Scheduling Split Intervals *

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Abstract

We consider the problem of scheduling jobs that are given as *groups* of non-intersecting segments on the real line. Each job J_j is associated with an interval, I_j, which consists of up to t segments, for some $t \geq 1$, and a positive weight, w_j; two jobs are in conflict if any of their segments intersect. Such jobs show up in a wide range of applications, including the transmission of continuous-media data, allocation of linear resources (e.g. bandwidth in linear processor arrays), and in computational biology/geometry. The objective is to schedule a subset of non-conflicting jobs of maximum total weight.

Our problem can be formulated as the problem of finding a *maximum weight independent set* in a *t-interval* graph (the special case of $t = 1$ is an ordinary interval graph). We show that, for $t \geq 2$, this problem is APX-hard, even for highly restricted instances. Our main result is a $2t$-approximation algorithm for general instances. This is based on a novel *fractional* version of the Local Ratio technique. One implication of this result is the first constant factor approximation for non-overlapping alignment of genomic sequences. Previously, the problem was considered only for proper union graphs, a restricted subclass of t-interval graphs, and the best approximation factor known was $(2^t + 1)/2$ [7]. Finally, a bi-criteria *polynomial time approximation scheme* is developed for the subclass of *t-union* graphs.

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1 Introduction

1.1 Problem Statement and Motivation

We consider the problem of scheduling jobs that are given as *groups* of non-intersecting segments on the real line. Each job J_j is associated with a *t-interval*, I_j, which consists of up to t disjoint segments, for some $t \geq 1$, and a positive weight, w_j; two jobs are in conflict if any of their segments intersect. The objective is to schedule on a single machine a subset of non-conflicting jobs whose total weight is maximum.

An instance of our problem can be modeled as the intersection graph of t-intervals, known as a *t-interval graph*. Each vertex in the graph corresponds to an interval that has been *"split"* into t parts, or segments, such that two vertices u and v intersect if and only if some segment in the interval corresponding to u intersects with some segment in the interval corresponding to v (an example is given in Figure 1). Note that 1-interval graphs are precisely interval graphs. For a given instance of our problem, we seek to find a *maximum weight independent set* (MWIS) in the resulting weighted t-interval graph, that is, a subset of non-adjacent vertices $U \subseteq V$, such that the weight of U is maximized.

We describe below several practical scenarios involving t-interval graphs.

Transmission of Continuous-media Data. Traditional multimedia servers transmit data to the clients by *broadcasting* video programs at pre-specified times. Modern systems allow to replace broadcasts with the allocation of video data streams to individual clients *upon request*, for some time interval (see, e.g., [32, 6]). In this operation mode, a client may wish to take a break, and resume viewing the program at some later time. This scenario is natural, e.g., for video programs that are used in remote education [13].

Suppose that a client starts viewing a program at time t_0. At time t_1 the client takes a break, and resumes viewing the program at t_2, till the end of the program (at t_3). This scenario can be described by a *split interval*, I, that consists of two segments: $I^1 = (t_0, t_1)$ and $I^2 = (t_2, t_3)$.

The scheduler may get many requests formed as split intervals; each request is associated with a profit which is gained by the system only if *all* of the segments corresponding to the request are scheduled. The goal is to schedule a subset of non-overlapping requests that maximizes the total profit, i.e., find a MWIS in the intersection graph of the split intervals.

Most of the previous work in this area describe analytic models (e.g., [30]) or experimental studies, in which VCR-like operations can be used by the clients (see [6, 11, 32, 44]); however, these studies focus on the efficient use of system resources while supporting such operations, rather than on the scheduling problem.

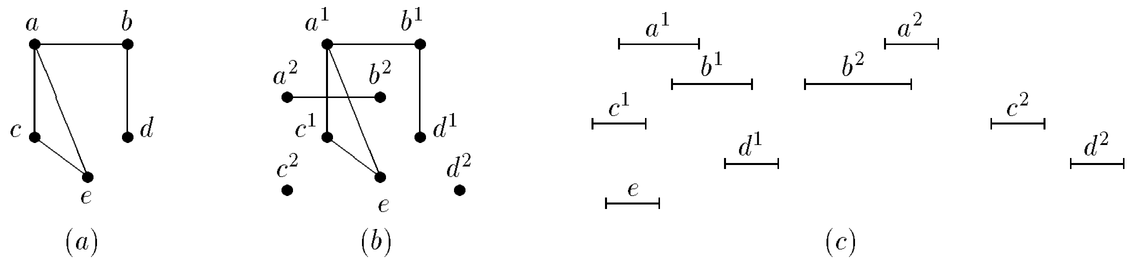

Figure 1: An example of a 2-interval graph (a), corresponding interval (segment intersection) graph (b), and interval system (c).

Linear Resource Allocation. Another application is linear resource allocation [22]. Requests for a linear resource can be modeled as intervals on a line; two requests for a resource can be scheduled together unless their intervals overlap. A disk drive is a linear resource when requests are for contiguous blocks [36]. A linear array network is a linear resource, since a request for bandwidth between processors i and j requires that bandwidth be allocated on all intervening edges. Consider a computer system that consists of a linear array network and a large disk. A scheduler must decide when to schedule requests, where each request may comprise distinct requests to these two linear resources, e.g., "a certain amount of bandwidth between processors 4 and 7, and a lock on blocks 1000-1200 of the disk". Two requests are in conflict if they overlap on the disk or in their bandwidth requirements. Thus, when the goal is to maximize the amount of requests satisfied by the system, we get an instance of the MWIS problem on a subclass of 2-interval graphs, known as *2-union* graphs (see in Section 2.1.)

