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Compton's Method for Proving Logical Limit Laws

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Abstract. Developments in the study of logical limit laws for both labelled and unlabelled structures, based on the methods of Compton (1987/1989), are surveyed, and a sandwich theorem is proved for multiplicative systems.

1. Introduction

All structures in this paper will be finite structures for a finite language, unless stated otherwise. Likewise, by the models of a sentence φ we mean the finite models.

In two papers ([**23**] 1987, [**26**] 1989) Compton introduced a new method to prove logical limit laws for classes $\mathcal{A}$ of relational structures, a method that, in retrospect, depended solely on the function $a(n)$ that counted the number of structures of size n for $n \geq 1$. He treated two different count functions $a(n)$ in parallel, namely:

- the *unlabelled* count function that counts the number of isomorphism types of size n, and
- the *labelled* count function that counts the number of labelled structures of size n, that is, the number of structures in $\mathcal{A}$ on the universe $\{1, \ldots, n\}$.

The study of logical limit laws started with the results of Glebskiĭ, Kogan, Liogon'kiĭ, and Talanov [**33**] 1969, and Liogon'kiĭ [**39**] 1970, where they proved: *the class $\mathcal{A}$ of all relational structures, for a finite language $\mathcal{L}$, has both a labelled and an unlabelled first-order 0–1 law.* Later Fagin [**31**], 1976, independently found a simpler proof in the interesting cases where at least one relation in $\mathcal{L}$ is not unary—here is an outline of the steps of his proof:

(a) Let Φ be a collection of first-order sentences consisting of: (i) axioms for the class $\mathcal{A}$ being considered, and (ii) for any finite structure $\mathfrak{S}_1$ and a one-element extension $\mathfrak{S}_2$, both in $\mathcal{A}$, there is a sentence in Φ that asserts: a substructure isomorphic to $\mathfrak{S}_1$ can be likewise extended to a substructure isomorphic to $\mathfrak{S}_2$. The sentences described in (ii) are called *extension axioms.*

(b) Prove that Φ axiomatizes a complete theory.

(c) For $\varphi \in \Phi$, prove that φ has labelled asymptotic probability $= 1$.

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(d) Conclude that every φ in the theory axiomatized by Φ has labelled asymptotic probability $= 1$.

(e) Conclude that $\mathcal{A}$ has a labelled first-order 0–1 law.

(f) Show that the asymptotic probability of a structure in $\mathcal{A}$ being automorphism-rigid (the only automorphism is the identity map) is $= 1$. [For graphs this was proved by Pólya.]

(g) Conclude that $\mathcal{A}$ has an unlabelled first-order 0–1 law. Furthermore, given a first-order sentence φ, the labelled and unlabelled asymptotic probabilities that φ holds are the same.

This method applied to a few other situations, for example, the class $\mathcal{A}$ of graphs. The first-order sentences about graphs that *almost always hold*, that is, have asymptotic probability $= 1$, are precisely the sentences true of the famous Erdős random graph, where the edges occur with probability $1/2$. Aside from these and similar examples, all the ingredients needed for Fagin's method of proving 0–1 laws are seldom available for classes $\mathcal{A}$.

Compton's 1987 paper, on first-order 0–1 laws for admissible classes $\mathcal{A}$, used the above idea of finding a set Φ of axioms for a complete theory that extended the first-order theory of $\mathcal{A}$, and then showing that the sentences in Φ had asymptotic probability $= 1$, in both the labelled and unlabelled settings. The axioms he added to the theory of $\mathcal{A}$ simply say: for any given connected structure $\mathfrak{S}$ in $\mathcal{A}$ and any $n \geq 0$, it is not the case that a structure has exactly n components isomorphic to $\mathfrak{S}$.

With this choice of Φ for an admissible class $\mathcal{A}$, Compton found a simple property that sufficed to obtain an unlabelled first-order 0–1 law, namely $a(n-1)/a(n) \to 1$ as $n \to \infty$. In the labelled setting, a more complicated condition, expressed as $\mathcal{A} \to \infty$, was used. When such a condition held, he could first show that another notion of probability, similar to one Dirichlet used to analyze the density of prime numbers in an arithmetical progression, gave a first-order 0–1 law. Then a Tauberian theorem was applied, to show that the same 0–1 law held for the asymptotic probability. The automorphism-rigid aspect (f) of Fagin's argument, which allowed one to deduce the unlabelled 0–1 law from the labelled 0–1 law, was not available for such admissible classes—indeed, the probability of being automorphism-rigid was 0.

Compton's 1989 paper lifted the first-order 0–1 laws of the 1987 paper to monadic second-order 0–1 laws (see Theorem 1.8 below). However, parts of the method of proof used in the 1987 paper for first-order 0–1 laws did not carry over to the MSO setting—the axiomatization Φ of the sentences that were almost always true was replaced by a structure theorem for the class $\mathcal{A}_\varphi$ of members of $\mathcal{A}$ that satisfied φ, where φ was any given MSO sentence. As will be described below, the structure theorem said that $\mathcal{A}_\varphi$ was a finite union of pairwise disjoint partition classes—this was proved using Ehrenfeucht-Fraïssé games. Then followed a proof of the existence of (an extended version of the previous) Dirichlet density for $\mathcal{A}_\varphi$, and then the application of a Tauberian theorem to obtain the asymptotic probability $\mathsf{Prob}(\varphi)$. Parallel and similar arguments gave a MSO 0–1 law in both the labelled and the unlabelled cases.

Compton also gave sufficient conditions for simply having a MSO limit law (see Theorem 1.9 below), where the asymptotic probability of each MSO sentence φ exists, but is not necessarily 0 or 1. In the labelled case he provided a single application of this theorem, namely to the class of permutations. In the unlabelled case, no examples were given where these conditions applied.

1.1. Counting functions, generating series. Let $\mathcal{A}$ be a class of relational structures. For labelled structures one uses the *exponential generating series*

$$\mathbf{A}(x) \ := \ \sum_{n\geq 1} \frac{a(n)}{n!} x^n.$$

If $a(n)$ counts unlabelled structures of size n then one uses the *ordinary generating series* for $\mathcal{A}$ given by

$$\mathbf{A}(x) \ := \ \sum_{n\geq 1} a(n) x^n.$$

In either case ρ_A denotes the *radius of convergence* of the series.

1.2. Compton's admissible classes. The class of all structures for a given relational language $\mathcal{L}$ is closed under the operation of disjoint union, as are many others—for the language of one binary relation, the class of graphs and the class of forests give two examples. As is well known, every graph is uniquely a disjoint union of connected graphs (its components), and connected graphs are indecomposable under disjoint union. Thus graphs have a unique decomposition into indecomposables.

Compton gave a natural generalization of the definition of 'connected', as used in graph theory, to relational structures, and noted that every relational structure is uniquely a disjoint union of connected structures, called the *components* of the structure.

The class of all $\mathcal{L}$-structures, the class of graphs and the class of forests are examples of classes closed under both disjoint union and extraction of components. Compton was perhaps the first to realize that classes of relational structures with these two closure properties enjoy a privileged status in combinatorics, and he introduced a name for them.

DEFINITION 1.1. Given a finite relational language $\mathcal{L}$, a class $\mathcal{A}$ of $\mathcal{L}$-structures (that is closed under isomorphism) is *admissible* if

(a) it is closed under disjoint union, and
(b) it is closed under extraction of components.

Note that the classes for which the first important logical limit laws were proved—namely for each $\mathcal{L}$ the class all $\mathcal{L}$-structures, as well as the class of graphs—are admissible classes $\mathcal{A}$ with $\rho_A = 0$.

Given $\mathcal{A}$, let $\mathcal{P}$ denote the subclass of connected members, with counting function $p(n)$ and generating series $\mathbf{P}(x)$. The bonus one has when working with an admissible class is called the *fundamental identity* in [**15**]. In the labelled case it is

$$1 + \mathbf{A}(x) \ = \ \exp\big(\mathbf{P}(x)\big), \tag{1.1}$$

and in the unlabelled case

$$1 + \mathbf{A}(x) \ = \ \prod_{n\geq 1} (1 - x^n)^{-p(n)} \ = \ \exp\Big(\sum_{m\geq 1} \mathbf{P}(x^m)/m\Big). \tag{1.2}$$

Clearly either of $\mathbf{P}(x)$ and $\mathbf{A}(x)$ determines the other. The fundamental identities are key tools when applying analytic methods (for example, the Cauchy Integral Theorem) to study the count functions $a(n)$ and $p(n)$.

Compton developed the theory of logical limit laws for the labelled and unlabelled cases in parallel. For φ an $\mathcal{L}$-sentence, $a_\varphi(n)$ denotes the (labelled or unlabelled) count function for $\mathcal{A}_\varphi$, the class of members of $\mathcal{A}$ which satisfy φ.

Let $\mathbb{L}$ be a logic for a finite relational language $\mathcal{L}$—we only consider first-order (FO) and monadic second-order (MSO) logic. A class $\mathcal{A}$ of $\mathcal{L}$-structures has an $\mathbb{L}$ *limit law* if, for every $\mathbb{L}$ sentence φ, the probability $\mathsf{Prob}(\varphi)$ that 'a randomly selected structure from $\mathcal{A}$ satisfies φ' is defined. Compton used *asymptotic probability* and denoted the probability of φ by $\mu(\varphi)$ in the labelled case, by $\nu(\varphi)$ in the unlabelled case. For the labelled case this means:

$$\mathsf{Prob}(\varphi) \ := \ \mu(\varphi) \ := \ \lim_{n\to\infty} \frac{a_\varphi(n)}{a(n)};$$

and similarly one has the unlabelled case by changing μ to ν, and using the unlabelled count functions. This definition assumes that $\mathsf{Spec}(\mathcal{A})$, the spectrum[1] of $\mathcal{A}$, is cofinite, that is, $a(n)$ is eventually positive. For an arbitrary class $\mathcal{A}$ of $\mathcal{L}$-structures, Compton ([**23**] p. 87, 1987) introduced the *generalized* asymptotic probability

$$\mu^*(\varphi) \ := \ \lim_{\substack{n\to\infty \\ a(n)\neq 0}} \frac{a_\varphi(n)}{a(n)}$$

in the labelled case; change the μ to ν for the unlabelled case.

Admissible classes $\mathcal{A}$ have a periodic spectrum, so the theory developed for the cases when $\mathcal{A}$ has a cofinite spectrum easily lifts to arbitrary admissible classes $\mathcal{A}$. In particular, given an admissible class $\mathcal{A}$, let $d = \gcd\{n : a(n) > 0\}$. Then the spectrum of $\mathcal{A}$ eventually agrees with $d \cdot \mathbb{N}$, the multiples of d. Thus, for a sentence φ, one has the labelled asymptotic probability of φ expressed by

$$\mathsf{Prob}(\varphi) \ := \ \mu^*(\varphi) \ = \ \lim_{n\to\infty} \frac{a_\varphi(nd)}{a(nd)},$$

provided the limit exists; and similarly for the unlabelled asymptotic probability. *The standing assumption is that $d = 1$*, unless explicitly stated otherwise. (For details on why it suffices to consider only the case $d = 1$, see [**15**], Chapter 3.)

In the 1987 paper [**23**], Compton used another notion of probability, but did not identify it as such until his 1989 paper ([**26**], p. 117) on MSO limit laws. In the labelled case he defined

$$\overline{\mu}(\varphi) \ := \ \lim_{x\to\rho_A} \frac{\mathbf{A}_\varphi(x)}{\mathbf{A}(x)}, \tag{1.3}$$

provided the limit exists, where $x \to \rho_A$ means that x approaches ρ_A from the left. (In order for the limit to exist one must have $\rho_A > 0$.) Likewise one can define $\overline{\nu}(\varphi)$, using ordinary generating functions—however Compton did not follow this obvious parallel. Instead he generalized $\overline{\mu}$ to the *extended asymptotic density* (defined after the next theorem), and then gave a parallel definition of $\overline{\nu}$. For the next theorem we will use the obvious definition of $\overline{\nu}$ that is parallel to (1.3).

THEOREM 1.2 (Compton [**26**] Proposition 3.1, 1989). *Suppose $\mathcal{A}$ is an admissible class with $\rho_A > 0$ and $\mathbf{A}(\rho_A) = \infty$. Then $\mu \subseteq \overline{\mu}$, that is, if φ is a MSO sentence and $\mu(\varphi)$ is defined, then $\overline{\mu}(\varphi)$ is defined and equals $\mu(\varphi)$. More generally, one has $\mu^* \subseteq \overline{\mu}$. Similar assertions hold for ν and $\overline{\nu}$.*

[1] See [**9**], this volume, for a detailed discussion of basic properties of spectra.

