Contemporary Mathematics

# Partitions and Permutation Groups

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Abstract. We show that non-trivial extremely amenable topological groups are essentially the same thing as permutation models of the Boolean prime ideal theorem that do not satisfy the axiom of choice. Both are described in terms of partition properties of group actions.

## 1. Introduction

The purpose of this paper is to point out that two results, in apparently quite different areas, become the same when reduced to their combinatorial essence. The first is the result of Herer and Christensen [**4**] on the existence of extremely amenable groups, with some additional information from Pestov [**10**]. The second is the result of Halpern [**2**] that the Boolean prime ideal theorem does not imply the axiom of choice, in a set theory allowing atoms, with some additional information from Halpern's proof.

Both of these results are equivalent to the existence of groups and filters of subgroups with particular combinatorial properties. In the case of extremely amenable groups, the equivalence in question is (almost) contained in the work of Kechris, Pestov, and Todorčević [**7**]. In the case of the Boolean prime ideal theorem, the required equivalence is (almost) contained my paper [**1**].

In Sections 2 and 3 of the present paper, we review background material about extremely amenable groups and about the Boolean prime ideal theorem. In Section 4, we define the relevant combinatorial properties of group actions. We prove some basic facts about these properties, and we also raise an open question. Finally, in Section 5, we recall the results from [**7**] and [**1**] mentioned above, and we make the minor modifications needed to establish the claimed equivalences.

This arrangement of the material tends to obscure the connection with the topics of this volume, finite combinatorics and model theory. Nevertheless, finite combinatorics is, as will become clear from the examples in Section 4, the essence of the group-theoretic properties discussed here. There are also implicit connections with model theory, not only in the direction of models of set theory, but also in the crucial role of the compactness theorem in the results proved in [**1**] and used here in Section 5.

2010 *Mathematics Subject Classification.* Primary 05C55; Secondary 03E25, 03E35, 22F05.
Partially supported by NSF grant DMS-0653696.

Throughout this paper all topological spaces (including topological groups) are assumed to be Hausdorff spaces unless the contrary is explicitly stated.

## 2. Extremely Amenable Groups

An *action* of a group $G$ on a set $X$ is a function $\alpha : G \times X \to X$ such that, abbreviating $\alpha(g, x)$ as $g \cdot x$ and writing 1 for the identity element of $G$, we have $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ and $1 \cdot x = x$ for all $g_1, g_2 \in G$ and all $x \in X$. Equivalently, $g \mapsto (x \mapsto \alpha(g, x))$ is a homomorphism from $G$ into the group of permutations of $X$. In this situation, we call $X$ (understood as being equipped with the action) a *$G$-set*.

When $G$ is a topological group and $X$ is a topological space, we call an action $\alpha$ continuous if it is continuous with respect to the product topology on $G \times X$ (and the given topology on $X$).

**Definition 2.1.** A topological group is *extremely amenable* if, whenever $G$ acts continuously on a compact Hausdorff space $X$, there is a fixed point in $X$, i.e., an $x \in X$ such that $g \cdot x = x$ for all $g \in G$.

The first question to ask about this concept was asked by Mitchell [**8**]: Are there non-trivial examples? (Of course the trivial group is extremely amenable, by the second equation in the definition of "action.") This question was answered affirmatively by Herer and Christensen [**4**] by a direct, hands-on construction. More natural examples were given by Pestov [**10**], and they had the following additional property.

**Definition 2.2.** A topological group has *small open subgroups* if every neighborhood of 1 includes an open subgroup.

Equivalently, the open subgroups constitute a neighborhood base at 1, so they determine the topology (because a neighborhood base at any point in a topological group can be obtained by simply translating a neighborhood base at 1).

Pestov gave, as an example of an extremely amenable group, $\mathrm{Aut}(\mathbb{Q}, <)$, the group of order-automorphisms of the rational number line. The topology is the subspace topology obtained by regarding $\mathrm{Aut}(\mathbb{Q}, <)$ as a subset of the product space $\mathbb{Q}^{\mathbb{Q}}$, which in turn has the product topology induced by the discrete topology on $\mathbb{Q}$. A neighborhood base at 1 is given by the pointwise stabilizers of finite sets, i.e., the subgroups of the form

$$\mathrm{Fix}(F) = \{\pi \in \mathrm{Aut}(\mathbb{Q}, <) : (\forall q \in F)\, \pi(q) = q\}$$

for finite subsets $F$ of $\mathbb{Q}$. Thus, $\mathrm{Aut}(\mathbb{Q}, <)$ has small open subgroups.

We summarize the preceding results of Herer, Christensen, and Pestov as follows.

**Theorem 2.3.** *There exists a non-trivial extremely amenable topological group with small open subgroups.*

## 3. Boolean Prime Ideal Theorem

The *Boolean prime ideal theorem* (BPI) is the assertion that every non-degenerate Boolean algebra has a prime ideal. ("Non-degenerate" means that the algebra has at least two elements; equivalently $0 \neq 1$.) BPI is often stated in terms of ultrafilters rather than prime ideals; it makes no difference because ultrafilters are just the complements of prime ideals.

The usual proof of BPI uses Zorn's Lemma to produce a maximal ideal and then checks that all maximal ideals are prime. Thus, BPI is a consequence of the axiom of choice,[1] and it is natural to ask whether it is equivalent to AC in the presence of the other axioms of Zermelo-Fraenkel set theory (ZF). This question was answered negatively by Halpern [**2**] in the context of ZFA, a version of ZF that allows the existence of atoms (objects that are not sets and therefore do not have elements but can be elements of sets). Later, Halpern and Lévy [**3**] improved the result to work with ZF (without atoms), but it is the original result of Halpern that will be relevant for us here.