Genomic Sequence Similarity. Bafna, Narayan and Ravi [2] considered determining the similarity between a pair of genetic sequences under large-scale mutational operations, including reversal and transposition. There are non-negative weights attached to pairs of (contiguous) subsequences that measure their similarity, e.g. derived from *local alignment*. This is represented as an intersection graph of two-dimensional axis-parallel boxes, where a pair of boxes is independent (or non-adjacent in the graph) if their projections on both axes are disjoint. The maximum global non-overlapping alignment of the sequences then corresponds to a maximum weight independent set in the intersection graph. More generally, multiple alignment of t sequences corresponds to the MWIS problem in t-union graphs. Previously, the problem was only considered in the case where the projections of input boxes did not contain one another, i.e., the case of proper t-union graphs. While making the problem easier, this restriction is not intrinsic to the biological problem.

Computational Geometry. The problem of finding an independent set among a set of multi-dimensional axis-parallel boxes is of independent interest in computational geometry. It corresponds to the MWIS problem in t-union graphs, a subclass of t-interval graphs.

1.2 Our Results

We provide a comprehensive study of the MWIS problem in t-interval graphs. In Section 2, we show that the problem is APX-hard even on highly-restricted instances, namely, on $(2, 2)$-union graphs.[1] In Section 3 we discuss some structural properties of t-interval graphs. In particular, we derive a bound on the inductiveness of a t-interval graph. As a corollary, we extend the best bound known on the chromatic number of t-interval graphs of Gyárfás [18]. We show this bound to be asymptotically optimal.

In Section 3.2, we study the MWIS problem on 2-interval graphs. We show that a simple greedy algorithm achieves the ratio $O(\min\{\log R, \log n\})$, where R is the ratio between the longest and shortest segment in the instance.

Our main result (in Section 4) is a $2t$-approximation algorithm for MWIS in any t-interval graph, for $t \geq 2$, which is based on a novel *fractional* version of the Local Ratio technique. (The Local Ratio technique was first developed in [4] and later extended by [3, 5].) We use the fractional Local Ratio technique to round a fractional solution obtained from a linear programming relaxation of our problem. Previously, the problem was considered only for proper t-union graphs, a restricted subclass of t-interval graphs, and the best approximation factor known was $(2^t + 1)/2$ [7]. We expect that our non-standard use of the Local Ratio technique will find more applications. Indeed, recently, this technique was used for obtaining improved bounds for MWIS in the intersection graph of axis parallel rectangles in the plane [29].

As we shall see, the MWIS in t-interval graphs properly includes the k-dimensional matching problem. For this unweighted problem the best known approximation factor is $k/2 + \epsilon$, for any $\epsilon > 0$ [24]. Hazan, Safra, and Schwartz [37] have recently shown that it is hard to approximate the k-dimensional matching problem within an $O(k/\log k)$ ratio unless P = NP. Thus, our results are close to best possible.

For the class of t-union graphs, we develop (in Section 5) a *bi-criteria* PTAS. Given $\epsilon > 0$, our scheme finds a subset of intervals of optimal weight and a schedule where each interval is delayed by at most $\epsilon T_{\mathcal{O}}$, assuming that there exists an optimal solution whose latest completion time is $T_{\mathcal{O}}$.

[1] See the definition in Section 2.1.

1.3 Related Work

We mention below several works that are related to ours.

Split interval graphs. Many NP-hard problems, including MWIS [15, 16], can be solved efficiently in interval graphs. Split interval graphs have a long history in graph theory [42, 17, 38, 43], and more recently, union graphs have been studied under the name of *multitrack* interval graphs [28, 19, 27]. We mention some of the main results. For any fixed $t \geq 2$, determining whether a given graph is a t-interval (t-union) graph is NP-complete [43] ([19], respectively). 2-union graphs contain trees [42, 28] and more generally all outerplanar graphs [27], while 3-interval graphs contain the class of planar graphs [38]. Graphs of maximum degree Δ are $\lceil \frac{1}{2}(\Delta+1) \rceil$-interval graphs [17]. The complete bipartite graph, $K_{m,n}$, is a t-interval and t-union graph for $t = \lceil (mn+1)/(m+n) \rceil$ [42, 19].

Union graphs, which constitute a sub-family of split interval graphs, were also considered in several papers. As mentioned earlier, Bafna *et al.* [2] considered the problem of finding a weighted independent set in t-union graphs in the context of an application coming from computational biology. The union graphs considered in [2] are *proper*, i.e., there is no containment between segments. For the weighted independent set problem in proper t-union graphs, the paper [2] shows that the problem is NP-hard and gives a $(2^t - 1 + 1/2^t)$-approximation algorithm. This is obtained by mapping the problem to finding weighted independent set in (2^t+1)-claw free graphs, noting that proper t-union graphs are (2^t+1)-claw free. The best factor known is $(2^t+1)/2$ due to Berman [7]. Recently, Berman et al. showed in [9] that a simple $O(n \log n)$ algorithm (based on the local ratio technique) yields a factor of 3 for proper 2-union graphs. This algorithm can be extended to yield a $(2^t - 1)$-approximation for proper t-union graphs.

Coupled-tasks and flow shop scheduling. The problem of scheduling 2-intervals (known as *coupled-task scheduling*) was considered in the area of machine scheduling, with the objective of minimizing the overall completion time, or *makespan* (see e.g. [33, 40]). Relaxed versions of the problem, that require only a lower bound on the time that elapses between the schedules of the two tasks of each job (also called *time-leg problems*) were studied, e.g., in [35, 12, 10].