There are admissible classes where $\rho_A > 0$ but $\mathbf{A}(\rho_A) < \infty$, for example the class of forests. To have the possibility of an approach to such cases, Compton immediately modified the definition of $\overline{\mu}$ ([**26**] p. 118, 1989) to the *extended asymptotic probability* in the labelled case:

$$\overline{\mu}(\mathcal{A}_\varphi) \;:=\; \lim_{j\to\infty} \lim_{x\to\rho_A} \frac{\mathbf{A}_\varphi^{(j)}(x)}{\mathbf{A}^{(j)}(x)}, \tag{1.4}$$

where $\mathbf{A}^{(j)}(x)$ denotes the jth derivative of $\mathbf{A}(x)$ with respect to x, etc. Compton points out that this is indeed an extension of the $\overline{\mu}$ defined in (1.3). With this new definition of $\overline{\mu}$ the next theorem shows: (i) $\mu \subseteq \overline{\mu}$ under weaker hypotheses than those in Theorem 1.2; and (ii) with the hypotheses of Theorem 1.2, the extended asymptotic probability $\overline{\mu}$ is always defined and is given by (1.3). Likewise, one has the parallel results for ν and $\overline{\nu}$.

THEOREM 1.3 (Compton [**26**] Proposition 3.2, Theorem 4.3, 1989). *Suppose $\mathcal{A}$ is an admissible class with $\rho_A > 0$. Then $\mu \subseteq \overline{\mu}$. If furthermore $\mathbf{A}(\rho_A) = \infty$ then $\overline{\mu}(\varphi)$ exists for any MSO sentence φ and it is given by* (1.3). *Similar assertions hold for ν and $\overline{\nu}$.*

It now seems clear that the more general definition of $\overline{\mu}$ given in (1.4) does not offer any advantages over the original definition given in (1.3) when it comes to proving MSO limit laws—see §2 for details.

A key concept regarding admissible classes $\mathcal{A}$ is $\mathcal{A} \to \lambda$, defined as follows (Compton [**23**] 1987, p. 74):

(a) (Labelled): For $0 \le \lambda \le \infty$, $\mathcal{A} \to \lambda$ holds iff $\displaystyle\lim_{n\to\infty} \frac{a(n-j)/(n-j)!}{a(n)/n!} = \lambda^j$, for any $j \in \mathsf{Spec}(\mathcal{P})$, where $a(n)$ is the labelled count function.

(b) (Unlabelled): For $0 \le \lambda \le 1$, $\mathcal{A} \to \lambda$ holds iff $\displaystyle\lim_{n\to\infty} \frac{a(n-j)}{a(n)} = \lambda^j$, for any $j \in \mathsf{Spec}(\mathcal{P})$, where $a(n)$ is the unlabelled count function.

This concept simplifies considerably for most values of λ.

LEMMA 1.4 (Compton [**23**] Proposition 4.1, 1987). *Suppose $\mathcal{A}$ is an admissible class. Then $\mathcal{A} \to \lambda$ implies $\rho_A = \lambda$. Furthermore,*

(a) (Labelled): *For $0 < \lambda < \infty$, $\mathcal{A} \to \lambda$ holds iff* $\displaystyle\lim_{n\to\infty} \frac{na(n-1)}{a(n)} = \lambda$.

(b) (Unlabelled): *For $0 < \lambda \le 1$, $\mathcal{A} \to \lambda$ holds iff* $\displaystyle\lim_{n\to\infty} \frac{a(n-1)}{a(n)} = \lambda$.

The first question posed by Compton at the end of [**23**] 1987, presented in §1.8 below as (Q1), was whether or not these results could be extended—for item (a) to the cases $\lambda \in \{0, \infty\}$, and for item (b) to the case $\lambda = 0$. The direction ($\Leftarrow$) is true in both items (a) and (b) for these extreme values of λ, so the question was whether or not ($\Rightarrow$) holds. The case $\lambda = 0$, that is, $\mathcal{A} \to 0$, occurs in the 'classical' examples of 0–1 laws mentioned in the beginning, namely graphs, etc.

A key observation concerning an admissible class $\mathcal{A}$ with component class $\mathcal{P}$ is that, for any choice of components $\mathfrak{S}_1, \ldots, \mathfrak{S}_r$ and any choice $j_1, \ldots, j_r$ of non-negative integers, there is a first-order sentence $\theta(\vec{\mathfrak{S}}, \vec{j})$ which says: *there are exactly j_i components isomorphic to $\mathfrak{S}_i$, for $1 \le i \le r$.* For $r = 1$ this becomes simply $\theta(\mathfrak{S}, j)$, expressing: *there are exactly j components isomorphic to $\mathfrak{S}$.* Compton gives explicit formulas for $\mathsf{Prob}(\theta(\vec{\mathfrak{S}},\vec{j}))$ in both the labelled and unlabelled settings ([**23**] 1987, Theorem 5.6). From this follows:

PROPOSITION 1.5. *Suppose $\mathcal{A}$ is an admissible class with a first-order limit law. Then $\mathcal{A} \to \rho_A$. If $\mathcal{A}$ has a first-order 0–1 law then:*

(a) (Labelled):
- $\rho_A \in \{0, \infty\}$.
- *If $\rho_A = 0$ then for any connected $\mathfrak{S}$ from $\mathcal{A}$, $\mathsf{Prob}(\theta(\mathfrak{S}, 0)) = 1$, and $\mathsf{Prob}(\theta(\mathfrak{S}, j)) = 0$ for $j \geq 1$. Thus the asymptotic probability that there is a component isomorphic to $\mathfrak{S}$ is 0.*
- *If $\rho_A = \infty$ then for any connected $\mathfrak{S}$ from $\mathcal{A}$, $\mathsf{Prob}(\theta(\mathfrak{S}, j)) = 0$ for $j \geq 0$. Thus given any n, the asymptotic probability that there are at most n components isomorphic to $\mathfrak{S}$ is 0.*

(b) (Unlabelled):
- $\rho_A \in \{0, 1\}$.
- *If $\rho_A = 0$ then for any connected $\mathfrak{S}$ from $\mathcal{A}$, $\mathsf{Prob}(\theta(\mathfrak{S}, 0)) = 1$, and $\mathsf{Prob}(\theta(\mathfrak{S}, j)) = 0$ for $j \geq 1$. Thus the asymptotic probability that there is a component isomorphic to $\mathfrak{S}$ is 0.*
- *If $\rho_A = 1$ then for any connected $\mathfrak{S}$ from $\mathcal{A}$, $\mathsf{Prob}(\theta(\mathfrak{S}, j)) = 0$ for $j \geq 0$. Thus given any n, the asymptotic probability that there are at most n components isomorphic to $\mathfrak{S}$ is 0.*

Suppose an admissible class $\mathcal{A}$ has a first-order 0–1 law, and $\rho_A = 0$. In both the labelled and unlabelled cases one has $\mathcal{A} \to 0$, and for any connected $\mathfrak{S}$ in $\mathcal{A}$, the asymptotic probability of $\mathfrak{S}$ appearing as a component is 0. Compton's papers offer no further general facts regarding admissible classes $\mathcal{A}$ with $\rho_A = 0$ and a first-order 0–1 law.

The development of a substantial theory by Compton was based on assuming $\rho_A > 0$. For a first-order 0–1 law that meant $\rho_A = \infty$ in the labelled case, and $\rho_A = 1$ in the unlabelled case. For $\rho_A = 1$ in the unlabelled case the conclusion $\mathcal{A} \to 1$ can be expressed simply as $\lim\limits_{n\to\infty} \dfrac{a(n-1)}{a(n)} = 1$. For $\rho_A = \infty$ in the labelled case, no simplification is currently known for $\mathcal{A} \to \infty$.

The method of Compton's 1989 paper [**26**], as stated at the beginning of §6 of [**26**], was for the case $\rho_A > 0$, and was: (i) to find conditions to show that the extended asymptotic probability $\overline{\mu}(\varphi)$, respectively $\overline{\nu}(\varphi)$, exists, and then (ii) to find conditions that showed the asymptotic probability $\mu(\varphi)$, respectively $\nu(\varphi)$, was defined. This method was first used for individual MSO sentences φ.

THEOREM 1.6 (Compton [**26**] Theorem 6.1, 1989). *Suppose $\mathcal{A}$ is an admissible class with $\rho_A > 0$, $\mathbf{A}(\rho_A) = \infty$ and $\mathcal{A} \to \rho_A$. Let φ be a MSO sentence. Then:*

(a) (Labelled): *If $\overline{\mu}(\varphi) \in \{0,1\}$ then $\mu(\varphi)$ exists and $= \overline{\mu}(\varphi)$.*

(b) (Unlabelled): *If $\overline{\nu}(\varphi) \in \{0,1\}$ then $\nu(\varphi)$ exists and $= \overline{\nu}(\varphi)$.*

It is easy to see that one can express 'is connected' by a sentence in MSO logic.

COROLLARY 1.7 (Compton [**26**] Corollary 6.2, 1989). *Given the hypotheses of Theorem 1.6, let φ be a MSO sentence that expresses 'is connected'. Then in the labelled case $\mu(\varphi) = 0$, and in the unlabelled case $\nu(\varphi) = 0$. In either case this says that the probability of being connected is 0.*

Next Compton proves the fundamental results on MSO 0–1 laws for admissible classes $\mathcal{A}$ with $\rho_A > 0$.

THEOREM 1.8 (Compton [**26**] Theorems 6.3 and 6.4, 1989). *Suppose $\mathcal{A}$ is an admissible class with $\rho_A > 0$. Then:*

(a) (Labelled): *$\mathcal{A}$ has a MSO 0–1 law iff $\mathcal{A}$ has a FO 0–1 law iff $\mathcal{A} \to \infty$. If (any of) these conditions hold then $\rho_A = \infty$ and $\mathsf{Prob}(\varphi) = \overline{\mu}(\mathcal{A}_\varphi)$, for φ a MSO sentence.*

(b) (Unlabelled): *$\mathcal{A}$ has a MSO 0–1 law iff $\mathcal{A}$ has a FO 0–1 law iff $\mathcal{A} \to 1$. If (any of) these conditions hold then $\rho_A = 1$ and $\mathsf{Prob}(\varphi) = \overline{\nu}(\mathcal{A}_\varphi)$ for φ a MSO sentence.*

This is not the end of the story about 0–1 laws for admissible classes with $\rho_A > 0$, for the simple reason that the last of the equivalent conditions in each of (a) and (b), concerning $\mathcal{A} \to$, is not so easy to verify in practice. With admissible classes it is usual that one knows a great deal more about the count function $p(n)$ for the components of the class than for the entire class; consequently much effort has been expended to find conditions on $p(n)$ that imply the desired condition on $a(n)$.

The more difficult, and even harder to apply, result was for MSO limit laws that were not 0–1 laws.

THEOREM 1.9 (Compton [**26**] Theorem 6.6, 1989). *Suppose $\mathcal{A}$ is an admissible class with $\rho_A > 0$, $\mathbf{A}(\rho_A) = \infty$ and $\mathcal{A} \to \rho_A$.*

(a) (Labelled): *If $\rho_A < \infty$ and for some $C, N > 0$ one has, for $n \geq N$, $\dfrac{a(n-i)/(n-i)!}{a(n)/n!} \leq C{\rho_A}^i$, for $0 \leq i \leq n$, then $\mathcal{A}$ has a labelled MSO limit law, that is, $\mu(\varphi)$ exists for all MSO sentences φ.*

(b) (Unlabelled): *If $\rho_A < 1$ and for some $C, N > 0$ one has for $n \geq N$, $\dfrac{a(n-i)}{a(n)} \leq C{\rho_A}^i$, for $0 \leq i \leq n$, then $\mathcal{A}$ has an unlabelled MSO limit law, that is, $\nu(\varphi)$ exists for all MSO sentences φ.*

Except for a result of Woods [**56**] (discussed in §3.2 below) that covers some interesting cases concerning a single unary function, after 20 years one finds that, for admissible classes $\mathcal{A}$, Theorem 1.9 is still the foundation result for proving MSO limit laws that are not 0–1 laws. However it is not an easy theorem to apply—Compton gave one labelled example, namely permutations, and no unlabelled examples based on it.

In the following examples and results, *trees* are rooted trees and *forests* are forests of rooted trees.

1.3. Compton's examples. Compton's examples were stated in his 1987 paper [**23**] for first-order logic, and then upgraded in the 1989 paper [**26**] to MSO logic.

An important tool for the labelled case is *Hayman-admissible functions*—this use of the word 'admissible' is not directly related to Compton's admissible classes. These functions have a complicated definition, and it can be difficult to establish that a function is Hayman admissible; however a few nice examples are known—see, for example, items (c) and (d) in §1.4.

THEOREM 1.10 (Compton [**23**] Theorem 4.2, 1987). *If $\mathbf{B}(x)$ is Hayman admissible then* $\lim_{n\to\infty} \frac{b(n-1)}{b(n)} = \rho_B$.

Compton's applications of Theorem 1.8 above relied primarily on the following two propositions. First there is the case where $\mathcal{A}$ is a finitely generated admissible class, that is, $\mathcal{A}$ has only finitely many indecomposable members. In the unlabelled case this is connected to the study of the famous Coin Problem.[2]

PROPOSITION 1.11 (Compton [**23**] Example 7.15, 1987). *Suppose $\mathcal{P}$ is finite (counting up to isomorphism), say $\mathcal{P}$ is represented by structures $\mathfrak{S}_1, \ldots, \mathfrak{S}_r$, with sizes $s_1, \ldots, s_r$. Then:*

(a) *In the labelled case one has $\mathcal{A} \to \infty$.*

(b) *In the unlabelled case the count function is asymptotic to a polynomial, namely*

$$a(n) \sim \frac{1}{(r-1)!} \frac{n^{r-1}}{s_1 \cdots s_r},$$

thus $\frac{a(n-1)}{a(n)} \to 1$ *as* $n \to \infty$.

