Contemporary Mathematics

Spectra and Systems of Equations

Jason P. Bell, Stanley N. Burris, and Karen Yeats

Abstract. Periodicity properties of sets of nonnegative integers, defined by systems $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ of equations, are analyzed. Such systems of set equations arise naturally from equational specifications of combinatorial classes—Compton's equational specification of monadic second order classes of trees is an important example.

In addition to the general theory of set equations and periodicity, with several small illustrative examples, two applications are given:

(1) There is a new proof of the fundamental result of Gurevich and Shelah on the periodicity of monadic second order classes of finite monounary algebras. Also there is a new proof that the monadic second order theory of finite monounary algebras is decidable.

(2) A formula derived for the periodicity parameter $\mathfrak{q}$ is used in the determination of the asymptotics for the coefficients of generating functions defined by well conditioned systems of equations.

1. Introduction

Logicians have developed the subject of finite model theory to study classes of finite *structures* defined by sentences in a formal logic. (In logic, a structure is a set equipped with a selection of functions and/or relations and/or constants.) Combinatorialists have been interested in classes of *objects* defined by equational specifications (the objects are not restricted to the structures studied by logicians). In recent years, there has been an increasing interest in the study of specifications involving several equations, and this article continues these investigations. Background, definitions, and an introduction to the topics are given in §1.1–§1.4. §1.5 is an outline, giving the order of presentation of the topics.

1.1. Combinatorial classes, generating functions and spectra.

A *combinatorial class* $\mathcal{A}$ is a class of objects with a function $\| \ \|$ that assigns a positive integer $\|\mathfrak{a}\|$, the *size* of $\mathfrak{a}$, to each object $\mathfrak{a}$ in the class, such that there are only finitely many objects of each size in the class.[1] (When counting the objects of a

2010 *Mathematics Subject Classification.* Primary 03C13; Secondary 05A15 .

The authors thank NSERC for supporting this project.

[1]This definition is essentially that used in the recently published book *Analytic Combinatorics* by Flajolet and Sedgewick [**19**]—it differs in that we do not allow objects of size 0. These authors have, for several years, kindly made pre-publication drafts of this book available; these drafts have played a singularly important role in our own education and investigations.

given size, it is often the case that one is counting modulo some natural equivalence relation, such as isomorphism.) The objects of $\mathcal{A}$ are said to be *unlabelled*.

Given a combinatorial class $\mathcal{A}$, let the *count function* $a(n)$ for $\mathcal{A}$ be the number of (unlabelled) objects of size n in $\mathcal{A}$. Two combinatorial classes $\mathcal{A}, \mathcal{B}$ are *isomorphic* if they have the same count function. Combinatorics is essentially the study of count functions of combinatorial classes, and generating functions provide the main tool for this study. The *(ordinary) generating function* $A(x)$ of $\mathcal{A}$ is the formal power series

$$A(x) \ := \ \sum_{n=1}^{\infty} a(n)x^n.$$

In the following, only unlabelled objects are considered, and thus the only generating functions discussed are ordinary generating functions.

Throughout this article, a power series and its coefficients will use the same letter—for power series the name is upper case, for coefficients the name is lower case, as with $A(x)$ and $a(n)$ above. When a combinatorial class is named by a single letter, say $\mathcal{A}$, the same letter will be used, in the appropriate size and font, to name its count function and its generating function, just as above with $\mathcal{A}$, $a(n)$, $A(x)$. For a combinatorial class with a composite name, say $\mathsf{MSet}(\mathcal{A})$, we can also use $\mathsf{MSet}(\mathcal{A})(x)$ to name its generating function.

The *spectrum* $\mathsf{Spec}(\mathcal{A})$ of a combinatorial class $\mathcal{A}$ is the set of sizes of the objects in $\mathcal{A}$, that is, $\mathsf{Spec}(\mathcal{A}) := \{\|\mathfrak{a}\| : \mathfrak{a} \in \mathcal{A}\}$. The *spectrum* of a power series $A(x)$ is $\mathsf{Spec}(A(x)) := \{n : a(n) \neq 0\}$, the support of the coefficient function $a(n)$. Thus $\mathsf{Spec}(\mathcal{A}) = \mathsf{Spec}(A(x))$.

1.2. The spectrum of a sentence.

In 1952 the *Journal of Symbolic Logic* initiated a section devoted to unsolved problems in the field of symbolic logic. The first problem, posed by Scholz [**28**], concerned the spectra of first order sentences. Given a sentence φ from first order logic, he defined the *spectrum* of φ to be the set of sizes of the finite models of φ. For example, binary trees can be defined by such a φ, and its spectrum is the arithmetical progression $\{1, 3, 5, \ldots\}$. Fields can also be defined by such a φ, with the spectrum being the set $\{2, 4, \ldots, 3, 9, \ldots\}$ of powers of prime numbers. The possibilities for the spectrum $\mathsf{Spec}(\varphi)$ of a first order sentence φ are amazingly complex.[2] The definition of $\mathsf{Spec}(\varphi)$ has been extended to sentences φ in any logic; we will be particularly interested in monadic second order (MSO) logic. If $\mathcal{A}$ is the class of finite models of a sentence φ, then $\mathsf{Spec}(\varphi) = \mathsf{Spec}(\mathcal{A}) = \mathsf{Spec}(A(x))$.

Scholz's problem was to find a necessary and sufficient condition for a set S of natural numbers to be the spectrum of some first order sentence φ. This led to considerable research by logicians—see, for example, the recent survey paper [**17**] of Durand, Jones, Makowsky, and More.

[2] Asser's 1955 conjecture, that *the complement of a first order spectrum is a first order spectrum*, is still open. It is known, through the work of Jones and Selman and Fagin in the 1970s, that this conjecture is equivalent to the question of whether the complexity class NE of problems decidable by a nondeterministic machine in exponential time is closed under complement. Thus the conjecture is, in fact, one of the notoriously difficult questions of computational complexity theory. Stockmeyer [**31**], p. 33, states that if Asser's conjecture is false then NP $\neq$ co-NP, and hence P $\neq$ NP.

1.3. Equational systems.

The study of equational specifications of combinatorial systems and equational systems defining generating functions is well established (see, for example, *Analytic Combinatorics* [**19**]; or [**2**]), but the corresponding study of spectra is new. Equational specifications $\boldsymbol{\mathcal{Y}} = \boldsymbol{\Gamma}(\boldsymbol{\mathcal{Y}})$ of combinatorial classes usually lead to systems of equations $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ defining the generating functions, and as will be seen, either of these usually lead to equational systems $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ defining the spectra. (See §5 for details.) A calculus of sets, to determine the spectrum of a combinatorial class defined by a single equation, was first introduced in 2006 (in [**2**]). That calculus is developed further here, in order to analyze spectra defined by systems of equations.

As an example of how this calculus of sets fits in, consider the class $\mathcal{P}$ of planar binary trees.[3] It is specified by the equation $\mathcal{P} = \{\bullet\} \cup \bullet/\mathsf{Seq}_2(\mathcal{P})$, which one can read as: *the class of planar binary trees is the smallest class $\mathcal{P}$ which has the 1-element tree '$\bullet$', and is closed under taking any sequence of two trees and adjoining a new root '$\bullet$'* . From the specification equation one finds that the generating function $P(x)$ of $\mathcal{P}$ satisfies $P(x) = x + x \cdot P(x)^2$, a simple quadratic equation that can be solved for $P(x)$. One also says that $P(x)$ is a solution to the polynomial equation $y = x + x \cdot y^2$. For the spectrum $\mathsf{Spec}(\mathcal{P})$, one has the equation $\mathsf{Spec}(\mathcal{P}) = \{1\} \cup (1 + 2 * \mathsf{Spec}(\mathcal{P}))$; thus $\mathsf{Spec}(\mathcal{P})$ satisfies the set equation $Y = \{1\} \cup (1 + 2 * Y)$ (see §2 for the notation used here). Solving this set equation gives the periodic spectrum $\mathsf{Spec}(\mathcal{P}) = 1 + 2 \cdot \mathbb{N}$.