An instance of our problem can be viewed as an instance of the *flow shop* problem, in which the segments and break times are represented by *tasks* that need to be processed on a set of $m = 2t+1$ machines. (The precise transformation is given in Section 5.) In general, the flow shop problem, where the objective is to minimize the makespan, is NP-complete even on three machines ([14]). The best result known is $O(\log^2(m\mu)/\log\log(m\mu))$-approximation algorithm, where μ is the maximum number of operations per job, and m is the number of machines ([39, 41]). Hall [20] gave a PTAS for this problem in the case where m is fixed (but arbitrary).

2 Preliminaries

2.1 Definitions and Notation

Let $\mathcal{I}$ be a collection of segments (or intervals) on the real line, partitioned into disjoint *groups* containing at most t segments, where $t \geq 1$. A t-interval graph $G = (V, E)$ is the intersection graph of the groups of segments. Each vertex $v \in V$ corresponds to a group of segments, and $(u, v) \in E$ if one of the segments belonging to the group of u intersects some segment belonging to the group of v. We call a vertex in a t-interval graph a *split interval*. Given a t-interval graph, we assume that each vertex can be mapped to a set of segments, i.e., we can say that a segment I belongs to a vertex v and denote it by (v, I). A t-interval graph is *proper* if no segment properly contains another segment.

In the subfamily of t-union graphs, the segments associated with each vertex can be labeled in such a way that for any two vertices u and v, the ith segment of u and the ℓth segment of v never intersect for $1 \leq i, \ell \leq t$, and $i \neq \ell$. Union graphs correspond also to certain geometric intersection graphs. The t segments are viewed as intervals on orthogonal axes, corresponding to a t-dimensional box; two boxes intersect if their projections on any of the t axes do. We further define subclasses of union graphs, where coordinates are all integral. In an (a, b)-union graph, all x-segments are of length a and y-segments of length b.

Given a graph $G = (V, E)$, we denote by $N(v)$ the set of neighbors of $v \in V$, and by $N[v]$ the closed neighborhood of v, $\{v\} \cup N(v)$. A $(k+1)$-*claw* is a subgraph consisting of a center vertex adjacent to $k+1$ mutually non-adjacent vertices. A graph is called $(k+1)$-*claw free* if it does not contain an m-claw, where $m \geq k+1$.

Finally, we define our performance measures. Denote by OPT an optimal algorithm. The *approximation factor* of an algorithm $\mathcal{A}$ is r if for every finite input instance I, $\mathcal{A}(I)/OPT(I) \geq 1/r$, where $\mathcal{A}(I)$ and $OPT(I)$ are the values of $\mathcal{A}$ and OPT on I. A *polynomial time approximation scheme (PTAS)* is an algorithm $\mathcal{A}$ which takes as input both the instance I and an error bound ϵ, has the performance guarantee $R_{\mathcal{A}}(I, \epsilon) \leq (1+\epsilon)$, and runs in time polynomial in $|I|$. A (β, ϵ) *bi-criteria PTAS* is a PTAS which is a β-approximation in one optimization criterion, and a $(1+\epsilon)$-approximation in the other criterion.

2.2 Hardness Results

The independent set problem in interval graphs is easy to solve exactly, since interval graphs always contain a *simplicial* vertex, i.e., a vertex whose neighborhood is a clique. In fact, most approximation algorithms for independent sets on geometric intersection graphs are based on a related relaxed property: there always exists a vertex whose neighborhood does not contain a large independent set. We first show that for general t-interval graphs this property does not hold.

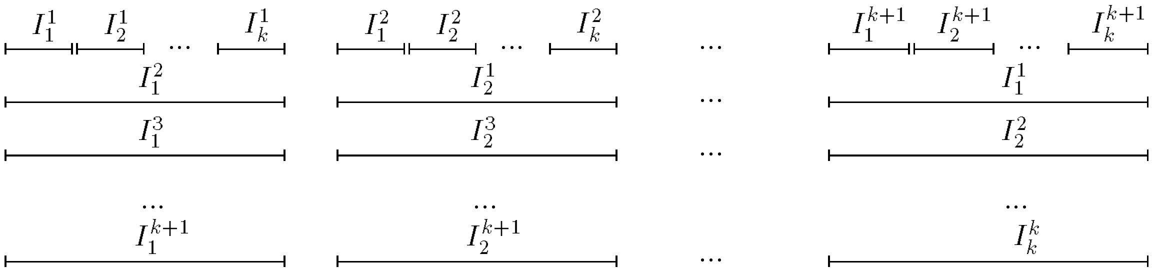

Figure 2: An example of a 2-interval graph, in which every vertex has k independent neighbors.

Observation 2.1 *For any $n \geq 2$, there exists a 2-interval graph G on n vertices, in which every vertex has $\Omega(\sqrt{n})$ independent neighbors.*

Proof: For a given $n \geq 2$, let $k = \lfloor (\sqrt{4n+1} - 1)/2 \rfloor$. We show how to construct a 2-interval graph, in which every vertex has k independent neighbors. We construct the graph from $(k+1)$ subsets of intervals; each subset consists of k intervals, and each interval is composed of two segments. We denote the jth interval in subset ℓ by I_j^ℓ.

The graph is constructed as follows. Proceeding from left to right, we place under the intervals of subset ℓ, $I_1^\ell, \ldots, I_k^\ell$ the ℓth intervals of subsets $1, \ldots, k+1$, i.e., $I_\ell^1, \ldots, I_\ell^{k+1}$, excluding $I_{\ell,\ell}$. This is repeated for $\ell = 1, \ldots, k$. Finally, under the intervals of the $(k+1)$-th subset, we place the intervals $I_{\ell,\ell}$, $1 \leq \ell \leq k$ (see Figure 2).