Halpern's proof used the Fraenkel-Mostowski-Specker technique of *permutation models* of ZFA, which we briefly review here. See [**6**, Chapter 4] for details. The method begins with a set $A$ of atoms, a group $G$ of permutations of $A$, and a normal filter $\mathcal{F}$ of subgroups of $G$. "Normal filter" means that $\mathcal{F}$ is a collection of subgroups of $G$, closed under supergroups, under finite intersections, and under conjugation by elements of $G$. One builds a universe $V(A)$ of sets and atoms by starting with the atoms in $A$ and iterating transfinitely the operation of forming arbitrary sets of already formed elements. That is, one defines, by induction on the ordinal $\xi$,

$$V_\xi(A) = A \cup \bigcup_{\eta<\xi} \mathcal{P}(V_\eta(A))$$

where $\mathcal{P}(X)$ means the set of all subsets of $X$. Then $V(A)$ is defined as the union of the sets $V_\xi(A)$ over all ordinals $\xi$.

Any permutation $\pi$ of $A$, in particular any element of $G$, naturally extends to an automorphism (preserving the membership relation $\in$) of $V(A)$ by the inductive definition

$$\pi(x) = \{\pi(y) : y \in x\} \quad \text{for all sets } x.$$

The *stabilizer* of an $x \in V(A)$ is defined as the subgroup $\{\pi \in G : \pi(x) = x\}$ of $G$.[2] An element $x \in V(A)$ is said to be *symmetric* if its stabilizer is in the given filter $\mathcal{F}$; it is *hereditarily symmetric* if, in addition, all elements of its transitive closure (i.e., the elements of $x$, their elements, their elements, etc.) are symmetric. The subuniverse $M = M(A, G, \mathcal{F}) \subseteq V(A)$ of hereditarily symmetric sets, with the membership relation $\in$ inherited from $V(A)$, satisfies ZFA. It is called a permutation model of ZFA, and, except in very special cases, it does not satisfy AC.

The method of permutation models is used to establish independence results between AC and its various weak forms by cleverly choosing $A$, $G$, and $\mathcal{F}$ so as to make certain weak axioms of choice hold in $M(A, G, \mathcal{F})$ while others fail. Halpern [**2**] showed that BPI holds while AC fails in a particular permutation model, the ordered Mostowski model [**9**] obtained by choosing the parameters as follows. $A$ is a countable set equipped with a dense linear ordering without endpoints, i.e., an order-isomorphic copy of $(\mathbb{Q}, <)$. $G$ is the group of order-automorphisms of $A$. $\mathcal{F}$ is the filter generated by the pointwise stabilizers $\mathrm{Fix}(F)$ of finite subsets $F$ of $A$.

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[1] BPI is one of the most useful consequences of AC. See for example the long list of its equivalent forms in the compendium by Howard and Rubin [**5**, Form 14].

[2] Note that, in the notation of the previous section, this would be $\mathrm{Fix}(\{x\})$, not $\mathrm{Fix}(x)$; the latter would be the smaller subgroup of those $\pi$ that fix every element of $x$ individually, whereas a $\pi$ in the stabilizer of $x$ could permute the elements of $x$ with each other.

We summarize Halpern's result as follows.

**Theorem 3.1.** *There is a permutation model of ZFA that satisfies the Boolean prime ideal theorem but not the axiom of choice.*

## 4. Ramsey Actions

The thesis to be explained in the rest of this paper is that Theorems 2.3 and 3.1 are essentially the same theorem. A certain similarity between them is already apparent in the specific examples used to prove them. Pestov's and Halpern's arguments both use $\mathrm{Aut}(\mathbb{Q}, <)$. Our thesis, however, is much stronger: *Any* example for either of these theorems is, after some minor normalizations, also an example for the other. To justify this claim, we must look at the combinatorial core of both theorems; that core is the subject of the present section.

We begin with the notion of a Ramsey action, introduced in [**1**]. The following is not literally the definition from [**1**], but the equivalence between the definitions is included in Proposition 4.2 below.

**Definition 4.1.** An action of a group $G$ on a set $X$ is a *Ramsey action* and is said to have the *Ramsey property*, and $X$ is a *Ramsey* $G$-set, if, for every finite subset $F$ of $X$ and every 2-coloring $c : X \to 2$, there is some $g \in G$ such that $c$ is constant on $g \cdot F$.

We use here and below the standard convention from set theory identifying a natural number with the set of smaller natural numbers, so 2 means $\{0, 1\}$. We also use the standard notation $g \cdot F$ for $\{g \cdot x : x \in F\}$. Thus, the definition of a Ramsey action says that, given a 2-coloring of $X$, every finite subset of $X$ can be translated (on the left, by an element of $G$) to a monochromatic set.

Before giving some examples of Ramsey actions, we point out some elementary facts about the concept. We begin with some equivalent reformulations of the definition.

**Proposition 4.2.** *For any action of a group $G$ on a set $X$, the following are equivalent.*

(1) *The Ramsey property.*
(2) *For each finite $F \subseteq X$ there exists a finite $Y \subseteq X$ such that, whenever $c : Y \to 2$, there is $g \in G$ such that $g \cdot F \subseteq Y$ and $c$ is constant on $g \cdot F$.*
(3) *For every finite $k$, the definition of Ramsey action is satisfied with $k$-colorings in place of 2-colorings.*
(4) *For every finite $k$, item* (2) *above is satisfied with $k$-colorings in place of 2-colorings.*

Proof. Some of the implications are trivial: (4) implies both (2) and (3), each of which implies (1).