If one drops the 'planar' condition, the situation becomes more complicated. The class $\mathcal{T}$ of binary trees has the specification $\mathcal{T} = \{\bullet\} \cup \bullet/\mathsf{MSet}_2(\mathcal{T})$, which one can read as: *the class of binary trees is the smallest class $\mathcal{T}$ which has the 1-element tree '$\bullet$', and is closed under taking any multiset of two trees and adjoining a new root '$\bullet$'.* From this specification equation one finds that the generating function $T(x)$ of $\mathcal{T}$ satisfies $T(x) = x + x \cdot \big(T(x)^2 + T(x^2)\big)/2$. This is not so simple to solve for $T(x)$; however, it gives a recursive procedure to find the coefficients $t(n)$ of $T(x)$, and one can compute the radius of convergence ρ of $T(x)$, and the value of $T(\rho)$, to any desired degree of accuracy (see *Analytic Combinatorics* [**19**], VII.22, p. 477). For the spectrum $\mathsf{Spec}(\mathcal{T})$, one has the equation $\mathsf{Spec}(\mathcal{T}) = \{1\} \cup (1+2*\mathsf{Spec}(\mathcal{T}))$; thus $\mathsf{Spec}(\mathcal{T})$ also satisfies the set equation $Y = \{1\} \cup (1 + 2 * Y)$, and the solution is again $\mathsf{Spec}(\mathcal{T}) = 1 + 2 \cdot \mathbb{N}$.

There is a long history of generating functions defined by a *single* recursion equation $y = G(x, y)$, starting with Cayley's 1857 paper [**8**] on trees, Pólya's 1937 paper (see [**23**]) that gave the form of the asymptotics for several classes of trees associated with classes of hydrocarbons, etc, right up to the present with the thorough treatment in *Analytic Combinatorics*.

Treelike structures have provided an abundance of recursively defined generating functions. Letting $t(n)$ be the number of trees (up to isomorphism) of size n, one has the equation

$$\sum_{n\ge 1} t(n)x^n \;=\; x \cdot \prod_{n\ge 1} (1 - x^n)^{-t(n)},$$

[3] We regard a tree as a certain kind of poset, with a largest element called the root. See §6.2 for a precise definition.

which yields a recursive procedure to calculate the values of $t(n)$. By 1875 Cayley had used this to calculate the first 13 coefficients $t(n)$, that is, the number of trees of size n for $n = 1, \ldots, 13$.

In 1937 Pólya (see [**23**]) would rewrite this equation as

$$T(x) \ = \ x \cdot \exp\left(\sum_{m=1}^{\infty} T(x^m)/m\right).$$

This allowed him to show that the coefficients $t(n)$ have the asymptotic form

$$t(n) \ \sim \ C\rho^{-n}n^{-3/2}, \tag{1.1}$$

for a suitable constant C, where ρ is the radius of convergence of $T(x)$.

Pólya's result has been generalized to show that any *well conditioned* equation $y = G(x, y)$ has a power series solution $y = T(x)$ whose coefficients satisfy the asymptotics in (1.1).[4] The determination of the constant C depends, in part, on knowing the periodicity parameter $\mathfrak{q}$ (defined in §2.4) for the spectrum of $T(x)$—for a generating function defined by a single equation, a formula for $\mathfrak{q}$ was given in [**2**].

The theory of generating functions defined by a system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ of several equations has built on the successes in the single equation case. If the system is *well conditioned* then the solution $\mathbf{y} = \mathbf{T}(x)$ is such that the $T_i(x)$ have the same radius of convergence $\rho \in (0, \infty)$, and they have the same periodicity parameter $\mathfrak{q}$. Drmota [**14**], [**15**] (see also [**3**]) showed that the coefficients $t_i(n)$ of the $T_i(x)$ satisfy the same asymptotics as in the 1-equation case, namely there are constants C_i such that

$$t_i(n) \ \sim \ C_i\rho^{-n}n^{-3/2}.$$

Such asymptotic expressions are understood to include the restriction *for* $n \in \mathsf{Spec}(T_i(x))$. As in the 1-equation case, the value of C_i depends partly on knowing the parameter $\mathfrak{q}$. The formula (4.2) shows how to find $\mathfrak{q}$ from the $G_i(x, \mathbf{y})$, and §7 shows how this is used to find the constants C_i in many well conditioned systems.

1.4. Monadic second order classes of trees and unary functions.

Let q be a positive integer, and let $\mathcal{T}_1, \ldots, \mathcal{T}_k$ be the minimal (nonempty) classes among the classes of trees defined by MSO sentences of quantifier depth q. The $\mathcal{T}_i$ are pairwise disjoint, and every class of trees defined by a MSO sentence of quantifier depth q is a union of some of the $\mathcal{T}_i$. In the 1980s, Compton [**11**], building on Büchi's 1960 paper [**6**] (on regular languages and MSO classes of m-colored chains), showed that the $\mathcal{T}_i$ have an equational specification $\boldsymbol{\mathcal{Y}} = \boldsymbol{\Gamma}(\boldsymbol{\mathcal{Y}})$. Following standard translation procedures, this gives an equational system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ that defines the generating functions $T_i(x)$ of the classes $\mathcal{T}_i$. In 1997, Woods [**33**] used the system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ to prove that the class of trees has a MSO limit law.[5] One can easily convert either of these systems—$\boldsymbol{\mathcal{Y}} = \boldsymbol{\Gamma}(\boldsymbol{\mathcal{Y}})$ and $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$—into an equational system $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ defining the spectra of the $\mathcal{T}_i$. From this one readily proves that every MSO class of trees has an eventually periodic spectrum. An easy

[4]See §7 for a discussion of well conditioned systems.

[5]Recently we have applied Compton's Specification Theorem to prove MSO 0–1 laws for many classes of forests. (See [**4**]).

argument then shows that the same holds for MSO classes of monounary algebras,[6] proving the Gurevich and Shelah result in [**20**]. Additionally, we use Compton's equational specification to give new proofs of the decidability results in [**20**].

1.5. Outline of the presentation.

§2 *Set Operations and Periodicity.* The basic operations $(\cup, +, \cdot, *)$ and laws, for a calculus of subsets of the nonnegative integers $\mathbb{N}$, are introduced; periodicity and the periodicity parameters $\mathfrak{c}$, $\mathfrak{m}$, $\mathfrak{p}$, $\mathfrak{q}$ are defined, and the fundamental results on periodicity are established.

§3 *Systems of Set Equations.* The set calculus of §2 is applied to the study of subsets of $\mathbb{N}$ defined by *elementary* systems $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ of set equations, with the main result being Theorem 3.11. Such systems have unique solutions among subsets of the positive integers $\mathbb{P}$. Conditions for periodicity of the solutions, and formulas for some of the periodicity parameters, are given.

§4 *Elementary Power Series Systems.* The development of §3 is paralleled for *elementary* systems $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ of power series equations (which are used to define generating functions). Elementary systems satisfy a special nonlinear requirement (see Definition 4.5) that guarantees a unique solution. The main result of §4 is Theorem 4.6, which gives criteria for generating functions defined by elementary systems to have periodic spectra. Furthermore, it gives formulas for some of the periodicity parameters.

§5 *Constructions, Operators, and Equational Specifications.* The equational specifications considered in this article use constructions built from compositions of a few basic constructions; it is routine to translate such specifications into systems of equations for the spectra. Also, under suitable conditions, one can translate equational specifications into equational systems defining generating functions.

§6 *Monadic Second Order Classes.* This section gives the aforementioned applications of Compton's Specification Theorem (Theorem 6.7) for monadic second order classes of trees.

§7 *Well Conditioned Systems.* The formulas, for the periodicity parameters in Theorem 4.6, are used to determine the asymptotics for the coefficients of generating functions defined by well conditioned systems.

