On symmetric indivisibility of countable structures

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Abstract. A structure $\mathcal{M} \leq \mathcal{N}$ is symmetrically embedded in $\mathcal{N}$ if any $\sigma \in \mathrm{Aut}(\mathcal{M})$ extends to an automorphism of $\mathcal{N}$. A countable structure $\mathcal{M}$ is *symmetrically indivisible* if for any coloring of M by two colors there exists a monochromatic $\mathcal{M}' \leq \mathcal{M}$ such that $\mathcal{M}' \cong \mathcal{M}$ and $\mathcal{M}'$ is symmetrically embedded in $\mathcal{M}$.

We give a model-theoretic proof of the symmetric indivisibility of Rado's countable random graph **[9]** and use these new techniques to prove that $\mathbb{Q}$ and the generic countable triangle-free graph $\Gamma_\triangle$ are symmetrically indivisible. Symmetric indivisibility of $\mathbb{Q}$ follows from a stronger result, that symmetrically embedded elementary submodels of $(\mathbb{Q}, \leq)$ are dense. As shown by an anonymous referee, the symmetrically embedded submodels of the random graph are not dense.

1. Introduction

Indivisibility of countable structures is a broad phenomenon. The ordered rationals $(\mathbb{Q}, \leq)$, Rado's countable random graph, Γ, and the generic countable K_n-free graph, Γ_n, for $n \geq 3$, known also as Henson graphs, are examples of countable structures which are indivisible under finite partitions: when partitioned to finitely many parts, one of the parts contains an isomorphic copy of the whole structure.

Indivisibility of $(\mathbb{Q}, \leq)$ is a triviality. That of Γ is well known, easy to prove and in fact true in a stronger form: whenever Γ is partitioned into finitely many parts, one of the parts itself is isomorphic to Γ **[1]**. Indivisibility of the generic triangle-free graph is somewhat harder and was proved in **[10]**. Indivisibility of all Henson graphs was proved in **[4]**.

In this paper we investigate indivisibility with respect to a stronger condition on the desired copy of the whole structure, which we now define. A substructure $\mathcal{M}$ of a structure $\mathcal{N}$ is *symmetrically embedded* in $\mathcal{N}$ if every automorphism of $\mathcal{M}$ extends to an automorphism of $\mathcal{N}$. Let us call a countable structure $\mathcal{N}$ *symmetrically*

1991 *Mathematics Subject Classification.* Primary 05C63, 11U09, 05D10; Secondary 03C64, 22F50, 20B27.

Key words and phrases. Rado's random graph, ordered rationals, generic triangle-free graph, automorphism, $\aleph_0$-categorical structure, stable embedding, indivisibility, symmetric indivisibility.

The first author was partially supportes by an Israel Science Foundation grant number 1156/10.

The second author thanks the Institute for Advanced Study, Princeton, NJ, for its hospitality during his work on this paper.

indivisible if for every partition of $\mathcal{N}$ into finitely many parts, one of the parts contains an isomorphic copy of $\mathcal{N}$ which is symmetrically embedded in $\mathcal{N}$.

The symmetric indivisibility of Rado's random graph was established recently in [**9**]. In this paper we investigate symmetric indivisibility in the ordered rationals and in Henson's generic triangle-free graphs. The main results are:

Theorem (Theorem 3.11 below). *The symmetrically embedded copies of $\mathbb{Q}$ are* dense *in all copies of $\mathbb{Q}$ in $\mathbb{Q}$, that is, for every $\mathcal{P} \leq \mathbb{Q}$ which is isomorphic to $\mathbb{Q}$ there exists $\mathcal{Q} \leq \mathcal{P}$ isomorphic to $\mathbb{Q}$ and symmetrically embedded in $\mathbb{Q}$.*

This of course implies the symmetric indivisibility of $\mathbb{Q}$.

Theorem (Theorem 5.30 below). *The generic triangle-free graph is symmetrically indivisible.*

The next major open problem (addressed in a sequel paper) concerning symmetric indivisibility in countable structures is, then:

Problem. Does symmetric indivisibility hold in all Henson graphs?

The symmetric indivisibility of Rado's graph Γ was established in [**9**] along the following scheme. (1) The composition of Γ with itself, denoted $\Gamma[\Gamma]$ is symmetrically embedded in Γ. (2) For every partition of $\Gamma[\Gamma]$ to two parts, one of the parts contains a symmetrically embedded copy of Γ.

In Section 2 we investigate further the scheme of using composition of structures, use it to prove the symmetric indivisibility of $(\mathbb{Q}, \leq)$, and indicate its limitations.

In Section 3 we develop a different strategy for proving symmetric indivisibility, by proving the density of symmetrically embedded copies of an indivisible structure. This suffices for a second proof of the symmetric indivisibility of $\mathbb{Q}$, and of $\mathbb{Q}$-trees, which are handled in Section 4.

However, this method too has limitations: most notably, density of symmetrically embedded copies is false in Rado's graph, as proved by an anonymous referee of this paper (see Theorem 5.3 below).

Finally, to obtain the symmetric indivisibility of the generic triangle-free graph, a combination of both approaches with a new ingredient is employed in Section 5. The new ingredient is the notion of a *stably embedded* sub-structure. A substructure $\mathcal{M} \leq \mathcal{N}$ is stably embedded in $\mathcal{N}$ if every subset $A \subseteq \mathcal{M}$ which is definable with parameters in $\mathcal{N}$ is also definable in $\mathcal{M}$ with parameters in $\mathcal{M}$. We prove that (1) the stably embedded copies of $\Gamma_\triangle$ are indestructible by finite partitions, and, (2) the symmetrically indivisible copies of the $\Gamma_\triangle$ are dense in the stably embedded copies.

Symmetric indivisibility of all Henson graphs can be proved along similar lines, but the proof of indestructibility of stably embedded copies, which follows the proof in [**4**], is quite involved, and will be presented elsewhere. We conclude the paper with the discussion of elementary symmetric indivisibility in Section 6.

Thus, the main progress on the methods of [**9**] is the improved understanding of the distribution of symmetrically embedded copies of a structure. The common feature to both density of symmetric copies of $\mathbb{Q}$ and the existence of symmetrically embedded copies of, say, Γ in stably embedded ones, is the following: it is possible to decrease the space of types realized over an infinite subset of a structure by thinning out the set. Since thinning out a set puts more elements in its complement — which may realize more types over the set — this phenomenon runs somewhat agains one's intuition.

In the case of the rationals a copy of $\mathbb{Q}$ can be thinned out so that up to conjugation only two types are realized over the remaining set. In the case of the random graph a stably embedded copy can be thinned out so that every vertex outside the remaining copy has only a finite set of neighbors inside it.

The main results in this paper are combinatorial, with the model-theoretic methods serving to obtain them. We did, though, include some results which are model-theoretic. Such are Proposition 2.14, Theorem 3.12, which looks ahead to Section 5, and is not needed in the sequel, and all of Section 6. These results are not needed for the rest of the paper and may be skipped without any harm to the comprehension of the main combinatorial results.

We tried to minimize the model theoretic jargon in the parts needed for the combinatorics. The concepts of definability and of stable embedding in the particular context of the random graph and of the generic triangle-free graphs are simpler than in their most general form, so (at least in our sincere opinion) a reader who is interested in symmetric indivisibility but is less interested in model theory can still follow all proofs without learning model theory thoroughly.

Acknowledgment. The authors are grateful to two anonymous referees for many valuable comments that improved this paper. We are especially grateful to one of the referees for proving Theorem 5.3 below and allowing us to include it here.

2. The composition of two structures

In **[9]** the symmetric indivisibility of the countable random graph, Γ, is proved in three steps:

(1) It is observed that the composition[1] $\Gamma[\Gamma]$ of Γ with itself embeds symmetrically into Γ.
(2) Any fibre and any section of $\Gamma[\Gamma]$ are isomorphic to Γ and symmetrically embedded in $\Gamma[\Gamma]$.
(3) Any coloring $c : \Gamma[\Gamma] \to \{0, 1\}$ contains either a monochromatic section or a monochromatic fibre.

This argument translates readily to show that $(\mathbb{Q}, \leq)$ is symmetrically indivisible, with a significant strengthening of (1): $\mathbb{Q}[\mathbb{Q}]$ does not only embed symmetrically into $\mathbb{Q}$, it is actually isomorphic to $\mathbb{Q}$. This stronger form of (1) is false for the random graph. As we shall see below, this difference between Γ and $\mathbb{Q}$ runs deeper, as in $\mathbb{Q}$ the symmetrically embedded submodels are dense (Theorem 3.11 below), while in Γ they are not dense (Theorem 5.3 below by an anonymous referee).

We give a now more general treatment of composition:

Definition 2.1. Let $\mathcal{L}$ be a relational first order language and $\mathcal{M}$ a structure for $\mathcal{M}$. The *composition* $\mathcal{M}[\mathcal{M}]$, of $\mathcal{M}$ with itself is the $\mathcal{L}$-structure whose universe

[1]Kojman and Geschke refer to the composition of two graphs as the wreath product. At the suggestion of one of the referees we replaced the terminology to one which seems somewhat more accurate.

is $M \times M$ and such that for $R(x_1, \dots, x_k) \in \mathcal{L}$ we set

$$R(M[M]) :=$$
$$\{((a, a_1), \dots, (a, a_k)) : a \in M, M \models R(a_1, \dots, a_k)\} \cup$$
$$\{((a_{1,1}, a_{1,2}), \dots, (a_{k,1}, a_{k,2})) : \bigwedge_{i \neq j}(a_{i,1} \neq a_{j,1}), M \models R(a_{i,1}, \dots, a_{j,1})\}$$

Recall that:

DEFINITION 2.2. A structure $\mathcal{M}$ is *transitive* if $\mathrm{Aut}(\mathcal{M})$ acts transitively on M.

As we will see, transitivity is a natural condition when studying (symmetric) indivisibility. We need the following technical definition:

DEFINITION 2.3. Call a theory (or a structure) in a relational language Δ-free if every relation is either binary or does not intersect any diagonal.

Note that the notion of Δ-freeness is a harmless technicality, since any n-ary relation, R, can be written as the union of an n-ary Δ-free relation with Δ-free relations of smaller arities (interpreted as the intersection of R with the various diagonals). At all events, the following is now obvious:

LEMMA 2.4. *If $\mathcal{M}$ is a transitive structure in a Δ-free relational language then every section and every fibre of $\mathcal{M}[\mathcal{M}]$ are isomorphic, with the induced structure, to $\mathcal{M}$.*

Where, naturally, a *fibre* of the composition $\mathcal{M}[\mathcal{M}]$ is a set of the form $\{(a, b) : b \in M\}$ for some fixed $a \in M$ and a *section* is a set of the form $\{(a, f(a)) : a \in M\}$ for some function $f : M \to M$.

Observe also that if $c : \mathcal{M}[\mathcal{M}] \to \{0, 1\}$ is any coloring then there exists either a 0-monochromatic fibre or a 1-monochromatic section. For if there is no 0-monochromatic fibre, we can always find a function $f : M \to M$ such that $c(a, f(a)) = 1$. Thus we get:

LEMMA 2.5. *Let $\mathcal{M}$ be a transitive structure in a Δ-free relational language $\mathcal{L}$. Let $W \subseteq \mathcal{M}$ be isomorphic to $\mathcal{M}[\mathcal{M}]$. Then $\mathcal{M}$ is indivisible. Moreover, under any coloring $c : M \to \{0, 1\}$ the monochromatic sub-structure of $\mathcal{M}$ isomorphic to $\mathcal{M}$ can be chosen either a section or a fibre of W.*

The above suggests the following definition:

DEFINITION 2.6. Let $\mathcal{M}$ be a structure in a Δ-free relational first order language. Say that $\mathcal{M}$ is *(symmetrically) self similar* if $\mathcal{M}[\mathcal{M}]$ embeds (symmetrically) into $\mathcal{M}$.

EXAMPLE 2.7. The structure $\mathcal{M} := (\mathbb{Q}, \leq)$ is symmetrically self-similar. The structure $\mathcal{M}[\mathcal{M}]$ is simply $(\mathbb{Q}^2, \leq_{\mathrm{lex}})$, which is a countable dense linear order with no endpoints. Since all such structures are isomorphic to each other we get that $\mathcal{M}[\mathcal{M}] \cong \mathcal{M}$.

The importance of the composition $\mathcal{M}[\mathcal{M}]$ to the study of symmetric indivisibility is that it has many copies of $\mathcal{M}$ which are symmetrically embedded:

LEMMA 2.8. *Let $\mathcal{M}$ be a transitive structure in a relational language and $\mathcal{M}_0 \leq \mathcal{M}[\mathcal{M}]$ a fibre or a section. Then $\mathcal{M}_0$ is symmetrically embedded in $\mathcal{M}$.*

PROOF. Let $\sigma \in \mathrm{Aut}(\mathcal{M}_0)$. Assume, first, that $\mathcal{M}_0$ is a fibre. Define $\tau : \mathcal{M}[\mathcal{M}]$ by

$$\tau(a) = \begin{cases} a & \text{if } a \notin \mathcal{M}_0, \\ \sigma(a) & \text{if } a \in \mathcal{M}_0. \end{cases}$$

Then, by the definition of $\mathcal{M}[\mathcal{M}]$ it follows that τ preserves all relations, and being bijective, we are done. So it remains to treat the case where $\mathcal{M}_0$ is a section.

In that case, let $f : \mathcal{M} \to \mathcal{M}$ be the function defining the section. We define $\tilde{\sigma} : \mathcal{M} \to \mathcal{M}$ by $\tilde{\sigma}(a) = \pi_1(\sigma(a, f(a)))$, where π_1 is the projection onto the first coordinate. Let $(a, f(a)) \in \mathcal{M}_0$, then — by transitivity — there exists $\sigma_a \in \mathrm{Aut}(\mathcal{M})$ such that $\sigma_a(f(a)) = \pi_2(\sigma(a, f(a)))$, where π_2 is the projection onto the second coordinate. Now we define:

$$\tau(a, b) = (\tilde{\sigma}(a), \sigma_a(b)).$$

We claim that τ satisfies the requirements. First, by definition, for $(a, f(a)) \in \mathcal{M}_0$ we have $\pi_1\tau(a, f(a)) = \tilde{\sigma}(a) = \pi_1(\sigma(a, f(a)))$ and $\pi_2\tau(a, f(a)) = \sigma_a(f(a)) = \pi_2(\sigma(a, f(a)))$, so τ extends σ. To see that $\tau \in \mathrm{Aut}(\mathcal{M}[\mathcal{M}])$ let R be an n-ary relation and $\bar{a}_i := (a_i, b_i)$, $1 \leq i \leq n$ such that $\mathcal{M}[\mathcal{M}] \models R(\bar{a}_1, \ldots, \bar{a}_n)$. Observe that either $a_i = a_j := a$ or $a_i \neq a_j$ for all $i \neq j \leq n$ (and by transitivity $n > 1$). In the former case $\tau(\bar{a}_i) = (\tilde{\sigma}(a), \sigma_a(b_i)))$, and as $\sigma_a \in \mathrm{Aut}(\mathcal{M})$ the definition of the composition gives $\mathcal{M}[\mathcal{M}] \models R(\tau(\bar{a}_1, \ldots, \bar{a}_n))$. In the latter case, $\pi_1\tau(\bar{a}_i) \neq \pi_1\tau(\bar{a}_j)$ for all $i \neq j$ and, again — since $\tilde{\sigma} \in \mathrm{Aut}(\mathcal{M})$ the definition of the composition implies $\mathcal{M}[\mathcal{M}] \models R(\tau(\bar{a}_1), \ldots, \tau(\bar{a}_2))$. □

Summing up all of the above we get:

COROLLARY 2.9. *Let $\mathcal{M}$ be a transitive symmetrically self similar structure, then $\mathcal{M}$ is symmetrically indivisible.*

PROOF. Apply Lemma 2.5 with a symmetrically embedded copy W of $\mathcal{M}[\mathcal{M}]$. Then apply the last lemma and the transitivity of the property of being symmetrically embedded. □

As an application we get, using Example 2.7:

COROLLARY 2.10. *The structure $(\mathbb{Q}, \leq)$ is symmetrically self-similar.*

The same argument can be used to show that:

COROLLARY 2.11. *The countable universal n-uniform hypergraph, $\mathcal{G}_n$, is symmetrically indivisible for all $n \geq 2$.*

The case $n = 2$ is done in **[9]** and the general case follows the same lines. Since any countable n-hypergraph embeds in $\mathcal{G}_n$ we readily get that $\mathcal{G}_n$ is self-similar. It will suffice, therefore, to show:

LEMMA 2.12. *Any countable n-uniform hypergraph embeds symmetrically into $\mathcal{G}_n$.*

PROOF. For a set A let us denote $F_k(A) := [A]^k$. Let G_n^0 be any countable n-hypergraph and G_n^{m+1} the n-hypergraph obtained by adjoining to G_n^m vertices $\{a_I : I \in [F_{n-1}(G_n^m)]^{<\omega}\}$ such that $G(b_1, \ldots, b_{n-1}, b_n)$ holds if and only if one of the two following options hold:

(1) $b_i \in G_n^m$ for all $1 \leq i \leq n$ and $G(b_1, \ldots, b_{n-1}, b_n)$ is satisfied in G_n^m, or
(2) $b_n \notin G_n^m$, in which case $b = a_I$ for some $I \in [F_{n-1}(G_n^m)]^{<\omega}$ and $\{b_1, \ldots, b_{n-1}\} \in I$.

