On Counting Generalized Colorings

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Abstract. The notion of graph polynomials definable in Monadic Second Order Logic, **MSOL**, was introduced by B. Courcelle, J.A. Makowsky and U. Rotics in 2001. It was shown later that the Tutte polynomial and generalizations of it, as well as the matching polynomial, the cover polynomial and the various interlace polynomials fall into this category.

In this article we present a uniform model theoretic framework for studying graph polynomials. In particular we study an infinite class of graph polynomials based on counting functions of generalized colorings definable in full second order logic **SOL**.

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1. Introduction

1.1. Graph invariants and graph polynomials. A *graph invariant* is a function from the class of (finite) graphs $\mathcal{G}$ into some domain $\mathcal{D}$ such that isomorphic graphs have the same image. Usually such invariants are uniformly defined in some formalism. If $\mathcal{D}$ is the two-element Boolean algebra we speak of *graph properties*. Examples are the properties of being connected, planar, Eulerian, Hamiltonian, etc. If $\mathcal{D}$ consists of the natural numbers, we speak of *numeric graph invariants*. Examples are the number of connected components, the size of the largest clique or independent set, the diameter, the chromatic number, etc. But $\mathcal{D}$ could also be a polynomial ring $\mathbb{Z}[\overline{X}]$ over $\mathbb{Z}$ with a set of indeterminates $\overline{X}$. Here examples are the characteristic polynomial, the chromatic polynomial, the Tutte polynomial, etc.

There are many graph invariants discussed in the literature, which are polynomials in $\mathbb{Z}[\overline{X}]$, but there are hardly any papers discussing classes of graph polynomials as an object of study in its generality. An outline of such a study was presented in [**49, 50**].

The results reported in this article are part of an ongoing research project which aims to develop a general theory of graph polynomials[1].

1.2. Graph polynomials as generating functions. We denote by **SOL** Second Order Logic and by **MSOL** Monadic Second Order Logic, where quantification of relations is restricted to unary relations. In [**46, 47, 48**] J.A. Makowsky introduced the **MSOL**-definable and the **SOL**-definable graph polynomials, the class of graph polynomials where the range of summation is definable in (monadic) second order logic. He has verified that all the examples of graph polynomials discussed in the literature, with the exception of the weighted graph polynomial of [**53**], are **SOL**-polynomials over some expansions (by adding order relations) of the graph, cf. also [**49**]. Actually the most prominent graph polynomials in the literature, such as the characteristic polynomial, the chromatic polynomial, the matching polynomial, the Tutte polynomial and the interlace polynomial, are **MSOL**-definable. In some cases this is straightforward, but in some other cases it follows from intricate theorems. However, since the publication of our conference version of this paper, [**42**], we have found new **SOL**-graph polynomials, which are provably not **MSOL**-definable, [**31**]. **SOL**-definable graph polynomials are polynomials by definition. They are generalizations of generating functions counting subgraphs with prescribed properties definable in **SOL**. Without a definability condition imposed, these polynomials will be called *subset expansion polynomials*. The first graph polynomials of this type which appeared in the literature were the generating matching polynomial and the independence polynomial, [**34**]. They are indeed **MSOL**-definable.

Traditionally, generating functions use standard monomials, i.e., products of powers of the indeterminates, and their coefficients have some specific combinatorial interpretations. The **SOL**-definable polynomials in [**47**] were defined like this. However, some graph polynomials use different representations, where the powers of the occurences of indeterminates raised to the power X^i are replaced by falling factorial $X_{(i)} = X \cdot (X-1) \cdot \ldots \cdot (X-i+1)$, the function $\binom{X}{i}$, or other combinatorially motivated polynomials. To accommodate these cases we use here, as in [**42**], wider

[1] See `http://www.cs.technion.ac.il/ janos/RESEARCH/gp-homepage.html`

classes of **SOL**-definable polynomials, the *standard (falling factorial, and Newton)* **SOL**-*polynomials*.

1.3. Graph polynomials arising from generalized colorings. There are other ways graph polynomials occur naturally, namely as parametrized graph invariants $P(G, k_1, \dots k_\alpha)$ with one or several parameters $k_1, \dots, k_\alpha$ ranging over non-negative integers. The best known examples are the chromatic polynomial and the Tutte polynomial. As a matter of fact, the chromatic polynomial was the first graph polynomial to appear in the literature, [**11**]. Our main focus in this paper, expanding on [**51, 42**], is the study of generalized colorings, where we have several simultaneous colorings using several color sets, subject to conditions formulated by a formula θ in **SOL**. It will become clear in Section 2 that θ has to be subject to certain semantic restrictions such as: invariance under permutation of the colors, the existence of a bound on the colors used, and independence from the colors not used.

The associated counting function $\chi_\theta(G, k_1, \dots, k_\alpha)$ counts the number of generalized colorings satisfying θ as a function of $(k_1, \dots, k_\alpha)$.

The starting point of our investigation is the following fact:

Proposition A. *Let $\overline{k} = (k_1, \dots, k_\alpha)$ be the cardinalities of the various color sets. For ϕ subject to the conditions above, the counting function $\chi_\phi(G, \overline{k})$ is a polynomial in $\overline{k}$.*

This will lead us to observe that many known graph theoretic counting functions are in fact graph polynomials, which previously were not recognized as such.

We shall compare the counting functions $\chi_\theta(G, \overline{k})$ of generalized colorings with **SOL**-polynomials. A natural question which now arises is under what conditions an **SOL**-polynomial can be viewed as counting generalized colorings, and vice versa. The purpose of this paper is to answer this question. We shall present a framework in which we can prove the following:

Theorem B. *Every* **SOL**-*polynomial (standard or* FF*) is a counting function of a generalized coloring of ordered graphs definable in* **SOL**.

Theorem C. *Every counting function of a generalized coloring of ordered graphs definable in* **SOL** *is an* **SOL**-*polynomial, where the choice of standard,* FF *or Newton* **SOL**-*polynomial depends on the exact definition of generalized coloring.*

1.4. A model theoretic view. In B. Zilber's study of the structure of models of totally categorical first order theories, [**59, 60, 61**], the growth function of the size of finite approximations of models generated by finite sets of indiscernibles plays an important rôle. It is proved there, for theories which are $\aleph_0$-categorical and ω-stable, that this growth function is a polynomial. The totally categorical theories are a special case of this. It turns out that our counting functions of generalized colorings can be seen in this framework. Although this connection between graph polynomials and model theory may be of little interest to the finite combinatorists, it will allow us to associate with a graph G an infinite structure $\mathfrak{M}(G)$, which can be viewed as the most general graph invariant.

1.5. Outline of the paper. We assume the reader is familiar with the basics of graph theory as, say, presented in [**23, 12**]. We also assume the reader is familiar the basics of model theory and finite model theory as, say, presented in [**26, 25, 43, 39, 54**].

Section 2 is a prelude to our general discussion. In it we discuss the chromatic polynomial and the bivariate matching polynomial and motivate our general approach. In Section 3 we introduce our notion of counting functions of generalized colorings definable in **SOL**. We prove they are polynomials in the number of colors and show examples of graph polynomials from the literature which fall under this class of graph polynomials. In Section 4 we give precise definition of the various versions of **SOL**-definable polynomials and prove Proposition A. In Section 5 we discuss the various choices of monomials as the basis of the **SOL**-definable polynomials. In Section 6 we state and prove Theorems B and C precisely. In Section 7 we discuss under which conditions graph polynomials are or are not **MSOL**-definable. In Section 8 we discuss how counting functions of generalized colorings fit into the framework of the model theory of totally categorical theories, cf. [**61**]. Section 9 presents conclusions and open problems.

An earlier version of this article was posted as [**51**] and a short version was published as [**42**].

2. Prelude: two typical graph polynomials

Before we introduce our general definitions, we discuss two typical graph polynomials studied in the literature, the classical *chromatic polynomial* $\chi(G;X)$ and the *bivariate matching polynomial* $M(G;X,Y)$. Both have a very rich literature. For an exhaustive monograph on the chromatic polynomial, cf. [**24**]. For the matching polynomial the reader may consult [**45, 32**].

We denote by $[n]$ the set $\{1,\ldots,n\}$ and by $\mathcal{G}$ the set of undirected (labeled) graphs of the form $G=(V,E)$ where $V=[n]$ for some $n\in\mathbb{N}$. Let $G\in\mathcal{G}$. A *vertex-k-coloring of* G is a function $f:[n]\to[k]$. The coloring f is a *proper vertex-k-coloring of* G if additionally it satisfies that, whenever $(u,v)\in E$, then $f(u)\neq f(v)$. $\chi(G;k)$ denotes the number of proper vertex-k-colorings of G. For a fixed graph G this defines a function $\chi_G:\mathbb{N}\to\mathbb{N}$ which can be proved to be a polynomial in k with integer coefficients. Therefore it can be interpreted as a polynomial in $\mathbb{Z}[X]$ or even $\mathbb{R}[X]$. We denote this polynomial by $\chi_G(X)$. We shall discuss several proofs of this below. We define its coefficients by

$$\chi(G,X)=\sum_i c_i X^i=\sum_i b_i X_{(i)} \tag{2.1}$$

where $X_{(i)}=X\cdot(X-1)\cdot\ldots\cdot(X-i+1)$, the falling factorial.

Let $M\subseteq E$ be a set of edges. M is a *matching* if it consists of isolated edges. We denote by $cov(M)$ the set of vertices $v\in V$ such that there is an $e=(u,v)\in M$. We note that $|cov(M)|=2|M|$. Consider the graph parameter $m_i(G)$ which counts the number of matchings of G which consist of i (isolated) edges. The *bivariate matching polynomial* is defined as

$$M(G;X,Y)=\sum_i m_i(G)X^iY^{|V|-2i} \tag{2.2}$$

For a fixed graph G this defines a function $M_G:\mathbb{N}^2\to\mathbb{N}$. Here M_G is a polynomial by definition. Two matching polynomials are obtained from $M(G;X,Y)$ as substitution instances: the *generating matching polynomial*

$$g(G,X) = M(G;X,1) \tag{2.3}$$

and the *defect matching polynomial* or *acyclic polynomial*

$$m(G,X) = M(G;-1,X). \tag{2.4}$$

We note that $\chi_G(X)$ and $M(G;X,Y)$ both really denote a family of polynomials indexed by graphs from $\mathcal{G}$. These families are, furthermore, uniformly defined based on some of the properties of the graph G. We are interested in various formalisms in which such definitions can be given.

2.1. Recursive definitions of $\chi(G;X)$ and $M(G;X,Y)$. The first proof that $\chi_G(X)$ is a polynomial used the observation that $\chi_G(X)$ has a recursive definition using the order of the edges, which can be taken as the order induced by the lexical ordering on $[n]^2$. However, the object defined does not depend on the particular order of the edges. For details, cf. [**10, 12**]. We shall also give a similar definition of $M(G;X,Y)$. The essence of the proof is as follows:

For $e = (v_1, v_2)$, we put

(i) $G - e = (V, E')$ with $E' = E - \{e\}$.
The operation of passing from G to $G - e$ is called *edge deletion.*

(ii) $G/e = (V', E')$ with $V' = V - \{v_2\}$
and $E' = (E \cap (V')^2) \cup \{(v_1, v) : (v_2, v) \in E\}$.
The operation of passing from G to G/e is called *edge contraction.*

(iii) $G \dagger e = (V', E')$ with $V' = V - \{v_1, v_2\}$ and $E' = E \cap (V')^2$.
The operation of passing from G to $G \dagger e$ is called *edge extraction.*

Remark 2.1. *If we want to ensure that in the resulting graph the universe $V' = [n-1]$ or $V' = [n-2]$ then a suitable relabeling is needed. We omitted this from the definitions to keep the notation simple.*

It is easy to verify that these operations commute.

Lemma 2.2. *Let e, f be two edges of G. Then we have*

(i) $(G - e) - f = (G - f) - e$,
$(G/e) - f = (G - f)/e$,
$(G - e)/f = (G/f) - e$ *and*
$(G/e)/f = (G/f)/e$.

(ii) *If e, f are disjoint edges, then we also have*
$(G \dagger e) \dagger f = (G \dagger f) \dagger e$,
$(G - e) \dagger f = (G \dagger f) - e$, *and*
$(G \dagger e)/f = (G/f) \dagger e$.

Next one notes the following recurrence relations, cf. [**24, 32**]:

Lemma 2.3. *Let $G = (V, E)$ be a graph and $e \in E$ an edge.*

$$\chi(G;X) = \chi(G - e;X) - \chi(G/e;X) \tag{2.5}$$

$$M(G;X,Y) = M(G - e;X,Y) + X \cdot M(G \dagger e;X,Y) \tag{2.6}$$

Furthermore, if $G = G_1 \sqcup G_2$ is the disjoint union of G_1 and G_2, then we have multiplicativity, i.e.

$$\chi(G_1 \sqcup G_2; X) = \chi(G_1; X) \cdot \chi(G_2; X) \tag{2.7}$$

$$M(G_1 \sqcup G_2; X, Y) = M(G_1; X, Y) \cdot M(G_2; X, Y) \tag{2.8}$$

Let $E_n = ([n], \emptyset)$. To compute the polynomials recursively we note that

$$\chi(E_n; X) = X^n \tag{2.9}$$

$$M(E_n, X, Y) = Y^n \tag{2.10}$$

Let $E = (e_0, e_1, \ldots, e_m)$ be the enumeration of the edges in lexicographic order. One can compute $\chi_G(X)$ and $M(G; X, Y)$ by eliminating edges in this order. It also turns out, using Lemma 2.2, that the result is *independent* of the ordering of the edges.

Other graph polynomials from the literature which satisfy similar recursive definitions are the Tutte polynomial and its many variations and substitution instances, [**12, 13**, Chapter X], the Cover polynomial for directed graphs, [**20**], and the various Interlace polynomials, [**2, 5, 6**]. A systematic study of polynomials which are defined recursively using edge and vertex eliminations may be found in I. Averbouch's thesis [**8**].