The implication from (1) to (2) is established by the following standard compactness argument. If we had a counterexample to (2), i.e., an $F$ for which no appropriate $Y$ exists, then the compactness theorem for sentential logic would apply to the following set of sentences. Let the sentential variables be [$x$ gets color $i$]

for each $x \in X$ and $i \in 2$, and let $\Sigma$ be the set of sentences

$$\bigvee_{i\in 2}[x \text{ gets color } i] \qquad \text{for each } x \in X,$$
$$\neg([x \text{ gets color } i] \wedge [x \text{ gets color } j]) \qquad \text{for each } x \in X \text{ and distinct } i, j \in 2,$$
$$\neg \bigvee_{i\in 2} \bigwedge_{x\in F} [g \cdot x \text{ gets color } i] \qquad \text{for each } g \in G.$$

Our choice of $F$ implies that each finite subset of $\Sigma$ is satisfiable. By compactness, $\Sigma$ is satisfiable, and any satisfying truth assignment describes a coloring which, together with $F$, witnesses the failure of (1). The same proof, with all occurrences of 2 changed to $k$, shows that (3) implies (4).

To complete the proof of the proposition, we show that (2) implies (4), by induction on $k$. Assume (2) and assume (4) for a certain $k \geq 2$. Given any finite $F \subseteq X$, use the induction hypothesis to find a finite $Y \subseteq X$ as in (4) for $k$. It does no harm to enlarge $Y$ so that it includes $F$. Then use (2), with $Y$ in the role of $F$, to find a finite $Z \subseteq X$ such that each 2-coloring of $Z$ is constant on a set of the form $g \cdot Y$. We shall show that each $(k+1)$-coloring $c$ of $Z$ is constant on a set of the form $g \cdot F$.

Given $c : Z \to k+1$, form $c' : Z \to 2$ by setting

$$c'(z) = \begin{cases} 0 & \text{if } c(z) \neq k \\ 1 & \text{if } c(z) = k. \end{cases}$$

By our choice of $Z$, we can fix some $g \in G$ such that $c'$ is constant on $g \cdot Y$. If the constant is 1, then $c$ is constant with value $k$ on $g \cdot Y$ and therefore on its subset $g \cdot F$. Assume, therefore, that the constant value of $c'$ on $g \cdot Y$ is 0. This means that $c$ maps $g \cdot Y$ into $k$ (rather than $k+1$), and therefore the function $c''$ defined by

$$c''(y) = c(g \cdot y)$$

maps $Y$ into $k$. By our choice of $Y$, we can fix some $h \in G$ such that $c''$ is constant on $h \cdot F$. Then $c$ is constant on $(gh) \cdot F$. □

Item (2) in the preceding proposition was used as the definition of "Ramsey property" in [**1**].

**Proposition 4.3.** *In any Ramsey action of a group $G$ on a set $X$, the action is transitive and, if $X$ has at least two elements, then it has infinitely many.*

Proof. Recall that transitivity means that, for any two elements $x, y \in X$, there is some $g \in G$ with $g \cdot x = y$. To prove that this is necessary in any Ramsey action, let $x$ and $y$ be given and apply the definition of Ramsey action with $F$ being $\{x, y\}$ and $c$ having value 0 at all points of the form $g \cdot x$ and value 1 at all other points. We get some $h \in G$ such that

$$c(h \cdot y) = c(h \cdot x) = 0,$$

and so $h \cdot y = g \cdot x$ for some $g \in G$. Then $(h^{-1}g) \cdot x = y$; this completes the proof of transitivity.

If $X$ were finite but had at least two points, then, in the definition of Ramsey action, we could take $F$ to be all of $X$ and take $c$ to be a non-constant function $X \to 2$. The definition requires $c$ to be constant on a set of the form $g \cdot X$, but the only such set, regardless of $g$, is $X$ itself, on which $c$ is not constant. □

In the definition of Ramsey action, if the requirements are satisfied for a particular finite set $F$, then they are obviously also satisfied for any subset of $F$. The following proposition begins with this observation and extends it in ways that will be useful in verifying some examples.

**Proposition 4.4.** *Let $G$ act on $X$, and let $\mathcal{F}$ be a family of finite subsets of $X$. Under any of the following hypotheses, to verify the Ramsey property, it suffices to verify the definition for $F \in \mathcal{F}$.*

1. *Each finite subset of $X$ is included in a member of $\mathcal{F}$.*
2. *For each finite $F \subseteq X$, there is some $g \in G$ such that $g \cdot F$ is included in a member of $\mathcal{F}$.*
3. *For each finite $F \subseteq X$ there is a bijection $d : X \to X$ such that $d(F)$ is included in a member of $\mathcal{F}$ and $G$ is closed under conjugation by $d$, i.e., considering each $g \in G$ as a permutation of $X$ via the action, we have $d^{-1} \circ g \circ d \in G$ for all $g \in G$.*

Proof. The first part is obvious and the second almost as obvious. In any event, both of these parts are subsumed by the third, which we now prove.

Assume (3) and assume that the definition of Ramsey action is satisfied when $F \in \mathcal{F}$. Given now an arbitrary finite $F \subseteq X$ and an arbitrary $c : X \to 2$, let $d$ be as in (3), and let $F' \in \mathcal{F}$ with $d(F) \subseteq F'$. Let $c' = c \circ d^{-1} : X \to 2$. By assumption, there is $g \in G$ such that $c'$ is constant on $g(F')$ and therefore on its subset $g(d(F))$. Then $c$ is constant on $d^{-1}(g(d(F)))$, which is, by (3), $g'(F)$ for a certain $g' \in G$. □

To connect Ramsey actions with traditional Ramsey theory, we give two examples. The first is a special case of the action relevant to the constructions by Pestov and Halpern mentioned in the preceding sections. The second example will be useful in connection with an open problem to be discussed later.