Consequently $\mathcal{A}$ has both a labelled and an unlabelled MSO 0–1 law.

Compton gave no examples of admissible classes with only finitely many components, likely because it was trivial to do so. Simple examples would be the class of graphs with components of bounded size; and forests with trees having bounded height and width.

Next there is a famous result on partitions:[3]

PROPOSITION 1.12 (Bateman and Erdős [**1**], 1956). *Suppose $p(n)$ is a non-negative integer for $n \geq 1$ with* $\gcd(\{n : p(n) > 0\}) = 1$, *and*

$$1 + \mathbf{A}(x) = \prod_{n\geq 1} (1 - x^n)^{-p(n)}.$$

If $p(n) \leq 1$, for $n \geq 1$, then $\lim_{n\to\infty} \frac{a(n-1)}{a(n)} = 1$.

1.4. Four examples with an unlabelled 0–1 law. Theorem 1.8 and Proposition 1.12 show that the following four examples of Compton have an unlabelled MSO 0–1 law—the status of a labelled law varies:

(a) *Permutations.*[4]

In the labelled setting one has $a(n) = n!$, so $1 + \mathbf{A}(x) = 1/(1-x)$, which gives

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[2]Suppose one has coins with values $d_1, \ldots, d_k$, where $\gcd(d_1, \ldots, d_k) = 1$. Sylvester knew that the possible values of combinations of such coins formed a cofinite subset of the positive integers; he used partial fractions over the complex numbers to find the asymptotics for the number of ways that one could select coins to create a total value $= n$. The Coin Problem (a.k.a. Frobenius's Problem) was to find a formula for the largest n such that one could not realize the value n with the given coins. Sylvester found the formula for $k = 2$. The problem is still open for $k \geq 3$.

[3]The topic being studied by Bateman and Erdős was: how many ways can one partition numbers into parts when the parts come from a fixed subset of the positive integers? In particular they were looking at the kth difference function of the count function $a(n)$, and succeeded in finding a simple necessary and sufficient condition for the kth difference to be eventually monotonic.

[4]In the 1987 paper Compton states that permutations do not have a labelled first-order 0–1 law. In the 1989 paper one finds that they do have a labelled MSO limit law.

$\mathcal{A} \to 1$. Thus permutations do not have a first-order labelled 0–1 law. But Theorem 1.9 above applies to show that permutations indeed have a labelled MSO limit law.

(b) *Forests of height* 1.
Note that the labelled $p(n) = n$, so $1 + \mathbf{A}(x) = \exp\left(xe^x\right)$, a Hayman-admissible function with $\rho_A = \infty$. Thus $\mathcal{A} \to \infty$, so $\mathcal{A}$ has a labelled MSO 0–1 law.

(c) *Forests whose trees are linear.* For the labelled case note that $p(n) = n!$, so $1 + \mathbf{A}(x) = \exp\left(x/(1-x)\right)$. Compton states that one can prove this is a Hayman-admissible function, and clearly $\rho_A = 1$. Thus $\mathcal{A} \to 1$, so $\mathcal{A}$ does not have a labelled first-order 0–1 law.[5]

(d) *Equivalence relations*, or *partitions*. In the labelled case $p(n) = 1$, so $1 + \mathbf{A}(x) = \exp\left(e^x - 1\right)$, a Hayman-admissible function with $\rho_A = \infty$. Thus $\mathcal{A} \to \infty$, so $\mathcal{A}$ has a labelled MSO 0–1 law.

REMARK 1.13. Since the unlabelled count function for $\mathcal{P}$ is $p(n) = 1$ in each of the four cases above, one could also use the famous 1917 result of Hardy and Ramanujan [**34**] on the number of partitions of an integer n, that is, the number of ways n can be expressed as a sum of positive integers. The fundamental identity in this case is

$$1 + \mathbf{A}(x) \;=\; \prod_{n\geq 1} (1-x^n)^{-1},$$

from which they prove

$$a(n) \;\sim\; \frac{\exp\left(\pi\sqrt{2n/3}\,\right)}{4\sqrt{3}n}.$$

It easily follows that $a(n-1)/a(n) \to 1$ as $n \to \infty$. However the result of Bateman and Erdős is much easier to prove, and leads to far more unlabelled 0–1 laws.

1.5. Partitions with selected subsets. The final example where Compton uses his methods to obtain a MSO 0–1 law is the class $\mathcal{A}$ of *partitions with selected subsets*.

In the labelled case one has $p(n) = 2^n - 1$, so $\mathbf{A}(x) = \exp\left(e^x(e^x-1)\right)$, a Hayman admissible function with $\rho_A = \infty$. Thus $\mathcal{A} \to \infty$, giving a labelled MSO 0–1 law.

For the unlabelled case $p(n) = n$. Compton appeals to an application of a difficult theorem of Meinardus (a theorem whose goal is to obtain results like those of Hardy and Ramanujan on partitions) to obtain asymptotics for $a(n)$ which show that $a(n-1)/a(n) \to 1$ as $n \to \infty$. Thus $\mathcal{A} \to 1$, so $\mathcal{A}$ has an unlabelled MSO 0–1 law as well.

1.6. First-order 0–1 laws that cannot be obtained by Compton's method. In his list of examples Compton states that the following admissible classes $\mathcal{A}$ have both labelled and unlabelled first-order 0–1 laws: *all $\mathcal{L}$-structures, directed graphs, oriented graphs* and *posets*. But these cannot be obtained by his method since in each case $\mathcal{A} \to 0$.

[5] The claim in [**23**] that the class of linear forests has a labelled first-order 0–1 law is incorrect.

1.7. Admissible classes without a first-order limit law. Compton also gives examples of admissible classes $\mathcal{A}$ that fail to have a first-order limit law, namely: the class of *binary forests* has neither a labelled nor an unlabelled limit law since $\mathcal{A}{\to}\rho_A$ fails in both cases; and likewise for the class of *binary forests with various orders* (pre-, post-, inorder).

1.8. Compton's status report and questions. In the 1987 paper Compton noted that the status of limit laws for several well-known classes was open. The following lists those examples, and updates what is known about them in the bulleted items.

Admissible classes for which the status of either the labelled or unlabelled first-order limit laws was not known at the time of publication of [**23**]:

(a) *Unlabelled* structures consisting of *a single unary function*, a.k.a. *functional digraphs*. (For the labelled case, Lynch [**40**], 1985, had proved a first-order limit law.)
  - [Woods [**56**] Corollaries 1.1 and 1.2, 1997, proved that a single m-colored unary function has a MSO limit law in both the labelled and unlabelled cases.]

(b) *Labelled or unlabelled Forests.*
  - [Woods [**55**] Theorems 1.1 and 6.5, 1997, proved that the class of m-colored trees (which is not an admissible class) has both labelled and unlabelled MSO limit laws (but not 0–1 laws). A MSO class of forests naturally corresponds to a MSO class of trees by simply adding a root to each forest. Using this, one sees that the class of m-colored forests also has both labelled and unlabelled MSO limit laws.]

(c) *Labelled or unlabelled Unary-Binary Forests.* This means every non-leaf has one or two immediate descendants.

(d) *Labelled or unlabelled Forests with pre- or post-order.*

(e) *Labelled or unlabelled Unary-Binary Forests with pre-, post-, or in-order.*

(f) *Labelled or unlabelled Acyclic Graphs.*
  - [McColm [**41**] 2002, proved that the class of *connected* acyclic graphs (also known as free trees) has a labelled MSO 0–1 law; and McColm [**42**] Corollary 2.1, 2004, proved that the same class has an unlabelled MSO 0–1 law. The situation for the class of all acyclic graphs is open.]

Compton also posed the following five questions about admissible classes $\mathcal{A}$ and first-order laws in [**23**]:

(Q1) For the labelled case there were two sub-questions, namely does:
$\lim_{n\to\infty} \dfrac{a(n-j)/(n-j)!}{a(n)/n!} = 0$, for $j \in \mathsf{Spec}(\mathcal{P})$, imply $\lim_{n\to\infty} \dfrac{na(n-1)}{a(n)} = 0$?
$\lim_{n\to\infty} \dfrac{a(n-j)/(n-j)!}{a(n)/n!} = \infty$, for $j \in \mathsf{Spec}(\mathcal{P})$, imply $\lim_{n\to\infty} \dfrac{na(n-1)}{a(n)} = \infty$?
In the unlabelled case suppose $\lim_{n\to\infty} \dfrac{a(n-j)}{a(n)} = 0$, for $j \in \mathsf{Spec}(\mathcal{P})$. Does this imply $\lim_{n\to\infty} \dfrac{a(n-1)}{a(n)} = 0$?
  - [There is no progress to report on (Q1).]

(Q2) In the labelled case, does $\rho_A = \infty$ imply $\mathcal{A} \to \infty$? (Compton conjectured the answer was "No".)

In the unlabelled case, does $\rho_A = 1$ imply $\mathcal{A} \to 1$? (Compton conjectured the answer was "No".)

- [A counterexample for the unlabelled case appeared in §4.4 of **[15]**.]

(Q3) Find easily verifiable conditions for $\lim_{n\to\infty} \frac{a(n-1)}{a(n)} = 1$. Does $p(n) = \mathrm{O}(n^k)$, for some k, suffice?

- [Answered in the affirmative by Bell **[4]** Theorem 1.8, 2001.]

Is there a similar result one could use in the labelled case?

- [Burris and Yeats **[19]** Theorem 7, 2008, show that $p(n) = \mathrm{O}\left(n^{\theta \cdot n}\right)$, $0 < \theta < 1$, guarantees a labelled MSO 0–1 law.]

(Q4) Are there natural examples of admissible classes with a labelled 0–1 law but not an unlabelled 0–1 law? (Permutations give an example with an unlabelled 0–1 law, but not a labelled 0–1 law.)

- [Forests of brooms with 2-colored handles provide an example, in Burris and Yeats **[19]**, 2008.]

(Q5) If the radius of convergence of the exponential generating series (for labelled structures) is 0, does it follow that there is a labelled 0–1 law iff there is an unlabelled 0–1 law?

- [There is no progress to report on (Q5).]

More questions were added in his 1989 paper:

(Q6) Is it true that every MSO sentence φ has a labelled probability iff it has an unlabelled probability?

- [There is no progress to report on (Q6).]

(Q7) If $\rho_A > 0$ and $\mathbf{A}(\rho_A) < \infty$, does it follow that the extended asymptotic probability $\overline{\mu}(\varphi)$ $[\overline{\nu}(\varphi)]$ of every MSO sentence φ exists?

- [There is no progress to report on (Q7).]

(Q8) If $\mathcal{A} \to \rho_A > 0$ and $\mathbf{A}(\rho_A) = \infty$, does it follow that the asymptotic probability $\mu(\varphi)$ $[\nu(\varphi)]$ of every MSO sentence φ exists?

- [There is no progress to report on (Q8).]

(Q9) Suppose $\mathcal{A} \to \rho_A > 0$, $\mathbf{A}(\rho_A) < \infty$ and φ is a MSO sentence. In the labelled, or unlabelled, case: Is there a modulus $m > 0$ such that

$$\lim_{q\to\infty} \frac{a_\varphi(mq+r)}{a(mq+r)} \quad \text{exists, for any } r \geq 0?$$

- [There is no progress to report on (Q9).]

In a second 1989 paper **[27]**, Compton surveyed various approaches to proving logical limit laws and posed several questions, the following being relevant to the topic discussed here:

(Q10) Develop techniques for proving FO and MSO limit laws for classes $\mathcal{A}$ whose generating series $\mathbf{A}(x)$ converges at its radius of convergence. Two specific examples mentioned, where the existence of a logical limit laws was not known, were forests and unit interval graphs.

- [As noted above, Woods **[55]** 1997 proved that m-colored trees (hence forests) have both labelled and unlabelled MSO limit laws.]

(Q11) Prove a 0–1 law for graphs in a logic strong enough to express Hamiltonicity.

- [There is no progress to report on (Q11).]

(Q12) Develop a theory of asymptotic probability for classes where direct product is an appropriate operation (such as in the class of groups).
- [In the unlabelled setting, the theory of *first-order* logical limit laws for multiplicatively admissible classes has been developed to the point where it completely parallels the work on additively admissible classes—these results are discussed in this article.
In the labelled setting it seems there is simply no literature on multiplicatively admissible classes, not even a statement of a fundamental identity.]

(Q13) Investigate asymptotic probability and 0–1 laws for classes of regular graphs.
- [There is no progress to report on (Q13).]

(Q14) Show that all first-order sentences have labelled asymptotic probability in a class of directed graphs with the amalgamation property and closed under substructures.
- [There is no progress to report on (Q14).]