2.2. Generating functions and explicit descriptions. The bivariate matching polynomial was defined by 2.2 as a *generating function*. This can be rewritten as

$$\begin{aligned} M(G; X, Y) &= \sum_i m_i(G) X^i Y^{|V|-2i} = \sum_{M \subseteq E} X^{|M|} Y^{|V|-|cov(M)|} \\ &= \sum_{M \subseteq E} \left(\sum_{C = V - cov(M)} X^{|M|} Y^{|C|} \right) \end{aligned} \tag{2.11}$$

where the summation is over all matchings $M \subseteq E$. The properties "M is a matching" and "$C = V - cov(M)$" can be expressed by formulas in Second Order Logic **SOL**. We call this an **SOL**-*polynomial* presentation for $M(G; X, Y)$. A formal definition will be given in Section 4.

In [**24**, Theorem 1.4.1] an explicit description of $\chi(G; X)$ is given: Let $a(G, m)$ be the number of partitions of V into m independent sets of vertices. Then

$$\chi(G; X) = \sum_m a(G, m) \cdot X_{(m)} = \sum_m b_m X_{(m)} \tag{2.12}$$

In other words, the coefficients b_m from Equation 2.1 have a combinatorial interpretation.

This again can be written as

$$\chi(G; X) = \sum_{P: indpart(P, A_P, V)} X_{(card(A_P))} \tag{2.13}$$

where $indpart(P, A_P, V)$ says that "P is an equivalence relation on V" and "each equivalence class induces an independent set" and "A_P consists of the first elements (with respect to the order on $V = [n]$) of each equivalence class". This can be expressed in **SOL**, and therefore is an **SOL**-*subset expansion* for $\chi(G; X)$ which uses an order relation on the vertices V of $G = (V, E)$. However, the choice of the particular order does not matter, the definition is *order invariant*.

Another explicit description for $\chi(G; X)$ is given in [**24**, Theorem 2.2.1]. It can be obtained from a two-variable dichromatic polynomial[2] $Z_G(X, Y)$ defined by

$$Z_G(X, Y) = \sum_{S: S \subseteq E} \left(\prod_{v: fcomp(v,S)} X \cdot \prod_{e: e \in S} Y \right) = \sum_{S: S \subseteq E} \left(X^{k(S)} \cdot \prod_{e: e \in S} Y \right)$$

where $fcomp(v, S)$ is the property "v is the first vertex in the order of V of some connected component of the spanning subgraph $< S : V >$ on V induced by S", and $k(S)$ is the number of connected components of $< S : V >$. Again this is *order invariant*. Now it is well known, [**55**], that

$$\chi(G; X) = Z_G(X, -1) \tag{2.14}$$

Hence, $\chi(G; X)$ is a *substitution instance of an order invariant* **SOL**-*subset expansion* of the graph G with an order on the vertices.

The bivariate matching polynomial has a presentation as an **SOL**-polynomial in the pure language of graphs, in other words, which does not use an order at all. It is natural to ask, whether such a presentation as an **SOL**-polynomial can also be found for the chromatic polynomial $\chi(G; X)$? We now show that this is not possible.

Proposition 2.4. *$\chi(G; X)$ has no presentation as an* **SOL**-*polynomial in the pure language of graphs.*

Proof. To see this, assume that

$$\chi(G; X) = \sum_{A \subseteq V^p : \phi_1(A)} X^{|A|} \tag{2.15}$$

or

$$\chi(G; X) = \sum_{A \subseteq V^p : \phi_2(A)} X_{(|A|)} \tag{2.16}$$

where A ranges over all subsets satisfying an **SOL**-property $\phi_1(A)$ and $\phi_2(A)$, respectively.

We set $X = 2$ and look at the graphs C_n, the 2-regular connected graphs on n vertices. This can be written as an **SOL**-formula $Cycle(G)$. Clearly, $\chi(C_n, 2) = 0$ if n is odd, and $\chi(C_n, 2) = 2$ if n is even. We first deal with Equation (2.15). So we have

$$\chi(C_{2n}; 2) = \sum_{A \subseteq V^p : \phi_1(A)} 2^{|A|} = 2 \tag{2.17}$$

and

$$\chi(C_{2n+1}; 2) = \sum_{A \subseteq V^p : \phi_1(A)} 2^{|A|} = 0 \tag{2.18}$$

It follows that $Cycle(G)$ and $\phi_1(A)$ imply that A is a singleton and uniquely defined on G. But this is a contradiction, because C_n has non-trivial automorphisms for $n \geq 3$.

[2] $Z_G(X, Y)$ is related to the Potts model in statistical mechanics and is related to the Tutte polynomial by a prefactor, cf. [**12**].

For Equation (2.16) the argument is similar, observing that $2^m = 2_{(m)}$ for $m \leq 2$ and $2_{(m)} = 0$ for $m \geq 3$. □

2.3. Recursive definitions vs presentations as SOL-polynomials. It is a recurrent theme in the literature about graph polynomials to look both for recursive definitions and for presentations as **SOL**-polynomials. In the classical literature these presentations are called subset expansions if the summation formula is true for all subsets, and spanning tree expansions if the formula requires that the spanning subgraph (V, A) is a tree. Sometimes both cases are called subset expansions. Good examples of polynomials which have both recursive definitions and subset expansions are the Tutte polynomial and its many variations and substitution instances, [**12, 13**, Chapter X], the Cover polynomial for directed graphs, [**20**], the various Interlace polynomials, [**2, 5, 6**], and all the polynomials defined recursively by vertex and edge eliminations studied in [**57, 7, 8**]. In all these cases the subset expansions turn out to be **SOL**-polynomials. We introduced the notion of "having a presentation as an **SOL**-polynomial" as a generalization which encompasses all the cases which we have encountered in the literature.

In [**30**], a general theorem is formulated and proved which states that, under rather general definitions, every recursively defined graph polynomial can be presented as an **SOL**-polynomial. The converse is likely not to be true. The recursive definitions here are more general than vertex and edge eliminations, and are based on local operations. The framework does cover all the above mentioned cases.

2.4. The bivariate matching polynomial counts colorings. Recall that a proper k-coloring of a graph $G = (V, E)$ is a function $f : V \to [k]$ such that $f^{-1}(i)$ induces an independent set of G. Alternatively, we can define it as a relation $r \subseteq V \times [k]$ satisfying the **SOL**-property $\varphi(r)$ saying "r is a total function such that $\{v \in V : \exists j\ r(v, j)\}$ induces an independent set".

A natural setting in which to interpret the formula $\varphi(r)$ is that of the two-sorted structure of the form $\mathfrak{M}_k = \langle V, [k]; E, r\rangle$, where V and $[k]$ are two universes, E is the edge relation of the graph, and $r \subseteq V \times [k]$ is a relation.

Now $\chi(G, k)$ can be written as

$$\chi(G, k) = \mid \{r \subseteq V \times [k] : \mathfrak{M}_k \models \varphi(r)\} \mid \tag{2.19}$$

In other words, $\chi(G, k)$ counts the number of φ-colorings of G.

We want to define $M(G; X, Y)$ in a similar way. We first do it for

$$g(G, X) = M(G; X, 1) = \sum_M X^{|M|} \tag{2.20}$$

where M ranges over all matchings of $G = (V, E)$. To do this we replace $\varphi(r)$ by $\varphi_1(r)$ which says that "$r \subseteq E \times [k]$ is a partial function the domain of which is a matching of G". In other words r is a partial edge-coloring such that for each $i \in [k]$ the set $\{e \in E : (e, i) \in r\}$ is an independent set of edges. For each matching $M \subseteq E$ there are $k^{|M|}$ many functions with domain M. Hence

$$g(G, k) = \mid \{r \subseteq E \times [k] : \mathfrak{M}_k \models \varphi_1(r)\} \mid = \sum_M X^{|M|} \tag{2.21}$$

This shows that $g(G; X)$ counts the number of φ_1-colorings of G.

We can obtain a similar presentation for $g^*(G,k) = \sum_M X_{(|M|)}$ by writing

$$g^*(G,k) = | \{ r \subseteq E \times [k] : \mathfrak{M}_k \models \varphi_2(r) \} | = \sum_M X_{(|M|)} \tag{2.22}$$

where $\varphi_2(r)$ is the formula "$\varphi_1(r)$ and r is injective". This shows that $g^*(G;X)$ counts the number of φ_2-colorings of G.

To interpret the bivariate polynomial $M(G;X,Y)$ as counting colorings we use two sorts of colors $[k_1]$ and $[k_2]$, the three-sorted structure $\mathfrak{M}_k = \langle V, [k_1], [k_2]; E, r_1, r_2 \rangle$, with two coloring relations $r_1 \subseteq E \times [k_1]$ and $r_2 \subseteq V \times [k_2]$ and a formula $\varphi_3(r_1, r_2)$ which says that
"$r_1 \subseteq E \times [k_1]$ is a partial function the domain M of which is a matching of G".
and "$r_2 \subseteq V \times [k_1]$ is a partial function with domain $V - cov(M)$".

2.5. Generalized chromatic polynomials. As stated earlier, the interpretation of $M(G;X,Y)$ as counting colorings will be generalized and formally defined in Section 3. Here we want to informally prepare our general definition.

Any relation $r \subseteq V^\alpha \times [k]$ on $\mathfrak{M}_k = \langle V, [k]; E, r \rangle$ satisfying a formula $\varphi(r)$ of **SOL** can be viewed as a coloring relation over the graph $G = (V,E)$. We denote by

$$\chi_\varphi(G,k) = | \{ r \subseteq V^\alpha \times [k] : \mathfrak{M}_k \models \varphi(r) \} | \tag{2.23}$$

the number of φ-colorings of G. What interests us here is when $\chi_\varphi(G,k)$ is a polynomial in k. It turns out that this is true under rather general conditions, as stated in Proposition A, in the introduction. The conditions are

(i) the coloring r is invariant under permutations of the colors, i.e., if π is a permutation of the colors, then r satisfies $\varphi(r)$ iff the composition r' of r with π satisfies $\varphi(r')$.
(ii) the number of colors used by each r satisfying $\varphi(r)$ is bounded by the size of V, and
(iii) the property $\varphi(r)$ is independent of the colors not used.

This is obviously the case for $\chi(G;X)$ and $M(G;X,Y)$ and, in general, is easily verified.

2.6. Previously unnoticed graph polynomials. The literature contains many papers on generalized colorings, and their authors are interested either in questions of extremal graph theory or in the complexity of deciding the existence of these colorings. Counting the number of generalized colorings is rarely studied. However, it turns out that counting the number of colorings with k colors very often gives rise to previously unnoticed graph polynomials. We list here a few examples, which we think deserve further investigations.

F. Harary introduced the notion of *P-colorings*, [**36, 38, 15, 16**]. Here P is any graph property. Given a graph $G = (V,E)$, a function $f : V \to [k]$ is a *P-k-coloring* if for all $i \in [k]$ in the range of f, the set $f^{-1}(i)$ induces a graph in P. If P is definable in **SOL**, there is a formula $\varphi(f)$ defining the P-colorings.

Examples of P colorings include:

(i) The proper k-colorings with P being the edgeless graphs.
(ii) The *convex colorings* [**52**], with P being the connected graphs. Convex colorings have applications in computational biology.
(iii) The G-free colorings studied in [**14, 1**], where P consists of G-free graphs.

(iv) The partitions of graphs into cographs, [**29**]. The family of cographs is the smallest class of graphs that includes K_1 and is closed under complementation and disjoint union. They can be characterized as the P_4-free graphs.

(v) the *mcc_t-colorings* defined in [**3**] and further studied in [**44**]. Here $t \in \mathbb{N}$ and $P = P_t$ consist of all graphs the connected components of which are of size at most t. For $t = 1$ these are just the proper colorings.

This list is far from being complete and the reader will easily find more examples of P-colorings.

It is easy to verify that P-colorings satisfy the three informal conditions required in Proposition A. The precise version of Proposition A is stated in Section 3 as Proposition 3.10.

COROLLARY 2.5. *Let P be a graph property and let $\chi_P(G, k)$ be the number of P-colorings with k colors. Then $\chi_P(G, k)$ is a polynomial in k.*

Note that the definability condition is not needed here. We only need that P is closed under isomorphisms.

REMARK 2.6. *For fixed t, $\chi_{P_t}(G, k)$ is a polynomial in k. However, it is not a bivariate polynomial in t and k.*

Harary's definition can be generalized in various ways, using edge colorings, rather than vertex colorings, or by requiring that the union of any s color classes induces a graph in P. An example of the latter with $s = 2$ and P the class of forests is the set of acyclic colorings introduced in [**33**] and further studied in [**4**]. Also these generalizations satisfy the three conditions required in Proposition A, hence they give rise to previously unnoticed graph polynomials.

Not all coloring properties in the literature can be formulated as variants of P-colorings.

(i) *Injective colorings* are vertex colorings such that for any three vertices $u, v, w \in V$ such that (u, v) and (u, w) are edges in E then $f(v) \neq f(w)$, [**35**].

(ii) A coloring f of a graph $G = (V, E)$ is *harmonious*, if it is a proper coloring and every pair of colors occurs at most once along an edge. Harmonious colorings were first studied in [**40**] and extensively studied in [**27**].

(iii) There are various notions of *rainbow colorings*, which all are edge colorings and impose some injectivity condition on certain configurations of edges. For example in [**18**], the condition is that any two vertices are joined by a path such that all edges on it have different colors.

It is easy to verify that these colorings are not P-colorings but still satisfy the three conditions required in Proposition A. Hence, counting injective colorings, harmonious colorings and rainbow colorings with k colors again gives rise to new graph polynomials.