Appendix A. *Proofs of preliminary material.*

Appendix B. *Büchi's Theorem.* This appendix shows how Büchi used minimal MSO_q classes to prove that MSO classes of m-colored chains can be identified with regular languages over a m-letter alphabet. Implicit in this proof is the fact that each MSO class of m-colored chains has a specification.

Acknowledgment. The authors are indebted to the two referees for detailed suggestions on how to improve the presentation.

2. Set Operations and Periodicity

2.1. Periodic sets.

Definition 2.1. $\mathbb{N}$ is the set of nonnegative integers, $\mathbb{P}$ is the set of positive integers. For $A \subseteq \mathbb{N}$:

[6] A *monounary algebra* $\mathfrak{a} = (A, f)$ is a set A, called the universe of the algebra, with a function $f : A \to A$. If one thinks of f as a binary relation instead of a unary function then one has the equivalent notion of a *functional digraph*.

(a) A is *periodic* if there is a positive integer p such that $p + A \subseteq A$, that is, $a \in A$ implies $p + a \in A$. Such an integer p is a *period* of A.
(b) A is *eventually periodic* if there is a positive integer p such that $p + A$ is eventually in A, that is, there is an m such that for $a \in A$, if $a \geq m$ then $p + a \in A$. Such a p is an *eventual period* of A.

Clearly every arithmetical progression and every cofinite subset of $\mathbb{N}$ is periodic; and every periodic set is eventually periodic. Finite subsets of $\mathbb{N}$ are eventually periodic; the only finite periodic set is $\emptyset$. As will be seen, periodicity seems to be a natural property for the spectra of combinatorial classes specified by a system of equations. The famous Skolem-Mahler-Lech Theorem (see, for example, *Analytic Combinatorics* [**19**], p. 266) says that the spectrum of every rational function $P(x)/Q(x)$ in $\mathbb{Q}(x)$ is eventually periodic, where the spectrum of $P(x)/Q(x)$ is the spectrum of its power series expansion. Consequently, polynomial systems $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$, with rational coefficients, that are linear in the variables y_i, and with a nonsingular Jacobian matrix $\partial\big(\mathbf{y} - \mathbf{G}(x, \mathbf{y})\big)/\partial\mathbf{y}$, have power series solutions $y_i = T_i(x)$ with eventually periodic spectra. However, much simpler methods give this periodicity result for the nonnegative $\mathbf{y}$-linear systems considered here.

If the spectrum of a combinatorial class $\mathcal{A}$ is eventually periodic then one has the hope, as in the case of regular languages and well behaved irreducible systems, that the class $\mathcal{A}$ decomposes into a finite subclass $\mathcal{A}_0$, along with finitely many subclasses $\mathcal{A}_i$, such that the $\mathsf{Spec}(\mathcal{A}_i)$ are arithmetical progressions $a_i + b_i \cdot \mathbb{N}$, and the generating functions $A_i(x)$ have well behaved coefficients (for example, monotone increasing, exponential growth, etc.) on $\mathsf{Spec}(\mathcal{A}_i)$.[7]

2.2. Set operations.

The calculus of set equations (for sets of nonnegative integers) developed in this section was originally extracted from work on the spectra of power series (see §4.2), to analyze the spectra of combinatorial classes. It uses the operations of union ($\cup$, $\bigcup$), addition ($+$), multiplication ($\cdot$), and star ($*$).

Definition 2.2. For $A, B \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ let

$$\begin{aligned} n + B &:= \{n + b : b \in B\} \\ n \cdot B &:= \{nb : b \in B\} \\ n * B &:= \begin{cases} \{0\} & \text{for } n = 0 \\ \underbrace{B + \cdots + B}_{n \text{ copies of } B} & \text{for } n > 0 \end{cases} \end{aligned} \qquad \begin{aligned} A + B &:= \{a + b : a \in A, b \in B\} \\ A * B &:= \textstyle\bigcup_{a \in A} a * B \end{aligned}$$

The values of these operations, when an argument is the empty set, are: $\emptyset + A = A + \emptyset = \emptyset$, $n \cdot \emptyset = \emptyset$, $\emptyset * B = \emptyset$, and $A * \emptyset = \{0\}$ if $0 \in A$, otherwise $A * \emptyset = \emptyset$.

The obvious definition of $A \cdot B$ is not needed in this study of spectra; only the special case $n \cdot B$ plays a role. The next lemma gives several basic identities needed for the analysis of spectra (all are easily proved).

Lemma 2.3. *For $A, B, C \subseteq \mathbb{N}$ and $m, n \in \mathbb{N}$*

$$\begin{aligned} A + (B \cup C) &= (A + B) \cup (A + C) \\ (A \cup B) * C &= A * C \cup B * C \end{aligned}$$

[7]The comments in this paragraph are related to Question 7.4 in Compton's 1989 paper [**10**] on MSO logical limit laws. (See [**1**] in this volume).

$$\begin{aligned}(A+B)*C &= A*C+B*C\\ m*(n*B) &= (m\cdot n)*B\\ n*(B+C) &= n*B+n*C\\ A*\big(B\cup C\big) &= \bigcup_{\substack{j_1,j_2\in\mathbb{N}\\ j_1+j_2\in A}} \big(j_1*B+j_2*C\big).\end{aligned}$$

It is quite useful that $*$ right distributes over both $\cup$ and $+$. However, neither left distributive law is generally valid; one only has a weak form of the left distributive law of $*$ over $+$, namely $n*(B+C)=n*B+n*C$.

2.3. Periodic and eventually periodic sets.

The following characterizations of periodic and eventually periodic sets are easily proved, if not well known.

LEMMA 2.4. *Let $A\subseteq\mathbb{N}$.*

(a) *A is periodic iff there is a finite set $A_1\subseteq\mathbb{N}$ and a positive integer p (a period for A) such that*

$$A\ =\ A_1+p\cdot\mathbb{N}$$

iff A is the union of finitely many arithmetical progressions.

(b) (Durand, Fagin, Loescher [**16**]; Gurevich and Shelah [**20**]) *A is eventually periodic iff there are finite sets $A_0, A_1\subseteq\mathbb{N}$ and a positive integer p (an eventual period of A) such that*

$$A\ =\ A_0\cup(A_1+p\cdot\mathbb{N})$$

iff A is the union of a finite set and finitely many arithmetical progressions.

REMARK 2.5. An infinite union of arithmetical progressions need not be eventually periodic. Let U be the union of the arithmetical progressions $a\cdot\mathbb{P}$, where a is a composite number. Then U consists of all composite numbers. Given any positive integer p, choose a prime number q that does not divide p. Then, by Dirichlet's theorem, the arithmetical progression $q^2+p\cdot\mathbb{N}$ has an infinite number of primes, thus $q^2+p\cdot\mathbb{N}$ is not a subset of U. Since $q^2\in U$, it follows that p is not an eventual period for U (one can choose q arbitrarily large). Thus U is not eventually periodic.

LEMMA 2.6. *Let $A,B\subseteq\mathbb{N}$.*

(a) *If A,B are eventually periodic, then so are $A\cup B$, $A+B$ and $A*B$.*
(b) *If A,B are periodic, then so are $A\cup B$ and $A+B$.*
(c) *Suppose A is periodic. Then $A*B$ is periodic iff $A*B\neq\{0\}$, which is iff neither $A\neq\emptyset$ and $B=\{0\}$ nor $0\in A$ and $B=\emptyset$ hold.*

PROOF. The results for $A\cup B$ and $A+B$ follow easily from Lemma 2.3 and Lemma 2.4. (The eventually periodic case is discussed in [**20**].)

