On the Equivalence Between the Primal-Dual Schema and the Local-Ratio Technique

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Technion

Local-Ratio Technique

Given: A set of feasibility constraints $\mathcal{F}$ on $x \in \{0,1\}^n$

A weight function $p \in (\mathbb{R}^+)^n$

Objective: Minimize (or Maximize) $p \cdot x$

Local-Ratio Theorem [BBFNS00]:
Let p and p_1 be weight functions. Then, if x is an r-approx w.r.t. $(\mathcal{F}, p_1)$ and $(\mathcal{F}, p - p_1)$, then x is an r-approx w.r.t. $(\mathcal{F}, p)$.

Intuition:
Find a *good* weight function p_1.
Solve recursively for $p - p_1$.

Definition:
A feasible solution x is a called *minimal* if $x \setminus \{j\}$ is not feasible for all $j \in x$.

A feasible solution x is a called *maximal* if $x \cup \{j\}$ is not feasible for all $j \notin x$.

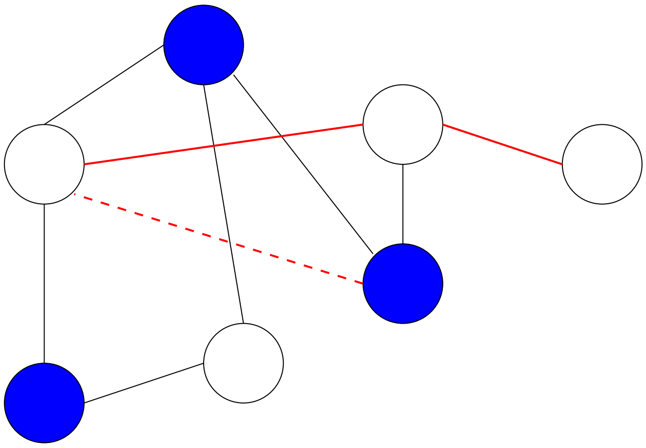
Generalized Vertex Cover

Instance: $G = (V, E)$, $p : V \cup E \to \mathbb{R}^+$

Solution: $C \subseteq V$ and $F \subseteq E$

$\forall e \in E \setminus F, \quad e \cap C \neq \emptyset$

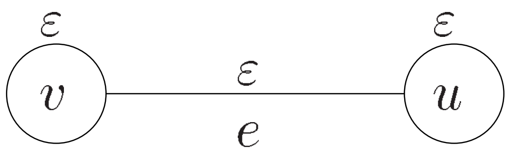
Measure: $\sum_{v \in C} p(v) + \sum_{e \in F} p(e)$



We pay for the blue vertices and red edges.

$$
\begin{array}{lll}
\min & \displaystyle\sum_{u \in V} p(u) x_u + \sum_{e \in E} p(e) x_e & \\
\text{s.t.} & x_u + x_v + x_e \geq 1 & \forall e = (u, v) \\
 & x_v, x_e \in \{0, 1\} & \forall v, e
\end{array}
$$

Generalized VC/Local-Ratio



$$p_1(t) = \begin{cases} \varepsilon, & t \in \{u, v, e\} \\ 0, & \text{otherwise} \end{cases}$$

Observation:

For any *minimal* solution x: $\quad \varepsilon \leq p_1 \cdot x \leq 2\varepsilon$

p_1 is a *2-effective weight function*

2-approx Algorithm:

1. For each edge $e = (u, v) \in E$
2. $\quad \varepsilon \leftarrow \min \{p(u), p(v), p(e)\}$
3. $\quad p \leftarrow p - p_1$
4. Return $\{u : p(u) = 0\}$,

   and $\{e = (u, v) : p(u), p(v) > 0\}$

Generalized VC/Primal-Dual

[GW97] Generic algorithm based on the Hitting Set problem

$$
\begin{array}{lll}
\max & \sum_{e \in E} y_e & \\
\text{s.t.} & \sum_{e: u \in e} y_e \leq p(u) & \forall u \in V \\
& y_e \leq p(e) & \forall e \in E \\
& y_e \geq 0 & \forall e \in E
\end{array}
$$

Algorithm:

1. $y \leftarrow 0$, $A \leftarrow \emptyset$
2. For each edge $e = (u, v)$:
3. $\quad y_e \leftarrow \min\{p(u) - \sum_{e': u \in e'} y_{e'},\ p(v) - \sum_{e': v \in e'} y_{e'},\ p(e)\}$
4. $\quad t \leftarrow \operatorname{argmin}\{p(u) - \sum_{e': u \in e'} y_{e'},\ p(v) - \sum_{e': v \in e'} y_{e'},\ p(e)\}$
5. $\quad A \leftarrow A \cup \{t\}$
6. For each graph element $t \in A$
7. $\quad$ If $A \setminus \{t\}$ is feasible then $A \leftarrow A \setminus \{t\}$
8. Output A