F. Harary [**37**] also introduced the notion of a *complete k-coloring* of a graph, and the associated chromatic number, which he named the *achromatic number*. A coloring f of a graph $G = (V, E)$ is *complete* if (i) it is a proper coloring and (ii) every pair of colors from $[k]$ occurs at least once along an edge. For a survey on this topic, cf. [**41**].

If f is a complete k-coloring, f cannot be a complete $(k+1)$-coloring, therefore complete colorings do not satisfy our conditions. Indeed, the number of complete k-colorings of a graph is not a polynomial in k, as it vanishes for k with $\mid E \mid < \binom{k}{2}$.

3. Counting generalized colorings

3.1. φ-colorings. Previously we only considered graph colorings; now we expand our discussion to include τ structures $\mathfrak{M}$, where τ is a finite vocabulary for relational structures. We shall also use the formalism of many-sorted structures. We think of many-sorted structures as having a single big universe (the union of the universes corresponding to the sorts), and with unary predicates whose interpretations are the universes of the sorts.

Let $\mathfrak{M}$ be a τ-structure with universe M. We assume without loss of generality that $M = [n] = \{1, \ldots, n\}$ for $n > 1$. We will assume that all our structures are ordered, i.e. there exists a binary relation symbol $\mathbf{R}_{\leq}$ in τ which is always interpreted as the natural linear ordering of the universe. To simplify notation, we omit the order relation from structures.

Let k be a natural number. We denote by $\mathfrak{M}_{F,k}$ the two-sorted structure

$$\mathfrak{M}_{F,k} = \langle \mathfrak{M}, [k], F \rangle$$

where $F : M \to [k]$ is a function. We think of $\mathfrak{M}_{F,k}$ as the colored structure induced by the function F on $\mathfrak{M}$. The set $[k]$ will be referred to as the color set. Note that the order relation does not extend to the second sort $[k]$.

Let $\mathbf{F}$ be a unary function symbol and let $\tau_{\mathbf{F}} = \tau \cup \{\mathbf{F}\}$, then $\mathfrak{M}_{F,k}$ is a $\tau_{\mathbf{F}}$-structure. On the other hand, every two-sorted $\tau_{\mathbf{F}}$-structure $\mathfrak{A}$ with second sort $[k]$ can be thought of as $\mathfrak{M}_{F,k}$ for a unique tuple $\langle \mathfrak{M}, [k], F \rangle$ consisting of a τ-structure $\mathfrak{M}$ with universe M, the set $[k]$ and a function $F : M \to [k]$.

Let $\mathcal{L}$ be a fragment of **SOL**. We will only be interested in formulas whose interpretations are invariant under the choice of a linear ordering of the universe.

Definition 3.1 ($\mathcal{L}(\tau_{\mathbf{F}})$-Coloring formulas). *We say $\varphi \in \mathcal{L}(\tau_{\mathbf{F}})$ is an* $\mathcal{L}(\tau_{\mathbf{F}})$-coloring formula *if φ does not quantify over the second sort, but instead all first and second order quantifiers are on the first sort only. We say it is an* $\mathcal{L}(\tau_{\mathbf{F}})$-coloring sentence *if φ is a sentence.*

Definition 3.2 (φ-colorings and coloring properties). *Let φ be an $\mathcal{L}(\tau_{\mathbf{F}})$-coloring sentence.*

(i) *$\mathfrak{M}_{F,k}$ is called a* φ-colored τ-structure *and F is called a* φ-coloring of $\mathfrak{M}$ *if $\mathfrak{M}_{F,k}$ is a two-sorted $\tau_{\mathbf{F}}$-structure such that*

$$\mathfrak{M}_{F,k} \models \varphi$$

(ii) *Let $\mathcal{P}_{\varphi}$ be the class*

$$\mathcal{P}_{\varphi} = \{\mathfrak{M}_{F,k} : \mathfrak{M}_{F,k} \models \varphi\}$$

of φ-colored τ-structures. Then $\mathcal{P}_{\varphi}$ is called a coloring property.

Example 3.3 (Proper coloring and variations). *Let τ_{Graphs} be the vocabulary consisting of one binary relation symbol $\mathbf{E}$ as well as the order relation $\mathbf{R}_{\leq}$. Let $G = (V, E, \leq)$ be a τ_{Graphs}-structure, where $V = [n]$.*

(i) *A function* $F : V \to [k]$ *is a* proper coloring, *if it satisfies*

$$\varphi_{proper} = \forall u, v(\mathbf{E}(u, v) \to \mathbf{F}(u) \neq \mathbf{F}(v))$$

The class $\mathcal{P}_{\varphi_{proper}}$ *is the class of tuples* $\langle G, [k], F\rangle$ *of graphs together with a color-set* $[k]$ *and a proper coloring* F*.*

(ii) *A function* $F : V \to [k]$ *is* pseudo-complete *if it satisfies*

$$\forall x, y \exists u, v(\mathbf{E}(u, v) \wedge \mathbf{F}(u) = x \wedge \mathbf{F}(v) = y).$$

We will now see that pseudo-complete colorings do not form a coloring property.

Coloring properties $\mathcal{P}_\varphi$ satisfy the following two properties:

Permutation property: Let $\pi : [k] \to [k]$ be a permutation and let F_π be the function obtained from F by applying π, i.e. $F_\pi(v) = \pi(F(v))$. Then

$$\mathfrak{M}_{F,k} \in \mathcal{P}_\varphi \text{ iff } \mathfrak{M}_{F_\pi,k} \in \mathcal{P}_\varphi.$$

In other words, $\mathcal{P}_\varphi$ is closed under permutation of the color-set $[k]$.

Remark 3.4. *It follows from the Permutation Property, that we can assume that* $Range(F)$ *is of the form* $[k_0]$ *for some* $k_0 \leq k$*.*

Extension property: For every $\mathfrak{M}$, F with $Range(F) = [k_0]$, $k' \geq k_0$

$$\mathfrak{M}_{F,k_0} \in \mathcal{P}_\varphi$$

iff

$$\mathfrak{M}_{F,k'} \in \mathcal{P}_\varphi$$

Namely, the extension property requires that increasing or decreasing the number of colors not in $Range(F)$ does not affect whether it belongs to the property.

Let φ be a $\mathcal{L}(\tau_{\mathbf{F}})$-coloring sentence. The class $\mathcal{P}_\varphi$ satisfies the permutation property because it is definable in **SOL**.

Proposition 3.5. $\mathcal{P}_\varphi$ *satisfies the extension property.*

Proof. We prove by induction a slightly stronger statement:

(*) Let $\mathfrak{A}_1$ be a τ-structure with universe A_1. Let $\mathfrak{A}$ be a two-sorted $\tau_{\mathbf{F}}$-structure $\langle \mathfrak{A}_1, A_2, F\rangle$ and let $Range(F) = F(A_1)$. Then for every formula $\theta \in \mathcal{L}(\tau_{\mathbf{F}})$ with no variable which ranges over the second sort A_2,

$$\mathfrak{A} \models \theta \text{ iff } \langle \mathfrak{A}_1, Range(F), F\rangle \models \theta$$

Basis: Let θ be an atomic formula. Assume first that θ does not contain $\mathbf{F}$. Any relation symbol, function symbol or constant symbol in θ is interpreted in $\mathfrak{A}$ over A_1 only. Since all the variables in θ range over the first sort only, the truth-value of θ does not depend on A_2.

On the other hand, if θ contains $\mathbf{F}$, then θ must be of the form $\mathbf{F}(x_1) = \mathbf{F}(x_2)$, where x_1 and x_2 are first order variables (which range over the first sort). In this case, the truth-value of θ depends only on the elements of A_2 which can be obtained as $\mathbf{F}(a)$ for some $a \in A_1$. I.e., θ depends only on $Range(\mathbf{F})$.

Closure: If θ_1, θ_2 satisfy (*), then clearly so does any Boolean combination of them. We need only deal with universal quantifiers $\forall x, \forall X$, as we get the existential quantifiers as their negation. Let $\theta = \forall z \theta'(z)$ where z is a first order or second order variable. We extend the vocabulary τ with symbol s_z as follows. The symbol s_z is a constant symbol $s_z = c_z$ if z is a first order variable. The symbol s_z is a relation symbol $s_z = R_z$ of arity ρ if z is a second order variable of arity of ρ. We note $\mathfrak{A} \models \theta$ iff for every interpretation a of s_z it holds that $\langle \mathfrak{A}, a \rangle \models \theta'$. By the induction hypothesis, this happens iff for every interpretation a of s_z it holds that $\langle \mathfrak{A}_1, a, Range(F), F \rangle \models \theta'$. The latter occurs iff $\langle \mathfrak{A}_1, Range(F), F \rangle \models \forall z \theta$.

□

REMARK 3.6. *The class $\mathcal{P}_1$ of structures $\mathfrak{M}_{F,k}$ where F is a proper coloring which uses all the colors in the color-set $[k]$ is not a coloring property, since $\mathcal{P}_1$ violates the extension property. Similarly, the class of pseudo-complete colorings is not a coloring property.*

DEFINITION 3.7 (Counting functions of φ-colorings). *Let φ be a $\mathcal{L}(\tau_{\mathbf{F}})$-coloring sentence. Let χ_φ be a function from pairs $\langle \mathfrak{M}, k \rangle$ which consist of a τ-structure $\mathfrak{M}$ and $k \in \mathbb{N}^+$ defined as follows. The function $\chi_\varphi(\mathfrak{M}, k)$ is*

$$| \{ F : \mathfrak{M}_{F,k} \in P_\varphi \} |$$

I.e., $\chi_\varphi(\mathfrak{M}, k)$ counts the number of φ-colorings of $\mathfrak{M}$ with k colors.

For example, treating k as an indeterminate, $\chi_{\varphi_{proper}}(G, k)$ is the chromatic polynomial of the graph G.

Denote by $c_\varphi(\mathfrak{M}, j)$ the number of φ-colorings of $\mathfrak{M}$ with color-set $[j]$ which use all the colors of $[j]$.

PROPOSITION 3.8 (Special case of Proposition A). *Let φ be an $\mathcal{L}(\tau_{\mathbf{F}})$-coloring sentence. For every $\mathfrak{M}$ the number $\chi_\varphi(\mathfrak{M}, k)$ is a Newton polynomial in k of the form*

$$\sum_{j=1}^{|M|} c_\varphi(\mathfrak{M}, j) \binom{k}{j}$$

PROOF. We first observe that any φ-coloring F uses at most $|M|$ of the k colors. By the permutation property, if F is a φ-coloring which uses j colors then any function obtained by permuting the colors is also a φ-coloring. Therefore, given k colors, the number of φ-colorings that use exactly j of the k colors is the product of $c_\varphi(\mathfrak{M}, j)$ and the binomial coefficient $\binom{k}{j}$. So

$$\chi_\varphi(\mathfrak{M}, k) = \sum_{j=0}^{|M|} c_\varphi(\mathfrak{M}, j) \binom{k}{j}$$

The right side here is a polynomial in k, because each of the binomial coefficients is. We also use that for $k < j$ we have $\binom{k}{j} = 0$. □

Remark 3.9.

(i) *Since for a coloring property φ the function $\chi_\varphi(\mathfrak{M}, k)$ is a polynomial, it is now defined not only for positive integer values of k, but rather for every $k \in \mathbb{R}$. Still, these instances of $\chi_\varphi(\mathfrak{M}, k)$ may have a combinatorial meaning. E.g., the chromatic polynomial $\chi_G(k)$ is well known to have meaningful evaluations for the negative integers. In particular, $\chi_G(-1)$ is the number of acyclic orientations of the graph G, see* [**56**].

(ii) *The restriction to coloring properties in Proposition 3.8 is essential. Denote by $\chi_{onto}(G, k)$ the number of functions $f : V \to [k]$ which are onto. Clearly, this is not a polynomial in k since, for $k >\mid V \mid$, it always vanishes, so it should vanish identically, if it were a polynomial.*

(iii) *The proof of Proposition 3.8 does not garantee that the coefficients of the power of k are integers. However, the proof of Proposition 3.10 below does garantee it.*

In fact, it holds that $\chi_\varphi(\mathfrak{M}, k)$ is in $\mathbb{Z}[k]$. For two functions $f_1, f_2 : [n] \to [k]$ we write $\sim_{perm}$ if there exists a permutation $\pi : [k] \to [k]$ such that for all $i \in [n]$ we have $\pi(f_1(i) = f_2(i)$. In other words, f_1 and f_2 are equivalent if they can be obtained from one another by applying some permutation of the color set $[k]$. Let $d_\varphi(\mathfrak{M}, j)$ be the number of colorings F with $Range(F) = [j]$.

Proposition 3.10 (Special case of Proposition A). *Let φ be an $\mathcal{L}(\tau_{\mathbf{F}})$-coloring sentence. For every $\mathfrak{M}$ the number $\chi_\varphi(\mathfrak{M}, k)$ is an* FF *polynomial in k, namely*

$$\sum_{j=1}^{|M|} d_\varphi(\mathfrak{M}, j) \cdot k_{(j)}$$

where $k_{(j)}$ is the falling factorial, $k_{(j)} = k \cdot (k-1) \cdots (k-j+1) = \binom{k}{j} \cdot j!$.

Recall that FF polynomials are polynomials where the monomials are falling factorials.

