the *minimum s-t cut* problem can be solved by max-flow techniques, particularly, by the Ford and Fulkerson algorithm for max-flow [Ford and Fulkerson 1956]. In this section we present a local ratio interpretation of the Ford and Fulkerson algorithm. Apart from the fact that the algorithm below solves an optimization problem optimally, it is also unique in that it is the first (and only, as far as we know) local ratio algorithm that uses negative weights in w_1[10]. Because of these negative weights, there are cases where zero weight arcs become positive again, and therefore we cannot delete zero-weight arcs as in previous algorithms.

In the algorithm below, we assume that for every arc (u, v) in the graph, the arc (v, u) exists as well. (If this is not the case we can simply add the missing arcs with zero weight.) The algorithm is based on the following observation. We say that an arc is *unsaturated* if its weight is strictly positive. A path is *unsaturated* if it consists of unsaturated arcs only. Let P be an unsaturated path from s to t and let $\overleftarrow{P}$ be its reverse (from t to s). Define

$$w_1(u,v) = \begin{cases} \epsilon & (u,v) \in P, \\ -\epsilon & (u,v) \in \overleftarrow{P}, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that w_1 is 1-effective because the cost (with respect to w_1) of every cut is precisely ϵ. To see this, let C be any minimal feasible solution, i.e., let C be a set of arcs defined by a cut $(S, \bar{S})$. Consider the path P. It starts at s and proceeds initially within S. At some point it crosses over to $\bar{S}$ and then possibly crosses the cut back and forth several times, ending finally in $\bar{S}$. It then continues to t within $\bar{S}$. Thus, if $k \geq 1$ is the number of times P crosses the cut from S to $\bar{S}$, then $k-1$ is the number of times it crosses the cut in the reverse direction. Thus C contains precisely k edges of P and $k-1$ edges of $\overleftarrow{P}$, and thus its weight with respect to w_1 is $\epsilon \cdot k - \epsilon(k-1) = \epsilon$. (See Figure 4 for an illustration.)

[10]Although we have allowed negative weights in Algorithm **LPDAG** (Section 7.1), that algorithm may be easily modified to avoid them. In contrast, negative weights are inherent in the present algorithm.

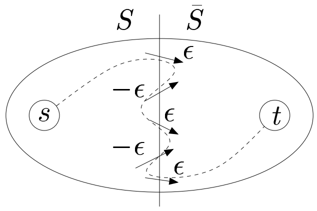

Fig. 4. The weight of a cut with respect to w_1.

Algorithm stCut(G, s, t, w)

1. Let S be the set of nodes reachable from s by unsaturated paths.
2. If $t \notin S$, return the cut $(S, \bar{S})$.
3. Else:
4. Let P be an unsaturated path from s to t and let $\overleftarrow{P}$ be its reverse.
5. Let $\epsilon = \min \{w(u, v) \mid (u, v) \in P\}$.
6. Define the weight functions $w_1(u, v) = \begin{cases} \epsilon & (u, v) \in P, \\ -\epsilon & (u, v) \in \overleftarrow{P}, \\ 0 & \text{otherwise}, \end{cases}$ and $w_2 = w - w_1$.
7. Return **stCut**(G, s, t, w_2).

Algorithm **stCut** is actually Ford and Fulkerson's algorithm [Ford and Fulkerson 1956]. Path P is an augmentation path, and w_2 defines the residual weights. As with the original algorithm, the time complexity of the algorithm depends on how P is chosen in Line 4. If it is chosen poorly, the algorithm might require pseudopolynomial time (or even fail to terminate if irrational weights are allowed).

7.3 Capacitated Vertex Cover

The *capacitated vertex cover* problem is similar to plain *vertex cover*, but now each vertex u has, in addition to its weight $w(u)$, also a *capacity* $c(u) \in \mathbb{N}$ that determines the number of edges it may cover. Specifically, vertex u may cover up to $c(u)$ incident edges. However, multiple "copies" of u can be used to cover additional edges[11]. Thus a feasible solution is an assignment of every edge to one of its endpoints, subject to the constraint that no edge is assigned to a zero-capacity vertex. Note that the presence of zero-capacity vertices may render the problem infeasible, but detecting this is easy: the problem is infeasible if and only if there is an edge whose two endpoints have zero capacity. We will assume henceforth that a feasible solution exists. Also note that the special case in which $c(u) \geq \deg(u)$ for every vertex u is the ordinary *vertex cover* problem.