Finally set $\mathcal{G} := \bigcup_{m\in\omega} G_n^m$. We claim that $\mathcal{G} \cong \mathcal{G}_n$ and that G_n^0 is symmetrically embedded in $\mathcal{G}$. To show that $\mathcal{G} \cong \mathcal{G}_n$ it suffices, of course to show that $\mathcal{G} \equiv \mathcal{G}_n$, since the latter is $\aleph_0$-categorical. But $\mathrm{Th}(\mathcal{G}_n)$ is axiomatized by stating that it is an n-hypergraph and that for all k_1, k_2 and distinct $\bar{a}_1, \ldots \bar{a}_{k_1}$, $\bar{b}_1, \ldots, \bar{b}_{k_2}$ all in $F_{n-1}(\mathcal{G}_n)$ there is an element d such that

$$(*) \qquad \bigwedge_{i=1}^{k_1} G(\bar{a}_i, d) \wedge \bigwedge_{i=1}^{k_2} \neg G(\bar{b}_i, d).$$

So assume that $\bar{a}_1, \ldots \bar{a}_{k_1}$, $\bar{b}_1, \ldots, \bar{b}_{k_2}$ all in $F_{n-1}(\Gamma)$ are distinct elements. Let $m \in \mathbb{N}$ be such that $\bar{a}_i, \bar{b}_j \subseteq G_n^m$ for all $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$. Let $I = \{\bar{a}_1, \ldots, \bar{a}_{k_1}\}$ and let $d := a_I \in G_n^{m+1}$. Then a_I witnesses $(*)$. To see that G_n^0 is symmetrically embedded in $\mathcal{G}$ it is enough to check that, in fact, given any isomorphism $\tau : G_n^m \to G_n^m$, τ can be extended to an automorphism of G_n^{m+1}. But this is obvious, since τ induces an automorphism of $(F_{n-1}(G_n^m), \subseteq)$ (which, for simplicity of notation, we will also denote τ), and it is clear by definition that extending τ by $a_I \mapsto a_{\tau(I)}$ for all $I \in F_{n-1}(G_n^m)$ we get an automorphism of G_n^{m+1}. □

We conclude this section by pointing out that the composition of structures allows us to construct new (symmetrically) indivisible structures from old ones. First, we slightly generalize the notion of composition:

Definition 2.13. Let $\mathcal{M}$, $\mathcal{N}$ be structures in a relational language, $\mathcal{L}$. The composition of $\mathcal{N}$ with $\mathcal{M}$, denoted $\mathcal{M}[\mathcal{N}]$ is the $\mathcal{L}$-structure whose universe is $M \times N$ where for an n-ary relation $R \in \mathcal{L}$ we set

$$\begin{aligned} &R(M[N]) := \\ &\{((a, a_1), \ldots, (a, a_k)) : a \in M, M \models R(a_1, \ldots, a_k)\} \cup \\ &\{((a_{1,1}, a_{1,2}), \ldots, (a_{k,1}, a_{k,2})) : \bigwedge_{i \neq j} (a_{i,1} \neq a_{j,1}), N \models R(a_{i,1}, \ldots, a_{j,1})\} \end{aligned}$$

Let $\mathcal{M}[\mathcal{N}]^s$ be $\mathcal{M}[\mathcal{N}]$ expanded by a new equivalence relation E_s interpreted as $\{((a, b), (a, c)) : a \in M, b, c \in N\}$.We call E_s the *standard equivalence relation* on the composition.

The following is now easy:

Proposition 2.14. *Let $\mathcal{N}$, $\mathcal{M}$ be countable structures in a Δ-free relational language $\mathcal{L}$. If $\mathcal{N}$ and $\mathcal{M}$ are both (symmetrically) indivisible (and the number of symmetrically embedded copies of $\mathcal{N}$ in $\mathcal{N}$ is finite up to conjugacy in* $\mathrm{Aut}(\mathcal{N})$*), then $\mathcal{M}[\mathcal{N}]^s$ is (symmetrically) indivisible.*

We remark that in the next Section it will be shown that the number of symmetrically embedded copies of $\mathbb{Q}$ in $\mathbb{Q}$ is finite. Thus, by this theorem, $\Gamma[\mathbb{Q}]^s$ is symmetrically indivisible, for example.

Proof. Let $c : M \times N \to \{0,1\}$ be any coloring. For each $a \in M$ denote $N_a = \{(a,b) : b \in N\}$ and $c_a : N \to \{0,1\}$ the coloring induced by c (i.e. $c_a(b) = c(a,b)$). Since N_a, with the induced structure, is isomorphic to $\mathcal{N}$ (through the natural projection $(a,b) \mapsto b$), we identify N_a with $\mathcal{N}$. We define a coloring $\tilde{c}$ of M as follows: For $a \in M$ define $\tilde{c}(a) = (0,i)$ if there is a (symmetric) $N'_a \subseteq N_a$ such that $c_a(N'_a) = \{0\}$ (and the conjugacy type of N'_a is the i-th in the finite list of conjugacy types of symmetrically embedded copies of $\mathcal{N}_n$ in itself). N'_a with the induced structure is isomorphic to $\mathcal{N}$ (and symmetrically embedded in N_a). Otherwise, let $c(s) = (1,j)$, when there is a 1-monochromatic copy of $\mathcal{N}$ (with the j-th conjugacy type in the finite list of types).

Because $\mathcal{M}$ is (symmetrically) indivisible we can find a $\tilde{c}$-monochromatic $M_0 \subseteq M$ such that, with the induced structure, $M_0 \cong \mathcal{M}$ (and M_0 is symmetrically embedded in $\mathcal{M}$). If $\tilde{c}(M_0) = \{(0,i)\}$ then, by definition, for all $a \in M_0$ we can find $N'_a \subseteq N_a$ such that $c(N_a) = \{0\}$, $N'_a \cong \mathcal{N}$ (and symmetrically embedded in N_a of conjugacy type i). Otherwise, because $\mathcal{N}$ is (symmetrically) indivisible we can find for each a some $N'_a \subseteq N_a$ such that $N'_a \cong \mathcal{N}$, $c(N_a) = \{1\}$ (and N_a is symmetrically embedded of conjugacy type j).

In any case we can define $W^c = \bigcup\{N'_a : a \in M_0\}$. Obviously W^c is c-monochromatic, and by construction $W^c \cong \mathcal{M}_0[\mathcal{N}] \cong \mathcal{M}[\mathcal{N}]$. So it remains only to verify that if $\mathcal{M}$ and $\mathcal{N}$ are both symmetrically indivisible every automorphism of W^c extends to $\mathcal{M}[\mathcal{N}]$.

Let $\alpha \in \operatorname{Aut}(W^c)$. So α induces, through the projection on the first coordinate, an automorphism α_1 on $\mathcal{M}_0$, which by symmetry of $\mathcal{M}_0$ in $\mathcal{M}$ we can extend, call the extension α_1, to the whole of $\mathcal{M}$. On the other hand, since α preserves the E_s structure, each fibre N'_a of W^c maps through α to the fibre $N'_{\alpha_1(a)}$. By construction, there is an automorphism, α_2^a of $\mathcal{N}$ such that $N'_a = N'_{\alpha_1(a)}$. So we can finally define $(\alpha_1 \times \alpha_2) : \mathcal{M}[\mathcal{N}]^s \to \mathcal{M}[\mathcal{N}]^s$ by:

$$(\alpha_1 \times \alpha_2)(a,b) = \begin{cases} (\alpha_1(a), b) & \text{if } a \notin \mathcal{M}_0, \\ (\alpha_1(a), \alpha_2^a(b)) & \text{if } a \in \mathcal{M}_0. \end{cases}$$

Then $(\alpha_1 \times \alpha_2) \in \operatorname{Aut}(\mathcal{M}[\mathcal{N}])$, and since $\alpha_1 \times \alpha_2$ extends α, this is what we needed.
. □

Observe that in the last proof the definability of the standard equivalence relation was only needed to prove the symmetric indivisibility of $\mathcal{M}[\mathcal{N}]$. It may be worth pointing out that in many cases, the standard equivalence relation E_s is definable (without parameters) in the composition. Observe, for example, that if we expand $\mathcal{M}$ by a binary relation $R(x,y)$ interpreted as equality then for the resulting structure $\mathcal{M}_=$ we have that $\mathcal{M}_=[\mathcal{N}]$ is $\mathcal{M}[\mathcal{N}]^s$.

Remark 2.15. Formally, we have only defined the composition $\mathcal{M}[\mathcal{N}]$ for structures in a common relational language. The general case (for relational languages) can be obtained by interpreting every relation in $\mathcal{L}(\mathcal{N})\setminus\mathcal{L}(\mathcal{M})$ as the empty relation in $\mathcal{M}$, and the other way around.

Finally, we point out that the converse of Proposition 2.14 is also true. I.e., if $\mathcal{M}$, $\mathcal{N}$ are structures in a relational language $\mathcal{L}$ and $\mathcal{M}[\mathcal{N}]^s$ is symmetrically indivisible then so are $\mathcal{M}$ and $\mathcal{N}$.

3. A new approach to symmetric indivisibility: density of symmetrically embedded structures

As we have seen in the previous section, $(\mathbb{Q}, \leq)$ is symmetrically indivisible because the composition of $\mathbb{Q}$ with itself is isomorphic to $\mathbb{Q}$. That proof, however, does not reveal the full nature of the situation. In fact, we will show that in $(\mathbb{Q}, \leq)$ symmetry and indivisibility can be viewed as independent phenomena.

The main result of this section is the *density of symmetrically embedded copies of* $\mathbb{Q}$ *in* $\mathbb{Q}$. Symmetric indivisibility follows directly from indivisibility and density of symmetrically embedded copies.

Definition 3.1. Let $\mathcal{M}$ be a (countable) structure. Say that the symmetrically embedded copies of $\mathcal{M}$ in itself are *dense* if for any $\mathcal{N} \leq \mathcal{M}$ which is isomorphic to $\mathcal{M}$ there exists $\mathcal{N}' \leq \mathcal{N}$ such that $\mathcal{N}'$ is isomorphic to $\mathcal{M}$ and $\mathcal{N}'$ is symmetrically embedded in $\mathcal{M}$.

In fact we obtain a stronger density result: we prove that there are 36 types of symmetrically embedded copies of $(\mathbb{Q}, \leq)$ in $\mathbb{Q}$ and that four of these types are dense.

We conclude this section with a study of the theory of pairs $(\mathbb{Q}, P)$, where P is a symmetrically embedded submodel of $\mathbb{Q}$ of a certain type.

Our proof of the density of the symmetrically embedded copies of $\mathbb{Q}$ is a construction. In order to get a picture of the direction our construction has to take we begin by analyzing the possible symmetrically embedded copies of $\mathbb{Q}$ up to conjugacy with respect to $\mathrm{Aut}(\mathbb{Q}, \leq)$. The first important structural result is Corollary 3.7. This part of the work culminates in Theorem 3.9, giving the full list of all 36 possible types of symmetrically embedded copies of $\mathbb{Q}$ in $\mathbb{Q}$.

We start with some terminology. Suppose $\mathcal{N} \leq \mathbb{Q}$ is a copy of $\mathbb{Q}$. Every real number $r \in \mathbb{R} \setminus \mathcal{N}$ satisfies over $\mathcal{N}$ one of the following *types*:

(1) The lower bound $LB(x) = \{x < q : q \in \mathcal{N}\}$.
(2) The upper bound $UB(x) = \{q < x : q \in \mathcal{N}$.
(3) The left infinitesimal of a point $q \in \mathcal{N}$, $L(q, x) = \{q < x\} \cup \{q' < x : q' \in \mathcal{M} \wedge q < q'\}$.
(4) The right infinitesimal of a point $q \in \mathcal{N}$, $R(q, x) = \{q < x\} \cup \{x < q' : q' \in \mathcal{M} \wedge q < q'\}$.
(5) The cut $(L, R, x) = \{q < x \wedge x < p : q \in L, p \in R\}$ over $\mathcal{N}$, where $L \cup R = \mathcal{N}$, $L \cap R = \emptyset$, $L \neq \emptyset \neq R$ and $p < q$ for all $p \in L$, $q \in R$.

If $\sigma \in \mathrm{Aut}(\mathcal{N})$ and t is a type, $\sigma(t)$ is gotten from t by replacing every parameter p in the type by $\sigma(p)$. Two types t and t' over $\mathcal{N}$ are *conjugate* if there is $\sigma \in \mathrm{Aut}(\mathcal{N})$ such that $\sigma(t) = t'$.

Claim 3.2. *Suppose* $\mathcal{N} \leq \mathbb{Q}$ *is a copy of* $\mathbb{Q}$*. Two types over* $\mathcal{N}$ *are conjugate if and only if they are of the same kind in the list (1)–(5) above.*

Proof. Direct inspection shows that two types of different kinds are not conjugate. If $p, q \in \mathcal{N}$ then any automorphism of $\mathcal{N}$ which takes p to q takes $L(p, x)$ to $L(q, x)$ and $R(p, x)$ to $R(q, x)$. Given two cuts (L, R, x) and (L', R', x), since the order types of all four sets is $\mathbb{Q}$, there is an automorphism of $\mathcal{N}$ carrying L to L' and R to R'. □

A type $t(x)$ is *realized* by a real $r \in \mathbb{R}$ iff r satisfies all formulas in t when substituted for x. For a copy $\mathcal{N} \leq \mathbb{Q}$ of $\mathbb{Q}$ it is possible for each of the types to be realized by a member of $\mathbb{Q}$. The type $LB(x)$ may be realized by a greatest rational number or there may not exist a greatest rational realization of it, and similarly for $UB(x)$.

Corollary 3.3. *If $\mathcal{M} \leq \mathbb{Q}$ is symmetrically embedded in $\mathbb{Q}$ and a type t over $\mathcal{M}$ is realized by some element of $\mathbb{Q}$, then all types of the same kind of t are realized in $\mathbb{Q}$.*

Proof. If t, t' are types of the same kind over a symmetrically embedded copy $\mathcal{M} \leq \mathbb{Q}$ and $q \in \mathbb{Q} \setminus \mathcal{M}$ realizes t, fix $\sigma \in \mathrm{Aut}(\mathcal{M})$ which carries t to t' and an extension $\sigma \subseteq \bar{\sigma} \in \mathrm{Aut}(\mathbb{Q})$. Now $\bar{\sigma}(q)$ satisfies t'. □

A cut (L, R) over a copy $\mathcal{N} \leq \mathbb{Q}$ may be realized by a single real number. If r realizes a cut over $\mathcal{N}$ and is the unique real number which realizes this cut, we call it a two-sided limit of $\mathcal{N}$. So r is a two-sided limit of $\mathcal{N}$ if $r \notin \mathcal{N}$ and $r = \inf\{q \in \mathcal{N} : q > r\} = \sup\{q \in N : q < r\}$.

Claim 3.4. *Suppose $\mathcal{N} \leq \mathbb{Q}$ is a copy of $\mathbb{Q}$. If $U \subseteq \mathbb{R}$ is an open interval and $|N \cap U| > 1$, there are continuum many irrationals $r \in U$ which are two-sided limits of N.*

Proof. Since $N \cap U$ contains a copy of $\mathbb{Q}$ there are continuum many cuts over $N \cap U$. If a cut over $N \cap U$ is realized by more than one real number, it is realized by a rational. Therefore, except for a countable set of cuts, every cut over $N \cap U$ is realized by a single irrational number. □

By this, almost all cuts over any copy $\mathcal{M} \leq \mathbb{Q}$ are not realized in $\mathbb{Q}$. Since realizing one cut over a symmetrically embedded copy $\mathcal{M} \leq \mathbb{Q}$ implies, by Claim 3.3, realizing all, we conclude:

Corollary 3.5. *If $\mathcal{N} \leq \mathbb{Q}$ is a symmetrically embedded copy of $\mathbb{Q}$ then no cuts over $\mathcal{M}$ are realized in $\mathbb{Q}$.*

Except for cuts, however, types over $\mathcal{N}$ which are realized by reals, are also realized by rational numbers. A convex subset of the rationals is a subset A that contains $\mathbb{Q} \cap (q, p)$ whenever $q < p$ belong to A.

Claim 3.6. *Suppose $\mathcal{N} \leq \mathbb{Q}$ is a copy of $\mathbb{Q}$.*

(1) *If the left [right] infinitesimal type $L(q, x)$ [$R(q, x)$] of some $q \in \mathcal{N}$ is realized by some real, then the set of rational numbers which realize it is a convex copy of $\mathbb{Q}$.*
(2) *If the type $LB(x)$ is realized by some real, then the set of rationals which realize it is either a convex copy of $\mathbb{Q}$, in case this set has no maximal element, or a convex copy of $\mathbb{Q} \cap (-\infty, 0]$, in case this set contains a maximal element. Similarly for $UB(x)$.*

Proof. Obvious. □

Corollary 3.7. *A copy $\mathcal{M} \leq \mathbb{Q}$ is symmetrically embedded if an only if no cuts over $\mathcal{M}$ are realized in $\mathbb{Q}$ and exactly one of the following conditions holds:*

- *For all $q \in \mathcal{M}$ neither $L(q)$ nor $R(q)$ are realized in $\mathbb{Q}$ (in which case $\mathcal{M}$ is an interval in $\mathbb{Q}$).*

- *For all $q \in \mathcal{M}$ both $L(q)$ and $R(q)$ are realized in $\mathbb{Q}$.*
- *For all $q \in \mathcal{M}$ the type $L(q)$ is realized in $\mathbb{Q}$ but $R(q)$ is not realized in $\mathbb{Q}$.*
- *For all $q \in \mathcal{M}$ the type $L(q)$ is not realized in $\mathbb{Q}$ but $R(q)$ is realized in $\mathbb{Q}$.*

In particular, a copy $\mathcal{M} \leq \mathbb{Q}$ of $\mathbb{Q}$ is symmetrically embedded if and only if for every $a < b$ in $\mathcal{M}$ the interval $(a, b) \cap M$ is symmetrically embedded in $\mathbb{Q}$.