Generalized VC/Primal-Dual

[BT98] Generic r-approx alg. for covering problems which uses inequalities

Observation:

Let $e = (u, v)$. For any *minimal* solution x:

$$1 \leq x_u + x_v + x_e \leq 2$$

$x_u + x_v + x_e \geq 1$ is 2-*effective*

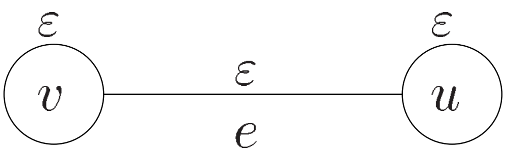
2-approx Algorithm:

1. For each edge $e = (u, v) \in E$
2. $\quad y_e \rightarrow \min\{p(u), p(v), p(e)\}$
3. $\quad p(u) \rightarrow p(u) - y_e$
4. $\quad p(v) \rightarrow p(v) - y_e$
5. $\quad p(e) \rightarrow p(e) - y_e$
6. Return $\{u : p(u) = 0\}$,

    and $\{e = (u, v) : p(u), p(v) > 0\}$

Remark: We do not modify the problem.

Equivalence



$$p_1 \text{ is 2-effective}$$
$$\equiv$$
$$x_u + x_v + x_e \geq 1 \text{ is 2-effective}$$

Lemma:
$\alpha x \geq \beta$ is an r-effective inequality
iff
α is an r-effective weight function
(with β as a witness).

Corollary:
The integrality gap of an integer program serves as a bound to the approximation ratio of a local-ratio algorithm.

Minimum s,t-cut

Instance:	digragh $G = (V, E)$,
	$s, t \in V$,
	$p : E \to \mathbb{R}^+$ (capacities)
Solution:	$F \subseteq E$ s.t. there is no path
	from s to t in $(V, E \setminus F)$
Measure:	$\sum_{e \in F} p(e)$

IP formulation:

$$\begin{array}{lll} \min & \sum_{e \in E} p(e) x_e & \\ \text{s.t.} & \sum_{e \in q} x_e \geq 1 & \forall q \in \text{Path}(s,t) \\ & x_e \in \{0,1\} & \forall e \in E \end{array}$$

where $x_e = 1$ iff e is in the solution F.

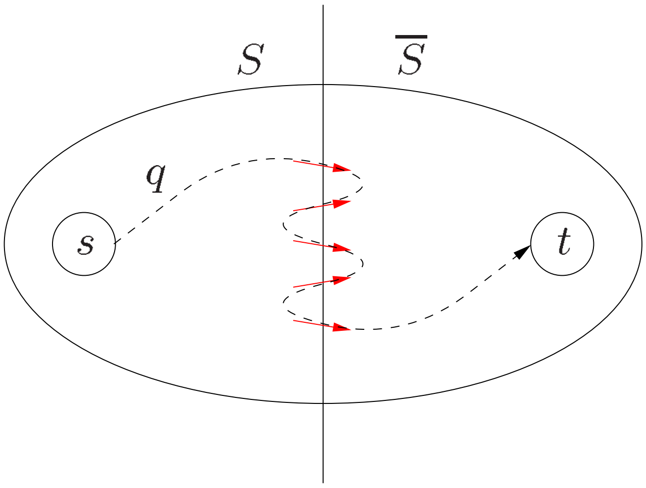
Observation: Minimal solution $\equiv$ Cut

Minimum s, t-cut

$$q \in \mathrm{Path}(s,t) \quad \text{and} \quad \bar{q} \in \mathrm{Path}(t,s)$$

Observation:

q passes through $(S : \overline{S})$: $k \geq 1$ times from S to $\overline{S}$, and $k-1$ times from $\overline{S}$ to S.



$$N^q(e) \triangleq \begin{cases} 1, & e \in q \\ -1, & e \in \bar{q} \\ 0, & \text{otherwise} \end{cases}$$