[11] Such capacities are called *soft*. With *hard* capacities only one copy of every vertex is allowed.

To define the cost of a feasible solution A, let us introduce some notation. Denote by $A(u)$ the set of edges assigned to vertex u and by $\alpha(u)$ the number of copies of u required to cover the edges $A(u)$. (If $A(u) = \emptyset$, then $\alpha(u) = 0$. Otherwise, $\alpha(u) = \lceil |A(u)|/c(u) \rceil$. Note that in the latter case the ratio $|A(u)|/c(u)$ is well defined, since $c(u)$ must be positive.) The cost of assignment A is $\sum_u \alpha(u)w(u)$. Thus, although feasible solutions are defined as assignments of edges to vertices, when considering their cost, they can be viewed as vectors of α values, and thus can be treated within the local ratio framework.

In the description of the recursive algorithm below, we use $N(u)$ to denote the set of vertex u's neighbors, we denote by V_+ the set of vertices with non-zero capacity and non-zero degree, and we denote $\tilde{c}(u) = \min\{\deg(u), c(u)\}$.

Algorithm CVC($G = (V, E), c, w$)

1. If $E = \emptyset$, return the assignment $A(v) = \emptyset$ for all $v \in V$.
2. Else, if there exists a vertex $u \in V_+$ such that $w(u) = 0$ do:
3.   Let G' be the graph obtained from G by deleting u.
4. $A \leftarrow \mathbf{CVC}(G', c, w)$.
5.   For every $v \in N(u)$ do:
6.   If $\deg(v) \le c(v)$ and $A(v) \neq \emptyset$ do:
7. $A(v) \leftarrow A(v) \cup \{(u, v)\}$.
8.   Else:
9. $A(u) \leftarrow A(u) \cup \{(u, v)\}$.
10.   Return A.
11. Else:
12.   Let $\epsilon = \min_{u \in V_+}\{w(u)/\tilde{c}(u)\}$.
13.   Define the weight functions $w_1(v) = \epsilon \cdot \tilde{c}(v)$ and $w_2 = w - w_1$.
14.   Return $\mathbf{CVC}(G, c, w_2)$.

Consider the recursive call made in Line 4. In order to distinguish between the assignment obtained by this recursive invocation and the assignment returned in Line 10, we denote the former by A' and the latter by A. Assignment A' is for graph G', and we denote the corresponding parameters α' and $\tilde{c}'$. Similarly, we use α and $\tilde{c}$ for the parameters of A and G.

To analyze the algorithm, we use a charging scheme by which the cost of the solution is charged to the graph edges. More specifically, we imagine that each edge is alloted two coins, which it may distribute between its endpoints. We aim to show that there is a way to distribute the coins so that the number of coins given to any vertex u is at least $\alpha(u)\tilde{c}(u)$. The key observation is the following.

Observation 12. *Suppose a recursive call is made in Line 4. Let $v \in N(u)$. Then:*

(1) If $\deg(v) \le c(v)$ *and* $A'(v) \neq \emptyset$*, then* $\alpha(v) = \alpha'(v) = 1$ *and* $\tilde{c}(v) = \tilde{c}'(v) + 1$*. Thus* $\alpha(v)\tilde{c}(v) = \alpha'(v)\tilde{c}'(v) + 1$.

(2) *If* $\deg(v) > c(v)$ *or* $A'(v) = \emptyset$, *then* $A(v) = A'(v)$ *and therefore* $\alpha(v) = \alpha'(v)$. *Moreover, there are two (not mutually exclusive) cases:* $\alpha(v) = \alpha'(v) = 0$ *or* $\deg(v) \geq c(v) + 1$. *In the latter case the degree of* v *in* G' *is at least* $c(v)$, *and therefore* $\tilde{c}(v) = \tilde{c}'(v) = c(v)$. *Thus in either case* $\alpha(v)\tilde{c}(v) = \alpha'(v)\tilde{c}'(v)$.