To show $A*B$ is eventually periodic in (a), there are finite sets A_0 and A_1, and a positive integer p, such that

$$A\ =\ A_0\cup(A_1+p\cdot\mathbb{N}).$$

Using the right distributive laws for $*$ over $+$ and $\cup$, we have

$$A*B\ =\ (A_0*B)\cup\big((A_1*B)+\mathbb{N}*(p\cdot B)\big). \tag{2.1}$$

For M a finite subset of $\mathbb{N}$, note that $M*B$ is eventually periodic because it is either $\emptyset$, or $\{0\}$; or it is a finite union of finite sums of B, and these operations preserve being eventually periodic. Also note that for M any subset of $\mathbb{N}$, $\mathbb{N} * M$ is eventually periodic because $\mathbb{N} * M \supseteq M + (\mathbb{N} * M)$, so either $\mathbb{N} * M$ is $\emptyset$, or $\{0\}$; or there is a positive integer p in M, in which case $p + \mathbb{N} * M \subseteq \mathbb{N} * M$. In the first and third cases, $\mathbb{N} * M$ is actually periodic. The right side of (2.1) results from applying operations, that preserve being eventually periodic, to eventually periodic sets, so $A * B$ is eventually periodic.

For item (c), choose a positive integer p such that $A \supseteq p + A$. Then

$$A * B \ \supseteq\ (p * B) + (A * B).$$

Consequently, one has $A * B$ being either $\emptyset$, or $\{0\}$; or there is a positive integer in B, and thus a positive integer q in $p * B$. $\emptyset$ is periodic, but $\{0\}$ is not. The third case, that there is a positive integer q in $p * B$, implies $A * B \supseteq q + (A * B)$, so $A * B$ is periodic.

□

2.4. Periodicity parameters.

For $A \subseteq \mathbb{N}$, for $n \in \mathbb{N}$, let

$$A - n \quad := \quad \{a - n : a \in A\}.$$

The next definition gives some important parameters for the study of periodicity, with the convention $\gcd(\{0\}) := 0$.

Definition 2.7 (Periodicity parameters). For $A \subseteq \mathbb{N}$, $A \neq \emptyset$, let

(a) $\mathfrak{m}(A) := \min(A)$
(b) $\mathfrak{q}(A) := \gcd\big(A - \mathfrak{m}(A)\big)$.[8]

If A is infinite and eventually periodic:

(c) $\mathfrak{p}(A)$ is the minimum of the eventual periods p of A.
(d) $\mathfrak{c}(A) := \min\big\{a \in A : \mathfrak{p}(A) \text{ is a period for } A \cap [a, \infty)\big\}$.

Remark 2.8. It is useful to note that $\mathfrak{q}(A) = \gcd\{a - b : a, b \in A\}$.

Proposition 2.9. *Let $A_1, A_2 \subseteq \mathbb{N}$ be nonempty, with $\mathfrak{m}_i := \mathfrak{m}(A_i)$, $\mathfrak{q}_i := \mathfrak{q}(A_i)$, for $i = 1, 2$. Then*

Set	$\mathfrak{m}$	$\mathfrak{q}$
$A_1 \cup A_2$	$\min(\mathfrak{m}_1, \mathfrak{m}_2)$	$\gcd(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{m}_2 - \mathfrak{m}_1)$
$A_1 + A_2$	$\mathfrak{m}_1 + \mathfrak{m}_2$	$\gcd(\mathfrak{q}_1, \mathfrak{q}_2)$
$A_1 * A_2$	$\mathfrak{m}_1 \mathfrak{m}_2$	$\begin{cases} \{0\} & \text{if } A_1 = \{0\} \\ \gcd(\mathfrak{q}_2,\ \mathfrak{q}_1 \mathfrak{m}_2) & \text{if } A_1 \neq \{0\}, \end{cases}$

where in the last item we assume $\mathfrak{m}_1 \leq \mathfrak{m}_2$.

Proof. The calculations for $\mathfrak{m}$ are clear in each case. The calculations for $\mathfrak{q}$ are also elementary, but slightly more delicate—see Appendix A. □

Definition 2.10. For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ let $A|_{\geq n} := A \cap [n, \infty)$. Likewise define $A|_{>n}$, $A|_{\leq n}$, and $A|_{<n}$.

[8]Although adjoining elements to the beginning of A can decrease $\mathfrak{q}$, they cannot increase it; $\mathfrak{q}(A)$ is always a divisor of $\mathfrak{q}(A|_{\geq \mathfrak{c}})$, the $\mathfrak{q}$ of the periodic part of A. Thus, smallness of $\mathfrak{q}$ gives little information about A (too sensitive to a finite initial segment), but largeness of $\mathfrak{q}$ does give information. Furthermore, $\mathfrak{q}$ is useful as we can obtain a formula for it under quite general conditions, but we can only obtain a formula for $\mathfrak{p}$ by adding constraints so that $\mathfrak{p} = \mathfrak{q}$ (and then using the formula for $\mathfrak{q}$).

LEMMA 2.11. *Suppose $A \subseteq \mathbb{N}$ is infinite and eventually periodic. Letting $\mathfrak{c} := \mathfrak{c}(A)$, $\mathfrak{q} := \mathfrak{q}(A)$, $\mathfrak{p} := \mathfrak{p}(A)$, one has the following:*

(a) (Gurevich and Shelah [**20**], Cor. 3.3) *The set of eventual periods of A is $\mathfrak{p} \cdot \mathbb{P}$.*
(b) *One has $\mathfrak{q} \mid \mathfrak{p}$, and $\mathfrak{p} = \mathfrak{q}$ iff $\mathfrak{p} \mid (A - \mathfrak{m})$.*
(c) *The following are equivalent:*
  (i) *$A = A_0 \cup (a + b \cdot \mathbb{N})$ for some finite A_0 and some $a, b \in \mathbb{N}$, that is, A is the union of a finite set and a single arithmetical progression.*
  (ii) *$A = A\big|_{<\mathfrak{c}} \cup (\mathfrak{c} + \mathfrak{p} \cdot \mathbb{N})$.*
  (iii) *One has $\mathfrak{p} = \mathfrak{q}\big(A|_{\ge \mathfrak{c}}\big)$.*
(d) *The following are equivalent:*
  (i) *One has $\mathfrak{c} + \mathfrak{p} \cdot \mathbb{N} \subseteq A \subseteq \mathfrak{m} + \mathfrak{p} \cdot \mathbb{N}$.*
  (ii) *$A = A\big|_{<\mathfrak{c}} \cup (\mathfrak{c} + \mathfrak{p} \cdot \mathbb{N}) \subseteq \mathfrak{m} + \mathfrak{p} \cdot \mathbb{N}$.*
  (iii) *One has $\mathfrak{p} = \mathfrak{q}$.*

PROOF. The proof of (a) in [**20**] is elementary, as are the proofs for (b)–(d). For completeness, the later can be found in Appendix A. □

REMARK 2.12. The spectra of combinatorial classes, whose generating functions are defined by well conditioned systems of equations (see §7), are quite well behaved; they are periodic sets, and $\mathfrak{p} = \mathfrak{q}$. Thus, by Lemma 2.11 (d), the spectra are cofinal subsets of arithmetical progressions.

The next lemma augments Lemma 2.11 (d), giving a simple condition that is sufficient to guarantee that A is a periodic set with $\mathfrak{p} = \mathfrak{q}$.

LEMMA 2.13. *Suppose $A \subseteq \mathbb{N}$ with $A|_{>0} \neq \emptyset$, and suppose there are integers $r \ge 0$ and $s \ge 2$ such that*

$$A \ \supseteq \ r + s * A.$$

Let $\mathfrak{c} := \mathfrak{c}(A)$, $\mathfrak{m} := \mathfrak{m}(A)$, $\mathfrak{p} := \mathfrak{p}(A)$ and $\mathfrak{q} := \mathfrak{q}(A)$.

Then A is a periodic set with $\mathfrak{p} = \mathfrak{q}$. If $(r, s) = (0, 2)$, that is, if $A \supseteq A + A$, then one has the additional conclusion that $\mathfrak{q} = \gcd(A)$.