**Example 4.5.** Fix a natural number $n$. Let $G$ be the group $\mathrm{Aut}(\mathbb{Q}, <)$. Its natural action on $\mathbb{Q}$ induces an action on the set $[\mathbb{Q}]^n$ of $n$-element subsets of $\mathbb{Q}$. We shall show that this is a Ramsey action. Every finite subset of $[\mathbb{Q}]^n$ is included in one of the form $[A]^n$ (the set of $n$-element subsets of $A$) for some finite $A \subseteq \mathbb{Q}$. So it suffices to check the definition of Ramsey action when $F$ is of the form $[A]^n$. Let $a$ be the cardinality of $A$, and, by the finite Ramsey theorem, let $M$ be a natural number so large that, whenever $[M]^n$ is partitioned into two pieces, there is $H \subseteq M$ of size $a$ with $[H]^n$ included in one of the pieces. Let $Z$ be a subset of $\mathbb{Q}$ of size $M$; we claim that $Y = [Z]^n$ satisfies the requirements in item (2) of Proposition 4.2. Indeed, if $c : [Z]^n \to 2$ then, by our choice of $M$, there is an $a$-element set $H \subseteq Z$ such that $c$ is constant on $[H]^n$. Since $A$ and $H$ are subsets of $\mathbb{Q}$ of the same finite size, there is an order-automorphism $g \in G$ sending $A$ to $H$. (In fact, $g$ can be chosen to be piecewise linear.) Then, in the action of $G$ on $[\mathbb{Q}]^n$, $g$ sends $[A]^n$ to $[H]^n$, on which $c$ is constant.

The preceding example, taken from [**1**], can be generalized to convert many other structural Ramsey results into statements about the Ramsey property of group actions. See, for example, the material in [**7**] leading from Proposition 4.2 to Theorem 4.7. The following example, also taken from [**1**], fits the same mold, but the partition theorem it uses, van der Waerden's theorem, is usually thought of as numerical rather than structural.

**Example 4.6.** Let $G$ be the group of affine permutations of $\mathbb{Q}$, i.e., transformations of the form $x \mapsto ax + b$ with $a \in \mathbb{Q} - \{0\}$ and $b \in \mathbb{Q}$. We claim that the natural action of $G$ on $\mathbb{Q}$ is a Ramsey action. Since every finite subset of $\mathbb{Q}$ is included in an arithmetical progression, it suffices to check the definition when $F$ is a finite arithmetical progression. By van der Waerden's theorem, whenever $c$ maps $\mathbb{Q}$ (or even just $\mathbb{N}$) to 2, there is an arithmetical progression $P$, of the same length as $F$, on which $c$ is constant. There is an affine permutation mapping $F$ onto $P$, and this completes the verification of the Ramsey property.

We turn next to some preservation results for the Ramsey property, constructing new Ramsey actions from old.

**Definition 4.7.** Let $\alpha : G \times X \to X$ and $\beta : G \times Y \to Y$ be two actions of the same group $G$. A *homomorphism* or *$G$-equivariant* map from the former to the latter is a function $p : X \to Y$ such that $p(\alpha(g, x)) = \beta(g, p(x))$ (more briefly, $p(g \cdot x) = g \cdot p(x)$) for all $g \in G$ and all $x \in X$. In this situation, the *fiber* over a point $y \in Y$ is the subset $p^{-1}(\{y\})$ of $X$.

Notice that, with notation as in this definition and with $H = \mathrm{Fix}(\{y\})$ being the stabilizer of the point $y$ in $Y$, the fiber $p^{-1}(\{y\})$ is invariant under the action of $H$ in $X$, so it makes sense to speak of the induced action of $H$ on the fiber.

**Proposition 4.8.** *Let $p : X \to Y$ be a surjective homomorphism of $G$-actions. If $X$ has the Ramsey property, then so does $Y$.*

Proof. Assume the Ramsey property for $X$, and let a finite set $F \subseteq Y$ and a 2-coloring $c$ of $Y$ be given. Choose, for each $u \in F$, some $v$ in its fiber, i.e., $p(v) = u$, and let $F'$ be the set of these chosen $v$'s. Let $c' = c \circ p : X \to 2$. Since $X$ has the Ramsey property, there is $g \in G$ such that $c'$ is constant on $g \cdot F'$. This means, in view of the definition of $c'$, that $c$ is constant on $p(g \cdot F') = g \cdot p(F') = g \cdot F$. □

**Proposition 4.9.** *Let $p : X \to Y$ be a surjective homomorphism of $G$-actions and let $y_0 \in Y$. Let $Z$ be the fiber over $y_0$, with the induced action of the stabilizer $H$ of $y_0$. If $X$ has the Ramsey property, then so does $Z$.*

Proof. Notice first that, since the action of $G$ on $X$ is transitive (by Proposition 4.3) and since $p$ is equivariant and surjective, the action of $G$ on $Y$ is also transitive. So we can fix, for each $y \in Y$, some element $m_y \in G$ with $m_y \cdot y_0 = y$.

Let a finite set $F \subseteq Z$ and a 2-coloring $c : Z \to 2$ be given. Define a 2-coloring $c' : X \to 2$ by

$$c'(x) = c({m_{p(x)}}^{-1} \cdot x).$$

This makes sense because

$$p({m_{p(x)}}^{-1} \cdot x) = {m_{p(x)}}^{-1} \cdot p(x) = y_0,$$

so ${m_{p(x)}}^{-1} \cdot x$ is in the domain $Z$ of $c$. (We could have required $m_{y_0}$ to be the identity element of $G$, and then $c'$ would extend $c$, but this is not needed for the proof.) By the Ramsey property of $X$, there is $g \in G$ such that $c'$ is constant on $g \cdot F$.