Proof. The fact that no cuts are realized over a symmetrically embedded copy has been proved above. Suppose $\mathcal{M} \leq \mathbb{Q}$ is a copy with no cuts realized in $\mathbb{Q}$ and one of the four possibilities in the claim holds. In the first case, any automorphism of $\mathcal{M}$ can be extended as the identity on $\mathbb{Q} \setminus \mathcal{M}$. In the other three, if $\sigma \in \mathrm{Aut}(M)$ and $\sigma(q) = p$ the set of left [right] infinitesimals of q can be carried to the set of left [right] infinitesimals of p by an order isomorphism. $\square$

Let us remark that something stronger holds for symmetrically embedded copies $\mathcal{M} \leq \mathbb{Q}$. We can find a *homomorphism* $\varphi : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathbb{Q})$ so that $\sigma \subseteq \varphi(\sigma)$ for all $\sigma \in \mathrm{Aut}(\mathcal{M})$. Extend via the identity above $\sup \mathcal{M}$ and below $\inf \mathcal{M}$. If right rational infinitesimals exist for members of $\mathcal{M}$, fix some $q_0 \in \mathcal{M}$ and fix an automorphisms φ^r_p from the set right rational infinitesimals of q to the set of right rational infinitesimal of p for each $p \in \mathcal{M}$ (with $\varphi^r_{q_0} = \mathrm{id}$) and extend each σ by moving the right infinitesimals of p to the right infinitesimals of $\sigma(p)$ via $\varphi^r_{p'} \circ (\varphi^r_p)^{-1}$. Similarly for left rational infinitesimals. It is easy to check that $\varphi(\sigma \circ \tau) = \varphi(\sigma) \circ \varphi(\tau)$ for $\sigma, \tau \in \mathrm{Aut}(M)$.

We get, then:

Corollary 3.8. *A copy $\mathcal{M} \leq \mathbb{Q}$ is symmetrically embedded in $\mathbb{Q}$ if an only if there is a homomorphism $\varphi : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathbb{Q})$ satisfying $\sigma \subseteq \varphi(\sigma)$ for all $\sigma \in \mathrm{Aut}(\mathcal{M})$.*

We now get:

Theorem 3.9. *Up to conjugacy in $\mathrm{Aut}(\mathbb{Q}, \leq)$ there are exactly 36 symmetrically embedded copies of $\mathbb{Q}$.*

Proof. There are 3 possibilities concerning $LB(x)$ over a copy $\mathcal{M} \leq \mathbb{Q}$: not realized, realized with a rational greatest realization and realized with no greatest rational realization, and similarly 3 independent possibilities for $UB(x)$. There are 4 independent possibilities concerning the realization of left and right infinitesimals. Any two symmetrically embedded copies are conjugate if and only if they satisfy the same possibility out of the potential 36 described.

In fact, all 36 possibilities do occur. $\mathbb{Q}$ itself is a copy of $\mathbb{Q}$ in $\mathbb{Q}$ with no upper or lower bounds and with neither left nor right rational infinitesimals. Upper and lower bounds can be realized with or without least upper/ greatest lower one.

So it only remains to check that only left, only right and both sides of rational infinitesimals can occur over symmetrically embedded copies. Let $\mathbb{Q}$ be presented as $\mathbb{Q}[\mathbb{Q}]$ and let $N_0 = \{(q, 0) : q \in \mathbb{Q}\}$, $N_1 = \{(q, p) : p \in \mathbb{Q} \wedge p \geq 0\}$ and $N_2 = \{(q, p) : p \in \mathbb{Q} \wedge p \leq 0\}$. These copies of $\mathbb{Q}$ are symmetrically embedded in $\mathbb{Q}$ and of the three required conjugacy types. $\square$

We proceed to proving the density result claimed at the beginning of this section. Our main technical tool is:

THEOREM 3.10. *For every substructure $\mathcal{N} \leq \mathbb{Q}$ isomorphic to $\mathbb{Q}$ there is an isomorphism $\varphi : \mathrm{conv}(\mathcal{N}) \to \mathbb{Q}[\mathbb{Q}]$ such that $\varphi^{-1}[\mathcal{N}] \cap \{(q,p) : p \in \mathbb{Q}\} \cong \mathbb{Q}$ for every $q \in \mathbb{Q}$.*

PROOF. Suppose that $\mathcal{N} \subseteq \mathbb{Q}$ is order isomorphic to $\mathbb{Q}$. By replacing $\mathbb{Q}$ with the convex hull $\mathrm{conv}(\mathcal{N})$ we may assume that $\mathrm{conv}(\mathcal{N}) = \mathbb{Q}$, for simplicity of notation.

Abusing notation, let us write $I < J$ for disjoint nonempty intervals of $\mathbb{R}$ if every point in I is smaller than any point in J, that is, we shall use $<$ for both the real ordering and the *interval ordering*. Let us fix an enumeration $\{q_n : n \in \mathbb{N}\}$ of $\mathbb{Q}$.

It will suffice to present $\mathbb{Q}$ as a disjoint union of intervals $\bigcup_n I_n$ such that:

(i) The mapping $q_n \mapsto I_n$ is an order isomorphism between the rational order and the interval order on $\{I_n : n \in \omega\}$;

(ii) $I_n \cap \mathcal{N}$ is isomorphic to $\mathbb{Q}$ for each n.

We define I_n by induction so that:

(1) $I_n = (a_n, b_n)$, $a_n < b_n$ are irrational numbers each of which is a two-sided limit of N, hence $\mathcal{N} \cap (a_n, b_n)$ is isomorphic to $\mathbb{Q}$.

(2) $m < n$ implies $I_n \cap I_m = \emptyset$.

(3) The type of I_{n+1} over $\{I_m : m \leq n\}$ with respect to the interval ordering is equal to the type of q_{n+1} over $\{q_m : m \leq n\}$ with respect to the rational ordering.

(4) If q is the first rational number (in some fixed enumeration of $\mathbb{Q}$) outside $\bigcup_{m \leq n} I_m$ with the type of $\{q\}$ over $\{I_m : m \leq n\}$ equal to the type of q_{n+1} over $\{q_m : m \leq n\}$ then $q \in I_{n+1}$.

To carry the induction let $I_0 = (a_0, b_0)$ be an arbitrary interval which satisfies (1). At step $n+1$ let q be the least rational in the enumeration which satisfies the hypothesis in (4). Find $a < a_{n+1} < q < b_{n+1} < b$, such that both a_{n+1} and b_{n+1} are two-sided limits of N satisfying the same type as q over $\{I_m : m \leq n\}$.

To prove that $\bigcup_n I_n = \mathbb{Q}$ suppose this is not so and let q be the least in the enumeration of $\mathbb{Q}$ which is outside this union. At some stage n_0 it holds that q is the least in the enumeration outside of $\bigcup_{m \leq n_0} I_m$. For some $n \geq m_0$ the type of $\{q\}$ over $\{I_m : m \leq n\}$ coincides with that of q_{n+1} over $\{q_m : m \leq n\}$ hence $q \in I_{n+1}$ contrary to our assumption. □

THEOREM 3.11. *For every copy $\mathcal{M} \leq \mathbb{Q}$ of $\mathbb{Q}$ there exists a copy $\mathcal{N} \leq \mathcal{M}$ of $\mathbb{Q}$ which is symmetrically embedded in $\mathbb{Q}$, bounded from both sides and with both right and left rational infinitesimals for every $q \in \mathcal{N}$. The existence of a greatest upper bound and of least upper bound of $\mathcal{N}$ in $\mathbb{Q}$ can be chosen at will.*

PROOF. Given $\mathcal{M} \leq \mathbb{Q}$ fix, by Theorem 3.10, a presentation $\mathrm{conv}(\mathcal{M}) = \bigcup I_q$ where $I_p < I_q \iff p < 1$ and $I_q \cap \mathcal{M}$ is isomorphic to $\mathbb{Q}$. Choose $p(q) \in I_q \cap \mathcal{M}$ for each q and let $N = \{p(q) : q \in \mathbb{Q}\}$.

$\mathcal{N}$ can be intersected with an open, closed or half-closed half-open interval with end-points in $\mathbb{Q}$ to arrange the existence of a greatest lower bound and of a least upper bound. □

Recalling Definition 3.1, the previous theorem can be described as the fact that the symmetrically embedded copies of $\mathbb{Q}$ of 4 of the 36 types of symmetrically embedded copies are dense.

For the next Theorem see below Definition 5.5 of a "stably embedded substructure" and Corollary 5.6 immediately following it.

THEOREM 3.12. *For any coloring* $c : \mathbb{Q} \to \{0,1\}$ *there exists a monochromatic submodel* $\mathcal{P} \leq \mathbb{Q}$ *such that the pair* $(\mathbb{Q}, \mathcal{P})$ *is* $\aleph_0$*-categorical and* $\mathcal{P}$ *is stably embedded in* $\mathbb{Q}$.

PROOF. Let $c : \mathbb{Q} \to \{0,1\}$ be any coloring. Since $\mathbb{Q}$ is indivisible, we can find $\mathcal{P}_0 \leq \mathbb{Q}$ monochromatic and bounded from both sides. By Theorem 3.10 we can find $\mathcal{P} \leq \mathcal{P}_0$ symmetrically embedded in $\mathbb{Q}$ and a family of disjoint open intervals $\{I_p\}_{p\in\mathcal{P}}$ whose union is $\operatorname{conv}\mathcal{P}_0$ and such that $I_p \cap P = \{p\}$ for all $p \in P$. By Corollary 5.6 $\mathcal{P}$ is stably embedded in $\mathbb{Q}$.

It remains to check that $(\mathbb{Q}, \mathcal{P})$ is $\aleph_0$-categorical. Let Q denote the convex hull of P in $\mathbb{Q}$.

Claim The structure $(\mathcal{Q}, \mathcal{P})$ is $\aleph_0$-categorical.

PROOF. To define an isomorphism between any two countable models $(\mathcal{Q}, \mathcal{S})$ and $(\mathcal{Q}', \mathcal{S}')$ first choose an isomorphism of $\mathcal{Q}$ with $\mathcal{Q}'$. Since the conjugacy class of S in $\mathbb{Q}$ is given (and equals that of S' in $\mathcal{Q}'$), we can now find an automorphism of $\mathcal{Q}'$ taking the image of S to S', which is all we need. $\square_{claim}$

Now to construct an isomorphism between any two models of $(\mathbb{Q}, \mathcal{N})$ first use the claim to construct an isomorphism between the convex hull of the smaller model. Then observe that each of the types at $\pm\infty$ (over the predicate) is realized in one model if and only if it is realized in all models. Moreover, the set of realizations of each such type is isomorphic to $\mathbb{Q}$, if it is non-empty. So the partial isomorphism can be extended. $\square$

4. A symmetrically indivisible $\mathbb{Q}$-tree

Traditionally, induced Ramsey theory is concerned with relational languages. But structures in non-relational languages can still be indivisible. In this section we give a new example of such a structure. The main point of this new example is that once the proof of indivisibility is obtained, symmetric indivisibility can be inferred from the "density" results which concluded the previous section.

Consider the language $\mathcal{L} := \{\leq, \wedge\}$ and the theory T (see Remark 4.3 below) stating that:

- $\leq$ is a partial order whose lower cones are linear (i.e. the universe is a tree).
- $\leq$ is dense with no endpoints.
- $\wedge$ is a binary function such that $z = x \wedge y$ implies $z \leq x, z \leq y$ and if $z < w \leq x$ then $w \not\leq y$.
- For any element z there are infinitely many elements $\{x_i\}$ pairwise $\leq$-incomparable such that $z = x_i \wedge x_j$ for all $i \neq j$.

We will sometimes refer to the relation $a \geq b$ as "a is over b" or "a is above b". We will denote $a \perp_b c$ if a is incomparable to c and $b = a \wedge c$, we will say that a, c are *orthogonal* over b. We will also call a *branch* over b a maximal chain of elements lying over b. The following observation will be useful:

Remark 4.1. For any natural number $n > 0$ let $\psi_n(z, x_0, \ldots, x_n)$ be the $\mathcal{L}$-formula stating that if $x_i \perp_z x_j$ for all $i \neq j$ then there exists $y > z$ such that $y \wedge x_i = z$ for all $i \leq n$. Then $T \models (\forall z \forall x_1, \ldots x_n)\psi_n$ for all n.

Proof. Let $\mathcal{M} \models T$ be any model. Let $\{x_1, \ldots, x_n\}$ and z, be elements in M as in the statement. Let $\{y_i\}_{i=0}^{n+1}$ in M be such that $y_i \wedge y_j = z$ for all $i \leq n+1$ (use the last axiom). Observe that for all $0 \leq i \leq n$ there exists at most one $0 \leq j \leq n+1$ such that $x_i \wedge y_j > z$. But then there exists $j \leq n+1$ such that $y_j \wedge x_i = z$ for all $0 \leq i \leq n$, as required. □

Remark 4.2. Let $\{a_i\}_{i \in I}$ be a set of elements pairwise orthogonal over b and let $\{c_i\}_{i \in I}$ be such that $c_i \geq a_i$ for all $i \in I$. Then $\{c_i\}_{i \in I}$ is a set of pairwise orthogonal elements over b.

Remark 4.3. The theory T is consistent.

Proof. We will construct a model for T in $\mathbb{Q}^{\mathbb{Q}}$. For a function $f \in \mathbb{Q}^{\mathbb{Q}}$ and $q \in \mathbb{Q}$ denote $f^{\downarrow q} := f|(-\infty, q]$. Let $\mathcal{F} := \{f^{\downarrow q} : f \in \mathbb{Q}^{\mathbb{Q}}, q \in \mathbb{Q}\}$. Let $\leq$ be the natural partial order on $\mathcal{F}$ given by $f \leq g$ if there exists $q \in \mathbb{Q}$ such that $f = g^{\downarrow q}$. Then $(\mathcal{F}, \leq)$ is a dense tree (or, rather, forest). Let $\mathcal{F}_0 := \{f \in \mathcal{F} : (\exists q) 0^{\downarrow q} \leq f\}$, where 0 is the constant function. Then $(\mathcal{F}_0, \leq)$ is a sub-tree. Say that $f \in \mathcal{F}_0$ is locally constant, if for all $r \in \mathbb{R}$ there exist rational numbers $a < r \leq b$ such that $f|(a, b]$ is constant. Let $\mathcal{F}_1 := \{f \in \mathcal{F}_0 : f \text{ is locally constant}\}$. The structure $(\mathcal{F}_1, \leq)$ will be our model (because the function $\wedge$ is definable in T from $\leq$, this suffices to determine the $\mathcal{L}$-structure). We will now verify that $\wedge$ is well defined in $(\mathcal{F}_1, \leq)$. Let $f, g \in \mathcal{F}_1$ and let $q(f, g) := \sup\{q \in \mathbb{Q} : f(q) = g(q)\}$. Note, first, that since $f, g \in \mathcal{F}_0$ we know that $q(f, g) \in \mathbb{R}$ (i.e. is not $-\infty$). Because f, g are locally constant, it follows that $q(f, g) \in \mathbb{Q}$. So we define $f \wedge g = f^{\downarrow q(f,g)}$. Verifying that $(\mathcal{F}_1, \leq, \wedge) \models T$ is now a triviality. □

Remark 4.4. The theory T is $\aleph_0$-categorical and has quantifier elimination in the language $\mathcal{L}$.

Proof. To see this, observe first that if $\mathcal{M} \models T$ and $A \subseteq M$ is finite then $\langle A \rangle$ — the substructure generated by A is finite. The proof is by induction on (n, m) — the number and length of the longest anti-chain in A. For let A be a set with n anti-chains of maximal length m. Let a be any member of an anti-chain of maximal length, and $a_1 < a$ be maximal in (the linearly ordered set) $\{a \wedge b : b \in A \setminus \{a\}\}$. Observe that $\langle A \rangle = \langle A_1 \rangle \cup \{a\}$ where $A_1 = A \cup \{a_1\} \setminus \{a\}$. But A_1 has fewer anti-chains of length m than A did. So the conclusion follows from the inductive hypothesis.