2. The presentation in "Number Theoretic Density and Logical Limit Laws", 2001

The interest in Compton's suggestion in (Q12) above, to develop a theory of asymptotic probability for classes where direct product is a natural operation for combining and decomposing structures, developed rapidly in the mid 1990s, following a lecture at the University of Waterloo by Compton on a first-order limit law for finite Abelian groups. In 2001 Burris published a book [**15**] where the first half was essentially an exposition of Compton's 1980s work on *additive* classes, that is, where the operation of combination is disjoint union, and the second half gave a parallel development for *multiplicative* classes, where the operation of combination is direct product. The book *only treated the unlabelled case*, mainly because a basic theory of labelled structures in the context of direct products did not (and still does not) exist.

Although Compton never quite said that his method was

> *to find criteria for logical limit laws for admissible classes $\mathcal{A}$ that depend only on the behavior of the generating series $\mathbf{A}(x)$,*

nonetheless this conclusion is strongly suggested by the fact that his results are of this form.[6] This view of Compton's method was adopted in [**15**], and it had strong consequences, which will be discussed in more detail below in §3.6. First it reduced Compton's method to showing that all partition classes $\vec{\gamma}\star\vec{\mathcal{P}}$, defined in §2.3, have asymptotic density in $\mathcal{A}$. This condition implies:

(a) $\displaystyle\lim_{n\to\infty}\frac{a(n-1)}{a(n)} = \rho_A$,

(b) the class of components $\mathcal{P}$ has density 0,

(c) $\rho_A > 0$, and

[6] This is not to say that Compton restricted his search for classes $\mathcal{A}$ with logical limit laws to those for which the laws were determined by the generating series $\mathbf{A}(x)$—in [**25**] he proved that posets have both a labelled and an unlabelled first-order 0–1 law by arguments that required more information than just knowing the generating series. Nonetheless, the remarkable feature of his logical limit laws, proved in the two papers ([**23**] 1987, [**26**] 1989) featured above, is that indeed they depend solely on the behavior of the generating series $\mathbf{A}(x)$, that is, solely on the number $a(n)$ of structures in $\mathcal{A}$ of each size n.

(d) $\mathbf{A}(\rho_A) = \infty$.

The last item was not known at the time of publication of [**15**]. The notation used in the following is mainly based on that of [**15**]. For $\mathcal{B} \subseteq \mathcal{A}$ let $B(x) := \sum_{n \le x} b(n)$. Then:

notation	meaning	name
$\delta(\mathcal{B})$	$\lim_{n\to\infty} \dfrac{b(n)}{a(n)}$	*local asymptotic density*
$\Delta(\mathcal{B})$	$\lim_{x\to\infty} \dfrac{B(x)}{A(x)}$	*global asymptotic density*
$\partial(\mathcal{B})$	$\lim_{x\to\rho_A} \dfrac{\mathbf{B}(x)}{\mathbf{A}(x)}$	*Dirichlet density*
$a(n)$ [or $\mathbf{A}(x)$] $\in \mathsf{RT}_\rho$	$\lim_{n\to\infty} \dfrac{a(n-1)}{a(n)} = \rho$	

Compton's method can only be used on admissible classes $\mathcal{A}$ that exhibit a medium to slow growth rate, since the radius of convergence of the generating series $\mathbf{A}(x)$ for the class $\mathcal{A}$ needs to be positive. For such classes the generating series (a formal power series) defines an analytic function in a neighborhood of 0. Fast growing classes, like the class of graphs (which has a first-order 0–1 law), require other means to establish their logical limit laws.

The 'Dirichlet density' ∂ is based on the original form of $\overline{\nu}$ used by Compton in [**26**] (see Theorem 1.2 above), not the extended asymptotic probability that requires differentiation. That is because item (d) above shows that there is no need to differentiate—admissible classes such as forests, which have $\mathbf{A}(\rho_A) < \infty$, simply cannot be handled by the method of Compton (as described above).

2.1. Classes closed under direct product and directly indecomposable factors. Multiplicative classes occur most naturally when one studies algebraic systems; for example, every finite group factors uniquely into a direct product of indecomposable groups. Likewise every finite lattice factors uniquely into a direct product of indecomposable lattices. When dealing with multiplicative classes, there were some key differences from the previous work on additive classes to take into account. First, finite structures may not have a unique factorization into indecomposables.[7] The second item was that the quotient $b(n)/a(n)$ was difficult to analyze—to obtain interesting results a certain smoothing out was needed, and this was done by replacing $b(n)$ by $B(x) := \sum_{n\le x} b(n)$, etc., when defining asymptotic density. The third item was that one needs Dirichlet series, not power series, for generating functions. The fourth item was that some multiplicative classes behaved like additive classes, which tended to be rather different from other multiplicative classes.

One has the definition of an admissible class in this context by replacing 'disjoint union' by 'direct product', and requiring unique factorization.

Definition 2.1. Given a finite language $\mathcal{L}$, a class $\mathcal{A}$ of $\mathcal{L}$-structures is *multiplicatively admissible* (or *admissible with respect to direct product*) if

(a) it is closed under direct product, and

[7] The question of unique factorization under direct product was studied intensely by Tarski, Jónsson and McKenzie (see [**44**]).

(b) every member of $\mathcal{A}$ of size at least 2 can be uniquely expressed as a direct product of members of $\mathcal{P}$, where $\mathcal{P}$ is the collection of directly indecomposable members of $\mathcal{A}$.

Multiplicatively admissible classes $\mathcal{A}$ split into two distinct types: (i) $\mathcal{A}$ is *discrete* if there is a positive integer λ such that the sizes of members of $\mathcal{A}$ are powers of λ, and (ii) $\mathcal{A}$ is *strictly multiplicative* if it is not discrete. Multiplicatively admissible classes that are discrete can be viewed as additively admissible classes by simply changing the notion of size—for such classes the previous results apply.

The generating series used for $\mathcal{B} \subseteq \mathcal{A}$ in the multiplicative case is the Dirichlet series $\mathbf{B}(x) := \sum_{n\geq 1} b(n)n^{-x}$. Thus for $\mathcal{A}$ a multiplicatively admissible class we have

$$\begin{aligned} \mathbf{A}(x) &:= \sum_{n\geq 1} a(n)n^{-x}, \quad a(1) = 1, \\ \mathbf{P}(x) &:= \sum_{n\geq 2} p(n)n^{-x}, \end{aligned}$$

and the fundamental identity for $\mathcal{A}$ is

$$\sum_{n\geq 1} a(n)n^{-x} \;=\; \prod_{n\geq 2} \left(1-n^{-x}\right)^{-p(n)} \;=\; \exp\Big(\sum_{m\geq 1} \mathbf{P}(mx)/m\Big). \tag{2.1}$$

$\mathbf{A}(x)$ is often called the *zeta function* for $\mathcal{A}$, and the product in the middle of (2.1) is called the *Euler product* for $\mathcal{A}$.

For $\mathcal{B} \subseteq \mathcal{A}$, α_B is the abscissa of convergence of $\mathbf{B}(x)$. The appropriate choice of asymptotic density for the multiplicative setting is

$$\Delta(\mathcal{B}) \;:=\; \lim_{x\to\alpha_A} \frac{B(x)}{A(x)},$$

provided the limit exists, where $x \to \alpha_A$ means that x approaches α_A from the right. This of course requires that $\alpha_A < \infty$. The notion of Dirichlet density $\partial(\mathcal{B})$ in the multiplicative setting is the obvious analog of the same in the additive setting, namely:

$$\partial(\mathcal{B}) \;:=\; \lim_{x\to\alpha_A} \frac{\mathbf{B}(x)}{\mathbf{A}(x)}.$$

The multiplicative analog of RT is RV, the well-known concept of *regular variation*. The notations $B(x) \in \mathsf{RV}_\alpha$ and $\mathbf{B}(x) \in \mathsf{RV}_\alpha$ both mean that $B(x)$ has *regular variation of index α at infinity*, that is,

$$\lim_{t\to\infty} \frac{B(tx)}{B(t)} \;=\; x^\alpha.$$

$B(x)$ is *slowly varying at infinity* if $B(x) \in \mathsf{RV}_0$.

A guiding principle in the post-1997 work with multiplicatively admissible classes was the belief that every local result for additive classes would have a global analog in the strictly multiplicative setting. Based on this principle, there were three conjectures for multiplicative classes posed in [**15**]:[8]

(C1) Does $\Delta(\mathcal{P}) = 0$ imply $\mathbf{A}(\alpha_A) = \infty$?
(C2) Does $P(x) \in \mathsf{RV}_0$ imply $A(x) \in \mathsf{RV}_0$?

[8]In [**15**] the conjectures were labelled Conjectures 9.70, 10.7 and 11.26.

(C3) Do $\alpha_A > 0$, $P_0(x) := (\log x)x^{-\alpha_A} P(x) \in \mathsf{RV}_0$ and $\liminf_{x\to\infty} P_0(x) > 1$ imply $(\star)$ holds?

All three were subsequently verified (conjecture C3 needed to be revised)—see §3 below.

2.2. Possible generating series for $\mathcal{P}$. In the unlabelled setting there is a tidy description of the possible generating series $\mathbf{P}(x)$ for the class of indecomposables $\mathcal{P}$ of admissible classes $\mathcal{A}$ in the additive [multiplicative] case, provided $\mathbf{P}(x)$ has a positive radius of convergence [a finite abscissa of convergence].

PROPOSITION 2.2. *Let $q(n)$ be a sequence of non-negative integers.*

(a) (Additive Case) *Suppose $\mathbf{Q}(x) := \sum_{n\geq 1} q(n)x^n$ has a positive radius of convergence ρ_Q. Then there is an admissible class $\mathcal{A}$ with $\mathbf{Q}(x)$ being the generating series for the subclass $\mathcal{P}$ of indecomposables.*

(b) (Multiplicative Case) *Suppose $\mathbf{Q}(x) := \sum_{n\geq 2} q(n)n^{-x}$ has a finite abscissa of convergence α_Q. Then there is an admissible class $\mathcal{A}$ with $\mathbf{Q}(x)$ being the generating series for the subclass $\mathcal{P}$ of indecomposables.*

PROOF. For (a) one has $\rho_Q > 0$ iff there is a positive integer M such that $q(n) \leq M^n$ for all $n \geq 1$. There are M^n M-colored chains (as posets) of size n. Let $\mathcal{P}$ be a subclass of M-colored chains with, up to isomorphism, exactly $q(n)$ members of size n, and let $\mathcal{A}$ be the closure of $\mathcal{P}$ under disjoint union.

For (b) one has $\alpha_Q < \infty$ iff there is a positive integer M such that $q(2) + \cdots + q(n) \leq n^M$ for all $n \geq 2$. There are n^M chains (as lattices) of size n with M constants. Let $\mathcal{P}$ be a subclass of such augmented chains with, up to isomorphism, exactly $q(n)$ members of size n, and let $\mathcal{A}$ be the closure of $\mathcal{P}$ under direct product. □

2.3. Partition Classes. Compton's method revolves around one key concept, that of a partition class. First, define $n \star \mathcal{B}$ in the additive [multiplicative] case to mean the collection of disjoint unions [direct products] of n structures from $\mathcal{B}$, repeats allowed among the n structures. Then for $\gamma \subseteq \mathbb{N}$ define $\gamma \star \mathcal{B}$ to be the union of the $n \star \mathcal{B}$ for $n \in \gamma$.

DEFINITION 2.3. A *partition class* of an admissible class $\mathcal{A}$ is a subclass of $\mathcal{A}$ of the form

$$\vec{\gamma} \star \vec{\mathcal{P}} \ := \ \begin{cases} \gamma_1 \star \mathcal{P}_1 + \cdots + \gamma_k \star \mathcal{P}_k & \text{additive case} \\ \gamma_1 \star \mathcal{P}_1 \times \cdots \times \gamma_k \star \mathcal{P}_k & \text{multiplicative case,} \end{cases}$$

where $\mathcal{P}_1, \ldots, \mathcal{P}_k$ is a partition of $\mathcal{P}$, the class of indecomposable members of $\mathcal{A}$, and the γ_i are subsets of $\mathbb{N}$ which are either finite or co-finite.

The importance of partition classes is made clear in the next two propositions—this is the part of Compton's method that belongs to model theory.

PROPOSITION 2.4. *Let $\mathcal{A}$ be an admissible class.*

($\mathfrak{a}$) *In the additive case, $\mathcal{A}_\varphi$ is always a union of finitely many pairwise disjoint partition classes $\vec{\gamma} \star \vec{\mathcal{P}}$, for φ a MSO sentence.*

[Compton [**26**] (in proof of) Lemma 4.1, 1989; for a presentation in the notation used here, see [**15**] Proposition 6.28, 2001]

($\mathfrak{m}$) *In the multiplicative case, $\mathcal{A}_\varphi$ is always a union of finitely many pairwise disjoint partition classes $\vec{\gamma} \star \vec{\mathcal{P}}$, for φ a FO sentence.*

[Burris and Sárközy [**17**] Theorem 3.4(a), 1997, following results of Burris and Idziak [**16**] Lemmas 10 and 11, 1996; for a presentation in the notation used here, see [**15**] Proposition 12.17, 2001]

Once φ is given, one can add a further restriction concerning which γ_i are actually needed for this proposition, namely there is a positive integer c_φ such that one can assume each γ_i is either one of $0, 1, \ldots, c_\varphi - 1$, or it is the set $\{n \geq c_\varphi\}$.