PROOF. Choose $a \in A|_{>0}$, and let $b := r + (s-2)a$. Then

$$A \ \supseteq \ b + A + A.$$

Choose $p \in b + A|_{>0}$. Then $A \supseteq p + A$ shows that A is periodic. By Lemma 2.11 (a), $\mathfrak{p} \mid (b + A)$, since all nonzero members of $b + A$ are periods of A. But then $\mathfrak{p} \mid (A - \mathfrak{m})$, so $\mathfrak{p} = \mathfrak{q}$ by Lemma 2.11 (b).

Now suppose $(r, s) = (0, 2)$. Then, for $a \in A$, one has $a + a \in A$. By Remark 2.8, $\mathfrak{q}$ divides their difference, that is, $\mathfrak{q}|a$. Thus $\mathfrak{q}|d := \gcd(A)$. Clearly $d \mid A - \mathfrak{m}$, so $d|\mathfrak{q}$. Thus $\mathfrak{q} = \gcd(A)$. □

3. Systems of Set Equations

For X a set, $\mathrm{Su}(X)$ is *the set of subsets of X*.

We will consider systems of set equations of the form

$$\begin{aligned} Y_1 &= G_1(Y_1, \ldots, Y_k) \\ &\vdots \\ Y_k &= G_k(Y_1, \ldots, Y_k), \end{aligned}$$

written compactly as $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$, with the $G_i(\mathbf{Y})$ having a particular form, namely

$$G_i(\mathbf{Y}) \;=\; \bigcup_{\mathbf{u}\in\mathbb{N}^k} \big(G_{i,\mathbf{u}} + u_1 * Y_1 + \cdots + u_k * Y_k\big), \tag{3.1}$$

where the $G_{i,\mathbf{u}}$ are subsets of $\mathbb{N}$. The system of equations (3.1) is compactly expressed by

$$\mathbf{G}(\mathbf{Y}) \;=\; \bigvee_{\mathbf{u}\in\mathbb{N}^k} \big(\mathbf{G}_{\mathbf{u}} + \mathbf{u} \circledast \mathbf{Y}\big), \tag{3.2}$$

where

$$\mathbf{u} \circledast \mathbf{Y} := u_1 * Y_1 + \cdots + u_k * Y_k.$$

Remark 3.1. $Y_1, \ldots, Y_k$ are variables that range over subsets of $\mathbb{N}$. Let the $G_{i,\mathbf{u}}$ be subsets of $\mathbb{N}$. The formal expression $\bigcup_{\mathbf{u}\in\mathbb{N}^k} \big(G_{i,\mathbf{u}} + u_1 * Y_1 + \cdots + u_k * Y_k\big)$ takes on a value (which is a subset of $\mathbb{N}$) by assigning set values $A_j \subseteq \mathbb{N}$ to the set variables Y_j.

The collection of sets $\mathrm{Su}(\mathbb{N})$ is closed under the familiar operations of union ($\cup$, $\bigcup$) and intersection ($\cap$, $\bigcap$). $\mathrm{Su}(\mathbb{N})^k$ is naturally viewed as a Boolean algebra, namely as the product of k copies of $\mathrm{Su}(\mathbb{N})$, and the inherited operations, corresponding to those just mentioned, are designated by the symbols $\vee$, $\bigvee$ and $\wedge$, $\bigwedge$. Thus when applied to k-tuples of subsets of $\mathbb{N}$, they act coordinatewise as $\cup$, $\bigcup$ and $\cap$, $\bigcap$, for example, $\mathbf{A} \vee \mathbf{B} := (A_1 \cup B_1, \ldots, A_k \cup B_k)$.[9] The notation $\mathbf{A} \le \mathbf{B}$ means $A_i \subseteq B_i$ for $1 \le i \le k$.

The expression $\bigvee_{\mathbf{u}\in\mathbb{N}^k} \big(\mathbf{G}_{\mathbf{u}} + \mathbf{u} \circledast \mathbf{Y}\big)$ is a formal expression such that when $\mathbf{Y}$ is assigned a k-tuple $\mathbf{A}$ from $\mathrm{Su}(\mathbb{N})^k$, the ith coordinate of $\bigvee_{\mathbf{u}\in\mathbb{N}^k} \big(\mathbf{G}_{\mathbf{u}} + \mathbf{u} \circledast \mathbf{A}\big)$ is $\bigcup_{\mathbf{u}\in\mathbb{N}^k} \big(G_{i,\mathbf{u}} + \mathbf{u} \circledast \mathbf{A}\big)$.

The notation $\mathbf{u} \circledast \mathbf{Y}$ is adopted for $u_1 * Y_1 + \cdots + u_k * Y_k$ since a natural definition of $\mathbf{u} * \mathbf{Y}$ would be the k-tuple $(u_1 * Y_1, \cdots, u_k * Y_k)$.

3.1. $\mathsf{Dom}(\mathbf{Y})$ and $\mathsf{Dom}_0(\mathbf{Y})$. This subsection defines the set operators we are interested in, and gives some useful lemmas about their behavior.

Definition 3.2. Let $\mathsf{Dom}(\mathbf{Y})$ be the set of $G(\mathbf{Y})$ of the form

$$G(\mathbf{Y}) \;=\; \bigcup_{\mathbf{u}\in\mathbb{N}^k} \big(G_{\mathbf{u}} + \mathbf{u} \circledast \mathbf{Y}\big),$$

where $G_{\mathbf{u}} \subseteq \mathbb{N}$, and let $\mathsf{Dom}_0(\mathbf{Y})$ be the set of $G(\mathbf{Y}) \in \mathsf{Dom}(\mathbf{Y})$ which map $\mathrm{Su}(\mathbb{P})^k$ into $\mathrm{Su}(\mathbb{P})$.

Lemma 3.3. *Suppose* $\mathbf{G}(\mathbf{Y}) \in \mathsf{Dom}(\mathbf{Y})^k$.

(a) *For* $\mathbf{A} \in \mathrm{Su}(\mathbb{N})^k$ *and* $1 \le i \le k$, *one has*

$$G_i(\mathbf{A}) \;=\; \bigcup_{\mathbf{u}\in\mathbb{N}^k} \Big(G_{i,\mathbf{u}} + \sum_{j\,:\,u_j>0} \big(u_j * A_j\big)\Big),$$

where the summation term is omitted in the case that all $u_j = 0$.

[9] When working with a product of structures, one usually uses the same symbols for the fundamental operations of the product as those used by the factors. For example, when working with rings, say $\mathfrak{r} = \mathfrak{r}_1 \times \mathfrak{r}_2$, one uses the symbols $+, \cdot$ as the fundamental operations for all three rings. However, the situation with products of Boolean algebras of sets is different—the inherited operations are no longer designated by $\cup$, $\bigcup$ and $\cap$, $\bigcap$, because these symbols have been given a fixed meaning in set theory. These fixed meanings lead, for example, to the fact that $\mathbf{A} \cup \mathbf{B}$ and $\mathbf{A} \vee \mathbf{B} := (A_1 \cup B_1, \ldots, A_k \cup B_k)$ are usually different.

(b) *If* $\mathbf{A} \in \mathrm{Su}(\mathbb{P})^k$ *then* $0 \in \mathbf{u} \circledast \mathbf{A} \;\Leftrightarrow\; \mathbf{u} = \mathbf{0}$.
(c) $\mathbf{G}(\mathbf{Y}) \in \mathsf{Dom}_0(\mathbf{Y})^k$ *iff* $\mathbf{G_0} := \big(G_{1,\mathbf{0}}, \ldots, G_{k,\mathbf{0}}\big) \in \mathrm{Su}(\mathbb{P})^k$.

Proof. (a) follows from the fact that $0 * A_j = \{0\}$, by Definition 2.2.