Therefore, is F is a cut $\quad \sum_{e \in F} N^q(e) = 1$

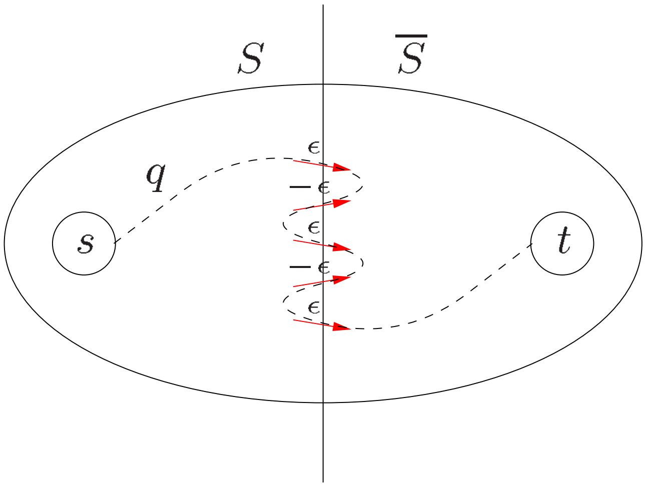
Minimum s, t-cut/Local Ratio

Local-Ratio Analysis of Ford & Fulkerson's alg.

Conclusion: N^q is 1-effective

Algorithm FF-LR:

1. Let $F = \{e : p(e) = 0\}$
2. If F is feasible then return
   a feasible minimal subset $F' \subseteq F$
3. Let q be a path from s to t in $(V, E \setminus F)$
4. $\varepsilon \leftarrow \min_{e \in q} \{p(e)\}$
5. Return FF-LR$(p - \varepsilon N^q)$



Remark: $p \equiv$ capacities of residual graph

Minimum s, t-cut/Primal-Dual

Conclusion: $\sum_{e \in E} N^q(e) x_e \geq 1$ is 1-effective

Algorithm FF-PD:

1. Let $F = \{e : p(e) = 0\}$
2. If F is feasible then return
   a feasible minimal subset $F' \subseteq F$
3. Let q be a path from s to t in $(V, E \setminus F)$
4. $y_q \leftarrow \min_{e \in q} \{p(e)\}$
5. Return FF-PD$(p - y_q N^q)$

Remarks:

- First local-ratio algorithm which uses negative weights
- First primal-dual algorithm (within the PD schema) which uses constraints with some negative coefficients

Interval Scheduling

Single *resource*

A set of *activities* $\mathcal{A}_1, \ldots, \mathcal{A}_m$

An *instance* $I \in \mathcal{A}_i$: $[s(I), e(I))$ and $p(I) \geq 0$

A feasible *schedule*:

(1) at most one instance of every activity

(2) at most one instance for all time instants t.

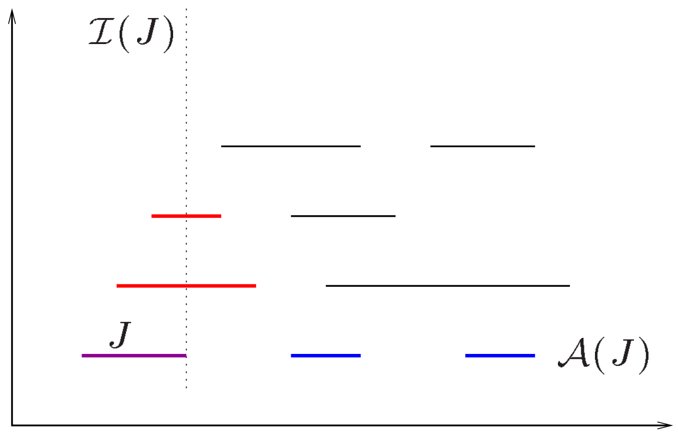
IP formulation:

$$\begin{array}{lll} \max & \sum_I p_I x_I & \\ \text{s.t.} & \sum_{I: s(I) \leq t < e(I)} x_I \leq 1 & \forall t \\ & \sum_{I: I \in \mathcal{A}_i} x_I \leq 1 & \forall i \in \{1, \ldots, m\} \\ & x_I \in \{0, 1\} & \forall i \; \forall I \in A_i, \end{array}$$

Interval Scheduling

[BBFNS00] - ????

Let J be an instance with minimum end-time.



$\mathcal{A}(J)$ - activity to which J belongs
$\mathcal{I}(J)$ - the set of instances intersecting J

$$p_1(I) = p(J) \cdot \begin{cases} 1, & I \in \mathcal{A}(J) \cup \mathcal{I}(J) \\ 0, & \text{otherwise} \end{cases}$$

$$\sum_{I : I \in \mathcal{A}(J) \cup \mathcal{I}(J)} x_I \le 2$$

are $\frac{1}{2}$-effective.

Our framework:

- Extends known frameworks for covering problems ([Bar00,BT98]) to non-covering problems (e.g., min-2SAT)
- Uses negative weights/coefficients
- Can be used for maximization problems

Remarks:

- Not all primal-dual algorithms fall within this framework (e.g., [JV99])
- Not all local-ratio algorithms fall within this framework (e.g., [BHNSS], [BCFN])