(3) *In all cases,* $\alpha(v) = \alpha'(v)$. *(This follows from the previous two observations.)*

LEMMA 13. *There exists a charging scheme (with respect to graph* G *and assignment* A*) in which every vertex* u *is given at least* $\alpha(u)\tilde{c}(u)$ *coins. Thus* $\sum_u \alpha(u)\tilde{c}(u) \leq 2|E|$.

PROOF. The proof is by induction on the recursion. For the base case $E = \emptyset$ it is trivial. For the inductive step there are two cases. If the recursive call is made in Line 14, the claim follows by the inductive hypothesis. Otherwise, the recursive call is made in Line 4, and by the inductive hypothesis there exists a charging scheme for (G', A'). We extend this scheme to (G, A) as follows. First note that the transition from (G', A') to (G, A) consists of adding a single vertex u and a collection of edges incident on it, and assigning each of these edges to either u or its other endpoint. Thus we need only take care of u and its neighbors; these are the only vertices whose α and $\tilde{c}$ values may change. All other vertices are already satisfied by the charging scheme for (G', A'). Let $v \in N(u)$. If $\deg(v) \leq c(v)$ and $A'(v) \neq \emptyset$, then by Observation 12, $\alpha(v)\tilde{c}(v) - \alpha'(v)\tilde{c}(v) = 1$. Edge (u, v) (which is assigned to v) passes one of its coins to v to cover the difference, and passes its other coin to u. If, on the other hand, $\deg(v) > c(v)$ or $A'(v) = \emptyset$, then by Observation 12, v is already satisfied by the charging scheme for (G', A'). Edge (u, v) (which is assigned to u) passes both its coins to u. Thus we ensure that all of u's neighbors are satisfied, and it remains to show that u is satisfied as well. Consider the coins given to u by its incident edges. Each edge assigned to u contributes two coins and each edge not assigned to u contributes one coin. Thus the number of coins is

$$\begin{aligned} |A(u)| + \deg(u) &\geq |A(u)| + \tilde{c}(u) \\ &= \left(\frac{|A(u)|}{\tilde{c}(u)} + 1\right) \cdot \tilde{c}(u) \\ &\geq \left(\frac{|A(u)|}{c(u)} + 1\right) \cdot \tilde{c}(u) \\ &> \alpha(u)\tilde{c}(u). \end{aligned}$$

□

THEOREM 14. *Algorithm CVC returns* 2*-approximate solutions.*

PROOF. The proof is by induction on the recursion. In the base case the algorithm returns an empty assignment, which is optimal for the empty graph. For the inductive step there are two cases. If the recursive invocation is made in Line 4, then by the inductive hypothesis the assignment A' it returns is 2-approximate with respect to G'. Clearly, the optimum value for G can only be greater than the optimum value for G', and because $w(u) = 0$ and $\alpha(v) = \alpha'(v)$ for all $v \in V \setminus \{u\}$ (by Observation 12), the cost of A equals the cost of A'. Thus A is 2-approximate with respect to G. If the recursive call is made in Line 14, then by the inductive hypothesis the assignment is 2-approximate with respect to w_2. By Lemma 13 its cost with respect to w_1 is at most $2\epsilon|E|$, and clearly the optimal cost is at least $\epsilon|E|$. Thus the assignment is 2-approximate with respect to w_1 too, and by the Local Ratio Theorem it is 2-approximate with respect to w as well. $\square$

7.4 Background

The problem of finding the longest path in a DAG (also called *critical path*) arises in the context of PERT (Program Evaluation and Review Technique) charts. For more details see Cormen et al. [1990, page 538] or Even [1979, pp. 138–142]. Algorithm **stCut**, the local ratio interpretation of Ford and Fulkerson's Algorithm [Ford and Fulkerson 1956], is from Bar-Yehuda and Rawitz [2004]. The algorithm for *capacitated vertex cover* (Algorithm **CVC**) is a local ratio version of the primal-dual algorithm given by Guha et al. [2002].