Proof. By proposition 3.8,

$$\chi_\varphi(\mathfrak{M}, k) = \sum_{j=1}^{|M|} c_\varphi(\mathfrak{M}, j) \binom{k}{j}$$

By the Permutation Property, F is a φ-coloring iff the functions which are $\sim_{perm}$ equivalent to F are φ-colorings. Therefore,

$$\chi_\varphi(\mathfrak{M}, k) = \sum_{j=1}^{|M|} d_\varphi(\mathfrak{M}, j) \cdot j! \binom{k}{j} = \sum_{j=1}^{|M|} d_\varphi(\mathfrak{M}, j) \cdot k_{(j)}$$

□

Remark 3.11. *There exists a coloring property $P_{no\ ext}$ which does not satisfy the extension property and yet $\chi_{no\ ext}(\mathfrak{M}, k)$ is a polynomial. Let $P_{no\ ext}$ consist of all structures $\mathfrak{M}_{F,k}$ which satisfy the following condition:*

- *Let γ_1, γ_2 and γ_3 be the least, second least and third least elements in the linear ordering of $M = [n]$. The function $F : M \to [k]$ satisfies either $F(\gamma_1) = F(\gamma_2) \neq F(\gamma_3)$ or $F(\gamma_1) = F(\gamma_3) \neq F(\gamma_2)$. If $F(\gamma_1) = F(\gamma_2)$ then F is onto and if $F(\gamma_1) = F(\gamma_3)$ then F is not onto.*

Let $F : M \to [k]$ be a function such that $F(\gamma_1) = F(\gamma_2) \neq F(\gamma_3)$ and F is onto. The addition of an unused color to $[k]$ (i.e., looking at F as a function from M to $[k+1]$) implies that F is no longer onto and yet $F(\gamma_1) \neq F(\gamma_3)$. Hence, $\mathfrak{M}_{F,k+1} \notin P_{no\ ext}$, so $P_{no\ ext}$ does not satisfy the extension property. On the other hand, the number of such structures equals the number of functions $F : [n] \to [k]$ for which $F(1) = F(2) \neq F(3)$, so $\chi_{no\ ext}(\mathfrak{M}, k) = k^{n-2}(k-1)$ is a polynomial.

3.2. Multi-colorings. To construct graph polynomials in several variables, we extend in this and the next subsections the definitions in order to deal with several color-sets.

Let $\mathfrak{M}$ be a τ-structure with universe M. Let $\mathfrak{M}_{F,\overline{k}}$ be the $(\alpha+1)$-sorted structure $\langle \mathfrak{M}, [k_1] \ldots, [k_\alpha], F\rangle$ with

$$F : M^m \to [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$$

We denote by $\tau_{\alpha,\mathbf{F}}$ the corresponding vocabulary.

We extend the definitions of $\mathcal{L}(\tau_{\alpha,\mathbf{F}})$-coloring formulas, φ-colorings and coloring properties naturally to $\mathcal{L}(\tau_{\alpha,\mathbf{F}})$-*multi-coloring formulas, φ-multi-colorings and multi-coloring properties.* Multi-coloring properties $\mathcal{P}_\varphi$ satisfy a version of the permutation and extension properties:

Permutation property: Let $\overline{\pi} = (\pi_1, \ldots, \pi_\alpha)$ be permutations of $[k_1], \ldots, [k_\alpha]$ respectively. Let $F : M^m \to [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$ and let $F_{\overline{\pi}}$ be the function obtained by applying the permutations $\pi_1, \ldots, \pi_\alpha$ on F. Then

$$\mathfrak{M}_{F,\overline{k}} \in \mathcal{P}_\varphi \text{ iff } \mathfrak{M}_{F_\pi,\overline{k}} \in \mathcal{P}_\varphi$$

Namely, $\mathcal{P}_\varphi$ is closed under permutations of the color-sets.

Extension property: For every $\mathfrak{M}$, $\overline{k} = k_1, \ldots, k_\alpha$, $\overline{k'} = k'_1, \ldots, k'_\alpha$, and F such that $k_1 \leq k'_1, \ldots, k_\alpha \leq k'_\alpha$,
we have

$$\mathfrak{M}_{F,\overline{k}} \in \mathcal{P}_\varphi$$

iff

$$\mathfrak{M}_{F,\overline{k'}} \in \mathcal{P}_\varphi$$

The multi-coloring properties $\mathcal{P}_\varphi$ satisfy the following property as well:

Non-occurrence property: Assume $F : M^m \to [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$ with $m_i = 0$. [3] Then for every $b \in \mathbb{N}$,

$$\langle \mathfrak{M}, [k_1], \ldots, [k_\alpha], F\rangle \in \mathcal{P}_\varphi$$

iff

$$\langle \mathfrak{M}, [k_1], \ldots, [b], \ldots, [k_\alpha], F\rangle = \mathfrak{M}_{(k_1,\ldots,k_{i-1},b,k_{i+1},\ldots,k_\alpha),F} \in \mathcal{P}_\varphi.$$

The extension property and the non-occurrence property require that increasing and decreasing the number of colors not used by F, respectively adding or removing unused color-sets, does not affect whether $\mathfrak{M}_{F,\overline{k}}$ belongs to the multi-coloring property $\mathcal{P}_\varphi$. The proofs that these properties hold for multi-coloring properties are similar to the one variable case.

We denote by $\chi_\varphi(\mathfrak{M}, k_1, \ldots, k_\alpha)$ the number of φ-multi-colorings with color-sets $[k_1], \ldots, [k_\alpha]$.

---

[3] For a set S the set S^0 is the singleton set which has as its unique element the empty tuple.

PROPOSITION 3.12 (Special case of Proposition A). *Let $\mathcal{P}_\varphi$ be a multi-coloring property with $\mathcal{L}(\tau_{\alpha,\mathbf{F}})$-coloring formula φ. For every $\mathfrak{M}$, $\chi_\varphi(\mathfrak{M}, k_1, \ldots, k_\alpha)$ is an* FF*-polynomial in $k_1, \ldots, k_\alpha$ of the form*

$$\sum_{j_1 \leq N_M} \sum_{j_2 \leq N_M} \cdots \sum_{j_\alpha \leq N_M} d_{\varphi(F)}(\mathfrak{M}, \bar{j}) \prod_{1 \leq \beta \leq \alpha} k_{\beta(j_\beta)}$$

where $\bar{j} = (j_1, \ldots, j_\alpha$, $d_{\phi(R)}(\mathfrak{M}, \bar{j})$ is the number of φ-multi colorings F with color-sets $[j_1], \ldots, [j_\alpha]$ which uses all the colors in every color-set up to permutations of the color sets, and N_M is a polynomial in $|M|$.

PROOF. Similar to the one variable case. □

EXAMPLE 3.13. *Recall that, in the prelude, $\chi_{mcc(t)}(G, k)$ denoted the number of vertex colorings for which no color induces a graph with a connected component of size larger than t. Let $\chi_{mcc}(G, k, t) = \chi_{mcc(t)}(G, k)$ be the counting function of multi-colorings satisfying the above condition, where t is considered a color-set. These multi-colorings do not satisfy the non-occurrence property. Indeed, we will now see that $\chi_{mcc}(G, k, t)$ is not a polynomial in t.*

Let $F : V \to [k]$ be any function. If $t \geq |V|$ then for every color $c \in [k]$ it holds that $|f^{-1}(c)| < t$ and, in particular, no color induces a graph which has a connected component larger than t. Therefore, for such t, $\chi_{mcc}(G, k, t) = k^{|V|}$. Since $\chi_{mcc}(G, k, t)$ does not always agree with $k^{|V|}$ on small values of t, $\chi_{mcc}(G, k, t)$ cannot be a polynomial in t.

This example shows the motivation for requiring the non-occurrence property of coloring properties.

3.3. Multi-colorings – the general cases.

3.3.1. *Multi-colorings with partial functions.* We extend our definition in two ways. First, we now allow $F \subseteq M^m \times [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$ to be a partial function. For this purpose **F** must now be a relation symbol. Second, we also allow several simultaneous coloring predicates $F_1, \ldots, F_s$ and the corresponding number of relation symbols. A coloring property P_φ will therefore consist of structures

$$\mathfrak{M}_{F_1,\ldots,F_s,k_1,\ldots,k_\alpha} = \langle \mathfrak{M}, [k_1], \ldots, [k_\alpha], F_1, \ldots, F_s \rangle$$

which satisfy φ and for which each F_i is a (possibly partial) function.

We may call multi-coloring properties and multi-coloring simply also coloring properties and colorings, if the situation is clear from the context. The permutation, extension and non-occurrence properties extend naturally to this case and Proposition A holds as well:

PROPOSITION 3.14 (Proposition A for several partial functions). *Let φ be an $\mathcal{L}(\tau_{\mathbf{F_1},\ldots,\mathbf{F_s}})$-multi-coloring sentence. For every $\mathfrak{M}$ the number $\chi_\varphi(\mathfrak{M}, k)$ of φ-multi-colorings with several partial functions is an* FF*-polynomial in k of the form*

$$\sum_{j_1 \leq N_M} \sum_{j_2 \leq N_M} \cdots \sum_{j_\alpha \leq N_M} d_{\varphi(\overline{F})}(\mathfrak{M}, \bar{j}) \prod_{1 \leq \beta \leq \alpha} k_{\beta(j_\beta)} \tag{3.1}$$

where $N_M \in \mathbb{N}$. Moreover, N_M is bounded by a polynomial in $|M|$.

3.3.2. *Multi-colorings with bounded relations.* Here we extend the multi-colorings with several partial functions by allowing several relations which are bounded in a certain way.

DEFINITION 3.15 (Bounded relations and multi-coloring properties).

(i) *We say a relation* $R \subseteq M^m \times [k_1]^{m_1} \times \cdots \times [k_\alpha]^{m_\alpha}$ *is* d-bounded if the set of tuples of colors used by R,

$$\{\bar{c} \mid \text{ there exists } \bar{x} \in M^m \text{ such that } (\bar{x}, \bar{c}) \in R\},$$

is of size at most $|M|^d$.

(ii) A $\varphi(\overline{\mathbf{R}})$-multi-coloring property P_φ is the class of τ-structures

$$\mathfrak{M}_{\overline{R},\overline{k}} = \langle \mathfrak{M}, [k_1], \ldots, [k_\alpha], \overline{R} \rangle$$

which satisfy φ. We say P_φ is *bounded* if there exists d such that each R in every structure $\mathfrak{M}_{\overline{R},\overline{k}}$ in P_φ is d-bounded.

Again, the permutation, extension and non-occurrence properties extend naturally and Proposition A holds. However, in this case we do not have a polynomial of the form of Equation (3.1). In particular, in this case we may have polynomials which do not belong to $\mathbb{Z}[\overline{k}]$.

PROPOSITION 3.16 (Proposition A for several d-bounded relations). *Let* $d \in \mathbb{N}$ *and let* φ *be an* $\mathcal{L}(\tau_{\mathbf{R_1},\ldots,\mathbf{R_s}})$*-multi-coloring sentence. For every* $\mathfrak{M}$ *the number* $\chi_\varphi(\mathfrak{M}, k)$ *of* φ*-multi-colorings with several* d*-bounded relations is a Newton polynomial in* k *of the form*

$$\sum_{j_1 \leq N_M} \sum_{j_2 \leq N_M} \cdots \sum_{j_\alpha \leq N_M} c_{\varphi(\overline{R})}(\mathfrak{M}, \bar{j}) \prod_{1 \leq \beta \leq \alpha} \binom{k_\beta}{j_\beta}$$

where $N_M \in \mathbb{N}$.

3.4. Closure properties for the general cases.

PROPOSITION 3.17 (Sums and products). *Let* ϕ, ψ *be coloring properties. Then there are* $\theta_1, \theta_2 \in \mathbf{SOL}$ *such that*

(i) $\chi_{\theta_1(\overline{\mathbf{F}}_3)}(\mathfrak{M}, \overline{k}, 1) = \chi_{\phi(\overline{\mathbf{F}}_1)}(\mathfrak{M}, \overline{k}) + \chi_{\psi(\overline{\mathbf{F}}_2)}(\mathfrak{M}, \overline{k})$

(ii) $\chi_{\theta_2(\overline{\mathbf{F}}_3)}(\mathfrak{M}, \overline{k}) = \chi_{\phi(\overline{\mathbf{F}}_1)}(\mathfrak{M}, \overline{k}) \cdot \chi_{\psi(\overline{\mathbf{F}}_2)}(\mathfrak{M}, \overline{k})$

PROOF. Since we are dealing with ordered structures, we can define $\varphi_{min}(\mathbf{F}')$ which requires that F' is a total function $F' : M \to [k']$, where $[k']$ is a new color set. Let

$$\begin{aligned} \theta_1(\overline{\mathbf{F}}_1, \overline{\mathbf{F}}_2, \mathbf{F}') \quad &= \quad \big(\varphi_{min}(\mathbf{F}') \wedge \phi(\overline{\mathbf{F}}_1) \wedge \mathbf{F}_2 = \emptyset\big) \vee \\ &\quad\quad \big(\mathbf{F}' = \emptyset \wedge \mathbf{F_1} = \emptyset \wedge \psi(\overline{\mathbf{F}}_2)\big) \end{aligned}$$

It holds that $\chi_{\theta_1}(\mathfrak{M}, \overline{k}, k') = (k')^{|M|} \cdot \chi_\phi(\mathfrak{M}, \overline{k}) + \chi_\psi(\mathfrak{M}, \overline{k})$. Taking k' to be 1, we get that $\chi_{\theta_1}(\mathfrak{M}, \overline{k}, 1)$ is the sum.

For the product we take $\chi_{\theta_2}(G, \lambda)$ with

$$\theta_2(\overline{\mathbf{F}}_3) = \big(\phi(\overline{\mathbf{F}}_1) \wedge \psi(\overline{\mathbf{F}}_2)\big)$$

where $\overline{\mathbf{F}}_3 = (\overline{\mathbf{F}}_1, \overline{\mathbf{F}}_2)$. □

3.5. The dichromatic polynomial $Z(G; X, Y)$**.** The dichromatic polynomial is sometimes considered a version of the Tutte polynomial, cf. [**55**]. To illustrate Theorem B we show how to convert $Z(G; X, Y)$ into a counting function of a φ-multi-coloring. We use the dichromatic polynomial in the following form:

$$Z(G; X, Y) = \sum_{A \subseteq E} X^{k(A)} Y^{|A|}$$

where $k(A)$ is the number of connected components of the spanning subgraph (V, A). For this purpose we look at the three-sorted structure

$$G_{(A,F_1,F_2),(k,l)} = \langle \mathfrak{G}, [k], [l], A, F_1, F_2 \rangle$$

with $A \subseteq E$, $F_1 : V \to [k]$ and $F_2 : A \to [l]$ such that $(u, v) \in A$ implies $F_1(u) = F_1(v)$. This is expressed in the formula $dichromatic(A, F_1, F_2)$. As we saw for the matching polynomial, this can be easily converted into a coloring property definable in **SOL**. Now we have

$$\chi_{dichromatic(A,F_1,F_2)}(G; k, l) = \sum_{A \subseteq E} k^{k(A)} l^{|A|}$$

which is the evaluation of $Z(G; X, Y)$ for $X = k, Y = l$.