Since $F \subseteq Z = p^{-1}(\{y_0\})$, we have, for all $g \cdot z \in g \cdot F$ that $p(g \cdot z) = g \cdot p(z) = g \cdot y_0$. Thus, if we write simply $m$ for $m_{g \cdot y_0}$, we have, for all elements $z \in F$,

$$c'(g \cdot z) = c(m^{-1} \cdot (g \cdot z)) = c((m^{-1}g) \cdot z).$$

As $z$ varies over $F$, the left side of this equation remains constant; therefore so does the right side, and this means that $c$ is constant on $(m^{-1}g) \cdot F$. Finally, note that $m^{-1}g$ is in $H$ as required, because

$$m^{-1}g \cdot y_0 = {m_{g \cdot y_0}}^{-1} \cdot (g \cdot y_0) = y_0.$$

□

We digress briefly to mention an open problem suggested by the two preceding propositions.

**Question 4.10.** Suppose $p : X \to Y$ is a surjective homomorphism of transitive $G$-actions. Suppose the action of $G$ on $Y$ has the Ramsey property, and suppose also that, for some (equivalently, for every) $y \in Y$, the action of its stabilizer on its fiber has the Ramsey property. Does it follow that the action of $G$ on $X$ has the Ramsey property?

Here is a simple special case.

**Question 4.11.** Let $G$ be the group of transformations of the rational plane $\mathbb{Q}^2$ of the form

$$(x, y) \mapsto (ax + b, \frac{y}{a} + c),$$

where $a \in \mathbb{Q} - \{0\}$ and $b, c \in \mathbb{Q}$. Does the natural action of this group on $\mathbb{Q}^2$ have the Ramsey property?

This is a special case of the previous question, as follows. The group maps horizontal lines $\mathbb{Q} \times \{y\}$ to horizontal lines; let $Y$ be the set of these lines with this action of $G$. The map $p$ sending each point in $\mathbb{Q}^2$ to the horizontal line through it is a surjective $G$-homomorphism from $\mathbb{Q}^2$ to $Y$. The action of $G$ on $Y$ is essentially the same as Example 4.6, so it has the Ramsey property. The fiber over 0 is the horizontal axis, and the stabilizer consists of the transformations which, in the notation displayed above, have $c = 0$. So the action of the stabilizer on the fiber is again that of Example 4.6. So we are in the situation of Question 4.10; a positive answer there would imply a positive answer to Question 4.11.

We can reformulate Question 4.11 by applying part (3) of Proposition 4.4 as follows. Consider dilations of $\mathbb{Q}^2$, i.e., transformations of the form $(x, y) \mapsto (rx, ry)$ with positive integer $r$. Conjugating an element of our group by such a dilation produces again an element of our group. Also, any finite $F \subseteq \mathbb{Q}^2$ can be moved, by such a dilation, to a subset of $\mathbb{Z}^2$ (i.e., the dilation can clear the denominators) and thus to a subset of $A^2$ for some interval $A$ of integers. The images of such a set $A^2$ under the elements of our group are *unit-area grids*, rectangular grids of unit-area cells; more precisely, they are sets of the form $B \times C$ where $B$ and $C$ are arithmetical progressions in $\mathbb{Q}$ whose differences are reciprocals of each other. Thus, Question 4.11 becomes: If $\mathbb{Q}^2$ is partitioned into two pieces, must one piece contain arbitrarily large unit-area grids?

A comment is perhaps in order about the use of the reciprocal $\frac{1}{a}$ as the multiplier of $y$ in the definition of our group. Had we used simply $a$ instead, the final form of the question would have been not about unit-area grids but about square grids, i.e., the differences in the arithmetical progressions $B$ and $C$ would have been equal rather than being reciprocals. This modified question would also be a special case of Question 4.10, but it would be less interesting because a positive answer follows easily from the Hales-Jewett theorem.

In contrast to Question 4.10, there is a situation where we can infer the Ramsey property for an action from the Ramsey property for some associated smaller actions.

**Proposition 4.12.** *Given Ramsey actions of $G$ on $X$ and of $H$ on $Y$, let $G \times H$ act on $X \times Y$ by*

$$(g, h) \cdot (x, y) = (g \cdot x, h \cdot y).$$

*This action also has the Ramsey property.*

Proof. By Proposition 4.4, we need only check the definition for $F$ of the form $A \times B$, where $A$ and $B$ are finite subsets of $X$ and $Y$, respectively. Let such sets be given, and let $c$ be an arbitrary 2-coloring of $X \times Y$.

By the Ramsey property of $X$, or rather by part (2) of Proposition 4.2, find a finite set $M \subseteq X$ such that every 2-coloring of $M$ is constant on $g \cdot A$ for some $g \in G$. Define a function $c'$ from $Y$ into the (finite) set of all functions $M \to 2$ by setting

$$c'(y)(x) = c(x, y).$$

By the Ramsey property of $Y$, or rather by part (3) of Proposition 4.2, there is $h \in H$ such that $c'(h \cdot b)$ is the same function, say $c'' : M \to 2$, for all $b \in B$. By our choice of $M$, there is $g \in G$ such that $g \cdot A$ is included in $M$ and $c''$ is constant on it, say with value $e$. Then we have, for all $a \in A$ and all $b \in B$,

$$c\big((g, h) \cdot (a, b)\big) = c(g \cdot a, h \cdot b) = c'(h \cdot b)(g \cdot a) = c''(g \cdot a) = e.$$

So $c$ is constant on $(g, h) \cdot (A \times B)$, as required. □

Of course, this proposition can be extended to apply to more than two factors, by routine induction.