PROPOSITION 2.5. *Suppose $\mathcal{A}$ is an admissible class of structures.*

($\mathfrak{a}$) *In the additive case, if some partition class of $\mathcal{A}$ does not have asymptotic density* [*in* $\{0,1\}$] *then there is an admissible class $\widehat{\mathcal{A}}$ such that (i) $\widehat{\mathbf{A}}(x) = \mathbf{A}(x)$ and (ii) $\widehat{\mathcal{A}}$ does not have a MSO limit* [0–1] *law.*

[Burris [**15**] Proposition 6.33, 2001].

($\mathfrak{m}$) *In the multiplicative case, if there is a partition class of $\mathcal{A}$ that does not have asymptotic density* [*in* $\{0,1\}$] *then there is an admissible class $\widehat{\mathcal{A}}$ such that (i) $\widehat{\mathbf{A}}(x) = \mathbf{A}(x)$ and (ii) $\widehat{\mathcal{A}}$ does not have a FO limit* [0–1] *law.*

[Burris and Sárközy [**17**] Theorem 3.4(b), 1997; Burris, Compton, Odlyzko and Richmond [**18**] Theorem 2.1, 1997].

Compton's method depends on:

(1) showing that $\mathcal{A}_\varphi$ is a finite union of partition classes of $\mathcal{A}$, and then
(2) finding conditions on $\mathbf{A}(x)$ and/or $\mathbf{P}(x)$ that guarantee
 ($\star$) every partition class of $\mathcal{A}$ has asymptotic density.[9]

The model-theoretic part of establishing Compton's method is item (1), which has been successfully completed (Propositions 2.4, 2.5 above)—in the additive case item (1) holds for MSO sentences; in the multiplicative case (1) holds for FO sentences. The proof of item (1) is based on the next two lemmas. For q a positive integer the expression $\mathfrak{S}_1 \equiv_q^{\mathsf{MSO}} \mathfrak{S}_2$ means that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ satisfy the same MSO sentences of quantifier depth q.

LEMMA 2.6 (Compton [**26**], 1989). *Let $\mathcal{A}$ be an admissible class of relational structures. Given a non-negative integer q, let $\equiv_q^{\mathsf{MSO}}$ partition $\mathcal{P}$ into the classes $\mathcal{P}_1, \ldots, \mathcal{P}_k$. There is a positive integer c_q such that:*

(a) *If $m_1, \ldots, m_k$ and $m'_1, \ldots, m'_k$ are such that for each i one has either $m_i = m'_i$ or $m_i, m'_i \geq c_q$ then*

$$\sum_{i=1}^{k} m_i \star \mathcal{P}_i \ \equiv_q^{\mathsf{MSO}} \ \sum_{i=1}^{k} m'_i \star \mathcal{P}_i,$$

meaning that one has the relation $\equiv_q^{\mathsf{MSO}}$ holding between any member $\mathfrak{S}$ of the left side and any member $\mathfrak{S}'$ of the right side.

[9] The property ($\star$) for the multiplicative case was expressed in Burris and Sárközy [**17**], 1997, by saying that $\mathcal{A}$ was 'loaded'. To say that every partition set had asymptotic density either 0 or 1 was expressed in Burris, Compton, Odlyzko and Richmond [**18**], 1997, by the phrase 'front loaded'. This terminology is no longer used.

(b) *Let C_q be the set of k-tuples $\vec{\gamma} := (\gamma_1, \ldots, \gamma_k)$ where each γ_i is one of the coefficients* $0, 1, \ldots, c_q - 1, (\geq c_q)$. *For $\vec{\gamma} \in C_q$ let*

$$\vec{\gamma} \star \vec{\mathcal{P}} \ := \ \sum_{i=1}^{k} \gamma_i \star \mathcal{P}_i,$$

and for φ a MSO sentence of quantifier depth q let

$$S_\varphi \ : = \{\vec{\gamma} \in C_q : \vec{\gamma} \star \vec{\mathcal{P}} \subseteq \mathcal{A}_\varphi\}.$$

Then

$$\mathcal{A}_\varphi \ = \ \bigcup_{\vec{\gamma} \in S_\varphi} \vec{\gamma} \star \vec{\mathcal{P}},$$

a union of finitely many disjoint partition classes.

PROOF. A routine application of Ehrenfeucht-Fraïssé games. □

The basic lemma for multiplicative classes, the analog of the above Lemma 2.6 (of Compton), was stated and proved in 1996 by Burris and Idziak ([**16**], Lemmas 10, 11) in the special context of directly representable equational classes. In 1997 it was used in the general multiplicative setting by Burris and Sárközy ([**17**], Lemma 3.1 and Theorem 3.4). For a clear statement and proof in the general setting see Burris [**15**], Proposition 12.17.

LEMMA 2.7. (Burris and Idziak [**16**], 1996) *Let $\mathcal{A}$ be a multiplicatively admissible class of structures. Given a first-order sentence φ, there is a positive integer c_φ and a partition of $\mathcal{P}$ into classes $\mathcal{P}_1, \ldots, \mathcal{P}_k$ such that $\mathcal{A}_\varphi$ is a finite union of disjoint partition classes $\vec{\gamma} \star \vec{\mathcal{P}}$, and each $\vec{\gamma}_i$ is one of* $0, 1, 2, \ldots, c_\varphi - 1, (\geq c_\varphi)$.

KEY IDEA OF PROOF. Choose a Feferman-Vaught sequence $\langle \Phi; \varphi_1, \ldots, \varphi_r \rangle$ for φ, and then choose Feferman-Vaught sequences $\langle \Phi_i; \varphi_{i,1}, \ldots, \varphi_{i,r_i} \rangle$ for the φ_i. Define the partition of $\mathcal{P}$ by letting two structures $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be equivalent if they satisfy the same $\varphi_{i,j}$. □

Thus when searching for conditions on admissible classes $\mathcal{A}$ which allow one to conclude, just by examining $\mathbf{A}(x)$ and/or $\mathbf{P}(x)$, that $\mathcal{A}$ has a logical limit law, the focus shifts entirely to item (2) above, namely showing that all partition classes have asymptotic density. This requires tools from analysis and combinatorics. When using Compton's method, it is Proposition 2.5 that says one needs to know *every* partition class has asymptotic density, even though it is clear that only countably many such classes can be defined by sentences φ.

The formulation of Compton's approach to asymptotic density for unlabelled structures, as simply requiring that ($\star$) holds, was used for both the additive and multiplicative cases in Burris's book [**15**], after being introduced for the multiplicative case by Burris and Sárközy [**17**]. In the additive case the main results are for local asymptotic density $\delta(\mathcal{A}_\varphi)$; in the multiplicative case they are for global asymptotic density $\Delta(\mathcal{A}_\varphi)$.

2.4. Number systems. Since Compton's method depends solely on the partition classes of $\mathcal{A}$ having asymptotic density in $\mathcal{A}$, it is convenient in the unlabelled setting to switch the focus from admissible classes to number systems. This is essentially what Compton did in the additive case by selecting representatives for the isomorphism classes. One can view these representatives as the 'numbers' of an additive number system, a system closed under addition with each member having a size and a unique decomposition into indecomposables. The size of the sum of two members is the sum of their sizes, justifying the name 'additive number system'. Likewise one has multiplicative number systems.

Number systems have fundamental identities of the forms we have seen, and, in a natural sense, they are determined by their fundamental identities. Generalized (or abstract) number systems have been studied intensely with regard to the prime number theorem. The goal was to analyze the classical prime number theorem for the integers—and more generally, Landau's prime ideal theorem, for the ideals of the integers of an arithmetic number field—to determine just how little was needed from the properties of the classical number systems in order to prove a prime number theorem. The pioneer in this work was Beurling[10] [**13**] 1937. In 1975 Knopfmacher [**37**] published his first book on the subject, *Abstract Analytic Number Theory*, with a strong emphasis on multiplicative number systems derived from well-known classes of structures such as groups and rings. Burris's book [**15**] adopts the Beurling-Knopfmacher framework of abstract number systems, but replaces the goal of proving a prime number theorem with the goal of proving that all partition sets have asymptotic density in the number system. It turned out that the conditions that had been found for proving a prime number theorem also sufficed to prove all partition sets have asymptotic density. (See Corollary 3.14.)

3. Further results regarding Compton's method

This section presents the main results on limit laws (up to June, 2011) for admissible classes, following the publication of Compton's papers in 1987/1989.

3.1. Directly representable equational classes. Burris and Idziak [**16**], 1996, published the first paper on logical limit laws for multiplicative systems. They showed that a finitely generated directly representable equational class[11] $\mathcal{A}$ has a *discrete* logical limit law, that is, there is a positive integer m such that the probability of a FO sentence φ holding in $\mathcal{A}$ is one of $\{0, 1/m, 2/m, \ldots, 1\}$. Given such an $\mathcal{A}$, let m_0 be the smallest choice of m. Then one always has a sentence φ such that $\mathsf{Prob}_{\mathcal{A}}(\varphi) = 1/m_0$. Furthermore, $\mathcal{A}$ has a 0–1 law iff $m_0 = 1$ iff $\mathcal{A}$ has unique factorization.

Thus, for example, the class of Boolean algebras has a FO 0–1 law. On the other hand, consider the equational class of Abelian groups of exponent 2 with two arbitrary constants; the smallest choice of m is $m_0 = 5$, so one could say that this class has a FO 0–$\frac{1}{5}$–$\frac{2}{5}$–$\frac{3}{5}$–$\frac{4}{5}$–1 law.

This paper was novel in that some of the classes it dealt with do not have unique factorization, and hence are not admissible. However, thanks to the detailed study of such equational classes by McKenzie [**43**], it is known that the possible factorizations of a finite algebra in such a class are 'well-behaved'.

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[10]During WWII the Swedish mathematician Beurling did brilliant work deciphering German codes—later he became a member of the Institute for Advanced Study in Princeton.

[11]These are finitely generated equational classes with only finitely many finite directly indecomposable members. Unfortunately they often lack the unique factorization property, and hence are not admissible classes—but some of these classes, such as Boolean algebras, are indeed admissible.

3.2. Unary functions. In response to Compton's work and questions on unary functions, Woods proved the following.

THEOREM 3.1 (Woods [**56**] Theorems 1.1 and 1.2, 1997). *Let $\mathcal{A}$ be an additively admissible class for which there are constants $C > 0$, $\beta > 0$ and $\alpha < 1$ such that:*

(a) (Labelled):

$$\frac{a(n)}{n!} \sim C\beta^n n^{-\alpha}, \quad \textit{and} \quad \frac{p(n)}{n!} = \mathrm{O}\big(\beta^n/n\big)$$

(b) (Unlabelled):

$$a(n) \sim C(1+\beta)^n n^{-\alpha}, \quad \textit{and} \quad p(n) = \mathrm{O}\big((1+\beta)^n/n\big).$$

Then $\mathcal{A}$ has a MSO limit law, and indeed the asymptotic probability of a MSO sentence φ is given by $\mathsf{Prob}(\varphi) = \partial(\mathcal{A}_\varphi)$.

Woods noted that one could replace the constant C by $L(n)$ where $L(x)$ is slowly varying at infinity. In addition to finding suitable hypotheses, the key step of Woods was to prove appropriate Tauberian theorems, to convert the existence of $\overline{\mu}(\varphi)$ to the existence of $\mu(\varphi)$; and likewise for ν. This theorem was applied to the class $\mathcal{A}$ of a single [partial] unary function with m colors.

3.3. Trees and forests. In a second paper in 1997, Woods [**55**] answered another question posed by Compton.

THEOREM 3.2. *Let $\mathcal{A}$ be the class of m-colored trees. Then $\mathcal{A}$ has both a labelled and an unlabelled MSO limit law.*

The proof started from Compton's observation that, given a positive integer q, the equivalence classes $\mathcal{A}_i$ of m-colored trees under the equivalence relation $\equiv_q^{\mathsf{MSO}}$ satisfy a fairly simple system of equations $\mathcal{A}_i = \Phi_i(\mathcal{A}_1, \ldots, \mathcal{A}_k)$. This gives a system of equations $y_i = \mathbf{G}_i(x, y_1, \ldots, y_k)$ that is solved by $y_i = \mathbf{A}_i(x)$, where the $\mathbf{A}_i(x)$ are generating series for the classes $\mathcal{A}_i$. A combination of extraordinary factors make it possible to show that the asymptotic density of the $\mathcal{A}_i$ exist—in particular the fact that the dependency digraph of the system has a single strong component that immediately dominates all other nodes in the digraph, and the fact that the Jacobian of the $\mathbf{G}_i$ with respect to the y_j is a stochastic matrix. This combination of properties is very rare, and allows for a precision attack using the Perron-Frobenius results on the dominant eigenvalue of a non-negative matrix. Finally, given a MSO sentence φ of quantifier depth q, one has $\mathcal{A}_\varphi$ equal to a union of some of the $\mathcal{A}_i$, so it also has asymptotic density in $\mathcal{A}$.