4. SOL-polynomials and subset expansion

We are now ready to introduce the **SOL**-polynomials, which generalize subset expansions and spanning tree expansions of graph polynomials as encountered in the literature.

4.1. SOL-polynomials. Let $\mathcal{R}$ be a commutative semi-ring, which contains the semi-ring of natural numbers $\mathbb{N}$. For our discussion $\mathcal{R} = \mathbb{Z}$ suffices, but the definitions generalize. Our polynomials have a fixed finite set of variables (indeterminates, if we distinguish them from the variables of **SOL**), $\mathbf{X}$. We denote by $card_{\mathfrak{M},\overline{v}}(\varphi(\overline{v}))$ the number of tuples $\overline{v}$ of elements of the universe that satisfy φ. We again assume τ contains a relation symbol $\mathbf{R}_{\leq}$ which is always interpreted as a linear ordering of the universe.

Let $\mathfrak{M}$ be a τ-structure. We first define the *standard (or geometric)* $\mathbf{SOL}(\tau)$-*monomials* inductively.

Definition 4.1 (standard **SOL**-monomials).

(i) *Let* $\phi(\overline{v})$ *be a formula in* $\mathbf{SOL}(\tau)$*, where* $\overline{v} = (v_1, \ldots, v_m)$ *is a finite sequence of first order variables. Let* $r \in \mathbf{X} \cup (\mathbb{Z} - \{0\})$ *be either an indeterminate or an integer. Then*

$$r^{card_{\mathfrak{M},\overline{v}}(\phi(\overline{v}))}$$

is a standard $\mathbf{SOL}(\tau)$*-monomial (whose value depends on* $card_{\mathfrak{M},\overline{v}}(\phi(\overline{v}))$*).*

(ii) *Finite products of standard* $\mathbf{SOL}(\tau)$*-monomials are standard* $\mathbf{SOL}(\tau)$*-monomials.*

Even if r *is an integer, and* $r^{card_{\mathfrak{M},\overline{v}}(\phi(\overline{v}))}$ *does not depend on* $\mathfrak{M}$*, the monomial stands as it is, and is not evaluated.*

The *falling factorial (*FF*)* $\mathbf{SOL}(\tau)$*-monomials* and the *Newton* $\mathbf{SOL}(\tau)$*-monomials* are defined similarly as follows:

DEFINITION 4.2 (FF **SOL**-monomials). *The* FF $\mathbf{SOL}(\tau)$*-monomials are defined as in Definition 4.1, except we replace the power*

$$r^{card_{\mathfrak{M},\overline{v}}(\phi(\overline{v}))}$$

with the falling factorial

$$r_{(card_{\mathfrak{M},\overline{v}}(\phi(\overline{v})))}$$

DEFINITION 4.3 (Newton **SOL**-monomials). *The Newton* $\mathbf{SOL}(\tau)$*-monomials are defined as in Definition 4.1, except we replace the power*

$$r^{card_{\mathfrak{M},\overline{v}}(\phi(\overline{v}))}$$

with the binomial coefficient

$$\binom{r}{card_{\mathfrak{M},\overline{v}}(\phi(\overline{v}))}$$

Note the degree of a monomial is polynomially bounded by the cardinality of $\mathfrak{M}$.

DEFINITION 4.4 (**SOL**-polynomials). *The polynomials definable in* $\mathbf{SOL}(\tau)$ *are defined inductively:*

(i) *standard (respectively* FF *respectively Newton)* $\mathbf{SOL}(\tau)$*-monomials are standard (respectively* FF *respectively Newton)* $\mathbf{SOL}(\tau)$*-polynomials.*
(ii) *Let* ϕ *be a* $\tau \cup \{\overline{\mathbf{R}}\}$*-formula in* **SOL** *where* $\overline{\mathbf{R}} = (\mathbf{R}_1, \ldots, \mathbf{R}_m)$ *is a finite sequence of relation symbols not in* τ*. Let* t *be a standard (respectively* FF *respectively Newton)* $\mathbf{SOL}(\tau \cup \{\overline{\mathbf{R}}\})$*-polynomial. Then*

$$\sum_{\overline{R}:\langle\mathfrak{M},\overline{R}\rangle\models\phi(\overline{R})} t$$

is a standard (respectively FF *respectively Newton)* $\mathbf{SOL}(\tau)$*-polynomial.*

For simplicity we refer to $\mathbf{SOL}(\tau)$-polynomials as **SOL**-polynomials when τ is clear from the context. Among the **SOL**-polynomials we find most of the known graph polynomials from the literature, cf. [**50**]. We will discuss the choice of basis of the **SOL**-polynomials in Section 5.

4.2. Properties of SOL-polynomials.

LEMMA 4.5.

(i) *Every indeterminate* $x \in \mathbf{X}$ *can be written as a standard,* FF *and Newton* **SOL***-monomial.*
(ii) *Every integer* c *can be written as a standard,* FF *and Newton* **SOL***-monomial.*

PROOF. The minimal element f_1 in the linear ordering of the universe is definable in **SOL**. For every $r \in (\mathbb{Z} - \{0\}) \cup \mathbf{X}$, the term r is a standard, FF and Newton **SOL**-monomial since

$$r = r^{card_{\mathfrak{M},v}(v=f_1)} = r_{(card_{\mathfrak{M},v}(v=f_1))} = \binom{r}{card_{\mathfrak{M},v}(v = f_1)}$$

□

LEMMA 4.6 (Normal form).

(i) *Let $P(\mathfrak{M})$ be a standard,* FF *or Newton* **SOL**$-$*polynomial. Then $P(\mathfrak{M})$ can be written in the form*

$$\sum_{R_1:\phi_1(R_1)} \cdots \sum_{R_s:\phi_s(R_1,\ldots,R_s)} \Phi(\mathfrak{M}, \overline{R}) \tag{4.1}$$

where $\phi_1, \ldots, \phi_s \in$ **SOL** *and $\Phi(\mathfrak{M}, \overline{R})$ is a standard,* FF *or Newton* **SOL**-*monomial respectively.*

(ii) *Let $\Phi(\mathfrak{M})$ be a standard* **SOL**-*monomial. Then $\Phi(\mathfrak{M})$ can be written in the form*

$$r_1^{card_{\mathfrak{M},\overline{v}}(\varphi_1(\overline{v}))} \cdots r_t^{card_{\mathfrak{M},\overline{v}}(\varphi_t(\overline{v}))} \tag{4.2}$$

where $\varphi_t \in$ **SOL** *and $r_1, \ldots, r_t \in (\mathbb{Z} - \{0\}) \cup$* **X** *are distinct.*

(iii) *Let $\Phi(\mathfrak{M})$ be an* FF *or Newton* **SOL**-*monomial. Then $\Phi(\mathfrak{M})$ can be written in the form of Equation (4.2), except we replace $r_i^{card_{\mathfrak{M},\overline{v}}(\varphi_i(\overline{v}))}$ with $r_{i(card_{\mathfrak{M},\overline{v}}(\varphi_i(\overline{v})))}$ or $\binom{r_i}{card_{\mathfrak{M},\overline{v}}(\varphi_i(\overline{v}))}$ respectively, and the r_i's might not be distinct.*

PROOF. (i) follows directly from the definitions. The proof of (ii) is as follows. From the definitions, it is easy to see that Φ is of the desired form, except $r_1, \ldots, r_t$ may not be distinct, i.e. $\Phi(\mathfrak{M})$ can be written as

$$r_1^{card_{\mathfrak{M},\overline{v}}(\varphi_{1,1}(\overline{v}))} \cdots r_1^{card_{\mathfrak{M},\overline{v}}(\varphi_{1,h_1}(\overline{v}))} \cdots \cdot r_t^{card_{\mathfrak{M},\overline{v}}(\varphi_{t,1}(\overline{v}))} \cdots r_t^{card_{\mathfrak{M},\overline{v}}(\varphi_{t,h_t}(\overline{v}))}$$

We will prove that

$$r_i^{card_{\mathfrak{M},\overline{v}}(\varphi_{i,1}(\overline{v}))} \cdots r_i^{card_{\mathfrak{M},\overline{v}}(\varphi_{i,h_i}(\overline{v}))}$$

can be written as $r_i^{card_{\mathfrak{M},\overline{v}}\theta(\overline{v})}$ for some formula θ. Let m be the maximum number of free variables in any of the $\varphi_{i,j}$. Without loss of generality, we may assume that all $\varphi_{i,j}$ have m free variables. We may do so because otherwise we can add free variables $v_q, \ldots, v_m$ to those $\varphi_{i,j}$ which have less m free variables. We then change $\varphi_{i,j}$ to require also that each new variable is equal to v_1, thus keeping the same number of tuples satisfying $\varphi_{i,j}$. The first element f_1 and the second element f_2 of the linear ordering of $\mathfrak{M}$ are definable in **SOL**. Let $\theta(v_1, \ldots, v_m, u_1, \ldots, u_t)$ be the formula such that $(a_1, \ldots, a_m, b_1, \ldots, b_{h_i})$ satisfies θ iff:

(i) exactly one of the b_j is f_1 and all other b_j's are f_2, and
(ii) if $b_j = f_1$ then $a_1, \ldots, a_m$ satisfy $\varphi_{i,j}$.

The formula θ uses the u_j variables to choose the formula $\varphi_{i,j}$. The tuples corresponding to the different $\varphi_{i,j}$ are disjoint which implies that

$$x_i^{card_{\mathfrak{M},\overline{v}}(\theta(\overline{v}))} = x_i^{card_{\mathfrak{M},\overline{v}}(\varphi_{i,1}(\overline{v}))} \cdots x_i^{card_{\mathfrak{M},\overline{v}}(\varphi_{i,h_i}(\overline{v}))}$$

The proof of (ii) follows. Finally, (iii) holds by definition. □

PROPOSITION 4.7. *The pointwise product of two standard,* FF *or Newton* **SOL**-*polynomials is again a standard,* FF *or Newton* **SOL**-*polynomial respectively.*

PROOF. We prove it for standard **SOL**-polynomials. The proof for FF-polynomials and Newton polynomials is identical. Let $P_1(\mathfrak{M})$ and $P_2(\mathfrak{M})$ be standard **SOL**-polynomials. Every **SOL**-polynomial can be written in the form of Equation (4.1). Without loss of generality, we may assume P_1 and P_2 have the same number of sums (otherwise we add dummy sums of the form $\sum_{U:U=\emptyset}$). We

proceed by induction on the number of summations in P_1 and P_2:
Basis: By definition $P_1(\mathfrak{M}) \cdot P_2(\mathfrak{M})$ is a **SOL**-monomial.
Step: For $i \in \{1,2\}$, let

$$P_i(\mathfrak{M}) = \sum_{R_i : \phi_i(R_i)} \Phi_i(\langle \mathfrak{M}, R_i \rangle)$$

Then

$$P_1(\mathfrak{M}) \cdot P_2(\mathfrak{M}) = \sum_{R_1, R_2 : \phi_1(R_1) \wedge \phi_2(R_2)} \Phi_1(\langle \mathfrak{M}, R_1 \rangle) \cdot \Phi_2(\langle \mathfrak{M}, R_2 \rangle)$$

By the induction hypothesis, this is a standard **SOL**-polynomial. □

Lemma 4.8. *Let τ be a vocabulary and let* $\mathbf{S}$ *be a relation symbol not in τ. Let $P(\mathfrak{M})$ be a standard,* FF- *or Newton* $\mathbf{SOL}(\tau)$*-polynomial and let $A \in \{\emptyset, M\}$. Let $P^A(\mathfrak{M}, S)$ be a graph polynomial which satisfies*

$$P^A(\mathfrak{M}, S) = \begin{cases} P(\mathfrak{M}) & S = A \\ 1 & \textit{otherwise} \end{cases}$$

Then $P^A(\mathfrak{M}, S)$ is a standard, FF- *respectively Newton* $\mathbf{SOL}(\tau \cup \{\mathbf{S}\})$*-polynomial.*

Proof. We prove the lemma for standard **SOL**-polynomials by induction on the structure of $P(\mathfrak{M})$. The proof for FF-**SOL**-polynomial and Netwon **SOL**-polynomial is similar.
Basis:

(i) Let $P(\mathfrak{M}) = r^{card_{\mathfrak{M},\overline{v}}(\varphi(\overline{v}))}$. Then $P^A(\mathfrak{M}, S) = r^{card_{\mathfrak{M},\overline{v}}(\varphi(\overline{v}) \wedge (S=A))}$ satisfies the conditions.
(ii) $P(\mathfrak{M}) = P_1(\mathfrak{M}) \cdot P_2(\mathfrak{M})$. Then $P^A(\mathfrak{M}) = P_1^A(\mathfrak{M}) \cdot P_2^A(\mathfrak{M})$ satisfies the conditions.