Now, let $\mathcal{M}_1, \mathcal{M}_2 \models T$ be countable models and $f : A_1 \to A_2$ a partial isomorphism between finite substructures of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Let $a \in M_1$ be any element. We will show that f can be extended to a. If $a_1 > a$ for some $a_1 \in A_1$ let c_a be the cut of a in the branch below a_1. Observe that if $d \in A_1$ and $d \wedge a_1 < a$ then $d \wedge a = d \wedge a_1$ and if $d \wedge a_1 \geq a$ then $d \wedge a = a$. In any case, if $b \in \mathcal{M}$ realizes $f(c_a)$ then $\{(a, b)\} \cup f$ is the desired extension of f (note that such b exists because the branch below $f(a_1)$ is dense with no minimum). If $a \geq a_1$ for some $a_1 \in A_1$, a_1 is maximal such and a is not bounded above in A, let $A_0 := \{b \in A : a \wedge b = a_1\}$. A similar argument as before will show that if we can find $b \in M_2$ such that $b \perp_{f(a_1)} f(a_0)$ for all $a_0 \in A_0$ then $\langle A_1, a \rangle = A_1 \cup \{a\}$ and $f \cup \{(a, b)\}$ is an extension of f. But such an element b exists in M_2 by the last claim. So it remains to extend the function f in the case where a is incomparable to A. In that case, let $a' := \max\{a \wedge a_1 : a_1 \in A\}$. Extend the function first to a' and then to a using the two previous cases. $\square$

Let $\mathcal{M} \models T$, a countable model. We will first show that $\mathcal{M}$ is indivisible. We split the proof in to several claims:

LEMMA 4.5. *Let $\mathcal{M} \models T$ and $c : M \to \{0, 1\}$ any coloring. Assume that for some point $a \in M$ there is no densely linearly ordered set $\{a_i\}_{i\in\omega}$ above a such that $c(a_i) = 1$ for all i, then for all $b > a$ there is a dense set $\{b_i\}_{i\in\omega}$ above b such that $c(b_i) = 0$ for all i.*

PROOF. The above is true for $(\mathbb{Q}, \leq)$ and is therefore true for M using the fact that it is a dense tree. $\square$

LEMMA 4.6. *Let $\mathcal{M} \models T$. For any $b_2 > b_1$, $b_i \in M$ let*

$$G_{b_1,b_2} := \bigcup \{b \in M : (\exists c)(b_1 < c < b_2 \text{ and } b \geq c)\}.$$

Then G_{b_1,b_2} is a submodel of $\mathcal{M}$ (and by quantifier elimination and $\aleph_0$-categoricity isomorphic to $\mathcal{M}$).

PROOF. It is obvious that G_{b_1,b_2} is a dense tree. Moreover, for all $b \in G_{b_1,b_2}$ if $c > b$ then $c \in G_{b_1,b_2}$. So every $b \in G_{b_1,b_2}$ has infinite sets $\{c_i\}_{i=1}^{\infty}$ pairwise orthogonal over b. So, by quantifier elimination, it remains to check that G_{b_1,b_2} is a substructure: if $a, b \in G_{b_1,b_2}$ there are $c_a, c_b \in (b_1, b_2)$ such that $c_a \leq a$ and $c_b \leq b$. Then $a \wedge b \geq \min\{c_a, c_b\}$. And since the right hand side is in G_{b_1,b_2} so is the left hand side. $\square$

The next lemma provides, given a countable model $\mathcal{M} \models T$ and a coloring of M in two colors, sufficient conditions for the existence of a monochromatic elementary substructure of $\mathcal{M}$:

LEMMA 4.7. *Let $\mathcal{M} \models T$ and $c : M \to \{0, 1\}$ any coloring. Assume that for every $a \in M$ there are $\{a_i\}_{i\in\omega}$ such that:*

1. *$\{a_i\}_{i\in\omega}$ is a dense linearly ordered set above a.*
2. *$c(a_i) = 1$ for all i.*

Then there exists $\mathcal{N} \leq \mathcal{M}$ such that $c(N) = \{1\}$.

PROOF. We construct $\mathcal{N}$ inductively as follows. Fix $f : \mathbb{N} \to \mathbb{N}$ such that $f^{-1}(n)$ is infinite for all $n \in \mathbb{N}$. This function will serve us in keeping track, at each stage of the construction, which element should be attended to. The requirement that $f^{-1}(n)$ is infinite for all n will assure that every element will be taken care of infinitely many times. Having said that, and in order to keep the notation cleaner, the function f will be suppressed from our notation. Fix $a_0 \in M$ such that $c(a_0) = 1$ and such that there is a dense linear order of elements $b < a_0$ with $c(b) = 1$. Now assume that we have constructed a finite substructure N_i such that:

1. For $a_1 < a_2 \in N_i$ the points $b \in M$ such that $a_1 < b < a_2$ and $c(b) = 1$ form a dense linear order.
2. $c(N_i) = \{1\}$.

We will now construct N_{i+1}. Let d be the first element of N_i (with respect to some fixed enumeration of M) which needs to be attended to (according to the function f) We proceed in two steps:

First, we fix some $a < d$ with $c(a) = 1$ as follows: if $d = \min\{x : x \in N_i\}$ we set $a' = -\infty$, otherwise let $a' < d$ be maximal in $\{x \in N_i : x < d\}$. It follows from (1) of the hypothesis that we can choose $a \in (a', d)$. so that $c(a) = 1$ and such that (1) remains true when extending N_i with a. Observe that if y is any element in N_i such that $y \wedge d \leq a'$, then $y \wedge a = y \wedge d$ and if $y \wedge d \geq d$ then $y \wedge a = a$. So setting $N^0_{i+1} = N_i \cup \{a\}$ we satisfy the inductive hypothesis.

Second, fix some $b \in M$ such that $b > d$. By Remark 4.1 we can choose b such that $b \wedge x \leq d$ for all $x \in N^0_{i+1}$ (and in particular $b \wedge a = a$). Moreover, if $b \wedge x < d$ for some $x \in N^0_{i+1}$ then $b \wedge x = b \wedge d$. By Lemma 4.2 and our assumption, we can choose b in such a way that, in addition, $c(b) = 1$. Observe that if $y \in N_i$ is an immediate successor (in N_i) of a (i.e., $N_i \cap (a, y) = \emptyset$) then $y \perp_a b$. So setting $N_{i+1} = N^0_{i+1} \cup \{b\}$, the inductive hypothesis is preserved for N_{i+1}.

Now let $N := \bigcup_{i\in\omega} N_i$. Then $\mathcal{N}$ (N with the induced structure) is an elementary substructure. To see this note, first, that $\mathcal{N}$ is indeed a substructure because all the N_i are, and it is obviously a tree. We have to show that it is a dense tree with no end points, and that the last axiom of T holds in $\mathcal{N}$.

That $\mathcal{N}$ is dense is easy. For let $a < b \in N$ and let i be such that $a, b \in N_i$. By the choice of the function f there will be some $i' \geq i$ such that b is the first element of $N_{i'}$ to be attended to. Therefore, in $N_{i'+1}$ there exists some $a < b' < b$, as provided by the first step of our construction. That $\mathcal{N}$ has no end points follows in a similar way.

To see that for all $a \in N$ there are $\{a_i\}_{i\in\omega}$ pairwise orthogonal over a we use the second step in the construction, combined with the fact that, as the N_i are substructures of M, if $a_1 \perp_a a_2$ in the sense of N_i (some i) then they are orthogonal in M, and therefore also in N_j for all $j \geq i$. □

The following is an immediate corollary of the lemma.

THEOREM 4.8. *Let $\mathcal{M} \models T$ be countable, then $\mathcal{M}$ is indivisible.*

PROOF. Let $c : M \to \{0, 1\}$ be any coloring. Because T is $\aleph_0$-categorical it will suffice to show that there exists a monochromatic $\mathcal{N} \leq \mathcal{M}$. By the last lemma, if for all $a \in M$ there exists a dense linear order of points greater than a colored 1, then we are done. So we may assume that there is a point $a \in M$ for which this is not the case. So by Lemma 4.5 for all $b > a$ there exists a dense linear order of points greater than b all colored 0. Using Lemma 4.6 with $b_2 > b_1 > a$ the desired conclusion follows, again, using the previous lemma. □

We will now start the analysis showing that $\mathcal{M} \models T$ countable is symmetrically indivisible. It will turn out that, in fact, the symmetric substructures of $\mathcal{M}$ are dense. The following claim is our main technical tool. It will provide criteria under which we can extend isomorphisms from sub-trees of $\mathcal{M}$ all the way to an automorphism of $\mathcal{M}$.

LEMMA 4.9. *Let $\mathcal{N}_1, \mathcal{N}_2 \leq \mathcal{M}$. And assume that the following hold:*

(1) *For every $a_i \in N_i$ there are $c < a_i < d$ such that $N_i \cap (c, d) = \{a_i\}$.*
(2) *For every $a < b \in N_i$ the interval $(a, b) \cap N_i$ is symmetrically embedded in (a, b).*
(3) *There are $r_i \in M$ $(i = 1, 2)$ which are upper lower bounds for N_i (so that $r_i < N_i$ and given any $x < N_i$ we have $x \leq r_i$).*
(4) *All branches in $\mathcal{N}_i$ are bounded in $\mathcal{M}$.*

(5) *If $C \subseteq \mathcal{M}$ is a maximal set of elements pairwise orthogonal over some $c \in N_i$ then $C \setminus N_i$ is infinite.*

Then any isomorphism $\sigma : \mathcal{N}_1 \to \mathcal{N}_2$ extends to an automorphism of $\mathcal{M}$.

PROOF. Assume $\mathcal{N}_1, \mathcal{N}_2$ satisfy the assumptions. Observe that $r_1 < N_1$ as provided in the assumptions is unique, so given $\sigma : \mathcal{N}_1 \to \mathcal{N}_2$ a partial isomorphism, we first extend σ by $r_1 \mapsto r_2$. This is clearly a partial isomorphism, since $x \wedge r_i = r_i$ for all $x \in N_i$.

CLAIM 4.10. *Let $G_i := \{x \in M : r_i \leq x\}$ and assume that we can find $\tilde{\sigma} : G_1 \to G_2$, a partial isomorphism extending σ. Then we can extend σ to* $\mathrm{Aut}(M)$.

PROOF. Since $(\mathcal{M}, r_1) \cong (\mathcal{M}, r_2)$ there is an isomorphism $\tau : (\mathcal{M}, r_1) \to (\mathcal{M}, r_2)$ with $\tau(r_1) = r_2$. Let $\mathcal{M}_i$ denote the induced structure on $M \setminus G_i$ expanded by a unary predicate P interpreted as $L_i := \{x \in M : x < r_i\}$, so that $\tau|_{\mathcal{M}_1}$ is an isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$.

Now, notice that if $a \in G_i$ and $b \in M_i$ then a, b are $\leq$-comparable if and only if $b < a$ and $b \in L_i$, so clearly $\tau|_{\mathcal{M}_1} \cup \tilde{\sigma}$ preserves the order, which implies it must be a homomorphism since the join operator is definable from it. So $\tau|_{\mathcal{M}_1} \cup \tilde{\sigma} \in \mathrm{Aut}(M)$, as required. □

Thus, it will suffice to extend σ to G_i (as above). We will show that given $d \in G_1$ we can find $d' \in G_2$ such that $\langle N_1, d\rangle \cong \langle N_2, d'\rangle$ and both structures satisfy the assumptions satisfied by $\mathcal{N}_i$. This will suffice, as a standard back and forth will then complete the argument.

So let $d \in G_1$ be any element. We now distinguish some cases:

Case I: d is bounded above and below in N_1. Let $a, b \in N_1$ be such that $a < d < b$. Since $(a, b) \cap N_1$ is symmetric in (a, b) it follows, by Remark 3.7, that d realizes a non-cut over $(a, b) \cap N_1$. So there exists (a unique element) $c(d) \in (a, b) \cap N_1$ such that either $d \models c(d)^+$ or $d \models c(d)^-$; we will assume without loss of generality that $d \models c(d)^+$. By the assumption (1) on $\mathcal{N}_2$ we can find $d^+ \in N_2$ such that $d^+ \models \sigma(c(d))^+$ (recall the notation used in Remark 3.7). By definition, extending σ by sending d to d^+ is an order preserving map from $N_1 \cup \{d\}$ to $N_2 \cup \{d\}$.

Now, let $x \neq c(d)$ in N_1, and consider $x \wedge b$. If $x \wedge b \leq c(d) < d$ then $x \wedge d = x \wedge b$; otherwise $x > d$ and $x \wedge d = d$. In either case we have $\langle N_1, d\rangle = N \cup \{d\}$, so we are done.

Case II: d is bounded from below but not bounded from above in N_1. In this case $B_d := (r_1, d) \cap N_1$ is a branch in N_1 which implies that so is $\sigma(B_d)$ in N_2. Since no branch in N_2 is unbounded in $\mathcal{M}$ we can find d' such that $d' > \sigma(B_d)$.

Now, given any $x \in N_1$ either $x < d$ and $x \wedge d = x$ or (since d is unbounded from above in N_1) there is some $x' \in (r_1, d)$ incomparable with x. In that case $d \wedge x = x' \wedge x$. Thus, once again, $\langle N_1, d\rangle = N_1 \cup \{d\}$ and setting extending σ by sending d to d' is an extension of our isomorphism.

Case III: d is bounded from above but not bounded from below in N_1. We will prove that under our hypothesis this case is impossible. Let $b > d$. Since the elements under b form a linear order, it follows that either $d > r_1$ or $d \leq r_1$.

The latter case implies that $d \notin G_1$ contradicting our hypothesis. In the former case, because r_1 is an upper lower bound there is some $a \in N_1$ such that $a \not\leq d$. In this case the $a \wedge b$ is an element in N_1 comparable to d (both are smaller than b) which by transitivity cannot be greater than d so it must be smaller than d, contradicting the non lower boundedness of d in N_1.

Case IV: d is not bounded from below or above in N_1. Let $C \subseteq N_1$ be a maximal set of elements pairwise independent (over r_1). Since $\sigma(C)$ is such a set in N_2 and $\sigma(C)$ is not maximal in M, we can find $d' \in N_2$ such that $d' \notin \sigma(C)$ and $\sigma(C) \cup \{d'\}$ is pairwise independent in N_2. It is easy to check that in this case extending σ by $d \mapsto d'$ is an isomorphism.

In all of the above cases, it follows easily that $\langle N_1, d\rangle$ satisfies all assumptions that $\mathcal{N}_1$ satisfied. So the proof is complete. □

So we are now ready to prove:

PROPOSITION 4.11. *Let $\mathcal{M} \models T$ be countable. Then the symmetric submodels of $\mathcal{M}$ are dense.*

PROOF. It will suffice to show that if $\mathcal{N} \leq \mathcal{M}$ there exists $\mathcal{N}' \leq \mathcal{N}$ such that the pair $(\mathcal{N}', \mathcal{M})$ satisfies the assumptions of Lemma 4.9 (with $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}'$). Notice that except for (3), all the assumptions of Lemma 4.9 are hereditary under the passage to sub-trees and given any $\mathcal{N}' \leq \mathcal{N}$ satisfying (1), (2), (4), and (5), the set $\{x \in N' \mid x > a\}$ is an elementary sub-tree of $\mathcal{N}'$ satisfying all the conditions of Lemma 4.9 for any $a \in \mathcal{N}'$. Therefore, it will suffice to show that for any given $\mathcal{N} \leq \mathcal{M}$ and one of the assumptions (1), (2), (4), or (5), of Lemma 4.9, there exists $\mathcal{N}' \leq \mathcal{N}$ satisfying that specific assumption.

To make notation easier, we will refer to the assumptions of Lemma 4.9 by their number without making explicit the lemma every time.

By the remark above we will assume that (3) holds for every tree that we refer to.

Our first goal is to find, for any given tree, a tree satisfying condition (5). Let $\{a_i\}_{i\in\omega}$ be an enumeration of N. Let $C_0 = \{a_0\}$ and $b_0 > a_0$ be any element. Let $F_0 := \{x \in N : x \geq b_0\}$. Observe that $N \setminus F_0 \cong N$. Assume that we have constructed

- A set C_i isomorphic to an initial segment of $\mathcal{N}$ (with respect to the above enumeration).
- A set F_i of *forbidden* elements such that $C_i \cap F_i = \emptyset$ and $\mathcal{N} \setminus F_i \leq \mathcal{N}$.

Let $p_{i+1} := \operatorname{tp}(a_0, \ldots, a_{i+1})$. Since $C_i \subseteq N \setminus F_i \cong \mathcal{N}$ we can find $c_{n+1} \in N \setminus F_i$ such that $\operatorname{tp}(C_i, c_{i+1}) = p_{i+1}$ and set $C_{i+1} = \langle C_i \cup \{c_{i+1}\}\rangle$. Since $N \setminus F_i$ is a substructure, $C_{i+1} \cap F_i = \emptyset$. If $c_{i+1} = \min C_{i+1}$ we set $F_{i+1} = F_i$. Otherwise, let $c \in C_{i+1}$ be such that $c < c_{i+1}$ and $(c, c_{i+1}) \cap C_{i+1} = \emptyset$ (i.e., c is the immediate predecessor in C_i of c_{i+1}). Let $b_{i+1} \in N \setminus F_i$ be any element such that $b_{i+1} \perp_c c_{i+1}$ and $b_{i+1} \wedge d \leq c$ for every $d \in C_{i+1}$. Let $F_{i+1} := F_i \cup G_{c,b_{i+1}}$. In any case $F_{i+1} \cap C_{i+1} = \emptyset$ and $N \setminus F_{i+1} \leq \mathcal{N}$.

Let $\mathcal{N}' := \bigcup_{i\in\omega} C_i$. By construction $\mathcal{N}' \leq \mathcal{N}$. Let $c \in \mathcal{N}'$ be any element, and $C \subseteq N'$ a maximal set of elements pairwise orthogonal over c. Let $i \in \omega$ be such that $c \in C_i$ and such that $c < c_{i+1}$ and $(c, c_{i+1}) \cap C_i = \emptyset$. Then $b_{i+1} \perp_c c'$ for any $c' \in C$. For assume that $c < d = c' \wedge b_{i+1}$ for some $c' \in C$. Then $d \in G_{d,b_{i+1}}$, implying that $c' \in F_{i+1}$, a contradiction. Since for every $c \in \mathcal{N}'$ there are infinitely many i with the above property, it follows that $\mathcal{N}'$ satisfies (5).