Thus, we get that any interval I_j^ℓ with $\ell \neq j$, intersects k non-intersecting intervals of subset j, and $I_{\ell,\ell}$ intersects k non-intersecting intervals of subset $k+1$.

Note that since $k(k+1) \leq n$, we may have some remaining intervals, which are not contained in any subset. We can place each such interval I under any of the subsets ℓ, $1 \leq \ell \leq k+1$, providing that interval k independent neighbors. $\square$

We note that we can modify the above construction to hold for 2-union graphs. We now give a hardness result for a highly restricted class of proper 2-union graphs.

Theorem 2.2 *The MWIS problem is APX-hard on $(2,2)$-union graphs.*

We first observe that $(2,2)$-union graphs include the class of graphs of maximum degree 3. The theorem follows from the fact that the (unweighted) MIS problem is APX-hard on degree-3 graphs (see [8, 23]).

The *linear arboricity* $la(G)$ of a graph G is the minimum number of classes in a partition of $E(G)$ such that each class induces a collection of paths (or a linear forest). Paths can be represented as special interval graphs, where each interval represents a length-2 half-closed interval between integral endpoints. Indeed, any path $\{v_0, v_1, v_2, \ldots\}$ can be represented by the intervals $[0,2), [1,3), [2,4)$, and so on. Thus, a union of a pair of linear forests can be represented as a $(2,2)$-union graph. Akiyama, Exoo and Harary [1] showed that $la(G) = 2$ for graphs of maximum degree three. The following lemma, and the above theorem, then hold.

Lemma 2.3 *$(2,2)$-union graphs include the class of degree-3 graphs.*

Segments of unit size, whose start points are integral, are called *unit segments*. The *k-dimensional matching problem* is defined as follows. The input is k-uniform k-partite hypergraph $H = (V_1, \ldots, V_k, E)$. The output is a matching of maximum cardinality.

For some $k > 1$, let $S = \{1, 2, \ldots, n\}$, and let C be a collection of subsets of S, where each subset is of size at most k. The *k-set packing* problem is that of finding a maximum cardinality sub-collection $C' \subseteq C$, such that the intersection of any two sets in C' is empty. In the *weighted* version, each subset has a weight, and we seek a sub-collection C' of maximum weight. Note that the k-set packing problem properly contains the k-dimensional matching problem as a special case.

Lemma 2.4 *The k-set packing problem is equivalent to MWIS in the special class of k-interval graphs of unit segments.*

Proof: There is a bijective mapping between unit segments and the set S, where $[i, i+1)$ maps to i, for all values of i. Thus, there is a bijective mapping between sets of up to k elements from S and sets of up to k unit segments. $\square$

Similarly, the k-dimensional matching problem is equivalent to MWIS in the special class of k-union graphs of unit segments. The former problem is NP-hard to approximate within factor $O(k/\log k)$ [37], while the best known ratio is $k/2 + \epsilon$, for any $\epsilon > 0$ [24] . We note that the 2-set packing problem is equivalent to the (polynomially solvable) edge cover problem, while 3-dimensional matching is APX-hard [34].

Corollary 2.5 *MWIS in $(1,1)$-interval graphs is polynomial solvable. MWIS in $(1,1,1)$-union graphs is APX-hard.*

The correspondence between $(1,1)$-union graphs to line graphs of bipartite graphs, and the resulting polynomial solvability of MWIS, was shown by Halldórsson *et al.* [22].

3 Greedy algorithms

3.1 Coloring t-Interval Graphs

For a t-interval graph G, let G^* denote the graph formed by the intersection of the segments of the intervals (see in Figure 1(c)). The clique number, $\omega(G^*)$, denotes the maximum number of segments crossing a point on the real line.

Theorem 3.1 *For any t-interval graph G, there is a vertex v in G such that*

$$d(v) \leq 2t(\omega(G^*) - 1).$$

Proof: Since each vertex in G corresponds to up to t vertices of G^*, $|V(G^*)| \leq t \cdot |V(G)|$, and since each edge in G corresponds to one or more edges in G^*, $|E(G)| \leq |E(G^*)|$. Since G^* is an interval graph, there is a simplicial ordering of the graph so that each vertex v_i has at most $\omega(G^*) - 1$ neighbors among the vertices $v_{i+1}, \ldots$. Thus, the number of edges in G^* is at most $(\omega(G^*) - 1)|V(G^*)|$; in fact, it must be strictly less, since the last few vertices have no later neighbors. It follows that the average degree of G is bounded by

$$\overline{d}(G) = \frac{2|E(G)|}{|V(G)|} \leq 2t\frac{|E(G^*)|}{|V(G^*)|} < 2t(\omega(G^*) - 1).$$

Hence, the minimum degree of G is at most $2t(\omega(G^*) - 1) - 1$. $\square$

This leads to a simple coloring algorithm: find a vertex v satisfying the lemma, color the remaining graph $G \setminus v$, and finally color v with the smallest color not used by previously colored neighbors. This results in a $2t(\omega(G^*) - 1)$-coloring.

The above gives a $2t$-approximation for coloring t-interval graphs via a greedy algorithm. Gyárfás [18] showed that the chromatic number of a t-interval graph G is at most $2t(\omega(G) - 1)$, where $\omega(G)$ is the clique number of the graph.

Corollary 3.2 *A greedy algorithm colors G using $2t(\omega(G^*) - 1)$ colors.*

Observe that this bound is obtained without knowledge of the underlying interval representation of G^*; this is important since deducing the representation is known to be NP-hard [43]. We show that this is about the best bound on $\chi(G)$ one can obtain in terms of $\omega(G^*)$, within a constant factor.