8. FRACTIONAL LOCAL RATIO

As we have seen, the standard local ratio approach is to use a weight decomposition that guarantees that the solution constructed by the algorithm will be r-approximate with respect to w_1 (for some desired r). In essence, the analysis consists of comparing at each level of the recursion the solution found in that level and an optimal solution for the problem instance passed to that level, where the comparison is made with respect to w_1 and with respect to w_2. Thus, in each level of the recursion there are potentially two optima (one with respect to w_1 and one with respect to w_2) against which the solution is compared, and in addition, different optima are used at different recursion levels. The *fractional local ratio* paradigm takes a different approach. It uses a single solution x^* to the *original* problem instance as the yardstick against which all intermediate solutions (at all levels of the recursion) are compared. In fact, x^* is not even feasible for the original problem instance but rather for a relaxation of it. Typically, x^* will be an optimal fractional solution to a linear programming[12] (LP) relaxation.

The fractional local ratio technique is based on a "fractional" version of the Local Ratio Theorem. Recall that both previous versions of the theorem (Theorems 2 and 9) apply to problems that can be formulated as finding an optimal vector $x \in \mathbb{R}^n$ subject to a set of feasibility constraints $\mathcal{C}$. In all previous applications we have assumed that the constraints describe the problem to be solved precisely, and in particular, one of the constraints is that x must be a 0-1 (or at least an integral) vector. In contrast, applications of the fractional local ratio technique consider relaxations of the actual problems—typically, LP relaxations of their integer programming (IP) formulations. This is the context in which we formulate the new version of the theorem. Particularly, we consider non-integral vectors, hence the term *fractional*. We precede the statement of the theorem with the following useful definition.

Definition 4. *Let $r \geq 1$ and let $w \in \mathbb{R}^n$ be a vector of weights in the context of an optimization problem. Let x and x^* be vectors in $\mathbb{R}^n$. Vector x is said to be r-*

[12] This section assumes familiarity with the concepts of integer programming, linear programming, and LP relaxations.

approximate relative to x^* *(with respect to* w*) if* $w \cdot x \leq r(w \cdot x^*)$ *(for a minimization problem) or* $w \cdot x \geq (w \cdot x^*)/r$ *(for a maximization problem).*

The theorem is nothing more than the following straightforward observation.

THEOREM 15 FRACTIONAL LOCAL RATIO. *Let* $w, w_1, w_2 \in \mathbb{R}^n$ *be weight vectors in the context of an optimization problem such that* $w = w_1 + w_2$. *Let* x^* *and* x *be vectors in* $\mathbb{R}^n$ *such that* x *is* r*-approximate relative to* x^* *with respect to* w_1 *and with respect to* w_2. *Then,* x *is* r*-approximate relative to* x^* *with respect to* w *as well.*

The first step in the fractional local ratio technique is to obtain an optimal solution x^* to an LP relaxation of the problem. This solution is then used to drive a recursive algorithm that constructs a feasible solution (to the actual problem, not the relaxation) in a manner quite similar to the standard local ratio method. At each level of the recursion, either the problem size is reduced (conceptually) or the weight function is decomposed. The novelty of the fractional approach is that in weight decomposition steps, the construction of w_1 is based on x^* and the analysis compares the weight of the solution returned with $w \cdot x^*$. Thus the fractional local ratio technique is a combination of LP rounding and the standard local ratio approach.

Intuitively, the drawback of the standard local ratio technique is that at each level of the recursion we attempt to approximate the weight of a solution that is optimal with respect to the weight function present at that level. The superposition of these "local optima" may be significantly worse than the "global optimum," i.e., the optimum for the original problem instance. Conceivably, we could obtain a better bound if at each level of the recursion we approximated the weight of a solution that is optimal with respect to the original weight function. This is the idea behind the fractional local ratio approach.

The best way to explain the technique is by means of an example. In the next section we demonstrate the technique on *maximum weight independent set*, and following that we apply it to *maximum weight independent set in* t*-interval graphs*, which is somewhat more involved.

8.1 Maximum Weight Independent Set

In the *maximum weight independent set* problem we are given a vertex-weighted graph $G = (V, E)$, and are asked to find a maximum weight *independent set* of vertices. An *independent set* of vertices is a subset of V in which no two vertices are adjacent.