To find a sub-tree which satisfies (4) we pursue a similar (though slightly simpler) construction. We only modify the construction of the sets F_i at stage i as follows: for every maximal $c \in C_i$, we choose some $b(c) \in N \setminus F_{i-1}$ with $b(c) > c$ and set

$$F_i := F_{i-1} \cup \{x : (\exists c \in \max\{C_i\})(x > b(c))\}$$

and the rest is precisely as above.

So it remains to take care of (1) and (2). We proceed as follows. Fix some $b_0 \in N$ and consider the interval $(r, b_0) \cap N$ (where $r < N$ is the upper lower bound of N). By Theorem 3.11 we can find an elementary substructure of $(r, b_0) \cap N$ symmetric in $\mathcal{M}$. Moreover, the construction in the proof of Theorem 3.11 assures that this can be done so that (1) is satisfied. So we may assume that $(r, b_0) \cap N$ is symmetrically embedded in $\mathcal{M}$. For every $b \in (r, b_0) \cap N$ let $\{c_b^i\}_{i \in \omega} \subseteq N$ an infinite set of elements pairwise orthogonal over b. Repeat the process for every interval of the form (b, c_b^i). Proceeding inductively in this way and using the "in particular" clause of Corollary 3.7, we get the desired conclusion. □

Summing up all of the above we get:

Theorem 4.12. *Let $\mathcal{M} \models T$ be a countable model. Then $\mathcal{M}$ is symmetrically indivisible.*

5. Symmetric indivisibility of the generic triangle-free graph

As we have seen, the symmetric indivisibility of $(\mathbb{Q}, \leq)$ can be viewed as the result of two separate phenomena — indivisibility and the density of symmetric substructures. This proof turned out to be more revealing than the one using composition construction. Moreover, the latter version is harder to generalize, and would not — for example — give any information on the symmetric indivisibility of the $\mathbb{Q}$-tree considered in the previous section.

In a similar way, the proof using the composition construction to show that Rado's random graph is symmetric does not give us any information, on, say, the situation with Henson graphs Γ_n, where the Henson graph for $n \geq 3$ is the generic countable K_n-free graphs (see **[5]** for the construction and basic properties of the Henson graphs).

In this section we give two alternative proofs of the symmetric indivisibility of the random graph. As before, the more revealing proof will be the one showing that symmetry and indivisibility seem not to be very closely related. It turns out that this strategy will allow us to prove, in addition, the symmetric indivisibility of the countable universal triangle free graph. In fact, the same methods give the analogous result for all Henson graphs, but there are more technicalities, and the details are deferred to a subsequent paper.

Before we embark on the proof, we set some standard terminology. A *neighborhood* in a graph G is a set of the form $R(a) := \{b : G \models R(b, a)\}$ and for a set $X \subseteq V$, the set of neighbors of a in X is $R(a, X) := \{b : b \in X \text{ and } G \models R(a, b)\}$. Throughout this section all graphs considered will have quantifier elimination in the pure language of graphs (i.e., the language containing a unique binary predicate).

In that situation, a *definable subset* of such a graph G is a finite boolean combination of neighborhoods and of singelton sets. We will introduce more notation and terminology as we proceed. We first return to Rado's random graph.

5.1. The strategy. Recall that Rado's random graph is the unique (up to isomorphism) countable graph Γ with the property that for any finite, disjoint $U, V \subseteq \Gamma$ there is a vertex v joined to every element in U and disjoint from every element in V. As in the case of $(\mathbb{Q}, \leq)$ it is well known (and very easy to show) that:

LEMMA 5.1. *The random graph,* Γ*, is indivisible.*

PROOF. It follows immediately from the axiomatisation of Γ that if $S \subseteq \Gamma$ is an infinite definable set then $S \cong \Gamma$. Fix some $c : \Gamma \to \{0, 1\}$. Then by the last remark we may assume that whenever $S \subseteq \Gamma$ is definable and infinite S is not monochromatic. It follows that for every such $S \subseteq \Gamma$ we can find a point $a_S \in S$ such that $c(a_S) = 1$. It is now obvious that the set

$$C := \{a_S : S \subseteq \Gamma, S \text{ infinite and definable}\} \cong \Gamma.$$

For assume that $a_1, \ldots, a_n, b_1, \ldots b_k \in C$ are distinct. Let

$$S := \{a \in \Gamma : \bigwedge_{i=1}^{n} R(a, a_i) \wedge \bigwedge_{i=1}^{k} \neg R(a, b_i)\}.$$

Then S is definable, and therefore $a_S \in C$ realizes S. Since $a_1, \ldots, a_m$ and $b_1, \ldots, b_k$ were arbitrary, this proves the claim. □

REMARK 5.2. An even simpler proof of a stronger version of the above lemma — namely that when partitioning Γ to two parts one of the parts is actually isomorphic to Γ — can be found in P. Cameron's blog, [**1**]: If $\mathcal{B} \cup \mathcal{R}$ is a partition of the random graph, Γ, and neither $\mathcal{B}$ nor $\mathcal{R}$ are isomorphic to Γ, there are sets $U_1, V_1 \subseteq \mathcal{B}$ and $U_2, V_2 \subseteq \mathcal{R}$ witnessing this. But in Γ there is some v connected to every vertex in $U_1 \cup U_2$ and not joined by an edge to any vertex in $V_1 \cup V_2$. However, our assumption implies that v cannot lie in neither $\mathcal{B}$ nor in $\mathcal{R}$ — a contradiction.

We chose to keep the proof of the weaker property, as the ideas appearing in it will serve us later on.

To prove that the random graph is symmetrically indivisible, one could now hope to show, as in the case of $(\mathbb{Q}, \leq)$, that the symmetrically embedded copies of Γ are dense. We were unable to prove this, apparently for a good reason... The following Theorem was proved and communicated to us by one of the two anonymous referees:

THEOREM 5.3 (Anonymous Referee). *Let* Γ *denote the countable random graph. Then the symmetrically embedded copies of* Γ *are not dense.*

PROOF. Let $\Gamma_0 \cong \Gamma$. Adjoin to Γ_0 two sets of vertices,

$$I := \{v_F : F \subseteq \Gamma_0 \text{ finite}\},\ J := \{w_F : F \subseteq \Gamma_0 \cup I \text{ finite}\}.$$

Extend the edge relation, R, from Γ_0 to $\Gamma := \Gamma_0 \cup I \cup J$, subject to the following conditions:

(1) I is an independent set of vertices.

(2) Edges connecting vertices in Γ_0 and I encode the membership relation, i.e., if $v \in \Gamma_0$ and $v_F \in I$ then $R(v, v_F)$ if and only if $v \in F$.
(3) Edges connecting vertices in $\Gamma_0 \cup I$ and J encode the non-membership relation, i.e., if $w \in \Gamma_0 \cup I$ and $w_F \in J$ then $R(w, w_F)$ if and only if $w \notin F$.
(4) Order $\Gamma_0 \cup I$ and let edges between vertices in J encode this ordering in the following way: for $a, b \in \Gamma_0 \cup I$ let $R(w_{\{a\}}, w_{\{a,b\}})$ if and only if $a < b$.
(5) Γ is the random graph, namely, whenever $F \subseteq \Gamma_0 \cup I \cup J$ is finite and p is a boolean combination of neighborhoods of elements in F, p is realized in Γ.

Conditions (1)–(4) are easily met. Starting with a minimal edge relation on Γ satisfying these conditions, we can satisfy condition (5) as follows: let $J_0 := \{w_F \in J : |F| < 3\}$ and $J_n := \{w_F \in J : |F| = n + 2\}$ for $n > 0$. For every finite $F \subseteq \Gamma_0 \cup I \cup \bigcup_{i=0}^{n} J_n$ and q as in condition (5) fix a realization of q in J_{n+1}.

We will show that no copy of Γ inside Γ_0 is symmetrically embedded in Γ. To this end define, for an arbitrary infinite set $X \subseteq \Gamma_0$,

$$S(X) := \{v \in \Gamma \setminus X : \forall F \subseteq X \text{ finite } \exists u(\neg R(u, v) \wedge F = R(u, X))\}.$$

We claim that $S(X) = (\Gamma_0 \setminus X) \cup I$. Given $w_F \in J$ let $\mathcal{F} := \{R(u, X) : u \in F\}$ and fix some finite $F' \subseteq X$ with $F' \notin \mathcal{F}$. Assume towards a contradiction that $w_F \in S(X)$. Then there would be some u such that $\neg R(u, w_F)$ and $F' = R(u, X)$. Since $F' \notin \mathcal{F}$ we know that $u \notin F$. By condition (3) above, as $\Gamma \models \neg R(u, w_F)$ it follows that $u \in J$. But $R(u, \Gamma_0)$ is co-finite, and therefore so is $R(u, X)$, a contradiction. On the other hand, if $v \in (\Gamma_0 \setminus X) \cup I$ and $F \subseteq X$ is any finite set then taking $u = v_F$ we know (conditions (1) and (2) above) that $\Gamma \models \neg R(u, v)$ and $\Gamma(u, X) = F$, implying that $v \in S(X)$. This shows that, indeed, $S(X) = (\Gamma_0 \setminus X) \cup I$, as claimed.

It follows that $S(X) \cup X = \Gamma_0 \cup I$, and thus if $\sigma \in \mathrm{Aut}(\Gamma)$ preserves X setwise it also preserves $\Gamma_0 \cup I$, and J. In particular, if $F \subseteq \Gamma_0 \cup I$ is finite, then $\sigma(w_F) = w_{\sigma(F)}$. Taking $a, b \in \Gamma_0 \cup I$ such that $a < b$ (with respect to the ordering fixed in condition (4) above) we see that

$$a < b \iff R(w_{\{a\}}, w_{\{a,b\}}) \iff R(\sigma(w_{\{a\}}), \sigma(w_{\{a,b\}})) \iff \sigma(a) < \sigma(b).$$

So any automorphism of Γ fixing an infinite subset of Γ_0 setwise preserves the order on $\Gamma_0 \cup I$. In particular, such automorphisms can only have trivial finite orbits. But the countable random graph does have automorphisms with non-trivial finite orbits. Thus, if $\Gamma_1 \subseteq \Gamma_0$ is isomorphic to Γ no automorphism of Γ_1 with non-trivial finite orbits can be extended to Γ. □

Thus, we have to modify the density argument in order for it to work in the present context. In $(\mathbb{Q}, \leq)$ the density of the symmetric copies of $\mathbb{Q}$ allowed us to give a proof where indivisibility and symmetry had nothing to do with each other. It turns out that allowing a little more interaction between these two requirements, essentially the same argument still works. Instead of proving the density of the symmetric copies of the random graph, Γ, among all sub-models we will restrict ourselves to a smaller class of sub-models, $\mathcal{C}$. If we can show a stronger version of indivisibility, namely one where the monochromatic sub-model can be found in $\mathcal{C}$, and then prove that the symmetric sub-models are dense in $\mathcal{C}$, our work will be done.

So it remains to choose the class $\mathcal{C}$. Looking under the lamppost is sometimes a good idea: Given any necessary condition, $\mathcal{P}$, for a sub-model to be symmetrically embedded we may as well consider the density of symmetric indivisibility in the class of models satisfying $\mathcal{P}$. It turns out that the model theoretic notion of *stable embeddedness* is a good such condition (though we do not know whether the symmetric sub-models of the random graph are, indeed, dense in the class of stably embedded sub-models, we do know it for the generic triangle free graph, Γ_Δ, and in fact for all Henson graphs Γ_n but this latter fact is not proved in the present paper).

5.2. Stable embeddedness. As we have seen, if $S \subseteq \Gamma$ is infinite and definable then $S \cong \Gamma$. However, such a set S cannot be symmetrically embedded in Γ. The main observation required in order to prove this is:

PROPOSITION 5.4. *Let $\mathcal{M}$ be a countable $\aleph_0$-categorical structure, $D \subseteq M^n$. Then the orbit of D under* $\mathrm{Aut}(\mathcal{M})$ *is countable if and only if D is definable.*

To see how this proposition is related to the symmetric embedding of S in Γ consider any point $a \in \Gamma \setminus S$. By the characterization of the random graph $R(a,S)$ is not an S-definable subset of S (i.e., $R(a,S)$ cannot be obtained as a finite boolean combination of neighborhoods of elements in S). By Proposition 5.4 $R(a,S)$ has an uncountable orbit in S under the action of $\mathrm{Aut}(S)$. But given $\sigma \in \mathrm{Aut}(S)$ such that $\sigma(R(a,S)) \neq R(a,S)$ and $\tilde{\sigma} \in \mathrm{Aut}(\Gamma)$ extending σ it must be that $\tilde{\sigma}(a) \neq a$. But a can only have a countable orbit in Γ (under the action of $\mathrm{Aut}(\Gamma)$), so it cannot be that every $\sigma \in \mathrm{Aut}(S)$ extends to an automorphism of Γ, i.e., S is not symmetrically embedded in Γ.

In model theoretic terminology, we would say that the set S of the previous paragraph is not *stably embedded* in Γ. Thus Proposition 5.4 suggests that, at least in the context of $\aleph_0$-categorical structures, stable embeddedness is a necessary condition for symmetric embeddedness (see Corollary 5.6). In this sub-section we will develop what little we need of stable embeddedness. Our treatment will be more general than necessary for the purposes of the present paper. Readers feeling uncomfortable with model theoretic terminology are referred to Proposition 5.8 for a proof of a special case of Proposition 5.4 sufficient for our purposes. A concrete treatment of stable embeddedness in the case of the random graph or the Henson graphs is given in the paragraphs concluding this sub-section, starting with Fact 5.12.

In the proof that an infinite co-infinite definable set S is not symmetrically embedded in Γ the only special role the random graph had to play was in showing that the neighborhood of some element in Γ, when intersected with S, was not definable in S. In order to generalize this argument we need the model theoretic notion of stable embeddedness:

DEFINITION 5.5. Let $\mathcal{M}, \mathcal{N}$ be structures (not necessarily in a common language) with $\mathcal{N} \subseteq \mathcal{M}^n$ for some $n \geq 1$. Then $\mathcal{N}$ is *stably embedded* in $\mathcal{M}$ if every $\mathcal{M}$-definable (with parameters from $\mathcal{M}$) relation on $\mathcal{N}$ is $\mathcal{N}$-definable (with parameters form $\mathcal{N}$).

In this terminology, the argument showing that infinite definable subsets of the random graph are not symmetrically embedded translates word for word to give:

Corollary 5.6. *Let $\mathcal{M}$ be an $\aleph_0$-categorical structure symmetrically embedded in a countable structure $\mathcal{N}$. Then $\mathcal{M}$ is stably embedded in $\mathcal{N}$.*

Before we proceed to discuss in more detail the notion of stable embeddedness, we return to the proof of Proposition 5.4. Since the proof is somewhat technical, we first give a proof in a special case that covers all cases of interest for us in this paper. Towards that end we recall the following:

Definition 5.7. A countable structure $\mathcal{M}$ has the *small index property* (SIP) if whenever a subgroup $G \leq \mathrm{Aut}(\mathcal{M})$ has a countable index in $\mathrm{Aut}(\mathcal{M})$ there exists a finite set A such that $\mathrm{Aut}(\mathcal{M}/A) \leq G$.

Thus we have:

Proposition 5.8. *Let $\mathcal{M}$ be a countable $\aleph_0$-categorical structure with SIP. Then $D \subseteq M^n$ has countable orbit under* $\mathrm{Aut}(\mathcal{M})$ *if and only if D is definable.*

Proof. Because $\mathcal{M}$ is $\aleph_0$-categorical, a set $D \subseteq M^n$ is definable if and only if there exists a finite set A such that $\sigma(D) = D$ for all $\sigma \in \mathrm{Aut}(\mathcal{M}/A)$. So assume that $G := \{\sigma : \sigma(D) = D\}$ has countable index in $\mathrm{Aut}(\mathcal{M})$. By SIP there is a finite set A such that $G \geq \mathrm{Aut}(\mathcal{M}/A)$, hence D is definable over A. □

By **[7]** the random graph and the Henson graphs (as well as many other $\aleph_0$-categorical structures) have the small index property.

We will now give an outline of the proof of Proposition 5.4:

Proof of Proposition 5.4. Of course, we only have to prove that if D is not definable its orbit is uncountable. The proof below, which we included for completeness of presentation's sake, is of a particular instance of Kueker-Reyes Theorem, which holds in larger generality (see Theorem 4.2.4 in **[6]**).

Given a non-definable subset D of $\mathcal{M}^n$, we construct a binary tree of elementary maps, whose branches correspond to automorphism of $\mathcal{M}$ with distinct images of D. For simplicity, we will assume that $D \subseteq \mathcal{M}$ (i.e., $n = 1$) — the general case being similar, with just a little extra care in the bookkeeping. We assume also that $\mathcal{M}$ is given in a language with quantifier elimination (i.e., that partial isomorphisms between finite substructures extend to automorphisms of $\mathcal{M}$).

Let σ be a finite partial isomorphism with domain A. We will show how to find σ_i for $i < 2$, such that:

1. $\sigma_i \supseteq \sigma$ is a finite partial isomorphisms.
2. For every choice of $\tilde{\sigma}_i$ such that $\sigma_i \subseteq \tilde{\sigma}_i \in \mathrm{Aut}(\mathcal{M})$ it holds that $\tilde{\sigma}_0(D) \neq \tilde{\sigma}_1(D)$.

Because D is not definable, there exists a type p over $A \cup \sigma(A)$ such that p has infinitely many realizations in D and and infinitely many realizations outside D.