COROLLARY 3.3. *Let $\mathcal{A}$ be the class of m-colored forests. Then $\mathcal{A}$ has both a labelled and an unlabelled MSO limit law.*

This follows easily, as noted by Compton. Extend the mapping $F \mapsto \bullet/F$, which attaches a root to a forest, to a mapping from a class $\mathcal{F}$ of forests to a class $\mathcal{T} = \bullet/\mathcal{F}$ of trees. If $\mathcal{F}$ is defined by a MSO sentence, then so is $\bullet/\mathcal{F}$. With this simple device one sees that the MSO limit law for trees gives a MSO limit law for forests. This argument lifts to m-colored trees and forests.

Woods' analysis shows that $\mathsf{Prob}(\varphi)$ is positive iff the radius of convergence of $\mathbf{A}_\varphi(x)$ is equal to that of $\mathbf{A}(x)$. At the end of the paper Woods noted that this leads to a labelled MSO limit law for connected acyclic graphs, and that McColm had informed him that the results on trees actually gave a labelled MSO 0–1 law for these graphs—McColm published his proof in [**41**] 2002. Woods ended his paper by asking if there is also an unlabelled MSO limit law for connected acyclic graphs, and if so, is it a 0–1 law. McColm published a positive answer, that indeed there is a 0–1 law, in [**42**] Corollary 2.1, 2004.

3.4. A presentation convention. Much of the research on finding ways to prove ($\star$) was motivated by the expectation that for every local result in the additive case there would be a corresponding global result in the strictly multiplicative case. So far this expectation has been well-founded—consequently for the rest of this section (on main results) each theorem has two parts, the ($\mathfrak{a}$) part and the ($\mathfrak{m}$) part:

- Item ($\mathfrak{a}$) is for the (unlabelled) additive case (with disjoint union);
- Item ($\mathfrak{m}$) is for the (unlabelled) strictly multiplicative case (with direct product);

and rarely there is a third part, the $L\mathfrak{a}$ part:

- Item ($L\mathfrak{a}$) is for the labelled additive case (with disjoint union).

To avoid being overly repetitious in stating conclusions, let it be noted that every result stated below, which concludes with the existence of a logical limit law for an admissible class $\mathcal{A}$ via Compton's method, can be strengthened to say:

furthermore the probability of each φ is equal to the Dirichlet density $\partial(\mathcal{A}_\varphi)$.

This can be quite useful when explicitly calculating $\mathsf{Prob}(\varphi)$.

3.5. Two consequences of $\mathcal{P}$ having density 0. While Burris was writing the book [**15**], he was fortunate to have Bell, a graduate student at UC San Diego, and Warlimont, a retired German professor living in South Africa, proving key results on the fundamental consequences of the condition ($\star$). It was easy to prove Theorem 3.7 below, that ($\star$) implies $\mathcal{P}$ has density 0; but to show that $\mathcal{P}$ having density 0 leads to important restrictions on the generating series was quite challenging.

THEOREM 3.4. *Let $\mathcal{A}$ be an admissible class. Then:*

($\mathfrak{a}$) $\delta(\mathcal{P}) = 0$ *implies* $\rho_A > 0$.

[Bell [**2**] 2000, Theorem 1(a)]

($\mathfrak{m}$) $\Delta(\mathcal{P}) = 0$ *implies* $\alpha_A < \infty$.

[Warlimont [**52**], 2001]

Bell actually proved a stronger result than that stated in ($\mathfrak{a}$), namely $\delta(\mathcal{P}) = 0$ implies $\limsup_{n\to\infty} \dfrac{p(n)}{a(n)} = 1$. Consequently, if an admissible class $\mathcal{A}$ with $\rho_A = 0$ has a MSO limit law then, since $\mathcal{P}$ is definable by a MSO sentence, it must be the case that $\delta(\mathcal{P}) = 1$, that almost all members of $\mathcal{A}$ are connected.

THEOREM 3.5. *Let $\mathcal{A}$ be an admissible class. Then:*

($\mathfrak{a}$) $\delta(\mathcal{P}) = 0$ *implies* $\mathbf{A}(\rho_A) = \infty$.

[Bell, Bender, Cameron and Richmond [**12**] Theorem 1, 2000]

($\mathfrak{m}$) $\Delta(\mathcal{P}) = 0$ *implies* $\mathbf{A}(\alpha_A) = \infty$.

[Warlimont [**53**], 2003]

REMARK 3.6. In an unpublished note ca. 2002, I. Ruzsa showed that one can easily derive the multiplicative results of the previous two theorems from the additive results.

3.6. Two consequences of the condition ($\star$). The paper [**17**] of Burris and Sárközy is the foundation paper on adapting Compton's method to multiplicative systems, followed by the paper [**18**] of Burris, Compton, Odlyzko and Richmond that treats the case of 0–1 laws in the multiplicative setting.

THEOREM 3.7. *Let $\mathcal{A}$ be an admissible class for which ($\star$) holds. Then:*

($\mathfrak{a}$) $\delta(\mathcal{P}) = 0$.

[Burris [**15**] Proposition 3.28, 2001]

($\mathfrak{m}$) $\Delta(\mathcal{P}) = 0$.

[Burris and Sárközy [**17**] 1997, Proposition 5.7(c)]

THEOREM 3.8. *Let $\mathcal{A}$ be an admissible class for which ($\star$) holds. Then:*

($\mathfrak{a}$) $\mathbf{A}(x) \in \mathsf{RT}_{\rho_A}$.

[Burris [**15**] Corollary 3.30, 2001]

($\mathfrak{m}$) $\mathbf{A}(x) \in \mathsf{RV}_{\alpha_A}$.

[Burris and Sárközy [**17**] 1997, Corollary 5.10]

Summarizing the last four theorems gives the following corollary.

COROLLARY 3.9. *Let $\mathcal{A}$ be an admissible class for which ($\star$) holds. Then:*

($\mathfrak{a}$) $\delta(\mathcal{P}) = 0$, $\rho_A > 0$, $\mathbf{A}(\rho_A) = \infty$, *and* $\mathbf{A}(x) \in \mathsf{RT}_{\rho_A}$.

($\mathfrak{m}$) $\Delta(\mathcal{P}) = 0$, $\alpha_A < \infty$, $\mathbf{A}(\alpha_A) = \infty$, *and* $\mathbf{A}(x) \in \mathsf{RV}_{\alpha_A}$.

3.7. Conditions for 0–1 laws.

LEMMA 3.10.

($\mathfrak{a}$) $\mathbf{A}(x) \in \mathsf{RT}_\rho$ *implies* $\rho_A = \rho$.

($\mathfrak{m}$) $\mathbf{A}(x) \in \mathsf{RV}_\alpha$ *implies* $\alpha_A = \alpha$.

For comparison and symmetry of presentation, the ($\mathfrak{a}$) items in the next two theorems repeat previously stated results of Compton.

THEOREM 3.11. *Let $\mathcal{A}$ be an admissible class.*

($\mathfrak{a}$) *If $\rho_A > 0$ then $\mathcal{A}$ has a MSO 0–1 law iff it has a FO 0–1 law iff* $\mathbf{A}(x) \in \mathsf{RT}_1$.

[Compton [**23**] Theorem 5.9, 1987, and Compton [**26**] Theorem 6.4, 1989]

($\mathfrak{m}$) *If $\alpha_A < \infty$ then $\mathcal{A}$ has a FO 0–1 law iff* $\mathbf{A}(x) \in \mathsf{RV}_0$.

[Burris [**15**] Theorem 10.2, 2001, following on Burris, Compton, Odlyzko and Richmond [**18**] Theorem 2.1(a), 1997].

3.8. Conditions for other laws.

THEOREM 3.12. *Let $\mathcal{A}$ be an admissible class.*

($\mathfrak{a}$) *Suppose $\mathbf{A}(x) \in \mathsf{RT}_\rho$ and there are $C, K > 0$ such that*

$$\frac{a(n-k)}{a(n)} \;\leq\; C\rho^k \text{ for } K \leq k \leq n.$$

Then ($\star$) holds, so $\mathcal{A}$ has a MSO limit law.

[Compton [**26**] Theorem 6.6 (ii), 1989; Burris [**15**] Theorem 6.31(c), 2001] (𝔪) *Suppose* $A(x) \in \mathsf{RV}_\alpha$, *and there are* $C, K > 0$ *such that*

$$\frac{A(x/k)}{A(x)} \leq Ck^{-\alpha} \text{ for } K \leq k \leq n.$$

Then $(\star)$ *holds, so* $\mathcal{A}$ *has a FO limit law.*

[Burris and Sárközy [**17**] Theorem 6.3, 1997; Burris [**15**] Proposition 12.19(b), 2001]

After finishing the paper [**17**] on multiplicative classes with Sárközy, Burris initiated a thorough study of what had been achieved in Compton's work on additive classes. It came as a bit of a shock to discover that Compton's result, Theorem 3.12 (𝔞) for additive classes, looked remarkably similar to the Burris and Sárközy result (𝔪) for multiplicative classes. Several useful corollaries to Theorem 3.12 (𝔪) had obvious analogs in additive systems—these analogs had not been noted by Compton. It was at this point that a parallel development of logical limit laws for additive and multiplicative classes seemed possible, and work was started on the book [**15**]—Chapters 1–6 are on additive classes, Chapters 7–12 on multiplicative classes, with Chapter 1 corresponding to Chapter 7, etc.

Corollary 3.13. *Let* $\mathcal{A}$ *be an admissible class.*

(𝔞) *Suppose there is a* $\beta \geq 1$ *such that* $a(n) \sim \beta^n \cdot \widehat{a}(n)$, $\widehat{a}(n) \in \mathsf{RT}_1$, *and* $\widehat{a}(n)$ *is eventually non-decreasing. Then* $(\star)$ *holds, so* $\mathcal{A}$ *has a MSO limit law.*

[Burris [**15**] Corollary 5.15, 2001]

(𝔪) *Suppose there is an* $\alpha \geq 0$ *and* $A(x) \sim x^\alpha \cdot \widehat{A}(x)$, *where* $\widehat{A}(x) \in \mathsf{RV}_0$ *and* $\widehat{A}(x)$ *is eventually non-decreasing. Then* $(\star)$ *holds, so* $\mathcal{A}$ *has a FO limit law.*

[Burris and Sárközy [**17**] Corollary 6.5, 1997; Burris [**15**] Corollary 11.17, 2001]

The first part of this corollary to be proved was item (𝔪), for multiplicative classes, by Burris and Sárközy. Later, working on the parallel development for the book [**15**], Burris made the routine translation of (𝔪) into (𝔞) to have the corresponding additive result, and realized that this gave a tool to find the first examples of MSO laws for unlabelled classes based on Theorem 3.12 (𝔞) proved by Compton a decade earlier—these examples were additive classes with non 0–1 MSO limit laws. Item (𝔞) of Corollary 3.13, combined with results of Knopfmacher, Knopfmacher and Warlimont [**38**], showed that if one had positive constants a, b, C with $1, b < a$ and $p(n) = Ca^n + \mathrm{O}(b^n)$, then $a(n)$ satisfies the hypotheses of (𝔞). (In this case $a = 1/\rho_A$.) The admissible class $\mathcal{A}$ of 2-colored linear forests has $p(n) = 2^n$, so $\mathcal{A}$ satisfies $(\star)$, and thus has a MSO limit law. Since $\rho_A = 1/2$, it cannot be a 0–1 law.

As a special case of Corollary 3.13 we have:

Corollary 3.14. *Let* $\mathcal{A}$ *be an admissible class.*

(𝔞) *Suppose there is a* $\beta \geq 1$ *and a constant* $C > 0$ *such that* $a(n) \sim C\beta^n$. *Then* $(\star)$ *holds, so* $\mathcal{A}$ *has a MSO limit law.*

(𝔪) *Suppose there is an* $\alpha \geq 0$ *and a constant* $C > 0$ *such that* $A(x) \sim Cx^\alpha$. *Then* $(\star)$ *holds, so* $\mathcal{A}$ *has a FO limit law.*

[Burris and Sárközy [**17**] Corollary 6.6, 1997]

The simple condition in item (𝔪) shows that the requirements of Beurling as well as of Knopfmacher, for a multiplicative prime number theorem, suffice to guarantee that all partition sets have asymptotic density. The class $\mathcal{A}$ of Abelian groups, the original multiplicative example studied by Compton, is covered by Corollary 3.14 ($\mathfrak{m}$) since Erdős and Szekeres [**30**] proved that $A(x) \sim Cx$, where $C := \prod_{n=2}^{\infty} \zeta(n)$.