Step: Let

$$P(\mathfrak{M}) = \sum_{\overline{R} : \langle \mathfrak{M}, \overline{R} \rangle \models \phi(\overline{R})} t$$

where t is a $(\tau \cup \{\overline{\mathbf{R}}\})$-polynomial. Let

$$P^A(\mathfrak{M}, S) = \sum_{\overline{R} : \langle \mathfrak{M}, \overline{R} \rangle \models \phi(\overline{R}) \wedge \mu(\overline{R}, S)} t^A$$

where $\mu(\overline{R}, S)$ says that either each relation in $\overline{R}$ is equal to $\emptyset$ or $S = A$ (or both). Then $P^A(\mathfrak{M}, S)$ satisfies the conditions. □

Proposition 4.9. *The pointwise sum of two standard,* FF *or Newton* **SOL**-*polynomials is again a standard,* FF *or Newton* **SOL**-*polynomial respectively.*

Proof. We prove it for standard **SOL**-polynomials. The proof for FF-polynomials and Newton polynomials is identical.

Let $P_1(\mathfrak{M})$ and $P_1(\mathfrak{M})$ be standard **SOL**-;polynomials. Let $P_1^{\emptyset}(\mathfrak{M}, S)$ and $P_2^M(\mathfrak{M}, S)$ be the **SOL**-polynomials given in Lemma 4.8. Then the sum of $P_1(\mathfrak{M})$ and $P_2(\mathfrak{M})$ is given by

$$\sum_{S \in \{\emptyset, M\}} P_1^{\emptyset}(\mathfrak{M}, S) \cdot P_2^M(\mathfrak{M}, S)$$

□

Proposition 4.10. *Let*

$$P_1(\mathfrak{M}) = \sum_{\overline{R}:\theta} \prod_{\overline{b}:\psi} \sum_{\overline{a}:\phi} P_2(\langle \mathfrak{M}, \overline{R}, \overline{a}, \overline{b} \rangle)$$

where $P_2(\mathfrak{A})$ *is a standard,* FF *or Newton* **SOL**-*monomial and the product and inner summation are on tuples of elements of the universe. It holds that* $P_1(\mathfrak{M})$ *is a standard,* FF *or Newton* **SOL**-*polynomial respectively.*

Proof. We can expand the product

$$\prod_{\overline{b}:\psi} \sum_{\overline{a}:\phi} P_2(\mathfrak{A}) = \sum_{f:\vartheta} \prod_{\overline{a},\overline{b}:\varphi} P_2(\mathfrak{A})$$

where ϑ says the relation f is a function

$$f : \{\overline{b} \mid \langle \mathfrak{A}, \overline{b} \rangle \models \psi\} \to \{\overline{a} \mid \langle \mathfrak{A}, \overline{a}, \overline{b} \rangle \models \phi\},$$

and $\varphi = \big(f(\overline{b}, \overline{a})\big) \wedge \psi \wedge \phi$. So the proposition holds. □

By induction the last proposition holds for functions defined by alternating $\prod$ and $\sum$, as long as all $\sum$ within the scope of a $\prod$ iterate over elements (and not over relations).

4.3. Combinatorial polynomials. As for the case of counting φ-multi-colorings, it is noteworthy that the following combinatorial invariants can be written as standard, FF and Newton **SOL**-polynomials.

Cardinality, I: The cardinality of a definable set

$$card_{\mathfrak{M},\overline{v}}(\varphi(\overline{v})) = \sum_{\overline{v}:\varphi(\overline{v})} 1$$

is an an **SOL**-polynomial.

Cardinality, II: Exponentiation of cardinalities

$$card_{\mathfrak{M},\overline{v}}(\varphi(\overline{v}))^{card_{\mathfrak{M},\overline{v}}(\psi(\overline{v}))} = \prod_{\overline{v}:\psi(\overline{v})} \sum_{\overline{u}:\varphi(\overline{u})} 1$$

is equivalent to a **SOL**-polynomial by proposition 4.10.

Factorials: The factorial of the cardinality of a definable set

$$card_{\mathfrak{M},\overline{v}}(\varphi(\overline{v}))! = \sum_{\pi:\varphi(\overline{v}) \xrightarrow{1-1} \varphi(\overline{v})} 1$$

is an **SOL**-polynomial.

5. Standard vs FF vs Newton SOL-polynomials

We have introduced three notions of **SOL**-polynomials: standard, FF, and Newton **SOL**-polynomials. The sets of monomials $X^i : i \in \mathbb{N}$ (powers of X) and $X_{(i)} : i \in \mathbb{N}$ (falling factorials of X) each form a basis of the polynomial ring $\mathbb{Z}[X]$ as a module over $\mathbb{Z}$. The sets of monomials $\binom{X}{i} : i \in \mathbb{N}$ (binomials of X) form a basis in the polynomial ring $\mathbb{Q}[X]$. Over $\mathbb{Q}$ each of these bases can be transformed into the other using linear transformations. In this section we discuss transformations of one basis into another using substitution by **SOL**-definable polynomials.

5.1. Standard vs FF polynomials. In the statement and proof of Proposition 3.10, the polynomial obtained is of the form

$$\sum_{j=1}^{|M|} d_{\varphi}(\mathfrak{M}, j) \cdot k_{(j)}$$

In the literature on graph polynomials mixed presentations also occur, e.g., the cover polynomial for directed graphs [**20**] is such a case.

We extend the definition of **SOL**-polynomials by allowing both monomials of the form

$$r^{card_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))}$$

and

$$r_{(card_{\mathfrak{M},\bar{v}}(\phi(\bar{v})))}$$

We call the polynomials obtained like this *extended* **SOL**-*polynomials.*

In the following we show that every extended **SOL**-polynomial on ordered structures can be written both as a standard **SOL**-polynomial and as a FF **SOL**-polynomial.

Proposition 5.1. *Let* $\Phi(\mathfrak{M}) = r^{card_{\mathfrak{M},v_1,\ldots,v_y}(\phi(\bar{v}))}$ *be a standard* **SOL**-*monomial with* $r \in \mathbf{X} \cup (\mathbb{Z} - \{0\})$. *There is a* FF **SOL**-*polynomial* Φ' *such that for all structures* $\mathfrak{M}$ *we have*

$$\Phi(\mathfrak{M}) = \Phi'(\mathfrak{M})$$

Proof. First assume r is a positive integer. The monomial $\Phi(\mathfrak{M})$ counts functions from the set

$$D_{\phi} = \{\bar{a} \mid \mathfrak{M} \models \phi(\bar{a})\}$$

to $[r]$. On the other hand, the monomial $(r)_{card_{\mathfrak{M},\bar{v}}(\phi(v))}$ counts *injective* functions from D_{ϕ} to $[r]$. Let $\Phi'(\mathfrak{M})$ be given by

$$\Phi'(\mathfrak{M}) = \sum_{A \subseteq V^y : \langle \mathfrak{M}, A \rangle \models \psi_1} \ \sum_{R \subseteq V^y : \langle \mathfrak{M}, A, R \rangle \models \psi_2} (r)_{card_{\mathfrak{M},\bar{v}}(\bar{v} \in A)}$$

where

(i) ψ_1 requires that A is a subset of D_{ϕ}, and
(ii) ψ_2 requires that R is an equivalence relation over D_{ϕ} such that for every two distinct tuples $\bar{a} \in A$ and $\bar{b} \in D_{\phi}$, if $\bar{a}$ and $\bar{b}$ belong to the same equivalence class in R, then $\bar{a} < \bar{b}$ with respect to the order on the structure $\mathfrak{M}$. Moreover, for every equivalence class in R there exists some $\bar{a} \in A$ which belongs to it.

Taking any injective function f from A to $[r]$, we may extend it to a function from D_{ϕ} to $[r]$ by assigning every $\bar{b} \in D_{\phi} - A$ with the same value as the $\bar{a} \in A$ for which $(\bar{a}, \bar{b}) \in R$. This extension is determined uniquely by f and R, and forms a bijection between the set of functions $g : D_{\phi} \to [r]$ and the set of triples (A, R, f) such that A and R satisfy ψ_1 and ψ_2 and $f : A \to [r]$ is injective. Hence, $\Phi'(\mathfrak{M}) = \Phi(\mathfrak{M})$. If $r \in \mathbf{X}$, we get that $\Phi'(\mathfrak{M})$ and $\Phi(\mathfrak{M})$ agree on every evaluation of r to a positive integer, and thus, by interpolation, $\Phi'(\mathfrak{M}) = \Phi(\mathfrak{M})$. Therefore, in particular $\Phi'(\mathfrak{M})$ and $\Phi(\mathfrak{M})$ also agree on all non-positive evaluations of r. $\square$

PROPOSITION 5.2. *Let* $\Phi(\mathfrak{M}) = r_{(card_{\mathfrak{M},v_1,\ldots,v_y}(\phi(\overline{v})))}$ *be a* FF **SOL**-*monomial for* $r \in \mathbf{X} \cup (\mathbb{Z} - \{0\})$. *There is a standard* **SOL**-*polynomial* Φ' *such that for all structures* $\mathfrak{M}$ *we have*

$$\Phi(\mathfrak{M}) = \Phi'(\mathfrak{M}).$$

PROOF. By definition,

$$(r)_{card_{\mathfrak{M},\overline{v}}(\phi(\overline{v}))} = \prod_{i=0}^{|D_\phi|-1} (r-i)$$

Therefore,

$$(r)_{card_{\mathfrak{M},\overline{v}}(\phi(\overline{v}))} = \prod_{\overline{a}:\phi(\overline{a})} \left(r - \sum_{\overline{b}:\phi(\overline{b})\wedge\overline{b}<\overline{a}} 1 \right)$$

where $\overline{b} < \overline{a}$ means $\overline{b}$ is smaller than $\overline{a}$ in the lexicographic order induced by the order on the elements of the structure $\mathfrak{M}$. By Proposition 4.10 we need only show that

$$r - \sum_{\overline{b}:\phi(\overline{b})\wedge\overline{b}<\overline{a}} 1 \tag{5.1}$$

is a standard **SOL**-polynomial with summation on elements only. The expression in (5.1) is given by

$$\sum_{\overline{b}:\phi(\overline{b})\wedge(\overline{b}\leq\overline{a})} r^{card_{\mathfrak{M},\overline{w}}(\overline{w}=\overline{a}\wedge\overline{b}=\overline{a})} \cdot (-1)^{card_{\mathfrak{M},\overline{w}}(\overline{w}=\overline{a}\wedge\overline{b}\neq\overline{a})}$$

Hence, $(r)_{card_{\mathfrak{M},\overline{v}}(\phi(\overline{v}))}$ is a standard **SOL**-polynomial. □

5.2. Newton polynomials. In Proposition 3.8 we used monomials of the form $\binom{X}{i}$. However, writing these as standard polynomials, they have rational but not integer coefficients; hence they are polynomials in $\mathbb{Q}[X]$. Furthermore, the coefficients of standard and FF **SOL**-polynomials are always integers by definition. Therefore, $\binom{X}{i}$ cannot be written as a standard or FF **SOL**-polynomial.

On the other hand, FF **SOL**-monomials can be written as Newton **SOL**-polynomials. To see this we note

$$X_{(|A|)} = |A|! \cdot \binom{X}{|A|} = \sum_{R\subseteq A^2} \binom{X}{|A|}$$

where R ranges over all permutations of A (as binary relations over A).

6. Equivalence of counting φ-colorings and SOL-polynomials

The following three theorems relate the counting functions of multi-colorings to **SOL** polynomials. Theorems 6.1 and 6.2 show that the class of the counting functions of **SOL**-definable $\varphi(\overline{F})$-multi-colorings with several partial functions $\overline{F}$ and the classes of standard and FF **SOL**-polynomials coincide.

THEOREM 6.1. *Let* $\mathcal{P}_\varphi(\overline{F})$ *be an* **SOL**-*definable multi-coloring property. The graph polynomial* $\chi_{\varphi(\overline{F})}(\mathfrak{M};\overline{k})$ *is both a standard and an* FF **SOL**-*polynomial.*

The proof of Theorem 6.1 is given in Subsection 6.1

Theorem 6.2 states that every standard or FF **SOL**-polynomial is an evaluation of some $\chi_\varphi(\mathfrak{M},\overline{k})$.

THEOREM 6.2. *Let $P(\mathfrak{M}; k_1, \dots, k_m)$ be either a standard or an* FF **SOL**-*polynomial. There exists an* **SOL**-*definable multi-coloring property $\mathcal{P}_\varphi$ with $m + l$ color-sets, $[k_1], \dots, [k_{m+l}]$, and $a_1, \dots, a_l \in \mathbb{Z}$ such that*

$$\chi_\varphi(\mathfrak{M}; k_1, \dots, k_m, a_1, \dots, a_l) = P(\mathfrak{M}; k_1, \dots, k_m)$$

where $\chi_\varphi(\mathfrak{M}; k_1, \dots, k_m, k_m, a_1, \dots, a_l)$ is obtained by evaluating the indeterminates $k_{m+1}, \dots, k_{m+l}$ to $a_1, \dots, a_l$ respectively in $\chi_\varphi(\mathfrak{M}; k_1, \dots, k_{m+l})$.

In fact, it will not be difficult to see that it is enough to have $l = 1$ with $a_{m+1} = -1$. The proof of Theorem 6.1 is given in Subsection 6.2

Theorem 6.3 shows that the class of counting functions of **SOL**-definable $\varphi(\overline{R})$-multi-colorings with bounded relations $\overline{R}$ and the class of Newton **SOL**-polynomials coincide.

THEOREM 6.3. *Let P be a function from the class of finite τ-structures to the ring $\mathbb{Q}[\overline{x}]$. The following statements are equivalent:*

(i) *P is a Newton* **SOL**-*polynomial.*
(ii) *P is an evaluation of the counting function $\chi_{\varphi(\overline{R})}(\mathfrak{M}; \overline{k})$ of an* **SOL**-*definable multi-coloring where the relations in $\overline{R}$ are bounded.*

The proof of a theorem similar to Theorem 6.3 is given in the conference version of this paper [**42**]. In Subsection 6.3 we motivate the need to extend partial functions to bounded relations in order to capture the Newton **SOL**-polynomials and sketch this direction of the proof. For the other direction of Theorem 6.3, one augments the proof of Theorem 6.1 given in Subsection 6.1 by using Proposition 3.8 instead of Proposition 3.10.