Fix $a_0 \in D \setminus A$ and $a_1 \notin D \cup A$ both realizations of p and fix some $b \notin \sigma(A)$ such that $b \models \sigma(p)$. Let $\sigma_0 = \sigma \cup \{(a_0, b)\}$ and $\sigma_1 = \sigma \cup \{(a_1, b)\}$. Each σ_i, for $i < 2$, is a finite partial automorphisms which extends σ, and whenever $\sigma_i \subseteq \tilde{\sigma}_i \in \mathrm{Aut}(\mathcal{M})$ it holds that $b \in \tilde{\sigma}_0(D) \setminus \tilde{\sigma}_1(D)$.

The rest is standard, so we will be brief: construct inductively for $\eta \in \{0,1\}^n$ partial isomorphisms σ_η such that:

1. $\sigma_{\langle\rangle}$ is the empty isomorphism, and the domain of σ_η is finite for all η.
2. If $\eta \in \{0,1\}^n$ then, with respect to some fixed enumeration of M, both the domain and range of η contain the first n elements of M.

(3) $\sigma_{\eta^\frown i} \supseteq \sigma_\eta$.
(4) If $\tilde{\sigma}_{\eta^\frown 0}, \tilde{\sigma}_{\eta^\frown 1} \in \mathrm{Aut}(\mathcal{M})$ are such that $\sigma_\eta \subseteq \tilde{\sigma}_{\eta^\frown i}$ then $\tilde{\sigma}_{\eta^\frown 0}(D) \neq \tilde{\sigma}_{\eta^\frown 1}(D)$.

By the argument of the previous paragraph this induction can be achieved. By (2) and (3) of the construction for any $\eta \in \{0,1\}^\omega$ the function $\sigma_\eta := \bigcup_{n\in\omega} \sigma_{\eta|_{0,\dots,n}}$ is an automorphism of $\mathcal{M}$, and by (4) of the constructing if $\eta_1, \eta_2 \in \{0,1\}^\omega$ are distinct then $\sigma_{\eta_1}(D) \neq \sigma_{\eta_2}(D)$, as required. □

We can now go back to the discussion of stable embeddedness. We start with a few examples:

Example 5.9.
(1) Let $\mathbb{N}^*$ be an $\aleph_1$-saturated elementary extension of $(\mathbb{N}, +, \cdot, 0, 1)$. Then $\mathbb{N}$ is symmetrically embedded in $\mathbb{N}^*$ (because $\mathbb{N}$ is rigid), but $\mathbb{N}$ is not stably embedded because by compactness and saturation for any set of primes $S \subseteq \mathbb{N}$ there exists in $\mathbb{N}^*$ an element n_S divisible precisely by those (finitary) primes in S. Thus $S := \{n \in \mathbb{N} : n | n_S\}$. Cardinality considerations now prevent $\mathbb{N}$ from being stably embedded.
(2) On the other hand, by quantifier elimination, it is easy to check that $(\mathbb{N}, \leq)$ is stably embedded in any elementary extension.
(3) If $\mathcal{M}$ is saturated, $P \subseteq \mathcal{M}$ is definable, then it is not hard to verify that P (with all the $\mathcal{M}$-induced structure) is stably embedded if and only if it is symmetrically embedded (see the Appendix of [**3**] for the proof).
(4) Stable embeddedness with respect to subsets is weaker than stable embeddedness with respect to relations: consider the structure $\mathcal{M}$ with two infinite disjoint predicates A, B and a unique generic ternary relation $R(x_1, x_2, y)$ holding only if x_1, x_2 are in A and y is in B. Then A is vacuously stably embedded with respect to subsets, but not stably embedded with respect to binary relations.

In the present paper, however, stable embeddedness will be needed only in contexts where we have quantifier elimination in a language containing only binary relations. In such circumstances the distinction between stable embeddedness for subsets and stable embeddedness for relations does not exist. We will also need a more convenient test for stable embeddedness, one that is easier to keep track of in inductive constructions. For that purpose we recall:

Definition 5.10. Let $\mathcal{M}$ be a structure and $A \subseteq M$ any set. The *type over A of an element $b \in M$ is definable* (over $B \subseteq A$) if for every formula $\varphi(x, \bar{y})$ (with $|\bar{y}| = n$, some $n \in \mathbb{N}$) the set $\varphi(b, A^n) := \{\bar{a} \in A^n : \mathcal{M} \models \varphi(b, \bar{a})\}$ is definable (over B).

It is now merely a question of unravelling the definitions to verify:

Remark 5.11. Let $\mathcal{M}, \mathcal{N}$ be structures with $N \subseteq M$. Then $\mathcal{N}$ is stably embedded in $\mathcal{M}$ if and only if every type over N is definable.

In the context of the random graph or the Henson graphs, the discussion of the present sub-section can be given in more concrete terms for those not at ease with the model theoretic terminology. From this point to the end of this subsection Γ will denote either the countable random graph or a Henson graph.

REMARK 5.12. Let $b \in \Gamma$ and $A \subseteq \Gamma$. The *type of b over A* is the set of formulas

$$\{R(x,a) : \Gamma \models R(b,a), a \in A\} \cup \{\neg R(x,a) : \Gamma \models \neg R(b,a), a \in A\}$$

REMARK 5.13.

(1) If the set A in the above definition is finite, then by quantifier elimination (or $\aleph_0$-categoricity, if you prefer) the type of every element over A is merely a definable set with parameters in A, i.e., it is a finite boolean combination of neighborhoods of elements in A.
(2) If A is finite as above, it is convenient to identify the type of an element b over A with the orbit of b under the action of $\mathrm{Aut}(\mathcal{M}/A)$. Viewed from this angle, $\aleph_0$-categoricity implies that $A \subseteq \Gamma$ is definable if an only if it is a finite union of types.
(3) In the present paper all types will be types of elements in Γ (over various sets of parameters).

In this context we have

FACT 5.14. *A set $A \subseteq \Gamma$ is stably embedded if and only if for every $b \in \Gamma$ there is a finite set $A_0 \subseteq A$ and types $p_1, \dots, p_n$ over A_0, such that $\{a \in A : \Gamma \models R(a,b)\}$ is the union of all realizations of $p_1, \dots, p_n$ in A.*

We can now return to our main question, the symmetric indivisibility of the random graph.

5.3. Extending partial isomorphisms. In this subsection we carry out one part of the strategy outlined in Subsection 5.1. We introduce the machinery for constructing, given $S \leq \Gamma$ — satisfying appropriate assumptions — symmetric submodels of Γ inside S. As we will see (in the case of the generic triangle free graph) stable embeddedness is an appropriate assumption for this machinery to work. For the random graph we have a short cut. We need some technical definitions and a lemma:

DEFINITION 5.15. (1) Let $C_i, C'_i \subseteq \Gamma$ (for $i = 1, 2, \dots, k$). By $\langle C_i \rangle_{i=1}^k \equiv \langle C'_i \rangle_{i=1}^k$ we mean that there exists a partial isomorphism $\sigma : \bigcup_{i=1}^k C_i \to \bigcup_{i=1}^k C'_i$ such that $\sigma(C_i) = C'_i$ for all i.
(2) Let $\mathcal{A} := \{\langle A_i, B_i \rangle\}_{i=0}^k$ be a sequence (of ordered pairs) of subsets of Γ, and let $A := \bigcup_{i=1}^k A_i$. Say that $\mathcal{A}$ extends partial automorphisms if:
  - $A_i \cap B_i = \emptyset$ for all $i \leq k$ and $i < j$ implies $A_i \subseteq A_j, B_i \subseteq B_j$.
  - $R(b, A) \subseteq A_i$ if $b \in B_i$ and $0 < i$. For $b \in B_0$, either $R(b, A) \subseteq A_0$ or $R(b, A) = A$.
  - Let $i < k$, $C, C' \subseteq B_i \cup A_i$ be such that $\langle C \cap A_i, C \cap B_i \rangle \equiv \langle A_i \cap C', B_i \cap C' \rangle$, σ a partial isomorphism witnessing this, and $b \in B_i$ such that $R(b, A) \subseteq C$. Then there exists $b' \in B_{i+1}$ such that $\sigma \cup \langle b, b' \rangle$ is a partial isomorphism.

As we will see below, the idea underlying the above definition is to allow an inductive construction of an increasing sequence of finite graphs, culminating in a pair of countable structures with the smaller symmetrically embedded in the larger. The definition is tailor made to suite our requirements in dealing with the random graph and, later on, with the Henson graphs, Γ_n.

LEMMA 5.16. *Assume that* Γ *is a graph in the language* $\mathcal{L} = \{R\}$, *and* $\mathcal{A} = \{\langle A_i, B_i\rangle\}_{i\in\omega}$ *a sequence of pairs of subsets of* Γ. *If* $\{\langle A_i, B_i\rangle\}_{i<k}$ *extends partial isomorphisms for each* $k \in \omega$ *and* $B_0 = \emptyset$, *then* $A := \bigcup_{i\in\omega} A_i$ *is symmetrically embedded in* $\Gamma' := A \cup \bigcup_{i\in\omega} B_i$.

PROOF. By a standard back and forth, it will suffice to show that if $C_1, C_2 \supseteq A$, $|C_i \setminus A| < \aleph_0$ and $\sigma : C_1 \to C_2$ is a partial isomorphism extending an automorphism of A then σ can be extended to any $c \in \Gamma$. Let $n \in \omega$ be such that $c \in B_n$. Without loss of generality $(C_1 \cup C_2) \setminus A \subseteq B_n$. So $R(c, A) \subseteq A_n$ (because $B_0 = \emptyset$ and therefore $c \notin B_0$). We may also assume (by increasing n if necessary) that $\sigma(R(c, A)) \subseteq A_n$. Denote $C_A = R(c, A)$, $C'_A := \sigma(R(c, A))$, $C_B := C_1 \setminus A$ and $C'_B := C_2 \setminus A$. Then, $\langle C_A, C_B\rangle \equiv \langle C'_A, C'_B\rangle$, as witnessed by σ. But $\langle A_i, B_i\rangle_{i=1}^{n+1}$ extends partial isomorphisms and applying this property to $C := C_A \cup C_B$, $C' := C'_A \cup C'_B$ and c we can find $c' \in B_{n+1}$ such that $(\sigma \restriction C) \cup \langle c, c'\rangle$ is a partial isomorphism.

It remains to show that $\sigma \cup \{\langle c, c'\rangle\}$ is a partial isomorphism. Indeed, we only have to show that if $b \in C_1$ then $\models R(c, b)$ if and only if $\models R(\sigma(b), c')$. If $b \in C$ this is obvious by the choice of c'. But, by assumption, if $b \in \Gamma \setminus C$ then $\models \neg R(c, b)$. Since $R(c', A) \subseteq C'_A$ and $\sigma(C_A) = C'_A$ the conclusion follows. □

Observe that the above proof gives a little bit more: it shows that if $\mathcal{A}$ extends partial isomorphisms then for any finite set C, if $\sigma : A \cup C \to A \cup C'$ is a partial isomorphism extending an automorphism of A, then σ can be extended to an automorphism. This suffices to prove:

LEMMA 5.17. *Let* Γ *be the random graph or a Henson graph. Let* $D \subseteq \Gamma$ *be a finite set and* $S \subseteq \Gamma$ *be such that for all* $n \in \mathbb{N}$, $b_1, \dots, b_n \in \Gamma \setminus D$ *and any formula* $\varphi(x)$ *with parameters in* S, *the formula* $\varphi(x) \wedge \bigwedge_{i=1}^n \neg R(x, b_i)$ *has a solution in* S *if and only if it has a solution in* Γ. *Then there exists* $\Gamma_0 \subseteq S$ *such that* $\Gamma_0 \cong \Gamma$ *and* Γ_0 *is symmetric in* Γ.

PROOF. Recall that the theory of Γ is $\aleph_0$-categorical and has quantifier elimination.

Let $D := \{d_1, \dots, d_n\} \subseteq \Gamma$ be as in the statement of the lemma. Assume that D is minimal with this property. Then for all $d_i \in D$ there are a formula $\varphi_i(x)$ and some $b_1, \dots b_k \in \Gamma \setminus D$ such that

$$\varphi_i(x) \cap \neg R(d_i, S) \cap \bigcap_{j=1}^{k} \neg R(b_j, S) = \emptyset, \quad \varphi_i(x) \cap \bigcap_{j=1}^{k} \neg R(b_j, S) \neq \emptyset.$$

Let $S_1 = \varphi_1(S) \cap \bigcap_{i=1}^k \neg R(b_i, S)$. Observe, first, that $S_1 \subseteq R(\Gamma, d_1)$ and that, moreover, S_1 satisfies the assumptions of the lemma (with respect to the same finite set D). Hence, replacing S with S_1 we may assume that $S \subseteq R(\Gamma, d_1)$. By induction we may now assume that S is such that $S \subseteq R(\Gamma, d)$ for all $d \in D$. For simplicity we may also assume that $S \cap D = \emptyset$.

We can now proceed with the proof under the above assumptions. Let $\{c_i\}_{i\in\omega}$ be some fixed enumeration of Γ and denote $C_n := \{c_0, \dots, c_{n-1}\}$. Let $a_0 \in S$ be any element, and set $A_0 = \{a_0\}$, $B_0 = D$. Assume that we have constructed an increasing sequence of sets A_n, B_n such that:

(1) $A_n \subseteq S$.
(2) $\langle A_i, B_i\rangle_{i=0}^n$ extends partial isomorphisms.

(3) $A_n \cap B_n \subseteq B_0$.
(4) $c_n \in B_n \cup A_n$.
(5) $A_n \cong C_m$, where $m = |A_n|$, witnessed by some partial isomorphism σ_n such that $\sigma_{n'} = \sigma_n|A_{n'}$ for $n' < n$.

We will now construct A_{n+1}, B_{n+1} such that the above inductive assumptions remain valid. Let $p = \sigma_n^{-1}(\mathrm{tp}(c_n/C_n)) \cup \{\neg R(x,b) : b \in B_n \setminus B_0\}$. By $\aleph_0$-categoricity of Γ and the finiteness of C_n we know that $p(x)$ is isolated, say by a formula $\varphi_p(x)$. Notice that by universality of the graphs we are considering (and the fact that in any of these graphs, given any consistent formula $\varphi(x)$ with parameters in A the formula $\varphi(x) \wedge \neg R(x,b)$ is consistent for $b \notin A$) $\varphi_p(x)$ is realized in Γ. Hence, our assumptions on S assure that we can find some $a \in p(S) \setminus A_n$. We let $A_{n+1} = A_n \cup \{a\}$.

Now we construct B_{n+1}. If $c_n \notin A_n$ we let $B_{n+1,0} = B_n \cup \{c_n\}$, otherwise $B_{n+1,0} = B_n$. Now, for all $C, C' \subseteq B_n \cup A_n$, partial isomorphism $\sigma : C \to C'$ and $b \in B_n$ as in Definition 5.15, we let $t_{CC'b} \in \Gamma$ be such that $\sigma \cup \langle b, t_{CC'b}\rangle$ is a partial isomorphism. This is possible because, as B_n and A_n are finite so are C and C' therefore, the type of $t_{CC'b}$ over $CC'b$ is isolated, hence realized in Γ. Moreover, by $\aleph_0$-categoricity of Γ the number of types to realize in this way is finite. Therefore, setting $B_{n+1} = B_{n+1,0} \cup \{t_{CC'b} : C, C', b \text{ as above}\}$, the set B_{n+1} is finite and the inductive assumptions still hold of $\langle A_i, B_i\rangle_{i=0}^{n+1}$.

Now let $\Gamma_0 := \bigcup_{n\in\omega} A_n$. Then $\Gamma_0 \cong \Gamma$ by (5) of the inductive hypothesis. So we only have to check that Γ_0 is symmetrically embedded in Γ. If $D = \emptyset$ then by Lemma 5.16 and (4) of the inductive assumptions, we are done. Otherwise, by assumption $S \subseteq R(\Gamma, d)$ for all $d \in D$, so $\mathrm{tp}(D/\Gamma_0) = \mathrm{tp}(D/\sigma(\Gamma_0))$ for all $\sigma \in \mathrm{Aut}(\Gamma_0)$. So, given $\sigma \in \mathrm{Aut}(\Gamma_0)$ we know that $\sigma' := \sigma \cup \mathrm{id}(D)$ is a partial isomorphism. By the observation following Lemma 5.16 we get that σ' can be extended to an automorphism of $\mathcal{M}$.

□

Remark 5.18. The above proof would work, essentially as written, for the class of $\aleph_0$-categorical graphs with quantifier elimination in the pure language of graphs. In this setting, however, we have to assume that in the theory any consistent formula $\varphi(x)$ is consistent with $\varphi(x) \wedge \neg R(x,b)$ provided b is not a parameter in $\varphi(x)$.