Consider the following weight function. For a given vertex v, let $N[v]$ denote the set consisting of vertex v and all its neighbors. Fix some vertex v and let $\epsilon > 0$. Define

$$w_1(u) = \begin{cases} \epsilon & u \in N[v], \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Delta = \max_{u \in V} \max\{1, \deg(u)\}$. Clearly, w_1 is Δ-effective since every feasible solution has weight at most $\epsilon \cdot \max\{1, \deg(v)\} \leq \epsilon \cdot \Delta$ and every maximal solution has weight at least ϵ. We can therefore use this weight function as the basis for a Δ-approximation algorithm in the standard local ratio framework. We can do better, however, by employing the fractional local ratio technique. Specifically, we can achieve an approximation guarantee of $\frac{1}{2}(\Delta+1)$, as we show now.

The first step is to solve the following LP relaxation of the natural IP formulation of the problem.

$$\text{Maximize} \sum_{u \in V} w(u) x_u \quad \text{subject to}$$

$$\begin{aligned} x_u + x_v &\leq 1 && \forall\, (u,v) \in E, \\ 0 \leq x_u &\leq 1 && \forall\, u \in V. \end{aligned}$$

Let x^* be an optimal (fractional) solution. We now run the recursive algorithm described next to obtain an independent set. Note that the algorithm does not contain problem-size reduction steps, as do other local ratio algorithms. Specifically, it does not delete vertices of non-positive weight. This is due to the nature of the analysis to follow; it is more convenient to simply ignore vertices of non-positive weight than to delete them.

Algorithm FracLRIS(G, w, x^*)

1. Let $V_+ = \{u \mid w(u) > 0\}$ and let $G_+ = (V_+, E_+)$ be the subgraph of G induced by V_+.
2. If $E_+ = \emptyset$, return V_+.
3. Let $v \in V_+$ be a vertex minimizing $\sum_{u \in N[v] \cap V_+} x^*_u$ and let $\epsilon = w(v)$.
4. Define the weight functions $w_1(u) = \begin{cases} \epsilon & u \in N[v] \cap V_+, \\ 0 & \text{otherwise,} \end{cases}$ and $w_2 = w - w_1$.
5. $S' \leftarrow$ **FracLRIS**(G, w_2, x^*).
6. If $S' \cup \{v\}$ is an independent set do:
7.   Return $S = S' \cup \{v\}$.
8. Else:
9.   Return $S = S'$.

For the analysis, let x denote the incidence vector of the solution S returned by the algorithm. We claim that x is $\frac{1}{2}(\Delta+1)$-approximate relative to x^* (or in other words, that $w \cdot x \geq 2/(\Delta+1)(w \cdot x^*)$), and thus S is $\frac{1}{2}(\Delta+1)$-approximate. As usual, the proof is by induction on the recursion, but to facilitate it we need an additional claim, namely, that $S \subseteq V_+$ (at all levels of the recursion). We prove both claims simultaneously. In the base case ($E_+ = \emptyset$) we have $S = V_+$ and $w \cdot x = \sum_{u \mid w(u)>0} w(u) \cdot 1 \geq \sum_{u \mid w(u)>0} w(u) \cdot x^*_u \geq \sum_u w(u) \cdot x^*_u = w \cdot x^*$. Since $w \cdot x \geq 0$, we get $w \cdot x \geq 2/(\Delta+1)(w \cdot x^*)$ even if $w \cdot x^*$ is negative. For the inductive step, let x' be the incidence vector of S' (obtained in Line 5), and let $V'_+ = \{u \mid w_2(u) > 0\}$. Then, by the inductive hypothesis and the definition of w_2, $S' \subseteq V'_+ \subseteq V_+$. Since $v \in V_+$, Lines 6–9 ensure that $S \subseteq V_+$. By the inductive hypothesis $w_2 \cdot x' \geq 2/(\Delta+1)(w_2 \cdot x^*)$. Since $w_2(v) = 0$, we get that $w_2 \cdot x = w_2 \cdot x'$, and therefore x is $\frac{1}{2}(\Delta+1)$-approximate relative to x^* with respect to w_2. In addition, Lines 6–9 also ensure that either $v \in S$ or else S contains some $v' \in N[v] \cap S' \subseteq N[v] \cap V_+$. In either case, S contains at least one vertex from $N[v] \cap V_+$ and thus $w_1 \cdot x \geq \epsilon$. The following lemma implies that $w_1 \cdot x^* \leq \frac{1}{2}\epsilon(\Delta+1)$. Hence, x is $\frac{1}{2}(\Delta+1)$-approximate relative to x^* with respect to both w_1 and w_2, and by the Fractional Local Ratio Theorem we get that x is $\frac{1}{2}(\Delta+1)$-approximate.