Given $\mathbf{A} \in \mathrm{Su}(\mathbb{P})^k$, (b) follows from

$$\begin{aligned} 0 \in \mathbf{u} \circledast \mathbf{A} \quad &\Leftrightarrow \quad 0 \in u_i * A_i, \quad \text{for } 1 \le i \le k,\\ &\Leftrightarrow \quad u_i = 0, \quad \text{for } 1 \le i \le k, \end{aligned}$$

the last assertion holding because $0 \notin A_i$, for any i, and Definition 2.2.

For (c), let $\mathbf{A} \in \mathrm{Su}(\mathbb{P})^k$. Then

$$\begin{aligned} \mathbf{G}(\mathbf{A}) \in \mathrm{Su}(\mathbb{P})^k \quad &\Leftrightarrow \quad 0 \notin G_i(\mathbf{A}), \quad \text{for } 1 \le i \le k,\\ &\Leftrightarrow \quad 0 \notin G_{i,\mathbf{u}} + \mathbf{u} \circledast \mathbf{A}, \quad \text{for } 1 \le i \le k,\ \mathbf{u} \in \mathbb{N}^k,\\ &\Leftrightarrow \quad 0 \notin G_{i,\mathbf{u}} \cap \mathbf{u} \circledast \mathbf{A}, \quad \text{for } 1 \le i \le k,\ \mathbf{u} \in \mathbb{N}^k,\\ &\Leftrightarrow \quad 0 \in \mathbf{u} \circledast \mathbf{A} \Rightarrow 0 \notin G_{i,\mathbf{u}}, \quad \text{for } 1 \le i \le k,\ \mathbf{u} \in \mathbb{N}^k,\\ &\Leftrightarrow \quad 0 \notin G_{i,\mathbf{0}}, \quad \text{for } 1 \le i \le k, \end{aligned}$$

the last line by item (b). □

$\mathbf{G}^{(n)}(\mathbf{Y})$ denotes the n-fold composition of $\mathbf{G}(\mathbf{Y})$ with itself, and $G_i^{(n)}(\mathbf{Y})$ is the ith component of this composition. Let

$$\mathbf{G}^{(\infty)}(\mathbf{Y}) \;:=\; \bigvee_{n \ge 0} \mathbf{G}^{(n)}(\mathbf{Y}),$$

that is, the ith component of $\mathbf{G}^{(\infty)}(\mathbf{Y})$ is $\bigcup_n G_i^{(n)}(\mathbf{Y})$. For $\mathbf{A}, \mathbf{B} \in \mathrm{Su}(\mathbb{N})^k$ let,

- $\min \mathbf{A} := (\min A_1, \ldots, \min A_k)$
- $\mathcal{N}(\mathbf{A}) := \{i : A_i = \emptyset\}$,

where $\min(\emptyset) := +\infty$. Recall that $\mathbf{A} \le \mathbf{B}$ expresses $A_i \subseteq B_i$, for $1 \le i \le k$.

Lemma 3.4. *Given* $\mathbf{G}(\mathbf{Y}) \in \mathsf{Dom}(\mathbf{Y})^k$*, and* $\mathbf{A}, \mathbf{B} \in \mathrm{Su}(\mathbb{N})^k$*, the following hold:*
(a) $\mathbf{A} \le \mathbf{B} \;\Rightarrow\; \mathbf{G}\big(\mathbf{A}\big) \le \mathbf{G}\big(\mathbf{B}\big)$.
(b) $\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{B}) \;\Rightarrow\; \mathcal{N}\big(\mathbf{G}(\mathbf{A})\big) = \mathcal{N}\big(\mathbf{G}(\mathbf{B})\big)$.
(c) $\mathbf{A} \le \mathbf{B} \;\Rightarrow\; \mathcal{N}\big(\mathbf{G}(\mathbf{A})\big) \supseteq \mathcal{N}\big(\mathbf{G}(\mathbf{B})\big)$.
(d) $\mathcal{N}\big(\mathbf{G}^{(k)}(\boldsymbol{\emptyset})\big) = \mathcal{N}\big(\mathbf{G}^{(k+n)}(\boldsymbol{\emptyset})\big)$, *for* $n \ge 0$.

Proof. Item (a) follows from the monotonicity of the set operations $\bigcup, +, *$ used in the definition of the $\mathbf{G}(\mathbf{Y})$ in $\mathsf{Dom}(\mathbf{Y})^k$.

Next observe that

$$\mathcal{N}\big(\mathbf{G}(\mathbf{A})\big) \;=\; \Big\{i : \big(\forall \mathbf{u} \in \mathbb{N}^k\big)\Big(G_{i,\mathbf{u}} = \emptyset \text{ or } \big(\exists j\big)\big(u_j > 0 \text{ and } A_j = \emptyset\big)\Big)\Big\}, \tag{3.3}$$

since from (3.2) one has $i \in \mathcal{N}\big(\mathbf{G}(\mathbf{A})\big)$ iff, for every $\mathbf{u} \in \mathbb{N}^k$, one has $G_{i,\mathbf{u}} + \mathbf{u} \circledast \mathbf{A} = \emptyset$, and this holds iff, for every $\mathbf{u} \in \mathbb{N}^k$, one has either $G_{i,\mathbf{u}} = \emptyset$, or for some j, $u_j * A_j = \emptyset$. Note that $u_j * A_j = \emptyset$ holds iff $u_j > 0$ and $A_j = \emptyset$.

Item (b) is immediate from (3.3).

Next note that $\mathbf{A} \le \mathbf{B}$ implies $\mathcal{N}(\mathbf{B}) \subseteq \mathcal{N}(\mathbf{A})$; this and (a) give (c).

To prove (d), note that from $\boldsymbol{\emptyset} \le \mathbf{G}(\boldsymbol{\emptyset})$ and (a) one has an increasing sequence

$$\boldsymbol{\emptyset} \le \mathbf{G}(\boldsymbol{\emptyset}) \le \mathbf{G}^{(2)}(\boldsymbol{\emptyset}) \le \cdots.$$

Then (c) gives the decreasing sequence

$$\{1,\ldots,k\} \;=\; \mathcal{N}(\mathbf{Ø}) \;\supseteq\; \mathcal{N}\big(\mathbf{G}(\mathbf{Ø})\big) \;\supseteq\; \mathcal{N}\big(\mathbf{G}^{(2)}(\mathbf{Ø})\big) \;\supseteq\; \cdots .$$

From (b) one sees that once two consecutive members of this sequence are equal, then all members further along in the sequence are equal to them. This shows the sequence must stabilize by the term $\mathcal{N}(\mathbf{G}^{(k)}(\mathbf{Ø}))$. □

LEMMA 3.5. *Suppose* $\mathbf{G}(\mathbf{Y}) \in \mathsf{Dom}(\mathbf{Y})^k$ *and* $\mathbf{A} \in \mathrm{Su}(\mathbb{N})^k$, *with* $\mathbf{A} \le \mathbf{G}(\mathbf{A})$. *Then*

$$\min \mathbf{G}^{(\infty)}(\mathbf{A}) \;=\; \min \mathbf{G}^{(k)}(\mathbf{A}).$$

In particular, $\min \mathbf{G}^{(\infty)}(\mathbf{Ø}) \;=\; \min \mathbf{G}^{(k)}(\mathbf{Ø})$.

PROOF. Use Lemma 3.4 (a) to show that the minimum stabilizes after at most k steps. The details are in Appendix A. □

3.2. The minimum solution of $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$.

PROPOSITION 3.6. *For* $\mathbf{G}(\mathbf{Y}) \in \mathsf{Dom}(\mathbf{Y})^k$, *the system of set equations* $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ *has a minimum solution* $\mathbf{S}$ *in* $\mathrm{Su}(\mathbb{N})^k$, *namely*

$$\mathbf{S} \;=\; \mathbf{G}^{(\infty)}(\mathbf{Ø}) \;:=\; \bigvee_{n\ge 0} \mathbf{G}^{(n)}(\mathbf{Ø}).$$

Furthermore, for $1 \le i \le k$, *one has* $S_i = \emptyset$ *iff* $G_i^{(k)}(\mathbf{Ø}) = \emptyset$.