Lemma 3.3 *For infinitely many t, there is a proper t-interval graph G such that $\omega(G) = (t - 1)\omega(G^*)$.*

Proof: Let p be a prime number and let $t = p+1$. We construct a t-interval graph G with $n = p^2$ intervals. The graph is based on a collection of nt segments, that are organized into $nt/p = p(p+1)$ groups of p identical segments. We refer to each group as a segment clique, and presume that it refers to a unique unit interval on the real line. We specify the graph by specifying the intervals contained in each segment clique.

Let $C_{i,j}$, $0 \le i < p$, $j \in \{0, \ldots, p-1, \infty\}$ be the collection of segment cliques. Let $I_{x,y}$, $0 \le x, y < p$ be the set of intervals. Define, for $i, j = 0, 1, \ldots, p-1$,

$$C_{i,j} = \{I_{x,\, ix+j \bmod p} : 0 \le x < p\},$$

and $C_{i,\infty} = \{I_{i,y} : 0 \le y < p\}$. Observe that for each i, there is exactly one $j < \infty$ such that $C_{i,j}$ contains $I_{x,y}$ (namely, where $j = y - ix \bmod p$). Hence, in addition to $C_{x,\infty}$, $I_{x,y}$ appears in exactly p segment cliques. Thus, the intersection graph is a t-interval graph.

Consider arbitrary distinct intervals $I_{x,y}$ and $I_{x',y'}$, where $x' \ge x$. We show that they are contained in the same clique and thus the corresponding nodes in the graph are adjacent. The lemma then follows. If $x = x'$, then both intervals are contained in $C_{x,\infty}$. Otherwise, let $x_0 = x' - x$ and $y_0 = y' - y \bmod p$. Let i be the solution to the linear modular equation $y_0 \equiv i \cdot x_0 (\mathrm{mod} p)$, which exists since p is prime and x_0 is non-zero. Let $j = y - ix \bmod p$. Then, $C_{i,j}$ contains $I_{x,y}$ since $y \equiv ix + j(\mathrm{mod} p)$. Also, it contains $I_{x',y'}$, since $y' = y + y_0 \equiv (ix + j) + (ix_0) \equiv ix' + j(\mathrm{mod} p)$. $\square$

3.2 Greedy Independent Set Algorithms

In this section, we study a greedy algorithm for the special case where $t = 2$, in order to motivate the use of more complicated techniques in later sections.

Recall from Observation 2.1 that, in 2-interval graphs, the neighborhood of every vertex may include many independent vertices. Thus, purely greedy methods are bound to fail. Consider, for instance, the optimal greedy algorithm for independent sets in interval graphs that iteratively adds the interval with the leftmost right endpoint. An analogous method for 2-interval graph could be to iteratively select the interval with the leftmost right endpoint of the *first segment*, among all intervals that do not intersect previously chosen intervals. This algorithm, which we call Sort-and-Select, cannot be expected to perform well on all 2-interval graphs. However, it performs well under certain circumstances, which allows us to partition the instance into well solvable subcases.

Theorem 3.4 *Let G be a 2-interval graph where*

- *the first segment is no shorter than the second, and*
- *the ratio between the shortest and longest second segment is at most 2.*

Then, the approximation factor of Sort-and-Select is 4.

Proof: Let I be the interval chosen first by Sort-and-Select. We claim that I intersects at most 4 independent intervals. Namely, the second segment of I is at most twice the length of the shortest segment in the graph; as a result, it intersects at most three independent segments/vertices. Also, since the first segment is furthest to the left of all segments in the graph, it does not intersect two independent vertices. Thus, among the intervals eliminated by the addition of I to the solution, the optimal solution can contain at most 4. By induction, the algorithm then achieves an approximation factor of 4. $\square$

Using the Local-Ratio technique, which is discussed in depth in the next section, one can obtain the same factor for the weighted case. Also, by a similar argument, one can argue a factor of 3 for the case of proper 2-interval graphs.

Given a general 2-interval graph, we first divide the intervals into those where the first segment is shorter than the second segment and those where the first segment is at least as long as the second. This gives us two instances, which can be viewed as symmetric by reversing the direction of the real line. Thus, by increasing the approximation factor by a factor of 2, we can assume that in our instance the first segments are no shorter than the second segments.

We can partition the instance into $\log R$ sub-instances, or *buckets*, where R is the ratio between the longest to shortest (S) second segment. Then, bucket G_i consists of intervals with second segments in the range $[2^{i-1}S, 2^iS]$, for $i = 1, 2, \ldots, \lceil \log R \rceil$. Each bucket satisfies the conditions of Theorem 3.4; thus, the largest of the independent sets found in each bucket by Sort-and-Select, is a $8 \log R$ approximation.

Note that we can represent the n second segments in the input by $2n$ endpoints on the line, and define the length of each segment as the number of endpoints that lie between its left and right endpoints plus one. Then, the maximal possible length of a segment is $2n-1$, and the number of buckets is $B = min\{\log R, \log(2n-1)\}$. Hence, we obtain the following result.

Theorem 3.5 *There is a greedy partitioning algorithm that approximates the maximum independent set in 2-interval graphs within a factor of* $O(\min\{\log R, \log 2n\})$.