LEMMA 16. *The choice of v in Line 3 ensures that* $\sum_{u \in N[v] \cap V_+} x^*_u \leq \frac{1}{2}(\Delta+1)$.

PROOF. It suffices to show that $\sum_{u \in N[s] \cap V_+} x^*_u \leq \frac{1}{2}(\Delta+1)$ is satisfied for some vertex $s \in V_+$. We show this for $s = \arg\max_{u \in V_+} x^*_u$. Note that $N[s] \cap V_+$ is the set consisting of s and its neighbors in G_+, and that $|N[s] \cap V_+| \leq \Delta + 1$. If s is an isolated vertex in G_+, it clearly satisfies the condition. Otherwise, if $x^*_s \leq 1/2$, then $x^*_u \leq 1/2$ for all vertices u in G_+, and the condition is again satisfied. Otherwise, $x^*_s > 1/2$ and thus $x^*_u < 1/2$ for all neighbors u of s in G_+. Fix any such neighbor s'. Then $\sum_{u \in N[s] \cap V_+} x^*_u = (x^*_s + x^*_{s'}) + \sum_{u \in (N[s] \cap V_+) \setminus \{s,s'\}} x^*_u < 1 + \frac{1}{2}(\Delta-1) = \frac{1}{2}(\Delta+1)$. □

8.1.0.1 *Remark.* The analysis above shows that v can actually be chosen (in Line 3) as *any* vertex satisfying $\sum_{u \in N[v] \cap V_+} x^*_u \leq \frac{1}{2}(\Delta+1)$—not necessarily the one minimizing $\sum_{u \in N[v] \cap V_+} x^*_u$. By the proof of Lemma 16, the simplest such choice is the vertex in $v \in V_+$ maximizing x^*_v.

8.2 Maximum Weight Independent Set in a t-interval Graph

Consider a teaching laboratory shared by several different academic courses, where each course has a schedule during which it requires the use of the laboratory for t (or less) sessions of varying duration. The problem is to maximize the utilization of the laboratory by admitting a subset of non-conflicting courses. This type of problem has a simple abstraction as the problem of finding a maximum weight independent set in a vertex-weighted *t-interval* graph. To define what a t-interval graph is, we need the following auxiliary definitions. A *t-interval system* is a collection $\{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n\}$ of nonempty sets of real intervals, where each $\mathcal{I}_i$ consists of t or less disjoint intervals. Given a t-interval system, the corresponding *intersection* graph is defined as follows. Each set $\mathcal{I}_i$, is represented by a vertex v_i, and there is an edge between two distinct vertices v_i and v_j if some interval in $\mathcal{I}_i$ intersects some interval in $\mathcal{I}_j$. A *t-interval* graph is the intersection graph of some t-interval system. The set system is then said to be a *realization* of the graph. (Note that a given t-interval graph may have many different realizations.) Figure 5 shows a 2-interval graph and one of its realizations.

In this section we present a $2t$-approximation algorithm for the problem of finding a maximum weight independent set in a vertex-weighted t-interval graph. Since even deciding whether a given graph is t-interval is NP-hard (for $t \geq 2$) [West and Shmoys 1984], we shall assume that a realization of the graph is given as part of the input. Given a realization of a t-interval graph, we identify each vertex with the corresponding set of intervals, and furthermore, we adopt the point of view that the intervals are not merely pairs of endpoints, but rather are objects in their own right.

What we mean by this is that if a certain interval $[a, b]$ belongs to two vertices, we do not view $[a, b]$ as a single interval shared by two vertices, but rather as two distinct intervals that happen to have identical endpoints. Thus, when viewed as sets of intervals, the graph vertices are always pairwise disjoint. In addition, for a given point p and vertex v, we denote by $p \in\in v$ the statement *p is contained in some interval belonging to v*. Finally, we assume for simplicity that the intervals are closed.