6.1. Proof of Theorem 6.1. We prove the theorem in the case of φ-colorings. The case of multiple indeterminates and several simultaneous functions is similar. Let $\mathcal{P}_\varphi$ be an **SOL**-definable coloring property. From Proposition 3.10 we know that for every $\mathfrak{M}$ the number of elements given by $\chi_{\varphi(R)}(\mathfrak{M}, k)$ is a polynomial in k of the form

$$\sum_{j=0}^{d \cdot |M|^m} d_\varphi(\mathfrak{M}, j) \cdot k_{(j)}$$

where $d_\varphi(\mathfrak{M}, j)$ is the number of φ-colorings F using all the colors in $[j]$. In other words, if $[k]$ was ordered, $d_\varphi(\mathfrak{M}, j)$ would count the number of φ-colorings F with a fixed set of j colors which are minimal lexicographically among φ-colorings F' obtained from F by permuting the color-set. The total number of colors used is bounded by $N = d\cdot \mid M \mid^m$. Hence we can interpret the set of colors used inside $\mathfrak{M}$ by the set $[M]^{d \cdot m}$. Since $\mathfrak{M}$ has a linear order $\leq_M$, a lexicographic order on $[M]^{d \cdot m}$ is definable in **SOL**.

We replace F by a relation where each occurrence of a color is substituted by a $(d \cdot m)$-tuple, and call this new relation S. We also modify the formula φ to a formula ψ by adding the requirement that all the colors used by S form an initial segment and that S is the smallest in the lexicographic order induced on the colors among its permutations. Let us denote by I_S the initial segment of this lexicographic ordering of the colors used by S. Clearly I_S is definable in **SOL** by a formula ρ.

We have that $\chi_{\varphi(R)}(\mathfrak{M}, k)$ is an FF **SOL**-polynomial given by

$$\sum_{S:\psi(S)} \sum_{I_S:\rho} x_{(card_{\mathfrak{M},\overline{v}}(\overline{v} \in I_S))} \tag{6.1}$$

By Proposition 5.2, $\chi_{\varphi(R)}(\mathfrak{M}, k)$ is a standard **SOL**-polynomial.

6.2. Proof of Theorem 6.2. We prove Theorem 6.2 first for standard **SOL**-monomials only in Lemma 6.4, then for **SOL**-polynomials.

LEMMA 6.4. *Every standard* **SOL**-*monomial* $\Psi(\mathfrak{M})$ *is an evaluation over* $\mathbb{Z}$ *of the counting function of* φ-*multi-colorings.*

PROOF. By Lemma 4.6, $\Psi(\mathfrak{M})$ is of the form

$$r_1^{card_{\mathfrak{M},\overline{v}}(\varphi_1(\overline{v}))} \cdots r_t^{card_{\mathfrak{M},\overline{v}}(\varphi_t(\overline{v}))}$$

$\varphi_1, \ldots, \varphi_t \in$ **SOL** and $r_1, \ldots, r_t \in (\mathbb{Z} - \{0\}) \cup \mathbf{X}$. Without loss of generality, assume $r_1 = x_1, \ldots, r_{t'} = x_{t'} \in \mathbf{X}$ and $r_{t'+1} = c_{t'+1}, \ldots, r_t = c_t \in \mathbb{Z} - \{0\}$. Then

$$x^{card_{\mathfrak{M},\overline{v}}(\varphi)(\overline{v})} = \chi_{\psi(F)}$$

where $\psi(F)$ counts functions F such that if $\overline{a}$ does not satisfy φ then $F(\overline{a}) = f_1$, where f_1 is the minimal element in the linear ordering of $\mathfrak{M}$. Similarly, it holds that $c^{card_{\mathfrak{M},\overline{v}}(\varphi(\overline{v}))} = \chi_{\psi(F,c)}$, i.e. $c^{card_{\mathfrak{M},\overline{v}}(\varphi(\overline{v}))}$ is obtained by evaluating $\overline{x}$ to $\overline{c}$ in $\chi_{\psi(F)}$. By Proposition 3.17, the set of counting functions for φ-multi-colorings is closed under finite product. □

Let $P(\mathfrak{M}, \overline{k})$ be a standard **SOL**-polynomial. As described in Subsection 3.3, we may assume P is given as follows:

$$P(\mathfrak{M}, \overline{k}) = \sum_{\overline{F}:\langle\mathfrak{M},\overline{F}\rangle \models \phi(\overline{F})} t(\langle\mathfrak{M}, \overline{F}\rangle)$$

where $\overline{F}$ is a tuple of functions and $t(\langle\mathfrak{M}, \overline{F}\rangle)$ is an **SOL**-polynomial. By induction on the structure of $t(\langle\mathfrak{M}, \overline{F}\rangle)$ is the evaluation of some counting function of φ-multi-colorings

$$t(\langle\mathfrak{M}, \overline{F}\rangle) = \chi_{\theta(\overline{F'})}(\langle\mathfrak{M}, \overline{F}\rangle, \overline{k}, \overline{a})$$

Then $P(\mathfrak{M}, \overline{k}) = \chi_{\theta(\overline{F},\overline{F'}) \wedge \phi(\overline{F})}(\mathfrak{M}, \overline{k}, \overline{a})$. Using Proposition 5.2, the case of FF **SOL**-polynomials follows.

6.3. From Newton SOL-polynomials to counting bounded relations. Now we prove, by example, direction (i) $\rightarrow$ (ii) of Theorem 6.3. Note that in this example the coloring relations cannot be replaced by partial functions.

Let

$$N(\mathfrak{M}) = \sum_{A \subseteq M} \binom{x}{card_{\langle\mathfrak{M},A\rangle,v}(v \in A)} = \sum_{A \subseteq M} \binom{x}{|A|} \tag{6.2}$$

We will show how to transform $N(\mathfrak{M})$ into a counting function of multi-colorings. The coloring property needs to consist of structures $\mathfrak{M}_{\overline{R},x}$ with the $\overline{R}$ bounded relations which are not necessarily partial functions.

The term $\binom{x}{|A|}$ counts the number of ways to choose a set of colors of size $|A|$ from $[x]$. Therefore, $N(\mathfrak{M})$ counts relations $R \subseteq A \times [x]$ such that there exists an $I \subseteq [x]$ for which $R = A \times I$ and $|I| = |A|$. It is not difficult to see that this can be expressed in **SOL** by a formula $\varphi_{choose}(R)$. Moreover, every such R is 1-bounded, so the multi-coloring property $P_{\varphi_{choose}}$ is bounded. However, R is not a (partial) function.

Let $N_\theta(\mathfrak{M})$ be the Newton polynomial obtained by adding a definability condition on A under the summation of Equation (6.2),

$$N_\theta(\mathfrak{M}) \sum_{A \subseteq M : \theta} \binom{x}{|A|}$$

Then

$$N_\theta(\mathfrak{M}) = \chi_{\varphi_{choose}(R) \wedge \theta(domain(R))}(\mathfrak{M}, x)$$

The extension to any Newton **SOL**-polynomial is not difficult.

7. MSOL-polynomials

An **SOL**-polynomial $P(\mathfrak{M})$ is an **MSOL**-polynomial if the summations are over unary relations and all the formulas involved are **MSOL**-formulas.

A simple example is the *independence polynomial* $Ind(G, X)$ with

$$Ind(G, X) = \sum_{A \subseteq V} X^{|A|}$$

where $G = (V, E)$ and A ranges over all independent sets of G. The condition of being an independent set can be expressed in **MSOL** where A is a free set variable.

If we look at the graph G as a two-sorted structure $G = (V, E; R)$, where we have a sort for vertices V and a sort for edges E, and an incidence relation R, then the matching polynomial $g(G, X)$ is also an **MSOL**-polynomial. In general, over ordered graphs, many classical graph polynomials, such as the dichromatic polynomial, the Tutte polynomial and the interlace polynomials, can be written as **MSOL**-polynomials. For the case of the various interlace polynomials this needs a proof, cf. [**21**]. In general, computing the coefficients of **SOL**-polynomials, and even **MSOL**-polynomials, can be hard, in fact $\sharp$**P**-hard. However, **MSOL**-polynomials are easy to compute for graph classes of bounded tree-width, cf. [**48, 47, 22**].

Definition 7.1. *A φ-multi-coloring is an* **MSOL**-*multi-coloring if the coloring relations $\overline{F} = (F_1, \ldots, F_k)$ in the formula φ are all unary functions and $\varphi \in \mathbf{MSOL}(\tau_{\overline{\mathbf{F}}})$.*

Inspecting the proof of 6.2 one can verify the following:

Proposition 7.2. *Every* **MSOL**-*polynomial $P(\mathfrak{M})$ is an evaluation of some* **MSOL**-*multi-coloring $\chi_\varphi(\mathfrak{M}, \overline{k})$.*

The converse is, unfortunately, not true. As an example, we look at the harmonious colorings $F : V \to [k]$, which are proper vertex colorings such that each pair of colors occurs at most once along some edge. This can be written as an **MSOL**-formula φ_{harm}. In [**31**, Theorem 10] it is shown that the counting function of harmonious colorings is not an **MSOL**-polynomial. Combining the results of [**28**] and [**31**] one can show that computing its coefficients is **NP**-hard even for trees.

Using the methods developed in [**31**] one can also show other graph polynomials are not **MSOL**-polynomials, e.g., the counting function of rainbow colorings.

8. Enter categoricity

In this section we present an even more general approach to graph polynomials, using advanced first order model theory, in particular the theory of categorical structures. We would like to remind the reader that in this section we require some background in model theory, which goes beyond what was needed in the previous sections. A good background reference is [**39**]. A bit more elementary and still providing necessary background on categoricity is the monograph [**54**].

We first describe a uniform method of attaching to each member G of a family of finite structures $\mathcal{G}$ an infinite structure $M(G)$. The reader can think of $\mathcal{G}$ as the class of finite graphs, but our construction works for arbitrary finite τ-structures.

In the simplest case, the structure $M(G) = M(G, D)$ depends on an infinite set $D = \mathbb{N}$. The structure $M(G, \mathbb{N})$ encodes the family of structures $\langle G, [j], F\rangle = G_{F,j}$ introduced in Section 3, but contains not only the coloring function F, but an infinite set of possible colorings, all first order definable in $M(G)$, and using the infinite set D as colors. The coloring functions (or relations) appear here as elements, and **SOL**-definability reduces to **FOL**-definability.

This approach is extended to definable sets in $M(G, \mathbb{N})$. Correspondingly, if instead of $\mathbb{N}$ we use k-many copies of $\mathbb{N}$ we get generalized multi-colorings. The novelty here is that we allow D to carry more structure, giving rise to a richer class of generalized colorings.

8.1. Background on categoricity. We quote from [**39, 54**]. We assume that all vocabularies are countable or finite. A *theory* $T \subseteq \mathbf{FOL}(\tau)$ is a consistent (satisfiable) set of first order sentences over the vocabulary τ. For a τ-structure $\mathfrak{M}$ we denote by $Th(\mathfrak{M})$ the set of $\mathbf{FOL}(\tau)$-sentences true in $\mathfrak{M}$.

Definition 8.1 (Background). *Let $T \subseteq \mathbf{FOL}(\tau)$ be a theory.*

- (i) *T is* complete*, if it is maximal consistent.*
- (ii) *T has the* finite model property*, if each finite subset of T has a finite model.*
- (iii) *Let κ be a cardinal (initial ordinal). T is κ-categorical if T has an infinite model and any two models of cardinality κ are isomorphic.*
- (iv) *An element $a \in M$ is* algebraic over $C \subseteq M$ *if there is τ-formula $\phi(x, \bar{c})$ with one free variable x and parameters $\bar{c}$ from C, such that the set*
$$\{b \in M : \mathfrak{M} \models \phi(b, \bar{c})\}$$
is finite and $\mathfrak{M} \models \phi(a, \bar{c})$.
- (v) *In a structure $\mathfrak{M}$ we define the* algebraic closure of a set $C \subseteq M$, *denoted by $acl(C)$, as the set of elements in M which are algebraic over C.*

Facts 1.

- (i) *If T is κ-categorical for some infinite κ, and has no finite models, then T is complete (Vaught's Test).*
- (ii) *If T is κ-categorical for some uncountable κ, then T is κ'-categorical for all uncountable κ' (Morley's Theorem).*
- (iii) *Hence there are two cases which can occur independently in all combinations: T is (or is not) ω-categorical, or T is (or is not) ω_1-categorical. A complete theory which is categorical in all infinite powers is called* totally categorical.

For the more complex notions related to the structure theory of totally categorical theories, such as *C-definable sets, minimal and strongly minimal sets, rank, dimension, ω-stability, etc.*, we refer the reader to the standard texts, e.g. [**9, 39, 54, 17**]. These notions are not used in our technical proofs, but they are mentioned in theorems needed in the proofs. Given a structure $\mathfrak{M}$ the rank of a subset $S \subseteq M$ is denoted by $\mathrm{rk}(S)$.

8.2. The Functor. Let $\mathcal{G} = \mathcal{G}(\tau_0)$ be a class of finite structures for a finite vocabulary τ_0. Let $D_1, \ldots, D_k$ be countable infinite structures for finite vocabularies $\tau_1, \ldots, \tau_k$, respectively.

For every $G \in \mathcal{G}$ we construct the structure $M(G, F, D_1, \ldots, D_k)$ with sorts $G, F, D_1, \ldots, D_k$, and with the vocabulary $\tau = \tau_0 \cup \tau_1 \cdots \cup \tau_k$ and an extra function symbol

$$\Phi : G \times F \to D_1 \times \ldots \times D_k$$

The sort F encodes all the functions from G to $D_1 \times \ldots \times D_k$. We think of these functions as colorings of elements (vertices) of G with a tuple of k colors from the color sets $D_1, \ldots, D_k$. If we wanted to color edges, given as pairs of vertices, or more general, tuples of elements of G, one has to modify our construction correspondingly.