**Corollary 4.13.** *Given Ramsey actions of groups $G_1, \dots, G_k$ on sets $X_1, \dots, X_k$, respectively, let the product group $G_1 \times \cdots \times G_k$ act on $X_1 \times \cdots \times X_k$ by*

$$(g_1, \dots, g_k) \cdot (x_1, \cdots, x_k) = (g_1 \cdot x_1, \dots, g_k \cdot x_k).$$

*This action also has the Ramsey property.*

Combining this corollary with Example 4.5, we find that the natural action of $(\mathrm{Aut}(\mathbb{Q}, <))^k$ on $([\mathbb{Q}]^n)^k$ or more generally on

$$[\mathbb{Q}]^{n_1} \times \cdots \times [\mathbb{Q}]^{n_k}$$

has the Ramsey property. There is a useful alternative way to look at this example. Choose a subset $C$ of $\mathbb{Q}$ of cardinality $k-1$. It cuts $\mathbb{Q} - C$ into $k$ intervals, each order-isomorphic to $\mathbb{Q}$. Furthermore, the pointwise stabilizer $\mathrm{Fix}(C)$ consists of order-preserving permutations that act independently on each of these intervals, so it is isomorphic to $(\mathrm{Aut}(\mathbb{Q}, <))^k$. Now let $F$ be a subset of $\mathbb{Q}$ that includes $C$ and has exactly $n_i$ additional elements in the $i^{\text{th}}$ interval for each $i$. Thus, the elements of $C$ occur in $F$ at positions $n_1 + 1$, $n_1 + n_2 + 2$, …, $n_1 + \cdots + n_{k-1} + k - 1$. Then the orbit of $F$ under the action of $\mathrm{Fix}(C)$ consists of all sets $F'$ having the same configuration as $F$ relative to the elements of $C$. This orbit can thus be identified with $[\mathbb{Q}]^{n_1} \times \cdots \times [\mathbb{Q}]^{n_k}$ by identifying any $F'$ with the list of its intersections with the $k$ intervals (and identifying each interval with $\mathbb{Q}$). Thus we find that the action of $\mathrm{Fix}(C)$ on the orbit of any finite $F \subseteq \mathbb{Q}$ that includes $C$ has the Ramsey property. This example is Halpern's main combinatorial lemma in [**2**].

The same result could be obtained as a consequence of Proposition 4.9 and Example 4.5, as follows. Given $k$ and $n_1, \dots, n_k$ as above, let $N = n_1 + \cdots + n_k + k - 1$; this is the cardinality of $F$ in the preceding discussion. There is an $\mathrm{Aut}(\mathbb{Q}, <)$-equivariant map $p : [\mathbb{Q}]^N \to [\mathbb{Q}]^{k-1}$ sending each $F \in [\mathbb{Q}]^N$ to the subset of its elements in positions $n_1 + 1$, $n_1 + n_2 + 2$, …,$n_1 + \cdots + n_{k-1} + k - 1$. So, in the notation above, it extracts $C$ from $F$. If we fix some $C \in [\mathbb{Q}]^{k-1}$, then its fiber $p^{-1}(\{C\})$ consists of those $F'$'s whose configuration relative to $C$ is the same as that of $F$. And the stabilizer of $C$ is $\mathrm{Fix}(C)$, because an order-preserving permutation that stabilizes $C$ must fix it pointwise. Since the action of $\mathrm{Aut}(\mathbb{Q}, <)$ on $[\mathbb{Q}]^N$ has the Ramsey property by Example 4.5, we may apply Proposition 4.9 to it, with $[\mathbb{Q}]^{k-1}$ and $C$ in the role of $Y$ and $y_0$. The fiber and stabilizer for which we thus obtain the Ramsey property are precisely the action considered in the preceding paragraph.

In the rest of this section, we reformulate some of the preceding material in terms of subgroups of $G$ rather than actions of $G$. If $H$ is a subgroup of $G$, which we shall write as $H \leq G$, then $G$ acts on the set $G/H = \{kH : k \in G\}$ of left cosets by $g \cdot (kH) = (gk)H$. This is a transitive action, and it is well known that every transitive action of $G$ is isomorphic to one of this form. Specifically, given any transitive action of $G$ on a set $X$ and given any element $x_0 \in X$, let $H$ be the stabilizer $\mathrm{Fix}(\{x_0\})$ of $x_0$. Then there is an isomorphism $G/H \to X$ defined by $kH \mapsto k \cdot x_0$. Thus, we may usually assume that transitive actions of $G$ are of the form $G/H$.

Furthermore, homomorphisms between transitive actions of $G$ are essentially given by inclusion relations between subgroups. Specifically, suppose $X$ and $Y$ are transitive $G$-sets, $p : X \to Y$ is $G$-equivariant, and $x_0 \in X$. Let $y_0 = p(x_0)$, and proceed as above to identify $X$ and $Y$ with $G/H$ and $G/K$, where $H$ and $K$ are the stabilizers of $x_0$ and $y_0$, respectively. Then $H \leq K$ and the homomorphism $G/H \to G/K$ that corresponds to $p$ via the identifications is simply $gH \mapsto gK$.

**Definition 4.14.** A subgroup $H$ of a group $G$ is a *Ramsey subgroup* of $G$ if the natural action of $G$ on $G/H$ has the Ramsey property.

Since every Ramsey action is transitive (Proposition 4.3) and therefore isomorphic to one of the form $G/H$, the study of Ramsey actions is essentially the same as the study of Ramsey subgroups.

In particular, we have the following succinct reformulation of Propositions 4.8 and 4.9.

**Corollary 4.15.** *Suppose $K \leq H \leq G$ and $K$ is a Ramsey subgroup of $G$. Then $K$ is a Ramsey subgroup of $H$, and $H$ is a Ramsey subgroup of $G$.*

Proof. Since $G/K$ has the Ramsey property and since we have a surjective $G$-homomorphism $G/K \to G/H : gK \mapsto gH$, Proposition 4.8 says that $G/H$ has the Ramsey property. Furthermore, the point $1H$ in $G/H$ has stabilizer $H$ and fiber $H/K$. So Proposition 4.9 says that $H/K$, as an action of $H$, has the Ramsey property. □

From this point of view, Question 4.10 simply asks about the converse of this corollary, namely whether "Ramsey subgroup" is a transitive relation.