4 A $2t$-approximation Algorithm

In this section we describe a $2t$-approximation algorithm for the maximum weight independent set problem in a t-interval graph $G = (V, E)$. The algorithm is based on rounding a fractional solution derived from a linear programming relaxation of the problem. The standard linear programming relaxation of the maximum weight independent set problem is the following. For each $v \in V$, let $x(v)$ be the linear relaxation of the indicator variable for v, i.e., whether v belongs to the independent set. Let $\mathbf{w}, \mathbf{x} \in \mathbb{R}^{|V|}$ be a weight vector and a relaxed indicator vector, respectively.

$$\begin{array}{lll} \text{maximize} & \mathbf{w} \cdot \mathbf{x} & \textit{subject to:} \\ \text{for each clique } \mathcal{C} \in G: & & \displaystyle\sum_{v \in \mathcal{C}} x(v) \leq 1 \end{array}$$

A feasible solution for the above linear program, whose value is an upper bound on the maximum weight independent set problem in the graph, can be obtained from the Lovász ϑ-function [31]. However, as we shall see, it is not necessary to optimize over all cliques in the case of t-interval graphs. We say that a clique $\mathcal{C}$ in the graph is an *interval clique* if for every vertex $v \in \mathcal{C}$, there is a segment (v, I) such that the intersection of $((v, I)|v \in \mathcal{C})$ is non-empty. We now further relax the maximum weight independent set problem and consider only interval cliques. For each vertex $v \in V$ and segment $I \in v$, let $x(v, I)$ denotes the value of segment I.

$$\begin{array}{lrcl} \text{(P)} \quad \text{maximize} \quad \mathbf{w} \cdot \mathbf{x} \quad \textit{subject to:} & & & \\ \text{for each interval clique } \mathcal{C}\text{:} & \displaystyle\sum_{(v,I) \in \mathcal{C}} x(v, I) & \leq & 1 \\ \text{for each } v \in V \text{ and } I \in v\text{:} & x(v, I) - x(v) & \geq & 0 \\ \text{for each } v \in V \text{ and } I \in v\text{:} & x(v), x(v, I) & \geq & 0 \end{array}$$

Notice that the number of interval cliques in a t-interval graph is linear in the number of segments, and therefore an optimal solution to (P) can be computed in polynomial time.

The heart of our rounding algorithm is the following lemma. It can be viewed as a fractional analog of Theorem 3.1.

Lemma 4.1 *Let* $\mathbf{x}$ *be a feasible solution to* (P)*. Then, there exists a vertex* $v \in V$ *satisfying:*

$$\sum_{u \in N[v]} x(u) \leq 2t$$

Proof: For two adjacent vertices u and v, define $y(u, v) = x(v) \cdot x(u)$. Define $y(u, u) = x(u)^2$. For a segment I, let $R(I)$ be the interval clique defined by the right endpoint of I ($I \in R(I)$). We prove the claim using a *weighted* averaging argument, where the weights are the values $y(u, v)$ for all pairs of adjacent vertices, u and v.

Consider the sum $\sum_{v \in V} \sum_{u \in N[v]} y(u, v)$. An upper bound on this sum can be obtained as follows. For each $v \in V$, consider all segments $I \in v$, and for each (v, I), add up $y(u, v)$ for all (u, J) that intersect with (v, I) (including (v, I)). In fact, it suffices to add up $y(u, v)$

only for segments (u,J) such that $(u,J) \in R(I)$, and then multiply the total sum by 2. This suffices since: (a) If, for segments (v,I) and (u,J), the right endpoint of I precedes the right endpoint of J, then (v,I) "sees" (v,J) and vice-versa. Since $y(u,v) = y(v,u)$, each of them contributes the same value to the other. (b) For segments (v,I) and (u,J), the constraints of (P) imply that $x(v,I) = x(v)$ and $x(u,J) = x(u)$. Hence, the mutual contribution of two segments (u,J) and (v,I) that intersect depends only on u and v, i.e., it is $y(u,v)$. Thus,

$$\sum_{v \in V} \sum_{u \in N[v]} y(u,v) \leq 2 \cdot \sum_{v \in V} \sum_{I \in v} \sum_{(u,J) \in R(I)} y(u,v)$$

Since

$$\sum_{(u,J) \in R(I)} y(u,v) \leq \ x(v) \cdot \sum_{(u,J) \in R(I)} x(u) \leq x(v)$$

we get that

$$\sum_{v \in V} \sum_{u \in N[v]} y(u,v) \leq 2t \cdot \sum_{v \in V} x(v).$$

Hence, there exists a vertex v satisfying

$$\sum_{u \in N[v]} y(u,v) \leq 2t \cdot x(v). \tag{1}$$

If we factor out $x(v)$ from both sides of (1) we obtain the statement of the lemma. □

We now define a fractional version of the Local Ratio technique. The proof of the next lemma is immediate.

Lemma 4.2 *Let* $\mathbf{x}$ *be a feasible solution to* (P). *Let* $\mathbf{w_1}$ *and* $\mathbf{w_2}$ *be a decomposition of the weight vector* $\mathbf{w}$ *such that* $\mathbf{w} = \mathbf{w_1} + \mathbf{w_2}$. *Let* $r > 0$. *Suppose that* $\mathbf{y}$ *is a feasible integral solution vector to (P) satisfying:* $\mathbf{w_1} \cdot \mathbf{y} \geq r(\mathbf{w_1} \cdot \mathbf{x})$ *and* $\mathbf{w_2} \cdot \mathbf{y} \geq r(\mathbf{w_2} \cdot \mathbf{x})$. *Then,*

$$\mathbf{w} \cdot \mathbf{y} \geq r(\mathbf{w} \cdot \mathbf{x}).$$

The rounding algorithm will apply a Local Ratio decomposition of the weight vector $\mathbf{w}$ with respect to an optimal solution $\mathbf{x}$ to linear program (P). The algorithm proceeds as follows.

1. Delete all vertices with non-positive weight. If no vertices remain, return the empty set.
2. Let $v' \in V$ be a vertex satisfying $\sum_{u \in N[v']} x(u) \leq 2t$. Decompose $\mathbf{w}$ by $\mathbf{w} = \mathbf{w_1} + \mathbf{w_2}$ as follows:

$$w_1(u) = \begin{cases} w(v') & \text{if } u \in N[v'], \\ 0 & \text{otherwise.} \end{cases}$$

(In the decomposition, the component $\mathbf{w_2}$ may be non-positive.)