PROOF. The sequence $\mathbf{G}^{(n)}(\mathbf{Ø})$ is monotone nondecreasing by Lemma 3.4 (a) since $\mathbf{Ø} \le \mathbf{G}(\mathbf{Ø})$. Suppose $a \in G_i^{(\infty)}(\mathbf{Ø})$. Then, for some $n \ge 1$,

$$a \in G_i^{(n)}(\mathbf{Ø}) \;=\; G_i\big(\mathbf{G}^{(n-1)}(\mathbf{Ø})\big) \;\subseteq\; G_i\big(\mathbf{G}^{(\infty)}(\mathbf{Ø})\big).$$

This implies $\mathbf{G}^{(\infty)}(\mathbf{Ø}) \le \mathbf{G}\big(\mathbf{G}^{(\infty)}(\mathbf{Ø})\big)$.

Conversely, suppose $a \in G_i\big(\mathbf{G}^{(\infty)}(\mathbf{Ø})\big)$. Then, for some $\mathbf{u} \in \mathbb{N}^k$,

$$a \in G_{i,\mathbf{u}} + \mathbf{u} \circledast \mathbf{G}^{(\infty)}(\mathbf{Ø}),$$

which in turn implies, for some $\mathbf{u} \in \mathbb{N}^k$ and $n \ge 1$,

$$a \in G_{i,\mathbf{u}} + \mathbf{u} \circledast \mathbf{G}^{(n)}(\mathbf{Ø}) \;\subseteq\; G_i^{(n+1)}(\mathbf{Ø}) \;\subseteq\; G_i^{(\infty)}(\mathbf{Ø}).$$

Thus $\mathbf{G}^{(\infty)}(\mathbf{Ø}) = \mathbf{G}\big(\mathbf{G}^{(\infty)}(\mathbf{Ø})\big)$, so $\mathbf{G}^{(\infty)}(\mathbf{Ø})$ is indeed a solution to $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$.

Now, given any solution $\mathbf{T}$, from $\mathbf{Ø} \le \mathbf{T}$ and Lemma 3.4 (a), it follows that, for $n \ge 0$, one has $\mathbf{G}^{(n)}(\mathbf{Ø}) \le \mathbf{G}^{(n)}(\mathbf{T}) = \mathbf{T}$, and thus $\mathbf{G}^{(\infty)}(\mathbf{Ø}) \le \mathbf{T}$, showing that $\mathbf{G}^{(\infty)}(\mathbf{Ø})$ is the smallest solution to $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$.

The test for $S_i = \emptyset$ is immediate from Lemma 3.5.

□

3.3. The dependency digraph for $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$.

In the study of systems $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ with $\mathbf{G}(\mathbf{Y}) \in \mathsf{Dom}(\mathbf{Y})^k$, it is important to know when Y_i depends on Y_j. This information is succinctly collected in the dependency digraph of the system.

DEFINITION 3.7. The *dependency digraph* D of a system $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ (with k equations) has vertices $1, \ldots, k$, and directed edges given by $i \rightarrow j$ iff there is a $\mathbf{u} \in \mathbb{N}^k$ such that $G_{i,\mathbf{u}} \neq \emptyset$ and $u_j > 0$.

The *dependency matrix* M of the system is the matrix of the digraph D.

If $i \rightarrow j \in D$ then we say "i depends on j", as well as "Y_i depends on Y_j". The transitive closure of $\rightarrow$ is $\rightarrow^+$; the notation $i \rightarrow^+ j$ is read: "i eventually depends on j". It asserts that there is a directed path in D from i to j. In this case one also says "Y_i eventually depends on Y_j". The reflexive and transitive closure of $\rightarrow$ is $\rightarrow^*$.

For each vertex i let $[i]$ denote the (possibly empty) *strong component* of i in the dependency digraph, that is,

$$[i] \ := \ \{j \ : \ i \rightarrow^+ j \rightarrow^+ i\}.$$

The system $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ is *irreducible* if the dependency digraph consists of a single strong component, that is, $i \rightarrow^+ j \rightarrow^+ i$, for all vertices i, j.

For a given system $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$, the following are easily seen to be equivalent:

(a) $i \rightarrow^+ j$.
(b) There is an $n \in \{1, \ldots, k\}$ such that $(M^n)_{i,j} > 0$.
(c) The (i, j) entry of $M + \cdots + M^k$ is > 0.

3.4. The main theorem on set equations.

Recall that $\mathbf{u} \circledast \mathbf{Y}$ is $u_1 * Y_1 + \cdots + u_k * Y_k$; and $\mathbf{G_u} + \mathbf{u} \circledast \mathbf{Y}$ is the k-tuple obtained by adding $\mathbf{u} \circledast \mathbf{Y}$ to each component of $\mathbf{G_u}$.

When a solution $\mathbf{S}$ of a system $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ gives the spectra S_i of generating functions, then 0 is excluded from the S_i, so one has the condition $0 \notin G_{i,\mathbf{0}}$, for $1 \leq i \leq k$, that is, $\mathbf{G}(\mathbf{Y}) \in \mathsf{Dom}_0(\mathbf{Y})^k$. One would like to assume that trivial equations $Y_i = Y_j$ have, after suitable substitutions into the other equations, been set aside. A much stronger condition that will appear in the study of spectra of generating functions is: $\mathbf{u} \circledast \mathbf{Y} = Y_j$ implies $0 \notin G_{i,\mathbf{u}}$, for any i. All of these restrictions on $\mathbf{G}(\mathbf{Y})$ are captured in the definition of *elementary* systems of set equations.

DEFINITION 3.8. $\mathbf{G}(\mathbf{Y})$ is *elementary* if $\mathbf{G}(\mathbf{Y}) \in \mathsf{Dom}(\mathbf{Y})^k$ and

$$0 \in G_{i,\mathbf{u}} \ \Rightarrow \ \sum_{j=1}^{k} u_j \geq 2, \quad \text{ for } 1 \leq i \leq k, \ \mathbf{u} \in \mathbb{N}^k.$$

A system $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ of set equations, where $\mathbf{G}(\mathbf{Y}) \in \mathsf{Dom}(\mathbf{Y})^k$, is *elementary* if $\mathbf{G}(\mathbf{Y})$ is elementary. If the system also satisfies $\mathcal{N}\big(\mathbf{G}^{(k)}(\boldsymbol{\emptyset})\big) = \emptyset$, that is, no coordinate of $\mathbf{G}^{(k)}(\boldsymbol{\emptyset})$ is the empty set, then one has a *reduced* elementary system.

One more fact is needed for the main theorem.

PROPOSITION 3.9. *Define* $d_k : \mathrm{Su}(\mathbb{N})^k \times \mathrm{Su}(\mathbb{N})^k \rightarrow \mathbb{R}$ *by*

$$d_k(\mathbf{A}, \mathbf{B}) \ := \ \begin{cases} 2^{-\min \bigcup_{i=1}^{k} \left(A_i \triangle B_i\right)} & \text{if } \mathbf{A} \neq \mathbf{B} \\ 0 & \text{if } \mathbf{A} = \mathbf{B}. \end{cases}$$