**Question 4.16.** Is a Ramsey subgroup of a Ramsey subgroup of $G$ necessarily a Ramsey subgroup of $G$?

To conclude this section, we consider a simplified yet equivalent version of a definition from [**1**].

**Definition 4.17.** Let $G$ be a group and $\mathcal{F}$ a normal filter of subgroups of $G$. We call $\mathcal{F}$ a *Ramsey filter* if it contains a group $H$ such that every subgroup of $H$ in $\mathcal{F}$ is a Ramsey subgroup of $H$.

The corresponding definition (of "Ramsey property" for normal filters) in [**1**] required not just one $H$ as in the present definition but many such $H$'s, enough to form a base for $\mathcal{F}$. That is, every subgroup in $\mathcal{F}$ had to include some $H \in \mathcal{F}$ all of whose subgroups in $\mathcal{F}$ are Ramsey subgroups of $H$. But by Corollary 4.15, this apparently stronger requirement actually follows from the definition given here. Once we have one $H \in \mathcal{F}$ with the required property, all its subgroups in $\mathcal{F}$ inherit that property, and these subgroups form a base for $\mathcal{F}$.

We shall need the following lemma about Ramsey filters.

**Lemma 4.18.** *Suppose $\mathcal{F}$ is a Ramsey filter of subgroups of $G$, and let $H$ be as in the definition of Ramsey filter. If the trivial subgroup $\{1\}$ is in $\mathcal{F}$, then $H = \{1\}$.*

Proof. It suffices to show that, if $H$ is any non-trivial group, then the trivial subgroup is not a Ramsey subgroup, i.e., the action of $H$ on itself by left translation is not a Ramsey action. Fix an element $a \in H - \{1\}$ and consider the left cosets $h\langle a\rangle$ of the cyclic subgroup $\langle a\rangle$ of $H$ generated by $a$. We can color the elements of $\langle a\rangle$ with at most three colors so that no two consecutive elements $a^n$ and $a^{n+1}$ have the same color. (If the order of $\langle a\rangle$ is even or infinite, two colors suffice, but a third color is needed if the order is odd.) Translate this coloring to each of the other cosets $h\langle a\rangle$ along some (chosen) $h$. The result is a coloring $c$ of $H$ such that no set of the form $h \cdot \{1, a\}$ is monochromatic. This means that the Ramsey property fails. □

## 5. Equivalence

This final section is devoted to establishing the main thesis of the paper.

**Thesis.** *The following are essentially the same.*

- *A non-trivial, extremely amenable topological group with small open subgroups*
- *A permutation model that satisfies the Boolean prime ideal theorem but not the axiom of choice*
- *A group with a Ramsey filter of subgroups*

Let us first consider topological groups $G$ with small open subgroups. In such a group, the open subgroups constitute a normal filter $\mathcal{F}$ of subgroups. (Normality comes from continuity of the group operations; any conjugate of an open subgroup is again an open subgroup.)

Conversely, if $\mathcal{F}$ is any normal filter of subgroups of a group $G$, then we obtain a topology on $G$ by declaring a set $A \subseteq G$ to be open if, for each $g \in A$, there is some $H \in \mathcal{F}$ such that $gH \subseteq A$. In other words, the neighborhood filter at any $g \in G$ is obtained by translating $\mathcal{F}$ on the left by $g$. (Translation on the right would produce the same filter, thanks to normality.) This topology makes $G$ a topological group with small open subgroups, and its filter of open subgroups is exactly $\mathcal{F}$.

To comply with our convention that topological spaces should be Hausdorff spaces, we should require that the intersection of all the groups in $\mathcal{F}$ contains only the identity element of $G$. This requirement makes little difference. If $\bigcap\mathcal{F}$ is not just $\{1\}$, then it is a non-trivial normal subgroup $N$ of $G$, and we should work with $G/N$ instead. In particular, $\mathcal{F}/N = \{H/N : H \in \mathcal{F}\}$ is a normal filter of subgroups of $G/N$, and the topology it defines makes $G/N$ a Hausdorff topological group with small open subgroups. Notice also that, in any continuous action of $G$ on a Hausdorff space, $N$ must act trivially, so there is an induced continuous action of $G/N$ on $X$. In particular, $G$ is extremely amenable if and only if $G/N$ is.

**Theorem 5.1.** *Let $G$ be a topological group with small open subgroups and let $\mathcal{F}$ be the filter of open subgroups of $G$. Then the following are equivalent.*

(1) *$G$ is extremely amenable and non-trivial.*
(2) *Every subgroup in $\mathcal{F}$ is a Ramsey subgroup of $G$, and $\{1\} \notin \mathcal{F}$.*

Notice that item (2) in this theorem says slightly more than "$\mathcal{F}$ is a Ramsey filter"; it requires that the $H$ in the definition of Ramsey filter must be $G$ itself, not some proper subgroup. This variation is necessary because of examples like the following. Consider an extremely amenable $G$ and the product $G \times (\mathbb{Z}/2)$. The latter is not extremely amenable, because it acts without fixed points on a two-element set. Yet the filter of open subgroups in $G \times (\mathbb{Z}/2)$ is simply the closure, under supergroups, of the filter of open subgroups of $G$ (identified with $G \times \{0\}$). It follows immediately that, if either of these is a Ramsey filter, then so is the other. So extreme amenability cannot correspond exactly to Ramseyness of the filter of open subgroups; the latter is preserved from $G$ to $G \times (\mathbb{Z}/2)$, and the former is not.

Proof of Theorem 5.1. This theorem is almost the same as Proposition 4.2 in [**7**]. The main difference is that, in [**7**], the group $G$ is assumed to be a subgroup of the group of permutations of $\mathbb{N}$, topologized as a subspace of the product $\mathbb{N}^{\mathbb{N}}$ (with $\mathbb{N}$ having the discrete topology). Inspection of the proof in [**7**] reveals, however, that this assumption about $G$ was used only in order to show that $G$ has small open subgroups. So the proof in [**7**] establishes that, for topological groups with small open subgroups, extreme amenability (i.e., our condition (1) without "non-trivial") is equivalent to our condition (2) without "$\{1\} \notin \mathcal{F}$."