Corollary 5.19. *The random graph is symmetrically indivisible.*

Proof. Let $c : \Gamma \to \{0,1\}$ and $S_1 = \{a \in \Gamma : c(a) = 1\}$. If S_1 meets every infinite definable set then set $S = S_1$. If $S_1 \cap \varphi(\Gamma) = \emptyset$ for some infinite definable set $\varphi(x, \bar{d})$ set $S = \{a \in \Gamma : c(a) = 0 \wedge \models \varphi(a, \bar{d})\}$. Either way S satisfies the assumptions of Lemma 5.17 (with $D = \emptyset$ in the former case and $D = \{\mathrm{dom}(\bar{d})\}$ in the latter). So the conclusion follows. □

Combined with Corollary 5.6 this gives:

Corollary 5.20. *The class of stably embedded submodels of the random graph is indestructible by finite partitions, namely under any coloring of the (vertices of the) random graph in two colors we can find a stably embedded monochromatic submodel.*

Remark 5.21. We do not know whether every stably embedded submodel of the random graph Γ contains a submodel symmetrically embedded in Γ. The proof of Corollary 5.19 can be viewed as a density result in a smaller class of stably embedded sub-models: those sub-models $\Gamma_0 \leq \Gamma$ such that for all but finitely many $b \in \Gamma \setminus \Gamma_0$ the set $R_b^{\Gamma_0}$ is finite in Γ_0.

5.4. The generic triangle free graph. The proof of Corollary 5.19 cannot be generalized to prove that the Henson graphs, Γ_n, are symmetrically indivisible, since it uses the fact that every infinite definable set contains an isomorphic copy of the whole universe. This is, of course, not true in Γ_n since for every $a \in \Gamma_n$ the set $R(a, \Gamma_n)$ does not admit K_{n-1}. Of course, $\Gamma_n[\Gamma_n]$ is not a K_n-free graph, so we cannot hope to use the composition construction either.

It turns out, however, that the ideas underlying our new proofs of the symmetric indivisibility of $(\mathbb{Q}, \leq)$ and the random graph can be adapted to prove the symmetric indivisibility of the generic triangle free graphs. Namely, once we show that Γ_n is indivisible in the stably embedded sub-models, we can prove that the symmetric sub-models are dense in the stably embedded ones, to get the desired conclusion.

The proof that Γ_n is indivisible is not as easy as in the previous cases. A simple generalization of the proof of Claim 5.1 gives only the following (see **[5]** for a proof along similar lines):

FACT 5.22. *Let $c : \Gamma_n \to \{0, 1\}$ be any coloring. Then there exists $i \in \{0, 1\}$ such that for any K_n-free graph G there exists $\sigma : G \hookrightarrow \Gamma_n$ such that $c(\sigma(G)) = \{i\}$.*

The indivisibility of $\Gamma_\triangle$, the generic triangle-free graph is proved by Komjáth and Rödl **[10]**. The proof that Γ_n is indivisible (for $n > 3$) is more technical, **[4]**. Inspecting the proof of Komjáth and Rödl we see that, in fact, we get a little more than indivisibility of $\Gamma_\triangle$. The proof assures that given $c : \Gamma_\triangle \to \{0, 1\}$ either there exists a monochromatic sub-model colored 0, or there is a stably embedded monochromatic sub-model colored 1. It turns out, however, that a natural modification of the proof shows the indivisibility of $\Gamma_\triangle$ in the stably embedded sub-models. The main part of the present subsection is dedicated to a self-contained proof of this fact. We then deduce the symmetric indivisibility of $\Gamma_\triangle$. Much of the work is done in the somewhat greater generality of arbitrary Henson graphs, Γ_n, the proof of whose symmetric indivisibility is postponed to a subsequent paper.

We start with some notation and conventions. We often enumerate the elements of a countable model Γ_n. In this case, we will think of the enumeration as an ordering of Γ_n and say that $a < b$ if a appears before b in this enumeration.

We also need a few technical remarks:

DEFINITION 5.23. A definable set $S \subseteq \Gamma_n$ is generic if it is of the form $\bigvee_{j=1}^{l} \psi(x, \bar{v}_j)$ and at least one of the formulas ψ_j is given by $\bigwedge_{i=1}^{k} \neg R(x, v_{j,i})$ for some $k \in \mathbb{N}$ and $v_{j,1}, \ldots, v_{j,k} \in \Gamma_n$.

Observe that a non-generic set in Γ_n may not admit K_{n-1}, so it is, in some sense "small". Generic sets, however, are nice:

REMARK 5.24. The intersection of finitely many generic sets in Γ_n is generic.

But more importantly:

REMARK 5.25. If $S \subseteq \Gamma_n$ is generic then S contains an isomorphic copy of Γ_n.

PROOF. It is enough to show that if S is of the form $\bigwedge_{i=1}^{k} \neg R(x, v_i)$ then, in fact $S \setminus \{v_1, \ldots, v_k\} \cong \Gamma_\triangle$, which is a triviality. □

But, in fact, we can show more:

LEMMA 5.26. *Assume that $S \subseteq \Gamma_n$ is generic. Then there exists $\Gamma_0 \subseteq S$ such that $\Gamma_0 \leq \Gamma_n$ and Γ_0 is stably embedded. Even more, we may choose Γ_0 such that $R(v, \Gamma_0)$ is finite for all $v \in \Gamma_n \setminus \Gamma_0$.*

PROOF. We may assume without loss of generality that S is of the form $\bigwedge_{i=1}^k \neg R(x, v_i)$. Let $\{v_i\}_{i\in\omega}$ be an enumeration of Γ_n. For simplicity of notation we will write $v < w$ if $v = v_j$, $w = v_i$ and $j < i$. We will construct Γ_0 by induction. Let $B_0 = \{v_1, \ldots, v_k\}$ and $a_0 \in S \setminus B_0$ the least such element (in our enumeration). Assume we have constructed $\{a_0, \ldots, a_n\}$ and sets $B_0, \ldots, B_n$ such that:

(1) $a_i \in S$ for all i and $(a_0, \ldots, a_n)$ is an increasing sequence.
(2) $(a_0, \ldots, a_k) \cong (v_0, \ldots, v_k)$.
(3) If $j < i$ and $v \in B_j$ then $\models \neg R(a_i, v)$.
(4) $B_{i+1} := \{v : a_i < v < a_{i+1}\}$.

We claim that if we can continue the construction (keeping the inductive assumptions) then our lemma will be proved. Indeed, taking $\Gamma_0 := \{a_i\}_{i\in\omega}$ we get (by condition (2)) that $\Gamma_0 \cong \Gamma_n$; by condition (4) $\Gamma_0 \cup \bigcup_{i\in\omega} B_i = \Gamma_n$ and by (3) for every $b \in \Gamma_n$, if $b \in B_i$ then $R(b, \Gamma_0) \subseteq \{a_0, \ldots, a_i\}$, so Γ_0 is stably embedded.

Thus, it remains to verify that the construction can be carried out, but this follows from the axiomatisation of Γ_n, as at stage n we only have finitely many conditions to realize, and these conditions can be expressed by a (single) formula which is merely a specialization of the universal quantifier in an explicit axiom of Γ_n. □

We will not use the above lemma explicitly, but the ideas appearing in the construction will play an important role in what follows. The following lemma is key in the proof:

LEMMA 5.27. *Assume that $S \leq \Gamma_\Delta$ and there exists a finite set D such that $R(v, S)$ is finite for all $v \in \Gamma_\Delta \setminus (S \cup D)$. Then there exists $\Gamma_0 \leq S$ stably embedded in Γ_Δ. Moreover, we can choose Γ_0 such that $R(v, \Gamma_0)$ is finite for all $v \notin \Gamma_0$.*

PROOF. By induction on $|D|$, it suffices to prove the claim for $D = \{v\}$. First, we show:

CLAIM 5.28. *Up to finitely many points, $S_0 := S \setminus R(v, S)$ is a model, i.e., there exists a finite set F such that $S_0 \setminus F \cong \Gamma_\Delta$.*

PROOF. If there exist $w_1, \ldots, w_k \in S$ such that $\left(\bigcup_{i=1}^k R(w_i, S)\right) \setminus R(v, S)$ is a finite set $\{s_1, \ldots s_l\}$, we are done: given an independent set $v_1, \ldots, v_n \in S_0$ and any $u_1, \ldots, u_m \in S_0$, all distinct from the w_i we can realize

$$\bigwedge_{i=1}^{l} x \neq s_i \bigwedge_{i=1}^{k} x \neq w_i \bigwedge_{i=1}^{k} \neg R(x, w_i) \wedge \bigwedge_{i=1}^{m} \neg R(x, u_i) \wedge \bigwedge_{i=1}^{n} R(x, v_i)$$

in Γ_Δ, and since all parameters in the above formula are in S, also in S. Thus, $S_0 \setminus \{w_1, \ldots, w_k, s_1, \ldots, s_l\}$ is a model. So we may assume that this is not the case. Let $w_1, \ldots, w_k$ be independent and $u_1, \ldots, u_r$ any elements, all distinct from the w_i. By our assumption we can find $u \in R(v, S)$ not connected to any of the w_i and different from all the u_i. In Γ_Δ there must be some u' such that u' is connected to u, to all the w_i and to none of the u_i. But $u' \notin R(v, \Gamma_0)$, since otherwise u, u' and v would form a triangle. Since $S \leq \Gamma_0$ such a u' can already be found in S. □

We let $B_0 = \{v\}$ if there are no $w_1, \dots, w_k, s_1, \dots, s_l \in S$ such that $R_v^S \subseteq \bigcup_{i=1}^k R(w_i, S) \cup \{s_1, \dots, s_l\}$ and $B_0 := \{w_1, \dots, w_k, s_1, \dots, s_l\}$ if $w_1, \dots, w_k, s_1, \dots, s_l$ witness that this is not the case. We let $S_0 = S \setminus (R(v, S) \cup B_0)$. From the proof of the last claim we get that $S_0 \leq \Gamma_\Delta$ (and not only up to finitely many points). We can now repeat the proof of Lemma 5.26 to obtain the desired conclusion. We give the details, as they are slightly more delicate than above. We let $a_0 \in S_0 \setminus B_0$ be the first element in some (fixed) enumeration of Γ_Δ, $\{v_i\}_{i \in \omega}$. Assume we have constructed $\{a_0, \dots, a_n\}$ and sets $B_0, \dots, B_n$ such that:

(1) $a_i \in S_0$ for all i and $(a_0, \dots, a_n)$ is an increasing sequence.
(2) $(a_0, \dots, a_n) \cong (v_0, \dots, v_n)$.
(3) If $j < i$ and $w \in B_j$ then $\models \neg R(a_i, w)$.
(4) $B_{i+1} := \{x : a_i < x < a_{i+1}\}$.

As before, it will suffice to show that the construction can be continued. Since the construction is given once we define a_{n+1}, at each stage we have to realize in S_0 a formula of the following form:

$$\varphi(x, \bar{a}) \wedge \bigwedge_{i=0}^{l} \neg R(x, b_i)$$

where $\bar{a} \subseteq S_0$, none of the b_i are in S_0, and $\varphi(x, \bar{a})$ has infinitely many solutions. By Claim 5.28, S_0 is a model so there is no problem realizing $\varphi(x, \bar{a})$ in S_0. We will need to work a bit harder to find a realization satisfying also the formulas $\neg R(x, b_i)$.

Notice that $\varphi(x, \bar{a}) \wedge \bigwedge_{b_i \in B_0} \neg R(x, b_i)$ has infinitely many solutions in S (if $v \in B_0$ use Claim 5.28, otherwise use the fact that S is a model), and obviously no such solution can be in $R(v, S)$. Setting

$$I = \{i : 1 \leq i \leq l \wedge b_i \in \Gamma_\Delta \setminus S\}$$

recall that, by assumption, $R(b_i, S_0)$ is finite for all $i \in I$. Therefore, it will suffice to check that

$$\varphi(x, \bar{a}) \wedge \bigwedge_{i \notin I} \neg R(x, b_i)$$

has infinitely many solutions in S_0. By definition $i \notin I$ implies that $b_i \in R_v^S$. Assume without loss of generality, that

$$\varphi(x, \bar{a}) = \bigwedge_{i=1}^{n} R(x, a_i) \wedge \bigwedge_{i=n+1}^{k} \neg R(x, a_i).$$

If there exists $u \in R_v^S$ such that $\bigwedge_{i=1}^n \neg R(u, a_i)$ then, as in the proof of Claim 5.28 any $u' \in S$ connected to u, to all the a_i with $1 \leq i \leq n$ and to none of the a_i with $n+1 \leq i \leq k$ is in S_0, and, as $S \leq \Gamma_\Delta$, we are done. So we may assume that there is no such u. But then

$$\bigwedge_{i=1}^{n} R(x, a_i) \wedge \bigwedge_{i=n+1}^{k} \neg R(x, a_i) \wedge \bigwedge_{b \in B_0} \neg R(x, b) \wedge \bigwedge_{i \notin I} \neg R(x, b_i)$$

is a formula in S, and it is clearly consistent. So it must have a solution in S, but such a solution must be in S_0. □

We can now show:

PROPOSITION 5.29. *Let $c : \Gamma_\Delta \to \{0,1\}$ be any coloring. Then there exists $\Gamma_0 \leq \Gamma_\Delta$ which is monochromatic and stably embedded in Γ_Δ.*

PROOF. This is a slight variation on the proof of **[10]** showing that Γ_Δ is indivisible. As already mentioned above, the proof in **[10]** assures that if there is no monochromatic sub-model colored 0 then there is a stably embedded monochromatic sub-model colored 1. The idea of the proof in **[10]** is to try and construct inductively a monochromatic sub-model colored 0. If the first attempt fails, this is because we ran into a formula $\varphi_0(x, \bar{c}_0)$ all of whose realizations are colored 1. We fix such a realization, r_0, and retry the construction of the 0-colored sub-model, starting above r_0 (with respect to some enumeration of Γ_Δ), collecting a new 1-colored vertex, r_1, if we fail again. We proceed with this construction until we either manage to construct a 0-colored sub-model or until we constructed a sequence $\{r_i\}_{i\in\omega}$ of 1-colored points. Collecting the vertices r_i with enough ingenuity we can assure that if all our attempts in constructing a 0-colored sub-model failed then $\{r_i\}_{i\in\omega}$ is a (stably embedded) 1-colored sub-model. Our main observation is that, essentially, the same proof works if instead of trying to construct any 0-colored sub-model we apply the construction of Lemma 5.27 to try and construct a stably embedded sub-model. Since the construction of the sequence $\{r_i\}_{i\in\omega}$ in case of failure is similar to that of **[10]**, this assures that indeed Γ_Δ is indivisible in the stably embedded sub-models. As the proof is not long we give the details.

Fix an enumeration $\{v_i\}_{i\in\omega}$ of Γ_Δ, and again we refer to this enumeration as inducing an ordering on the vertices of Γ_Δ. We try to construct a monochromatic copy of Γ_Δ colored 0, as provided by Lemma 5.26, the only difference with respect to the construction there, is that — in addition — we require that all the elements in the construction of Γ_0 are colored 0. If we succeed, we are done. So we may assume that we have failed, at some finite stage j_0. Thus, at stage j_0 we have constructed a set $Y_0 = \{y_{0,0}, \ldots, y_{0,j_0}\}$ isomorphic to an initial segment of Γ_Δ and a set $B_0 = \bigcup_{k=0}^{j_0} B_{0,k}$ such that

(1) $Y_0 \cup B_0$ is an initial segment of Γ_Δ (not only isomorphic to one).
(2) For every $b \in B_{0,k}$ if $j > k$ then $\models \neg R(b, y_{0,j})$.
(3) If z is such that $\operatorname{tp}(Y_0, z) = \operatorname{tp}(v_0, \ldots, v_{j_0+1})$ and $\models \bigwedge_{b\in B_0} \neg R(b, z)$ then $c(z) = 1$.

We fix some z_0 as in (3) above and such that $z_0 > B_0 \cup Y_0$ and restart the construction anew, above z_0. That is, we do not give up after one failure, or indeed after any finite number of failures. We keep trying over and over again, keeping track of the information we gathered: each failure gives us a formula all of whose realizations are colored 1. So each failure could, if we are careful enough, advance us one more step towards the construction of a monochromatic sub-model colored 1. We will now give the technical details explaining how this is done. Assume we have constructed for all $i < n$ sets Y_i isomorphic to an initial segment of Γ_Δ, $B_i = \bigcup_{k=0}^{j_i} B_{i,k}$ and elements z_i satisfying conditions (1)-(3) above (for the index i, rather than 0, of course), and such that, in addition:

(4) For all $j < i$, if $b \in B_j$ then $\models \neg R(b, y_{i,k})$ for all $k < j_i$.
(5) $Y_j > Y_i \cup B_i \cup \{z_i\}$ for all $i < j$.

We now fix $y_{n,0} > Y_{n-1} \cup B_{n-1} \cup \{z_{n-1}\}$, and start all over again: set $B_{n,0} := \{v : v < y_{n,0}\}$ and try to construct inductively a 0-colored sequence $y_{n,0} < y_{n,1} \ldots$ with

$(y_{n,0},\dots,y_{n,i}) \cong (v_0,\dots,v_i)$ and a collection of sets $B_{n,i} = \{v : y_{n,i-1} < v < y_{n,i}\}$ for $0 < i$ satisfying (2) above. If we do not get stuck, i.e., if we manage to construct $\{y_{n,i}\}_{i\in\omega}$ as above, then setting $\Gamma_0 := \{y_{n,i}\}_{i\in\omega}$ the construction gives $\Gamma_0 \le \Gamma_\Delta$ which is monochromatic 0, and such that for all $v \ge y_{n_0}$, if $v \notin \Gamma_0$ then $R(v,\Gamma_0)$ is finite. By the previous lemma we can find $\Gamma_1 \le \Gamma_0$ which is stably embedded in Γ_Δ, and we are done.

So we may assume that the construction fails at some stage j_n. In that case we choose z_n satisfying the following conditions:

(1) If $\mathrm{tp}(v_n/v_0,\dots,v_{n-1})$ is generic we choose z_n such that $\mathrm{tp}(Y_n,z_n) = \mathrm{tp}(v_0,\dots,v_{j_n})$ and $\models \bigwedge_{i=0}^{n-1} \neg R(z_i,z_n)$.