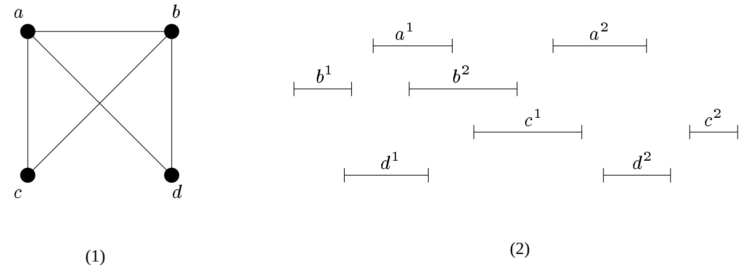

Fig. 5. (1) A 2-interval graph; (2) one of its realizations.

The algorithm is nearly identical to the algorithm for *maximum weight independent set* in general graphs presented in the previous section; the only difference is that in the first step we solve the following LP relaxation, rather than the one used there. Let R denote the set of right endpoints of intervals in the system. The linear program is:

$$\text{Maximize} \sum_{u \in V} w(u) x_u \quad \text{subject to}$$

$$\begin{aligned} \sum_{u \mid p \in\in u} x_u \leq 1 \qquad & \forall\, p \in R, \\ 0 \leq x_u \leq 1 \qquad & \forall\, u \in V. \end{aligned}$$

In the second step we run Algorithm **FracLRIS** on the optimal (fractional) solution found, as before. The analysis is similar, except that Lemma 16 is replaced by the following lemma, which provides an approximation guarantee of $2t$.

LEMMA 17. *The choice of v in Line 3 ensures that $\sum_{u \in N[v] \cap V_+} x_u^* \leq 2t$.*

PROOF. To reduce clutter, we restrict the discussion to the graph G_+. Thus, *vertex* means *vertex in* V_+; $\sum_u$ stands for $\sum_{u \in V_+}$; $N[v]$ stands for $N[v] \cap V_+$; etc. Also, If I is an interval belonging to vertex v, we define $x_I^* = x_v^*$. In general, we will use I and J to denote intervals belonging to vertices.

To prove the lemma it suffices to show that there exists a vertex s such that $\sum_{u \in N[s]} x_u^* \leq 2t$. Clearly, every isolated vertex satisfies this condition, so we assume that G_+ contains no isolated vertices. Thus, an equivalent claim to the above is that there exists a vertex s satisfying $x_s^* \sum_{u \in N[s]} x_u^* \leq 2t x_s^*$ (note that $x_s^* > 0$ since s is a vertex in G_+), and to prove this claim it suffices to show that $\sum_s x_s^* \sum_{u \in N[s]} x_u^* \leq 2t \sum_s x_s^*$.

Let us denote by $I \backsim J$ the statement *interval I and interval J intersect*, and by $J \backsim_R I$ the statement *interval J contains the right endpoint of interval I.* (Note that $I \backsim I$ and $I \backsim_R I$ for all I.) Then,

$$\begin{aligned}\sum_s x_s^* \sum_{u \in N[s]} x_u^* &\leq \sum_{I \backsim J} x_I^* x_J^* \\ &\leq 2 \sum_{J \backsim_R I} x_I^* x_J^* \\ &= 2 \sum_s \sum_{I \in s} \sum_{J \backsim_R I} x_I^* x_J^* \\ &= 2 \sum_s x_s^* \sum_{I \in s} \sum_{J \backsim_R I} x_J^*.\end{aligned}$$

(In the right-hand-side of the first inequality, I and J are ordered, so every pair of distinct intersecting intervals appears twice.) The second inequality follows from the fact that if two intervals intersect, one of them necessarily contains the right endpoint of the other. To complete the proof, note that the feasibility constraints ensure that $\sum_{J \backsim_R I} x_J^* \leq 1$ for all intervals I, and thus $\sum_{I \in s} \sum_{J \backsim_R I} x_J^* \leq t$ for all s (as every vertex contains at most t intervals). □

8.3 Background

The algorithm for maximum weight independent set in t-interval graphs (Algorithm **FracLRIS**) is taken from Bar-Yehuda et al. [2002], where it is also shown that the problem is APX-hard for all $t \geq 2$, even for highly restricted instances. (Note that a 1-interval graph is an ordinary interval graph, in which finding a maximum weight independent set is a polynomial-time solvable problem.) Prior to that paper the problem was considered only on proper union graphs [Bafna et al. 1996]—a restricted subclass of t-interval graphs—and the approximation factor achieved was $(2^t - 1 + 1/2^t)$.

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