To ensure that the elements of F encode all functions, we require that Φ satisfies the following conditions:

(i) $\exists f \in F$
(ii) $\forall f, f' \in F([\forall g \in G \; \Phi(g, f) = \Phi(g, f')] \to f = f')$
(iii) $\forall f \in F \forall g \in G \forall \overline{d} \in D_1 \times \ldots \times D_k \exists f' \in F$
$\big((\forall g'(g \neq g' \to \Phi(g', f) = \Phi(g', f')) \wedge \Phi(g, f') = \overline{d})\big)$

(i) says that F is not empty, (ii) says that the elements of F are functions, and (iii) says that every one point modification of a function in F is again a function in F. Because G is finite, this ensures that all functions from G to $D_1 \times \ldots \times D_k$ are in F. In other words we have the canonical identification

$$\Phi^\star : F \leftrightarrow (D_1 \times \cdots \times D_k)^G$$

and fixing an enumeration of G we may identify the right-hand-side with the Cartesian power

$$(D_1 \times \cdots \times D_k)^{|G|}$$

We write $f(g)$ instead of $\Phi(g, f)$ and so identify elements $f \in F$ with functions $G \to D_1 \times \cdots \times D_k$.

Remark 8.2. *By the virtue of the construction, given $D_1, \ldots, D_k$, the isomorphism type of $M(G, F, D_1, \ldots, D_k)$ depends only on G. Obviously, G can be recovered from $M(G, F, D_1, \ldots, D_k)$. So, $M(G, F, D_1, \ldots, D_k)$ can be seen as the complete invariant of G. In particular, every definable subset S of F is an invariant of G.*

Proposition 8.3. *$M(G, F, D_1, \ldots, D_k)$ is definable using parameters in the disjoint union $D_1 \sqcup \cdots \sqcup D_k$.*

Proof. Obviously $M(G, F, D_1, \ldots, D_k)$ is definable in the disjoint union of G, F and $D_1, \ldots, D_k$. But as G is finite, one can interpret this sort using $|G|$ constants. □

Corollary 8.4.

(i) *Assume that the theory of each* D_i *is* ω*-categorical. Then the theory* $\mathrm{Th}[M(G, D_1, \ldots, D_k)]$ *is* ω*-categorical.*

(ii) *Assume that the theory of each* D_i *is strongly minimal. Then the theory* $\mathrm{Th}[M(G, D_1, \ldots, D_k)]$ *is* ω*-stable with* k *independent dimensions. If* $k = 1$ *then the theory is categorical in uncountable cardinals.*

Theorem 8.5 (B. Zilber). *Any theory satisfying the conclusions of (i) and (ii) has the finite model property. Moreover any countable model* M *can be represented as a union of an increasing chain of finite substructures* M_i (logically) approximating M*, i.e.,*

$$M = \bigcup_{i=1}^{\infty} M_i$$

Proof. This follows from Theorem 7 of [**19**], where also more details may be found. □

Remark 8.6. *The finite model property takes a very simple form for a strongly minimal structure* D*. Namely,* D *has the finite model property if and only if* $\mathrm{acl}\,(X)$ *is finite for any finite* $X \subseteq D$*.*

8.3. Counting functions for definable sets. A very important consequence of the finite model property is the possibility to introduce a stronger *counting function* on definable sets.

We prove here the existence of the counting polynomials in a special case, for the theory $\mathrm{Th}[M(G, D_1, \ldots, D_k)]$. The more general case of ω-categorical ω-stable theories can be found in [**19**, Proposition 5.2.2.]. The more special case of theories categorical in all infinite cardinals has been proved in [**59, 60**] and can be found in [**61**]. The proof under the special assumptions needed in this paper is really elementary and does not require any model-theoretic terminology if one assumes the D_i's to be just sets. It really is a slight generalization of the proof given for Proposition 3.8.

Theorem 8.7. *Let* $M = M(G, D_1, \ldots, D_k)$*. Assume the finite model property holds in the strongly minimal structures* $D_1, \ldots D_k$*. Then for every finite* $C \subseteq M$ *and any* C*-definable set* $S \subseteq M^\ell$ *there is a polynomial* $p_S \in \mathbb{Q}[x]$ *and there is a number* n_S *such that for every finite* $X \subseteq M$ *with* $C \subseteq X$*,*

(i) *if* $|D_i \cap \mathrm{acl}\,(X)| = x_i \geq n_S$*, we have* $|S \cap \mathrm{acl}\, X| = p_S(x_1, \ldots, x_k)$*;*

(ii) $\mathrm{rk}(S) = \deg(p_S)$*, the degree of the polynomial;*

(iii) *if* $g(S){=}T$ *for some automorphism* g *of* M *then* $p_S = p_T$ *and* $n_S = n_T$*.*

Furthermore, if $C = \emptyset$ *we can take* $n_S = 0$*.*

Proof. We construct the polynomial for a given S by induction on $\mathrm{rk}(S)$.

W.l.o.g. we may assume that S is an atom over C, that is defined by a principal type over C.

Let $f = \langle f_1, \ldots, f_\ell \rangle \in S$. Recall that each f_i is determined by the values of $f_i(g) \in D_1 \times \cdots \times D_k$, for $g \in G$. Denote $f_{im}(g)$ the mth co-ordinate of $f_i(g)$, an element of D_m.

Suppose $f_{im}(g) \in \mathrm{acl}\,(C)$ for all $i \leq \ell$, $m \leq k$ and $g \in G$. Then $f \in \mathrm{acl}\,(C)$. Since S is an atom, $S \subseteq \mathrm{acl}\,(C)$ and hence

$$|S \cap \mathrm{acl}\,(X)| = |S \cap \mathrm{acl}\,(C)|$$

is a constant, independent of X. So, we are done in this case.

We may now assume that $f_{11}(g_0) \notin \mathrm{acl}\,(C)$. So, we have the partition

$$S = \bigcup_{a \in D_1 \smallsetminus \mathrm{acl}\,(C)} S_a, \quad S_a = \{f \in S : f_{11}(g_0) = a\}$$

Since $D_1 \smallsetminus \mathrm{acl}\,(C)$ is an atom over C (use the strong minimality of D_1) and, of course $G \subseteq \mathrm{acl}\,(\emptyset)$, the subgroup of the automorphism group of M fixing C acts transitively on $D_1 \smallsetminus \mathrm{acl}\,(C)$. Hence all the fibers S_a are conjugated by automorphisms over C and have the same Morley rank. The latter implies by the addition formula for ranks that

$$\mathrm{rk}(S_a) = \mathrm{rk}(S) - 1$$

So, we may apply the induction hypothesis. By (iii) we get that

$$p_{S_a} = p_0, \text{ for all } a \in D_1 \smallsetminus \mathrm{acl}\,(C)$$

for some polynomial p_0. By (ii) $\deg(p_0) = \mathrm{rk}(S_a) = \mathrm{rk}(S) - 1$.

Let $c_0 = |\mathrm{acl}\,(C)|$. So,

$$|(D_1 \smallsetminus \mathrm{acl}\,(C)) \cap \mathrm{acl}\,(X)| = (x_1 - c_0).$$

We further calculate

$$S \cap \mathrm{acl}\,(X) = \bigcup_{a \in (D_1 \smallsetminus \mathrm{acl}\,(C)) \cap \mathrm{acl}\,(X)} S_a \cap \mathrm{acl}\,(X) = (x_1 - c_0) \cdot p_0(x_1, \ldots, x_k)$$

□

8.4. Generalized chromatic polynomials revisited. In the light of Theorem 8.7 let us look first at the generalized colorings of Section 3.

We discuss them for the class $\mathcal{G}(\tau)$ of finite (purely relational) τ-structures. We denote by $\mathbf{SOL}^n(\tau)$ the set of $\mathbf{SOL}(\tau)$-formulas where all second order variables have arity at most n. Let $\phi(\overline{\mathbf{R}}, \mathbf{F}) \in \mathbf{SOL}^n(\tau)$ define a notion of generalized coloring where $\overline{\mathbf{R}}$ is a list of relation parameters, and $\mathbf{F}$ denotes the coloring function. So the generalized chromatic polynomial on a τ-structure $\mathfrak{A}$ is defined as

$$\chi_{\phi(\overline{\mathbf{R}}, \mathbf{F})}(\mathfrak{A}, k) = |\ \{(\overline{R}, F) : \langle \mathfrak{A}, \overline{R}, F, [k] \rangle \models \phi(\overline{R}, F)\}\ |$$

We first expand $\mathfrak{A}$ so that quantification over relations becomes quantification over elements. So for each $\ell \leq n$ we add the set $\wp(A^\ell)$ with the corresponding membership relation $\in_\ell$. We define the $\tau^\star$-structure

$$\mathfrak{A}^\star = \langle \mathfrak{A}, \wp(A^\ell) \in_\ell, \ell \leq n \rangle$$

and apply our functor $M(\mathfrak{A}^\star, \mathbb{N})$ to it with $D_1 = \mathbb{N}$. Let $\tau^\sharp$ be the vocabulary of $M(\mathfrak{A}^\star, \mathbb{N})$.

Now the formula $\phi(\overline{\mathbf{R}}, \mathbf{F}) \in \mathbf{SOL}^n(\tau)$ has a straightforward translation

$$\phi^\sharp(\bar{c}_{\overline{R}}, d_F) \in \mathbf{FOL}(\tau^\sharp)$$

where the function symbol F becomes a variable d_F, the relation symbols $\overline{\mathbf{R}}$ become variables $\bar{c}_{\overline{\mathbf{R}}}$ of the appropriate sorts. Furthermore, it has no additional parameters. It follows that $C = \emptyset$.

Let $X \subseteq \mathbb{N}$ be finite. Due to Theorem 8.7(iii), w.l.o.g., $C = [k]$ for some $k \in \mathbb{N}$. Let $\mathfrak{A}^\star_k$ be the substructure of $M(\mathfrak{A}^\star, \mathbb{N})$ with universe $acl([k])$ and $F_k \subseteq F$ be its part of the sort F. We now easily verify that:

(i) $\mathfrak{A}^\star_{\mathfrak{k}}$ contains all of $\mathfrak{A}^\star$.

(ii) F_k consists exactly of all functions f with range $Rg(f) \subseteq [k]$.
(iii) For $S = \{(\bar{c}, d) \in M(\mathfrak{A}^\star, \mathbb{N}) : M(\mathfrak{A}^\star, \mathbb{N}) \models \phi^\sharp(\bar{c}, d)\}$
we have that $| S \cap \mathrm{acl}([k]) | = p_S(k)$ is a polynomial for every $k \geq n_S = 0$.
(iv) $\chi_{\phi(\overline{\mathbf{R}}, \mathbf{F})}(\mathfrak{A}, k) = | \{(\bar{c}, d) \in \mathfrak{A}^\star_k : M(\mathfrak{A}^\star, \mathbb{N}) \models \phi^\sharp(\bar{c}, d)\} | = p_S(k)$.

This proves Theorem B for the case of generalized chromatic polynomials in one variable.

8.5. Proof of Theorem B. To prove Theorem B in its full generality proceed as before. We observe the following points:

- For multi-colorings we use several copies of $\mathbb{N}$ as strongly minimal sets.
- If the generalized colorings are relations $r \subseteq G^\alpha \times \mathbb{N}^\beta$ the proof still works, provided $M(G, \mathbb{N}, \ldots, \mathbb{N})$ is ω-stable. This is where we use, in our definition of generalized multi-coloring that, for each $\bar{x} \in G^\alpha$, the set $r_{\bar{a}} = \{\bar{b} \in \mathbb{N}^\beta : r(\bar{a}, \bar{b})\}$ is bounded by a fixed finite number d. Without this restriction ω-categoricity is violated.

8.6. The full generality. The general theorem allows for more complicated strongly minimal structures to be used for D_1. A simple example would consist of a countable set of disjoint copies of a fixed finite structure such as a finite field $GF(p^q)$. The colors then would be pairs (n, a) where $n \in \mathbb{N}$ and $a \in GF(p^q)$. We could request that a graph coloring f of a graph $G = (V, E)$ satisfies, say,

$$[((u, v) \in E \wedge f(u) = (n_u, a_u) \wedge f(v) = (n_v, a_v)) \rightarrow$$
$$(n_u \neq n_v \wedge a_u + a_v = 0)]$$

It seems possible that such colorings may be useful in modeling wiring conditions when labeled graphs model network devices.

9. Conclusions

Starting with the classical chromatic polynomial we have introduced multi-colorings of graphs. We have shown that the corresponding counting functions are always polynomials. We have then shown that the class of counting functions of multi-colorings is very rich and covers virtually all examples of graph polynomials which have been studied in the literature. Additionally, it gives rise to counting graph invariants which previously were not recognized to be graph polynomials.

Motivated by the class of **SOL**-definable graph polynomials introduced in [**47**], we introduced variations of **SOL**-definable polynomials using different bases of the polynomial ring: the standard basis, the falling factorial bases and the Newton polynomials. We have then shown that the class of **SOL**-graph polynomials coincides with the class of **SOL**-definable generalized chromatic polynomials. This, along with the extensive scope of the class, suggests that the frameworks presented in this paper are natural for the study of graph polynomials.

Finally, we have constructed functors which map graphs (or other finite relational structures) into $\aleph_0$-categorical ω-stable structures of rank k which in a precise sense encode all **SOL**-graph polynomials in k indeterminates.

Theorems B and C can also be used to analyze the complexity of evaluations of **SOL**-definable polynomials at integer points. They fit nicely into the framework developed by S. Toda in his unpublished thesis and in [**58**].

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