3. Solve the problem recursively using $\mathbf{w_2}$ as the weight vector. Let $\mathcal{I}'$ be the independent set returned.
4. If $\mathcal{I}' \cup \{v'\}$ is an independent set, return $\mathcal{I} = \mathcal{I}' \cup \{v'\}$. Otherwise, return $\mathcal{I} = \mathcal{I}'$.

Clearly, the set $\mathcal{I}$ is an independent set. We now analyze the quality of the solution produced by the algorithm.

Theorem 4.3 *Let* $\mathbf{x}$ *be an optimal solution to linear program* (P). *Then, it holds for the independent set* $\mathcal{I}$ *computed by the algorithm that* $w(\mathcal{I}) \geq \frac{1}{2t} \cdot \mathbf{w} \cdot \mathbf{x}$

Proof: The proof is by induction on the number of recursive calls. At the basis of the recursion, the independent set returned is optimal (and hence a $2t$-approximation), since no vertices remain. Clearly, the first step in which vertices of non-positive weight are deleted cannot decrease the RHS above. We now prove the inductive step. Let $\mathbf{y}$ and $\mathbf{y}'$ be the indicator vectors of the sets $\mathcal{I}$ and $\mathcal{I}'$, respectively. Assume that $\mathbf{w_2} \cdot \mathbf{y}' \geq (1/2t) \cdot \mathbf{w_2} \cdot \mathbf{x}$. Since $w_2(v') = 0$, it also holds that $\mathbf{w_2} \cdot \mathbf{y} \geq (1/2t) \cdot \mathbf{w_2} \cdot \mathbf{x}$. From Step (4) of the algorithm it follows that at least one vertex from $N[v']$ belongs to $\mathcal{I}$. Hence, $\mathbf{w_1} \cdot \mathbf{y} \geq (1/2t) \cdot \mathbf{w_1} \cdot \mathbf{x}$. Thus, by Lemma 4.2, it follows that

$$\mathbf{w} \cdot \mathbf{y} \geq \frac{1}{2t} \cdot \mathbf{w} \cdot \mathbf{x}$$

We have thus proved that $\mathcal{I}$ is a $2t$-approximate solution to the MWIS problem. $\square$

We now outline an alternative way of using Lemma 4.1 to obtain the same approximation factor. Given an optimal solution $\mathbf{x}$ to linear program (P), a *multicoloring* of V by a set X is a mapping $\psi : V \to X$ such that $|\psi(v)| = x(v)$ for each vertex v, and $x(v) \cap x(u) = \emptyset$ for each edge $(u, v) \in E(G)$. Since $\mathbf{x}$ is a feasible solution to (P), a repeated application of Lemma 4.1 results in a multicoloring with values in the closed interval $[0, 2t]$.

To view this as a multicoloring, it may be easier to discretize the instance within any desired precision by multiplying the $x(v)$'s by a sufficiently large integer L. Then the values assigned are positive integers in the range $1, \ldots, 2tL$. A continuous viewpoint is to assign each vertex a collection of contiguous segments; if we use Lemma 4.1 to assign the values one by one, we can always guarantee that a vertex v can be mapped to segments from $[0, 2t]$ of combined length $x(v)$ without overlapping any of the segments to which its neighbors are mapped to. In fact, by always mapping a vertex to the smallest available values, we need never use more than n disjoint segments for any vertex.

Let $0 = z_0 < z_1 < \cdots < z_{k-1}$ denote the values where the multicoloring changes, and let $z_k = 2t$. Thus, the coloring remains unchanged in the segment $[z_i, z_{i+1})$, $i = 0, \ldots, k-1$. Consider the sets $S_i = \{v \in V : x_i \in \psi(v)\}$, for $i = 0, \ldots, k-1$. Since ψ is a multicoloring, the S_i's are independent sets in G. Let $\mathcal{I}$ be the set S_i of maximum weight, $\sum_{v \in S_i} w(v)$.

Theorem 4.4 *$w(\mathcal{I})$ is a $2t$-approximate independent set.*

Proof: Observe that the amount of color values to which vertex v is mapped is $x(v)$, and we can represent them by $\sum_{S_i \ni v}(z_i - z_{i+1}) = x(v)$. We have that

$$\begin{aligned}\sum_{v \in V} w(v)x(v) &= \sum_{v \in V} w(v) \sum_{S_i \ni v}(z_{i+1} - z_i) = \sum_{S_i}(z_{i+1} - z_i) \sum_{v \in S_i} w(v) \\ &= \sum_{i=0}^{k-1}(z_{i+1} - z_i)w(S_i) \leq \sum_{i=0}^{k-1}(z_{i+1} - z_i)w(\mathcal{I}) = 2tw(\mathcal{I}).\end{aligned}$$

$\square$

5 A Bi-criteria Approximation Scheme for Union Graphs

Recall that MWIS is APX-hard already on $(2,2)$-union graphs. We consider below the larger subclass of t-union graphs in which the possible number of segment lengths is bounded by some constant. For this subclass we develop a *bi-criteria* PTAS, which finds an MWIS by allowing some delays in the schedule.