That the conditions remain equivalent when we adjoin "non-trivial" to the first and "$\{1\} \notin \mathcal{F}$" to the second is immediate from Lemma 4.18 once we observe that the $H$ there will be all of $G$ in our present situation. □

We now turn to permutation models of BPI. Recall, from Section 3, that a permutation model $M = M(A, G, \mathcal{F})$ is specified by giving a set $A$ of atoms, a group $G$ of permutations of $A$, and a normal filter $\mathcal{F}$ of subgroups of $G$. We can make two normalizations of these data without affecting $M$.

First, we can assume that each atom in $A$ is symmetric. The reason is that, if any atom $a$ were not symmetric then neither $a$ nor any set with $a$ in its transitive closure would be in the permutation model $M$. The model would be unchanged if we simply deleted all non-symmetric atoms from $A$. We intend to delete from $A$ all the non-symmetric atoms, but, in order to do so, we must take care of some technicalities. Normality of $\mathcal{F}$ easily implies that the set $A'$ of surviving atoms is closed under the action of $G$. It is possible, though, that $G$ is no longer a group of permutations of $A'$ because there might be non-trivial elements of $G$ that fix all the atoms in $A'$. In that case, those elements form a normal subgroup $N$ of $G$, and we should replace $G$ by $G/N$ and $\mathcal{F}$ by the filter generated by the images under the projection $G \to G/N$ of the subgroups in $\mathcal{F}$. All these changes do not affect the model $M$, so we may assume henceforth that all atoms are symmetric.

Second, we can assume that each group in $\mathcal{F}$ occurs as the stabilizer of some element of $M$. To see this, suppose it were not the case, and let $\mathcal{F}'$ be the family of those groups $H \in \mathcal{F}$ that do occur as stabilizers of elements of $M$. We claim that $\mathcal{F}'$ is a normal filter of subgroups of $G$ and that the associated permutation model $M(A, G, \mathcal{F}')$ is the same as $M$.

To verify the claim, suppose first that $H \in \mathcal{F}'$ and $H \leq K \leq G$. So $H$ is the stabilizer of some $x \in M$. Then $M$ contains all the elements $g(x)$ for $g \in G$ (by an inductive proof, based on the facts that, for any $y$, the stabilizer of $g(y)$ is the conjugate, by $g$, of the stabilizer of $y$ and that $\mathcal{F}$ is normal), and it contains the orbit $\{g(x) : g \in K\}$, whose stabilizer is $K$. Thus $K \in \mathcal{F}'$, and we have shown that $\mathcal{F}'$ is closed under supergroups.

The rest of the verification of the claim is even easier. $\mathcal{F}'$ is closed under finite intersections, because the stabilizer of an ordered pair $(x, y)$ is the intersection of the stabilizers of $x$ and of $y$. So $\mathcal{F}'$ is a filter of subgroups of $G$. It is normal because, as already mentioned, the conjugate by any $g \in G$ of the stabilizer of any $x \in M$ is the stabilizer of $g(x)$, which is also in $M$. Since $\mathcal{F}' \subseteq \mathcal{F}$, anything symmetric with respect to $\mathcal{F}'$ is also symmetric with respect to $\mathcal{F}$; therefore $M(A, G, \mathcal{F}') \subseteq M$. The converse inclusion is immediate from the definition of $\mathcal{F}'$.

This completes the verification of the claim and thus the justification of the second normalization. Summarizing, we have arranged, without altering the model $M$, that

- all atoms are symmetric, and
- every group in $\mathcal{F}$ occurs as the stabilizer of some element of $M$.

The first of these normalizations and the fact that $G$ is a group of permutations of $A$ together imply that the intersection of all the groups in $\mathcal{F}$ is only the trivial group (so that $\mathcal{F}$ induces a Hausdorff topology on $G$). Indeed, given any element $g \in G$ other than the identity, there is an atom $a$ moved by $g$, and then the stabilizer of $a$ is a group in $\mathcal{F}$ that doesn't contain $g$.

**Theorem 5.2.** *With the normalizations above, the following are equivalent.*

(1) *The model $M(A, G, \mathcal{F})$ satisfies the Boolean prime ideal theorem but not the axiom of choice.*
(2) *$\mathcal{F}$ is a Ramsey filter of subgroups of $G$ not containing the trivial subgroup $\{1\}$.*

Proof. This is almost part of Theorem 2 of [**1**]. The relevant part says that $M = M(A, G, \mathcal{F})$ satisfies BPI if and only if $\mathcal{F}$ is a Ramsey filter of subgroups of $G$. It remains only to check under what circumstances $M$ satisfies the axiom of choice.

It is well known (see for example the end of Section 4.1 in [**6**]) that a set $x \in M$ admits a well-ordering in $M$ if and only if the subgroup $\mathrm{Fix}(x)$ of permutations in $G$ that fix all elements of $x$ is in $\mathcal{F}$. If AC holds in $M$, then the set $A$ of atoms can be well-ordered in $M$, and $\mathrm{Fix}(A)$ is clearly $\{1\}$. So we get $\{1\} \in \mathcal{F}$. Conversely, if $\{1\} \in \mathcal{F}$ then everything in $V(A)$ is symmetric, so $M = V(A)$, and $M$ satisfies AC. (The last step uses AC in the meta-theory.)

Thus, the requirement in item (1) of the theorem that $M$ should not satisfy AC corresponds exactly to the requirement in part (2) that $\{1\} \notin \mathcal{F}$. □

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