(2) If $\models R(v_n,v_i)$ for some $i<n$ let i_0+1 be minimal such that this is true. We choose z_n such that $\mathrm{tp}(Y_{i_0},z_n) = \mathrm{tp}(v_0,\dots,v_{j_{i_0}})$ and $\mathrm{tp}(z_0,\dots,z_n) = \mathrm{tp}(v_0,\dots,v_n)$.

By construction of the Y_i, if z_n exists then $c(z_n)=1$. Clearly, if case (1) holds, we have no problem finding z_n satisfying the requirements. So we check what happens in case (2). We will not be able to find such z_i only if we are required to connect z_n to two vertices, w_1,w_2 which are already connected. Obviously, w_1,w_2 cannot both be in Y_{i_0} or both in $\{z_0,\dots,z_{n-1}\}$, since z_n is required to satisfy a consistent type over both sets. So $w_1 \in Y_{i_0}$ and $w_2 = z_j$ for some j. By minimality of i_0+1, since $\models R(z_j,w_1)$, it follows that $\models R(z_{i_0+1},z_j)$. Surely, we can find z_n such that $\mathrm{tp}(z_0,\dots,z_n) = \mathrm{tp}(v_0,\dots,v_n)$. Then, $z_n \models R(x,z_{i_0+1}) \wedge R(x,z_j)$, creating a triangle in Γ_Δ, which is impossible.

Thus, either at some stage we manage to construct our 0-colored copy of Γ_Δ, and we can conclude using Lemma 5.27, or $\Gamma_0 := \{z_i\}_{i\in\omega} \cong \Gamma_\Delta$ is a 1-colored elementary substructure of Γ. So it remains only to check that in the latter case Γ_0 is stably embedded. So let $v \in \Gamma_\Delta \setminus \Gamma_0$. If $v \notin \bigcup Y_i$ then $R(v,\Gamma_0)$ is finite, because $v \in B_i$ for some i, and therefore $R(v,\Gamma_0) \subseteq \{z_0\dots,z_i\}$. So we may assume that $v \in Y_i$ for some i. In that case $z_j \in R_v^{\Gamma_0}$ if one of two cases happen:

(1) $j<i$ and $\Gamma_\Delta \models R(z_j,v)$ or
(2) $i<j$ and $\bigwedge_{k\le i} \neg R(z_j,z_k)$ and $\mathrm{tp}(z_j,v_0,\dots,v_{j_i}) = \mathrm{tp}(v_{j_i+1},v_0,\dots,v_{j_i})$

Since each of the two conditions is clearly definable (given v) with parameters in Γ_0, we are done. □

Combining this with the earlier results of this section we get:

Theorem 5.30. *The generic countable triangle free graph is symmetrically indivisible.*

Proof. Let $c : \Gamma_\Delta \to \{0,1\}$ be any coloring. Let $S \le \Gamma_\Delta$ be a monochromatic stably embedded sub-model, as provided by the last proposition. Then, if we can show that S satisfies the assumptions of Lemma 5.17 (with $D=\emptyset$), the conclusion will follow. Indeed, let $\varphi(x)$ be any formula over S and $b_1,\dots,b_n \in \Gamma_\Delta \setminus S$. Because S is stably embedded the set $B := \bigwedge \neg R(S,b_i)$ is a definable subset of S. Observe that the set $\bigvee R(\Gamma_\Delta,b_i)$ does not contain a copy of Γ_Δ, so it is non-generic. Therefore $S \setminus B = \bigvee R(S,b_i)$ cannot contain a model, and is therefore non-generic. Thus, since B is definable and its complement (in S) does not contain a model, B itself must be generic. But the intersection of two definable sets, one of which is generic is never empty, thus, $\varphi(B) \ne \emptyset$, as required. □

6. Elementary Symmetric indivisibility

In the examples studied in the previous sections the structure of interest had quantifier elimination in a natural language, suggesting a natural interpretation of the notion of indivisibility. But in general, indivisibility would depend in a significant way on our choice of language. For example, the structure $(\mathbb{N}, \leq)$ is trivially indivisible, but after adding a binary predicate $S(x, y)$ interpreted as "y is the immediate successor of x", it is no longer indivisible. From the model theoretic view point, in order to allow a more general discussion of indivisibility, one needs to strengthen the definition:

Definition 6.1. Let $\mathcal{M}$ be a (countable) structure. Say that $\mathcal{M}$ is *elementarily indivisible* if for any coloring of M in two colors there exists a monochromatic elementary substructure $\mathcal{N} \prec \mathcal{M}$ isomorphic to $\mathcal{M}$.

This definition releases the analysis of indivisibility from the dependence on the choice of language. As was pointed out to us by the referee, in most examples of interest indivisibility is studied in structures not only admitting quantifier elimination in a natural language, but which are also $\aleph_0$-categorical. In such cases the isomorphism requirement in the definition of indivisibility is automatic. In the abstract setting of elementary indivisibility, it may be of interest to study also *weak elementary indivisibility*, where the isomorphism condition is dropped from Definition 6.1.

In this section we study the basic properties of elementary indivisibility. Unfortunately, the results of this study seem mostly negative in the sense that most of our classification attempts ran into counter-examples constituting the main bulk of this section. These examples suggest that our understanding of elementary indivisibility is still far from being satisfactory. Most annoyingly, we were unable to settle the following:

Question 6.2. Is there an elementarily indivisible structure that is not symmetrically indivisible?

Since the notions of elementary divisibility and weak elementary divisibility coincide in the context of $\aleph_0$-categorical structures, it is natural to ask whether these notions coincide. To address this question we show, first, that $\aleph_0$-categoricity is not a necessary condition for elementary indivisibility:

Example 6.3. Let $\mathcal{C}$ be the class of all complete finite graphs in ω-many edge colors (so every $G \in \mathcal{C}$ is colored in finitely many colors, but there are infinitely many non-isomorphic structures on two elements). Clearly, $\mathcal{C}$ has the amalgamation property, but for any $G_0 \leq G_i$ $(i = 1, 2)$, colored graphs in $\mathcal{C}$ there are countably many possible amalgams of G_1 with G_2 over G_0 whose universe is $G_1 \amalg_{G_0} G_2$. Let $\mathcal{G}$ be the Fraïssé limit of $\mathcal{C}$ with respect to all such amalgams. More precisely, for every finite $G_1 \leq \mathcal{G}$ and $G_2 \in \mathcal{C}$, if $G_0 \hookrightarrow G_2$ then for an amalgam $G \in \mathcal{C}$ of G_1 with G_2 over G_0 there is an embedding $g : G_2 \hookrightarrow \mathcal{G}$, such that $g \restriction G_0 = \mathrm{id}$ and $G \cong G_1 \cup g(G_2)$ (with the induced structure).

Observe that this last property implies that if $G_1 \leq \mathcal{G}$ and $G_1 \leq G_2 \in \mathcal{C}$ then there is an embedding of G_2 into $\mathcal{G}$ over G_1. Thus, every complete countable graph on ω colors embeds into $\mathcal{G}$. In particular $\mathcal{G}[\mathcal{G}] \hookrightarrow \mathcal{G}$. So $\mathcal{G}$ is self similar and thus indivisible, but clearly not $\aleph_0$-categorical.

In fact, using the same construction as in the proof of Corollary 2.11 we can easily show that $\mathcal{G}$ is symmetrically self-similar and therefore symmetrically indivisible.

We get:

COROLLARY 6.4. *Elementary indivisibility is stronger than weak elementary indivisibility.*

PROOF. Let $\mathcal{G}_0$ be the graph obtained in the previous example with the color removed from one edge $e := (v_1, v_2)$. Then $\mathcal{G}_0$ is weakly elementarily indivisible, since $\mathcal{G}_0 \equiv \mathcal{G}$. It is not elementarily indivisible because we can separate v_1 from v_2 (and no other edges are colorless). $\square$

REMARK 6.5. The example in the above corollary also shows that a theory T may have two countable models, one elementarily indivisible and the other not.

Another corollary of the above example was supplied to us by the referee:

COROLLARY 6.6. *There exists an elementarily indivisible structure in a finite relational language which is not $\aleph_0$-categorical.*

PROOF. Let $\mathcal{G}$ be the colorful graph of Example 6.3. Let $\chi : [G]^2 \to \mathbb{N}$ be the function taking an edge e to its color $n \in \mathbb{N}$. For any binary relation S on $\mathbb{N}$ we define a 4-ary relation R^S on G by setting $R^s(x_1, y_1, x_2, y_2) \iff S(\chi(x_1, y_1), \chi(x_2, y_2))$. We let $\mathcal{G}^S$ be the structure with the same universe G, and whose unique relation is R^S.

Let S be the standard order on $\mathbb{N}$. We claim that $\mathrm{Aut}(\mathcal{G}^S) = \mathrm{Aut}(\mathcal{G})$. Obviously, $\mathrm{Aut}(\mathcal{G}) \subseteq \mathrm{Aut}(\mathcal{G}^S)$. For the other inclusion, we note that if $\alpha \in \mathrm{Aut}(\mathcal{G}^S)$ then α induces an automorphism of $(\mathbb{N}, S)$. Indeed, if e_1, e_2 are any two edges with $\chi(e_1) = \chi(e_2)$ then $\mathcal{G}^S \models \neg(S(e_1, e_2) \vee S(e_2, e_1))$, implying that $\chi(\alpha(e_1)) = \chi(\alpha(e_2))$ so α induces a bijection $\tilde{\alpha} : \mathbb{N} \to \mathbb{N}$, which is, by definition, an automorphism. Because $(\mathbb{N}, S)$ is rigid, $\tilde{\alpha} = \mathrm{id}$ for all α, thus, $\alpha \in \mathrm{Aut}(\mathcal{G})$.

Since $\mathcal{G}$ is indivisible so is $\mathcal{G}^S$. Since they have the same automorphism groups and $\mathcal{G}$ is not $\aleph_0$-categorical neither is $\mathcal{G}^s$. $\square$

We were also unable to find answers to either of:

QUESTION 6.7. Is there a rigid elementarily indivisible structure? Is an elementarily indivisible structure homogeneous?

The only result we have in this direction is the observation:

REMARK 6.8. If $\mathcal{M}$ is weakly elementarily indivisible then $|S_1(\emptyset)| = 1$, i.e., there are no formulas in one variable (with no parameters), except the formula $x = x$.

PROOF. If $p_1, p_2 \in S_1(\emptyset)$ choose $\varphi(x) \in p_1$ such that $\neg\varphi(x) \in p_2$ and color $\varphi(M)$ red and $\neg\varphi(M)$ blue. Since $\mathcal{M} \models (\exists x, y)(\varphi(x) \wedge \neg\varphi(y))$, neither $\varphi(M)$ nor $\neg\varphi(M)$ can contain any elementary substructure of $\mathcal{M}$ (let alone one which is isomorphic to $\mathcal{M}$). $\square$

Another restriction on elementarily indivisible structures is:

PROPOSITION 6.9. *If $\mathcal{M}$ is elementarily indivisible, then for every $A \subseteq M$, if $|A| < \aleph_0$ then $|\operatorname{acl}(A)| < \aleph_0$.*

Proof. We will show that if $|\operatorname{acl}(A)| = \aleph_0$ for some finite $A \subseteq M$ then M is not elementarily indivisible. So let $A \subseteq M$ be as in the assumption. Let $\{A_i\}_{i\in\omega}$ enumerate realizations of $\operatorname{tp}(A/\emptyset)$ (with respect to some enumeration of the elements of A). To show that $\mathcal{M}$ is divisible it will suffice to show that there is a coloring of M with the property that $\operatorname{acl}(A_i)$ is not monochromatic for all i. This can be done using a simple diagonalisation: Color A_0 blue and choose $a_0 \in \operatorname{acl}(A_0)$. Color a_0 red. Assume that for all $i < n$ we colored A_i and a point $a_i \in \operatorname{acl}(A_i)$ in such a way that $A_i \cup a_i$ is not monochromatic, we will now color A_n. Let $A = \bigcup_{i<n}(A_i \cup a_i)$. Then A is finite. Choose $a_n \in \operatorname{acl}(A_n)\setminus A$ (this is possible because $\operatorname{acl}(A_n)$ is infinite and A is finite). Color $A_n \cup a_n$ so that it is not monochromatic. Since $\bigcup_{i<\omega} A_i = m$ this defines a coloring of M, and we are done. □

We do not know whether Proposition 6.9 remains true for weakly elementarily indivisible structures. Observe, however, that if A is a singleton, then by Proposition 6.8 the same proof can be carried out for weakly elementarily indivisible structures. The following is an immediate corollary, which remains true for weakly indivisible structures:

Corollary 6.10. *If $\mathcal{M}$ is weakly elementarily indivisible there are no non-trivial $\emptyset$-definable equivalence relations (on M) with finite classes.*

We get

Corollary 6.11. *If $\mathcal{M}$ is weakly elementarily indivisible then* $\operatorname{acl}(a) = a$ *for every $a \in M$. In particular, there are no $\emptyset$-definable non-trivial definable unary functions.*

Proof. Note that if $x \in \operatorname{acl}(y)$ then $y \in \operatorname{acl}(x)$, for $\operatorname{acl}(x) \subseteq \operatorname{acl}(y)$. But $\operatorname{tp}(x) = \operatorname{tp}(y)$ (by Proposition 6.8), so equality must hold. By Claim 6.9 (see the discussion preceding Corollary 6.10 for the weakly elementarily indivisible case), if for some (equivalently, all) $a \in M$ we would have $\operatorname{acl}(a) \supsetneq a$ then $x \sim y$ if and only if $x \in \operatorname{acl}(y)$ would be a non-trivial equivalence relation with finite classes, contradicting 6.10. □

As we have seen in Section 4 it is not true that in an elementarily (symmetrically) indivisible structure the algebraic closure operator is totally trivial. We give some more examples:

Example 6.12.

(1) Let A be an infinite affine space over $\mathbb{F}_2$. Then A is symmetrically indivisible. This is an immediate corollary of Hindman's theorem on finite sums, [**8**]: Fix a coloring c of A in two colors $\{0,1\}$ and a point $a \in A$. Let V_a be the vector space obtained by localizing A in a. By Hindman's theorem there exists an infinite set $X \subseteq V_a$ and $i \in \{0,1\}$ such that for all finite $F \subseteq X$ we have $c(\sum_{x\in F} x) = i$. Since V_a is a vector space over $\mathbb{F}_2$, this implies that X, a generate a monochromatic (except, possibly at a), infinite dimensional sub-vector space $U_a \subseteq V_a$. Let $B \subseteq A$ be any infinite dimensional affine subspace of U_a not containing a. Then B is monochromatic and symmetric in A.
(2) It is apparently well known that the above is not true (in a strong sense) for affine spaces over any other (finite) field, see Theorem 7 of [**11**].

(3) The following was suggested to us by H. D. Macpherson: Let $G = [\mathbb{N}]^2$. Define a graph on G by setting $E := \{\{a,b\},\{a,c\} : c \neq b\}$. Let c be any coloring of G in two colors. By Ramsey's theorem there is an infinite set $I \subseteq \mathbb{N}$ such that $c|[I]^2$ is constant. Thus, the induced graph on $[I]^2$ is monochromatic. Obviously, $[I]^2$ is isomorphic to G (with any bijection $f : \mathbb{N} \to I$ inducing an isomorphism). Note that every permutation of $\mathbb{N}$ induces an automorphism of G. But the other direction is also true. Define, an equivalence relation $\sim$ on E by $(x,y) \sim (w,z)$ if the graph induced on $\{x,y,w,z\}$ is complete. Observe that if $x = \{n,m\}$ and $y = \{n,k\}$ then $(\{n',m'\},\{n',k'\})$ is equivalent to (x,y) if and only if $n = n'$. This gives a bijection $f : E/\sim \to \mathbb{N}$, and since E is a definable equivalence relation, it follows that any automorphism σ of G induces, through f, a permutation f_σ on $\mathbb{N}$, which - in turn - induces σ. So $[I]^2$ is symmetric in G. It is not hard to check that $\mathrm{Th}(G)$ has quantifier elimination, and thus $[I]^2 \leq G$.

The following question remains open:

Question 6.13. Let $\mathcal{M}$ be a symmetrically indivisible structure in a language $\mathcal{L}$. Let $\mathcal{L}_0 \subseteq \mathcal{L}$. Is $\mathcal{M} \restriction \mathcal{L}_0$ symmetrically indivisible?

It is obvious, of course, that a reduct of an indivisible structure is indivisible, but since a reduct of a symmetric sub-structure need not be symmetric (see the example below), it is not clear what one should expect as an answer to this question.

Example 6.14. Let Γ_0 be a countable random graph in the language $\mathcal{L} = \{R\}$. Expand the language by constants $\{c_i\}_{i\in\omega}$ enumerating Γ_0. Let $\Gamma_0 \leq \Gamma$ be a countable elementary extension. Assume that in Γ there is some infinite co-infinite $S \subseteq \omega$ and an element $b \in \Gamma$ such that $\Gamma \models R(b, c_i)$ if and only if $i \in S$.

Because Γ_0 is rigid, it is symmetric in Γ. If the set $\{c_i : i \in S\}$ is not Γ_0-definable, then its orbit under $\mathrm{Aut}(\Gamma_0)|\mathcal{L}$ is uncountable (Proposition 5.4). So as pure random graphs, Γ_0 is not symmetric in Γ.

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