Let c_i denote the number of distinct lengths of the i-th segment, $1 \leq i \leq t$, where t is some constant. Recall that, in the flow shop problem, we are given a set of n jobs, $J_1, \ldots, J_n$ that need to be processed on m machines, $M_1, \ldots, M_m$; each job, J_j, consists of m operations, $O_{j,1}, \ldots, O_{j,m}$, where $O_{j,i}$ must be processed without interruptions on the machine M_i, for $p_{j,i}$ time units. Any machine, M_i, can either process a *single* operation at a time, or an *unbounded* number of operations; in the latter case we call M_i a *non-bottleneck* machine. Each job may be processed by at most one machine at any time. For a given schedule, let C_j be the completion time of J_j. The objective is to minimize the *maximum completion time* (or makespan), given by $C_{max} = \max_j C_j$. Denote by C^*_{max} the optimal makespan.

An instance of our problem can be transformed to an instance of the flow shop problem, where each job has $2t+1$ operations, and the machines M_{2i+1}, $0 \leq i \leq t-1$, are non-bottleneck machines. In our transformation, we apply some ideas from [25, 21, 26]. We represent each t-interval, I_j, as a job J_j, where each segment is associated with an "operation" of the job. In addition, we simulate the breaks with operations of the same lengths that need to be processed on non-bottleneck machines. Similarly, to include the release time r_j of I_j, we add to J_j the operation $O_{j,1}$, whose length is equal to r_j; the machine M_1 is a non-bottleneck machine. Thus, if I_j has t segments, J_j has $2t$ operations.

Recall that, in a union graph, each interval has a due date, d_j, that is equal to its release time plus the sum of its processing times and break times. To simulate these due dates we define a *delivery time*, q_j, for each job, J_j. Let $q_j = -d_j$. We add to J_j the operation

$O_{j,(2t+1)}$, where $p_{j,(2t+1)} = q_j$, and M_{2t+1} is a non-bottleneck machine. Our objective then is to minimize the maximum *delivery completion time*, given by $\max_j\{C_j + q_j\} = \max_j\{C_j - d_j\}$. This is equivalent to minimizing the maximum *lateness* of any job, given by $L_j = C_j - d_j$. Hence, our objective can be viewed as minimization of $L_{max} = \max_j L_j$.

Denote by $T_{\mathcal{O}}$ the maximum completion time of an optimal solution for the MWIS instance. Since we look for a MWIS that can be scheduled with maximum lateness at most $\epsilon T_{\mathcal{O}}$, we slightly modify the definition of *lateness*. Let $\tilde{d}_j = d_j - T_{\mathcal{O}}$; then, for any j, $\tilde{d}_j \leq 0$. By setting $q_j = -\tilde{d}_j$, we get that all the delivery times are positive. The maximum lateness is now given by $L_{max} = \max_j\{C_j - d_j + T_{\mathcal{O}}\}$. Indeed, for any job J_j, $C_j \geq d_j$, therefore $L_{max} \geq T_{\mathcal{O}}$, and since in any optimal schedule there are no "late" jobs, the minimal lateness is $L^*_{max} = T_{\mathcal{O}}$.

Our scheme uses as procedure a PTAS for finding a $(1+\epsilon)$-approximation for the flow shop makespan problem with a fixed number of machines (see, e.g., [20]). We represent a t-interval I_j by a $(2t+1)$-vector $(p_{j,1}, \ldots, p_{j,2t+1})$, where $p_{j,1}$ is the release time, $p_{j,2i}$ $(p_{j,2i+1})$, is the length of the i-th segment (break), $1 \leq i < t$, and $p_{j,2t+1}$ $(= q_j)$ is the delivery time of the corresponding job, J_j.

We summarize below the steps of our scheme, which gets as parameters the value of $T_{\mathcal{O}}$ and some $\epsilon > 0$.

1. We scale the parameter values for J_j; that is, we divide the processing and release times by $T_{\mathcal{O}}$, and round each release time down and each break time up to the nearest multiple of ϵ.

2. We guess $\mathcal{O}$, the number of intervals scheduled by OPT;

3. We guess the subset $S_{\mathcal{O}}$ of $\mathcal{O}$ intervals of maximal weight, scheduled by OPT. This is done by guessing the set of vectors representing $S_{\mathcal{O}}$, among which we choose the subset of intervals of maximum weight.

4. Using a PTAS for minimizing the makespan in the flow shop instance of $S_{\mathcal{O}}$, we find a schedule of $S_{\mathcal{O}}$ for which $L_{max} \leq (1+\epsilon)L^*_{max}$.

Note that due to the above rounding, we need to add ϵ to the release times; also, each break time may delay the optimal completion time by ϵ, therefore, by taking $\epsilon' = \epsilon/2t$ we guarantee that the delay of each interval is at most $(1+\epsilon)$ times $T_{\mathcal{O}}$. Finally, we set $L_j = L_j - T_{\mathcal{O}}$; thus, the maximum lateness of any job in our schedule is equal to at most $\epsilon T_{\mathcal{O}} = \epsilon C'^{*}_{max}$.

For the complexity of the scheme, note that Steps 1. and 2. take linear time, and since the possible number of vectors $(p_{j,1}, \ldots p_{j,2t+1})$ is $(2t/\epsilon)^t \prod_{i=1}^{t} c_i$, we can guess $S_{\mathcal{O}}$ in $O(n^{(t^t \prod_{i=1}^{t} c_i)/\epsilon^t})$ steps. This is multiplied by the complexity of the PTAS for flow shop.

Theorem 5.1 *Let $t \geq 1$ be some fixed constant. Given a t-union graph with constant number of distinct segment lengths, let $\mathcal{W}$ be the weight of an optimal MWIS, whose latest completion time is $T_{\mathcal{O}}$. Then, for any $\epsilon > 0$, there is a PTAS that schedules an independent set of weight at least $\mathcal{W}$, such that* any *interval is late by at most $\epsilon T_{\mathcal{O}}$.*

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