3.9. Combining admissible classes with a 0–1 law.

THEOREM 3.15. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be admissible classes.*

($\mathfrak{a}$) *If each $\mathcal{A}_i$ has a MSO 0–1 law and a positive radius of convergence, then $\mathcal{A}_1 + \cdots + \mathcal{A}_n$ has a MSO 0–1 law.*

[Stewart, see [**15**] Theorem 4.15, 2001]

($\mathfrak{m}$) *If each $\mathcal{A}_i$ has a FO 0–1 law and a finite abscissa of convergence, then $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ has a FO 0–1 law.*

[Odlyzko, see [**15**] Theorem 10.5, 2001]

3.10. More conditions for 0–1 laws. The search continued for user-friendly ways to apply Theorem 3.11, resp. Theorem 3.12, to obtain interesting examples of admissible classes that had logical 0–1 laws, resp. logical limit laws.

THEOREM 3.16. *Let $\mathcal{A}$ be an admissible class. Then:*

($\mathfrak{a}$) $p(n) = \mathrm{O}(n^c)$ *implies* $\mathbf{A}(x) \in \mathsf{RT}_1$*, showing that $\mathcal{A}$ has a MSO 0–1 law.*[12]

[Bell [**4**] Theorem 1.5, 2001]

(L$\mathfrak{a}$) $p(n) = \mathrm{O}\big(n^{\theta n}\big)$ *for some* $0 < \theta < 1$ *implies* $\mathcal{A} \to \infty$*, and thus $\mathcal{A}$ has a labelled MSO 0–1 law.*

[Burris and Yeats [**19**] Theorem 7, 2008]

($\mathfrak{m}$) $P(x) = \mathrm{O}\big((\log x)^c\big)$ *implies* $\mathbf{A}(x) \in \mathsf{RV}_0$*, and thus $\mathcal{A}$ has a FO 0–1 law.*

[Bell [**4**] Theorem 1.8, 2001]

Compton's question (Q3) above is completely answered by Theorem 3.16. After being informed of item ($L\mathfrak{a}$), Compton asked in an e-mail if, in view of his Theorem 1.10 above, the condition on $p(n)$ actually implies $\mathbf{A}(x)$ is Hayman admissible. This is still open.

THEOREM 3.17. *Let $\mathcal{A}$ be an admissible class. Then:*

($\mathfrak{a}$) $\mathbf{P}(x) \in \mathsf{RT}_1$ *implies* $\mathbf{A}(x) \in \mathsf{RT}_1$*, and thus $\mathcal{A}$ has a MSO 0–1 law.*[13]

[Bell and Burris [**6**] Theorem 9.1, 2003]

($\mathfrak{m}$) $\mathbf{P}(x) \in \mathsf{RV}_0$ *implies* $\mathbf{A}(x) \in \mathsf{RV}_0$*, and thus $\mathcal{A}$ has a FO 0–1 law.*

[Bell [**5**] Theorem 17, 2004]

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[12] Bell's proof followed in part from his previous study of the Bateman and Erdős paper, as an undergraduate at the University of Waterloo. Shortly after proving that polynomially bounded $p(n)$ led to MSO 0–1 laws, he proved the nearly 50-year old conjecture (in the Bateman-Erdős paper) about an error term for $a^{(k+1)}(n)/a^{(k)}(n)$, where $a^{(k)}(n)$ is the kth difference of $a(n)$. This result was in turn considerably generalized in [**8**], to give estimates for the error term when $p(n)$ is polynomially bounded.

[13] One lemma in the proof of this result showed that $\mathbf{P}(x) \in \mathsf{RT}_1 \Rightarrow e^{\mathbf{P}(x)} \in \mathsf{RT}_1$, which turned out to answer a conjecture of Durrett, Granovsky and Gueron [**29**] connected with problems in coagulation and fragmentation.

In particular this result shows that there are admissible classes $\mathcal{A}$ with a 0–1 law that have $a(n)$ growing much faster than in the cases covered by Theorem 3.16. This is most easily seen by viewing the fundamental identity as a mapping Θ taking the generating series $\mathbf{P}(x)$ to the generating series $\mathbf{A}(x)$. Theorem 3.17 says Θ preserves RT_1 [RV_0] in the additive [multiplicative] setting. Consider the sequence of series $\mathbf{P}(x), \Theta(\mathbf{P}(x)), \Theta^2(\mathbf{P}(x)), \ldots$ From Proposition 2.2, one can regard each of these series as the generating series for the indecomposables of an admissible class with a 0–1 law.

In the additive case, for example, if one starts with $p(n) = 1$ for all n, then the $\Theta^n(\mathbf{P}(x))$ give the partition hierarchy, with the case $n = 1$ being the generating series for partitions of integers (see [**7**]). One can realize this sequence of generating series by looking at the admissible classes $\mathcal{F}_n$ of forests of height at most n. Examining the asymptotics for this hierarchy quickly led to the fact that conditions like $p(n) \sim a\exp\left(bn^c\right)$, where $a, b, c > 0$ and $c < 1$, imply a MSO 0–1 law.

After proving Theorem 3.17, there was the question as to just how far one could have $p(n)$ deviate from RT_1 and still have a 0–1 law. In the additive case this was to a certain extent answered by the sandwich theorems in the 2004 paper of Bell and Burris. The first sandwich theorem in the additive case, and its recently proved multiplicative analog, are stated in the next theorem.

THEOREM 3.18 (Sandwich Theorem). *Let $\mathcal{A}_0$ and $\mathcal{A}$ be admissible classes.*

($\mathfrak{a}$) *Suppose $a_0(n) \in \mathsf{RT}_1$ and $p_0(n) \le p(n) = \mathrm{O}(a_0(n))$. Then $\mathbf{A}(x) \in \mathsf{RT}_1$, so $\mathcal{A}$ has a MSO 0–1 law.*

[Bell and Burris [**7**] Theorem 4.4, 2004]

($\mathfrak{m}$) *Suppose $A_0(x) \in \mathsf{RV}_0$ and $P_0(x) \le P(x) = \mathrm{O}(A_0(x))$. Then $\mathbf{A}(x) \in \mathsf{RV}_0$, so $\mathcal{A}$ has a FO 0–1 law.*

[This is a new result—the proof is in §4]

This gives a great deal of freedom to $p(n)$ as n increases. For example, if in the additive setting one has $p_0(n) = 1$ for all n then $a_0(n) \in \mathsf{RT}_1$, indeed,

$$a_0(n) \;\sim\; \frac{\exp\left(\pi\sqrt{2n/3}\right)}{4\sqrt{3}n}.$$

Thus for any constant $C > 0$, if $1 \le p(n) \le C \cdot \mathsf{part}(n)$, where $\mathsf{part}(n)$ is the number of partitions of n, then one has $a(n) \in \mathsf{RT}_1$, and any associated admissible class $\mathcal{A}$ will have a MSO 0–1 law. A natural example that has a MSO 0–1 law based on ($\mathfrak{a}$) is the class $\mathcal{G}_n$ of acyclic graphs of diameter at most n.

There is a second sandwich theorem in the additive case that allows the condition $p_0(n) \le p(n)$ to fail for finitely many values of n, that is, $p(n)$ is eventually greater or equal to $p_0(n)$. It requires a considerable strengthening of the other hypotheses.

THEOREM 3.19 (Eventual Sandwich Theorem). *Let $\mathcal{A}$ be an admissible class. Then:*

($\mathfrak{a}$) *Suppose $p_0(n) \in \mathsf{RT}_1$, $p_0(n) \le p(n)$ for n sufficiently large, $p(n) = \mathrm{O}(a_0(n))$, and $\sum_n (p(n) - p_0(n)) \ge 0$. Then $\mathbf{A}(x) \in \mathsf{RT}_1$, so $\mathcal{A}$ has a MSO 0–1 law.*

[Bell and Burris [**7**] Theorem 5.3, 2004]

($\mathfrak{m}$) (*So far there is no analog for the multiplicative case.*)

Thus in the example for the previous theorem, with $p_0(n) = 1$, one can allow $p(n) = 0$ for finitely many values of n provided the places where $p(n)$ exceeds 1 compensates for the places where it took on the value 0.

3.11. Combining admissible classes with general limit laws. Compton's method for proving logical limit laws, based solely on properties of the generating functions, succeeds precisely when $(\star)$ holds, that is, all partition sets have asymptotic density. What happens if one combines admissible classes $\mathcal{A}_i$ which satisfy $(\star)$? (Combining admissible classes with a 0–1 law was considered in Theorem 3.15.)

The definitive study on this topic is due to Yeats. Unlike other results in this article, this cannot be reduced to the study of admissible classes $\mathcal{A}_i$ that satisfy our standing assumption that the $d_i = 1$, where $d_i := \gcd \mathsf{Spec}(\mathcal{A}_i)$. For the next theorem (and only for the next theorem), the d_i are unrestricted in the $(\mathfrak{a})$ part; and in the $(\mathfrak{m})$ part one allows discrete as well as the usual strictly multiplicative classes.

THEOREM 3.20. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be admissible classes satisfying $(\star)$. For $i = 1, 2$ let $d_i = \gcd \mathsf{Spec}(\mathcal{A}_i)$.*

$(\mathfrak{a})$ *Let ρ_i be the radius of convergence of $\mathcal{A}_i$, with $\rho_1 \le \rho_2$. Then $\mathcal{A}_1+\mathcal{A}_2$ satisfies $(\star)$ iff $d_1 \mid d_2$ or $\rho_1 = \rho_2$.*

[Yeats [**57**] Theorem 57, 2002]

$(\mathfrak{m})$ *Let α_i be the abscissa of convergence of $\mathcal{A}_i$, with $\alpha_1 \ge \alpha_2$. Then $\mathcal{A}_1 \times \mathcal{A}_2$ satisfies $(\star)$ iff*

- *$\alpha_1 = \alpha_2$, or*
- *$\alpha_1 > \alpha_2$ and $\mathcal{A}_1$ is strictly multiplicative, or*
- *$\alpha_1 > \alpha_2$ and $\mathcal{A}_1, \mathcal{A}_2$ are both discrete and λ_2 is a power of λ_1.*

[Yeats [**57**] Theorem 68, 2002]

3.12. More conditions for general limit laws.

THEOREM 3.21. *Let $\mathcal{A}$ be an admissible class. Then:*

$(\mathfrak{a})$ *$\rho_A > 0$, $p(n) \in \mathsf{RT}_{\rho_A}$ and $\liminf_{n\to\infty} n{\rho_A}^n p(n) > 1$ imply $(\star)$ holds, so $\mathcal{A}$ has a MSO limit law.*

[Bell and Burris [**6**] Theorem 9.3, 2003]

$(\mathfrak{m})$ *$\alpha_A > 0$, $P(x) \sim x^{\alpha_A} P_0(x) / \log x$ for some non-decreasing $P_0(x) \in \mathsf{RV}_0$, and $\lim_{x\to\infty} P_0(x) \in (1/\alpha_A, \infty)$ imply $(\star)$ holds, so $\mathcal{A}$ has a FO 0–1 law.*

[Bell [**5**] Theorem 18, 2004]

This theorem is a favorite for finding logical limit laws that are not 0–1 laws—in the additive setting it is more general than our original method for finding such laws using the asymptotic results of Knopfmacher, Knopfmacher and Warlimont mentioned in §3.8. An interesting example is the class of forests of planted plane trees of height at most h.

4. The multiplicative sandwich theorem

The multiplicative analogue of the additive sandwich theorem has not appeared in the literature. In this chapter we fill this gap by proving the following result.

THEOREM 4.1. *Suppose that*

$$\mathbf{A}_0(s) = \sum_{n=1}^{\infty} a_0(n) n^{-s} = \prod_{j\geq 2}(1-1/j^s)^{-p_0(j)}$$

has the property that $p_0(n)$ *is a nonnegative integer for all* n *and* $A_0(x)$ *is slowly varying at infinity. If*

$$\mathbf{A}(s) = \sum_{n=1}^{\infty} a(n) n^{-s} = \prod_{j\geq 2}(1-1/j^s)^{-p(j)}$$

has the property that

$$P_0(x) \leq P(x) = \mathrm{O}(A_0(x))$$

for all $x \geq 1$*, then* $A(x)$ *is slowly varying at infinity. Thus any multiplicatively admissible class* $\mathcal{A}$ *with generating series* $\mathbf{A}(x)$ *will have a first-order 0–1 law.*

We give a streamlined proof that uses Theorem 3.16 ($\mathfrak{m}$). It is possible to give a slightly longer proof that does not use this result. Let $\mathbf{R}(s) = \sum_{n\geq 1} r(n)n^{-s}$ be a Dirichlet series with non-negative coefficients. Note that

$$-\frac{d}{ds}\mathbf{R}(s) = \sum_{n\geq 1} r(n)\log(n) n^{-s}.$$

Let $\overline{R}(x)$ denote the global count function of $-\frac{d}{ds}\mathbf{R}(s)$; that is,

$$\overline{R}(x) = \sum_{n\leq x} r(n)\log n. \tag{4.1}$$

We need a few simple estimates about the global count functions of products of Dirichlet series. These results are not optimal.

LEMMA 4.2. *Let* $\mathbf{A}(s)$ *be a Dirichlet series with nonnegative coefficients and suppose that* $A(x)$ *is slowly varying at infinity. If*

$$\mathbf{B}(s) = \mathbf{A}(s)\prod_{j=1}^{r}(1-1/m_j^s)^{-1},$$

then we have the following:

(i) $B(x)$ *is slowly varying at infinity and* $A(x) = \mathrm{o}(B(x))$,

(ii) *if* $r = 1$, $\overline{B}(x) \geq A(x)(\log x)/2$ *for* x *sufficiently large;*

(iii) *if* $r > 1$, $A(x)\log(x) = \mathrm{o}(\overline{B}(x))$.

PROOF. We first consider (i) in the case that $\mathbf{B}(s) = \mathbf{A}(s)(1-1/m^s)^{-1}$. In this case $B(x) = \sum_{j\leq \log x/\log m} A(x/m^j)$, and so $B(x) - B(x/m) = A(x)$. To show that $B(x)$ is slowly varying at infinity, it is therefore sufficient to show that $A(x) = \mathrm{o}(B(x))$. Note, however, that $A(x/m)/A(x) \to 1$ as $x \to \infty$; hence if k is fixed we see that

$$\liminf_{x\to\infty}\frac{B(x)}{A(x)} \;\geq\; \lim_{x\to\infty}\frac{A(x)+A(x/m)+\cdots+A(x/m^{k-1})}{A(x)} \;=\; k.$$

Thus $A(x) = \mathrm{o}(B(x))$. This demonstrates (i) in the case that $r = 1$. The general case follows by induction on r.

To prove (ii), we have $r = 1$. Again assume that

$$\mathbf{B}(s) = \mathbf{A}(s)(1 - 1/m^s)^{-1}.$$

Fix a positive number x. For $n \le x$, let c_n denote the largest nonnegative integer satisfying $m^{c_n} \le x/n$. Then $x/m < nm^{c_n} \le x$. Moreover

$$b(nm^{c_n}) \;\ge\; a(nm^{c_n}) + a(nm^{c_n - 1}) + \cdots + a(n).$$

Note that

$$\begin{aligned}
\overline{B}(x) &= \sum_{n \le x} b(n) \log n \\
&\ge \sum_{\substack{n \le x \\ m \nmid n}} b(nm^{c_n}) \log(nm^{c_n}) \\
&\ge \sum_{\substack{n \le x \\ m \nmid n}} b(nm^{c_n}) \log(x/m) \\
&\ge (\log(x/m)) \sum_{\substack{n \le x \\ m \nmid n}} (a(n) + a(nm) + \cdots + a(nm^{c_n})) \\
&\ge \frac{\log(x)}{2} \sum_{k \le x} a(k) \quad \text{(for } x \text{ sufficiently large)} \\
&= \log(x) A(x)/2.
\end{aligned}$$

For (iii), suppose that $r \ge 2$. Let $\mathbf{A}_0(s) = \prod_{j=1}^{r-1}(1 - 1/m_j^s)^{-1}\mathbf{A}(s)$. By (i), $\mathbf{A}_0(s)$ is slowly varying at infinity. By (ii), we have $\overline{B}(x) \ge \log(x)A_0(x)/2$ for x sufficiently large. But by (i), $A(x) = \mathrm{o}\big(A_0(x)\big)$. Thus we obtain (iii). □

For functions $\mathbf{B}(s) = \prod_{j=1}^{r}(1 - 1/m_j^s)^{-1}$, the global count functions are well understood (Bell [**3**] Theorem 3.5, 2001):

$$\mathbf{B}(s) = \prod_{j=1}^{r}(1 - 1/m_j^s)^{-1} \quad \Rightarrow \quad B(x) = C \cdot (\log x)^r + \mathrm{O}\big((\log x)^{r-1}\big) \tag{4.2}$$

where

$$C \;=\; \frac{1}{r!(\log 2)\cdots(\log r)}. \tag{4.3}$$

We introduce the concept of *domination*, which will be used in giving a criterion for the product of two Dirichlet series to have a global count function that is slowly varying at infinity.

DEFINITION 4.3. *Let* $\mathbf{F}(s) = \sum_{n \ge 1} f(n)/n^s$ *and* $\mathbf{G}(s) = \sum_{n \ge 1} g(n)/n^s$ *be two Dirichlet series with nonnegative coefficients. We say* $\mathbf{F}(s)$ *is* dominated by $\mathbf{G}(s)$ *if* $F(x) = \mathrm{o}(G(x))$.

LEMMA 4.4. *Suppose* $\mathbf{R}(s)$ *is a Dirichlet series with nonnegative coefficients. Then either* $R(x)$ *is bounded or* $\mathbf{R}(s)$ *is dominated by* $-\mathbf{R}'(s)$.

PROOF. Let $\varepsilon > 0$. Then

$$\overline{R}(x) \;=\; \sum_{n \le x} r(n) \log n$$

$$\begin{aligned}
&\geq \sum_{e^{1/\varepsilon}\leq n\leq x} r(n)\log n\\
&\geq \frac{1}{\varepsilon}\sum_{e^{1/\varepsilon}\leq n\leq x} r(n)\\
&\geq \frac{1}{\varepsilon}\left(R(x)-R(\exp(1/\varepsilon))\right)\\
&= \frac{1}{\varepsilon}\cdot R(x)+\mathrm{O}(1).
\end{aligned}$$

It follows that either $R(x)$ is bounded or

$$\liminf_{x\to\infty}\frac{\overline{R}(x)}{R(x)} \;\geq\; \frac{1}{\varepsilon}.$$

□

LEMMA 4.5. *Let*

$$\mathbf{A}(s) \;=\; \sum_{n\geq 1} a(n)n^{-s} \text{ and } \mathbf{B}(s) \;=\; \sum_{n\geq 1} b(n)n^{-s}$$

be two Dirichlet series with nonnegative coefficients. If $\mathbf{A}(s)$ is slowly varying at infinity and $\mathbf{B}(s)$ is dominated by $\mathbf{C}(s) = \mathbf{A}(s)\mathbf{B}(s)$, then $\mathbf{A}(s)\mathbf{B}(s)$ is also slowly varying at infinity.

PROOF. Fix $\varepsilon > 0$. Then we have $C(x) = \sum_{n\leq x} A(x/n)b(n)$. Since $\mathbf{A}(s)$ is slowly varying at infinity, we can find a positive number $M \geq 1$ such that $A(2x) - A(x) < \varepsilon A(2x)$ for $x > M$. Furthermore, as $\mathbf{B}(s)$ is dominated by $\mathbf{C}(s)$ we have

$$2A(2M)B(2x) \leq \varepsilon C(2x)$$

for all x sufficiently large. Consequently,

$$\begin{aligned}
C(2x)-C(x) &= \sum_{n\leq 2x} A(2x/n)b(n) - \sum_{n\leq x} A(x/n)b(n)\\
&= \sum_{n\leq x}\left(A(2x/n)-A(x/n)\right)b(n) + \sum_{x<n\leq 2x} A(2x/n)b(n)\\
&\leq \sum_{x/M\leq n\leq x} A(2x/n)b(n) + \varepsilon\sum_{n<x/M} A(2x/n)b(n) + A(1)B(2x)\\
&\leq A(2M)B(x) + \varepsilon\sum_{n\leq 2x} A(2x/n)b(n) + A(1)B(2x)\\
&\leq \varepsilon C(2x) + 2A(2M)B(2x)\\
&\leq 2\varepsilon C(2x),
\end{aligned}$$

for all x sufficiently large. The result follows. □

LEMMA 4.6. *Let $\mathbf{H}(s) = \sum_{n\geq 1} h(n)n^{-s}$ be a Dirichlet series with nonnegative coefficients such that $h(1) = 0$ and let $\mathbf{A}(s) = \exp(\mathbf{H}(s))$. If $-\mathbf{H}'(s)$ is dominated by $\mathbf{B}(s)$, then either $\mathbf{A}(s)$ is dominated by $\mathbf{C}(s) = \mathbf{B}(s)\mathbf{A}(s)$ or $A(x)$ is uniformly bounded.*

PROOF. Let $\varepsilon > 0$. Since $-\mathbf{H}'(s)$ is dominated by $\mathbf{B}(s)$, there is $M > 0$ such that $\overline{H}(x) \le \varepsilon B(x)$ for $x > M$.

We note that $-\mathbf{A}'(s) = -\mathbf{H}'(s)\mathbf{A}(s)$ and hence

$$\begin{aligned}
\overline{A}(x) &= \sum_{n \le x} \overline{H}(x/n)a(n) \\
&= \sum_{n<x/M} \overline{H}(x/n)a(n) \;+ \sum_{x/M \le n \le x} \overline{H}(x/n)a(n) \\
&\le \varepsilon \sum_{n<x/M} B(x/n)a(n) \;+ \sum_{x/M \le n \le x} \overline{H}(x/n)a(n) \\
&= \varepsilon C(x/M) \;+ \sum_{x/M \le n \le x} \overline{H}(x/n)a(n) \\
&\le \varepsilon C(x/M) \;+\; \overline{H}(M)A(x) \\
&\le \varepsilon C(x/M) \;+\; \varepsilon B(M)A(x) \\
&\le \varepsilon C(x) \;+\; \varepsilon B(M)A(x).
\end{aligned}$$

By Lemma 4.4, either $\overline{A}(x)/A(x) \to \infty$ or $A(x)$ is uniformly bounded. If $A(x)$ is uniformly bounded, we are done. Otherwise, we have $C(x)/A(x) \to \infty$, so $\mathbf{A}(s)$ is dominated by $\mathbf{C}(s)$. □

PROOF OF THEOREM 4.1. Let $\mathbf{P}_0(s)$ and $\mathbf{P}(s)$ denote respectively the Dirichlet generating series for $p_0(n)$ and $p(n)$. If $\sum_n p_0(n) < \infty$, then $A_0(x)$ is polylog bounded, by Lemma 4.2, and hence $\overline{P}(x)$ is polylog bounded. This case follows from Theorem 3.16 ($\mathfrak{m}$).

Now consider the case that $\sum_n p_0(n) = \infty$. Then there exist distinct integers n_1 and n_2 with $p_0(n_1) + p_0(n_2) \ge 2$. Let

$$\mathbf{P}_1(s) \;=\; \sum_n p_1(n)/n^s \;:=\; \mathbf{P}_0(s) - 1/n_1^s - 1/n_2^s \tag{4.4}$$

and

$$\mathbf{P}_2(s) \;=\; \sum_n p_2(n)/n^s \;:=\; \mathbf{P}(s) - \mathbf{P}_1(s). \tag{4.5}$$

Similarly, we define

$$\mathbf{A}_1(s) \;=\; \sum_n a_1(n)/n^s \;:=\; \prod_j (1 - 1/j^s)^{-p_1(j)} \tag{4.6}$$

and

$$\mathbf{A}_2(s) \;=\; \sum_n a_2(n)/n^s \;:=\; \prod_j (1 - 1/j^s)^{-p_2(j)}. \tag{4.7}$$

Since $\mathbf{P}(s) = \mathbf{P}_1(s) + \mathbf{P}_2(s)$, we have

$$\mathbf{A}(s) \;=\; \mathbf{A}_1(s)\mathbf{A}_2(s).$$

Our goal is to show that $A_1(x)$ is slowly varying at infinity and that $\mathbf{A}_2(s)$ is dominated by $\mathbf{A}(s)$. Once we show this, we can use Lemma 4.5 to infer that $A(x)$ is also slowly varying at infinity.

We note that $\mathbf{A}_1(s) = \mathbf{A}_0(s)(1 - 1/n_1^s)^{-1}(1 - 1/n_2^s)^{-1}$. Since $A_0(x)$ is slowly varying at infinity, $A_1(x)$ is also slowly varying at infinity by part (i) of Lemma

4.2. Furthermore, $A_0(x)\log(x) = \mathrm{o}(\overline{A}_1(x))$ by part (iii) of Lemma 4.2. Note that $\mathbf{A}_2(s) = \exp(\mathbf{H}_2(s))$, where

$$\mathbf{H}_2(s) \;=\; \sum_{n\geq 1} h_2(n)n^{-s}$$

and

$$h_2(n) \;=\; \sum_{j^\ell = n} p_2(j)/\ell.$$

It follows that

$$\begin{aligned}
\overline{H}_2(x) &= \sum_{n\leq x} h_2(n)\log n \\
&= \sum_{n\leq x} \log(n) \sum_{j^\ell=n} p_2(j)/\ell \\
&\leq \sum_{2\leq j\leq x} \sum_{\ell\leq \log\, x/\log\, j} \log(j^\ell)p_2(j)/\ell \\
&= \sum_{2\leq j\leq x} \sum_{\ell\leq \log\, x/\log\, j} (\log\, j)p(j) \\
&\leq \log(x)P(x) \\
&\leq C\log(x)A_0(x) \quad \text{for some } C > 0 \\
&= \mathrm{o}(\overline{A}_1(x)).
\end{aligned}$$

Thus we see that $-\mathbf{H}_2'(s)$ is dominated by $\mathbf{A}_1(s)$. It follows that $\mathbf{A}_2(s)$ is dominated by $\mathbf{A}_1(s)\mathbf{A}_2(s) = \mathbf{A}(s)$. The result follows. □

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