Then $\big(\mathrm{Su}(\mathbb{N})^k, d_k\big)$ *is a complete metric space, and, for* $\mathbf{A}_1, \mathbf{A}_2, \ldots$ *a Cauchy sequence in this space, one has*

$$\lim_{j\to\infty} \mathbf{A}_j \;=\; \bigvee_{n\geq 1} \bigwedge_{m\geq n} \mathbf{A}_m,$$

that is, the i*th coordinate of* $\lim_j \mathbf{A}_j$ *is*

$$\bigcup_{n\geq 1} \bigcap_{m\geq n} A_{i,m}.$$

$\big(\mathrm{Su}(\mathbb{P})^k, d_k\big)$ *is a complete subspace.*

Proof. It is straightforward to verify that $\big(\mathrm{Su}(\mathbb{N}), d_1\big)$ is a metric space. A sequence $A_1, A_2, \ldots$ of subsets of $\mathbb{N}$ is a Cauchy sequence in this space iff, for any $a \in \mathbb{P}$, there is an $b \in \mathbb{P}$ such that for $m, n \geq b$, one has $A_m\big|_{\leq a} = A_n\big|_{\leq a}$. Then, for $A_1, A_2, \ldots$ a Cauchy sequence in this space, one has

$$\lim_{j\to\infty} A_j \;=\; \bigcup_{n\geq 1} \bigcap_{m\geq n} A_m,$$

so $\big(\mathrm{Su}(\mathbb{N}), d_1\big)$ is a complete metric space.[10] For $\mathbf{A}, \mathbf{B} \in \mathrm{Su}(\mathbb{N})^k$ one has

$$d_k(\mathbf{A}, \mathbf{B}) \;=\; \max\big(d_1(A_1, B_1), \ldots, d_1(A_k, B_k)\big).$$

Thus $\big(\mathrm{Su}(\mathbb{N})^k, d_k\big)$ is also a complete metric space, and for $\mathbf{A}_1, \mathbf{A}_2, \ldots$ a Cauchy sequence in this space,

$$\lim_{j\to\infty} \mathbf{A}_j \;=\; \bigvee_{n\geq 1} \bigwedge_{m\geq n} \mathbf{A}_m \;=\; \Big(\lim_{j\to\infty} A_{1,j}, \ldots, \lim_{j\to\infty} A_{k,j}\Big),$$

where the $\lim_j A_{i,j}$, for $1 \leq i \leq k$, are calculated in $\big(\mathrm{Su}(\mathbb{N}), d_1\big)$.

Given a Cauchy sequence $\mathbf{A}_n$ from $\mathrm{Su}(\mathbb{P})^k$, it is routine to check that $\lim_n \mathbf{A}_n \in \mathrm{Su}(\mathbb{P})^k$. □

An important collection of Cauchy sequences is given in the following corollary.

Corollary 3.10. *Let* $\mathbf{A}_1, \mathbf{A}_2, \ldots$ *be a nondecreasing sequence in* $\mathrm{Su}(\mathbb{N})^k$*, that is,* $\mathbf{A}_1 \leq \mathbf{A}_2 \leq \cdots$*. Then* $\mathbf{A}_n$ *is a Cauchy sequence in* $\big(\mathrm{Su}(\mathbb{N})^k, d_k\big)$*, and*

$$\lim_{n\to\infty} \mathbf{A}_n \;=\; \bigvee\nolimits_{n\geq 1} \mathbf{A}_n,$$

that is, $\big(\lim_n \mathbf{A}_n\big)_j = \bigcup_n A_{j,n}$*, for* $1 \leq j \leq k$*.*

Theorem 3.11. *Let* $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ *be an elementary system of* k *set equations. Then the following hold:*

[10] In descriptive set theory it is well known that the Cantor ternary set, as a topological subspace of the real line, is homeomorphic to the infinite product $2^{\mathbb{N}}$, where 2 has the discrete topology. Consequently the set $2^{\mathbb{N}}$ is called *the Cantor set*, and the topological space $2^{\mathbb{N}}$ is called *the Cantor space*. It is metrizable space using the metric (see Exercise 105 in [**24**])

$$\delta(s,t) \;:=\; \begin{cases} 2^{-min\{n\,:\,s(n)\neq t(n)\}} & \text{if } s \neq t \\ 0 & \text{if } s = t. \end{cases}$$

The natural map from the set $2^{\mathbb{N}}$ to $\mathrm{Su}(\mathbb{N})$ converts $\big(2^{\mathbb{N}}, \delta\big)$ into $(\mathrm{Su}(\mathbb{N}), d_1)$.

(a) *There is a unique solution* $\mathbf{T} \in \mathrm{Su}(\mathbb{P})^k$*, and it is given by*

$$\mathbf{T} \;=\; \lim_{n\to\infty} \mathbf{G}^{(n)}(\mathbf{A}),\ \textit{for any } \mathbf{A} \in \mathrm{Su}(\mathbb{P})^k.$$

Setting $\mathbf{A} = \boldsymbol{\emptyset}$ *one has*

$$\mathbf{T} \;=\; \lim_{n\to\infty} \mathbf{G}^{(n)}(\boldsymbol{\emptyset}) \;=\; \mathbf{G}^{(\infty)}(\boldsymbol{\emptyset}).$$

Let $\mathfrak{m} := \mathfrak{m}(\mathbf{T})$ *and* $\mathfrak{q} := \mathfrak{q}(\mathbf{T})$.

(b) $T_i = \emptyset$ *iff* $G_i^{(k)}(\boldsymbol{\emptyset}) = \emptyset$*, that is,* $i \in \mathcal{N}\big(\mathbf{G}^{(k)}(\boldsymbol{\emptyset})\big)$.

For the remaining items, we assume the system is reduced elementary; in particular this means all T_i *are nonempty subsets of* $\mathbb{P}$.

(c) *If* $[i] \neq \emptyset$ *then* T_i *is periodic. If also there is a* $j \in [i]$ *such that, for some* $\mathbf{u} \in \mathbb{N}^k$*, one has* $G_{j,\mathbf{u}} \neq \emptyset$ *and* $\sum_{\ell\in[i]} u_\ell \geq 2$*, then* $\mathfrak{p}_i = \mathfrak{q}_i$*, so* T_i *is the union of a finite set with a single arithmetical progression, in particular*

$$T_i \;=\; T_i|_{<\mathfrak{c}_i} \cup \big(\mathfrak{c}_i + \mathfrak{q}_i \cdot \mathbb{N}\big) \;\subseteq\; \mathfrak{m}_i + \mathfrak{q}_i \cdot \mathbb{N}.$$

(d) *Suppose* $[i] = \emptyset$ *and the ith equation can be written in the form*

$$Y_i \;=\; P_i \;+\; \bigcup_{\mathbf{Q}\in\mathcal{Q}_i} \sum_{j=1}^{k} Q_j * Y_j,$$

with P_i *[eventually] periodic, and* $\mathcal{Q}_i$ *a finite set of* k*-tuples* $\mathbf{Q} = (Q_1, \ldots, Q_k)$ *of [eventually] periodic subsets* Q_j *of* $\mathbb{N}$*, and, for* $i \to j$*, one has* T_j *being [eventually] periodic. Then* T_i *is [eventually] periodic.*

(e) *The periodicity parameters* $\mathfrak{m}, \mathfrak{q}$ *of the solution* $\mathbf{T}$ *can be found from* $\mathbf{G}^{(k)}(\boldsymbol{\emptyset})$ *and the* $\mathbf{G}_\mathbf{u}$ *via the formulas:*

$$\mathfrak{m}_i := \mathfrak{m}(T_i) \;=\; \min\Big(G_i^{(k)}(\boldsymbol{\emptyset})\Big) \tag{3.4}$$

$$\mathfrak{q}_i := \mathfrak{q}(T_i) \;=\; \gcd\Bigg(\bigcup_{j\,:\,i\to^* j} \bigcup_{\mathbf{u}\in\mathbb{N}^k} \Big(G_{j,\mathbf{u}} + \mathbf{u} \circledast \mathfrak{m} - \mathfrak{m}_j\Big)\Bigg). \tag{3.5}$$

Thus, if $[i] \neq \emptyset$*, all* $\mathfrak{q}_j$ *with* $j \in [i]$ *are equal.*

(f) *One has* $\mathfrak{q}_i \mid \mathfrak{q}_j$ *whenever* $i \to j$.

Proof. First observe that since $\mathbf{G}(\mathbf{Y})$ is elementary, the mapping $\mathbf{G} : \mathrm{Su}(\mathbb{P})^k \to \mathrm{Su}(\mathbb{P})^k$ is a contraction map on the metric space $\big(\mathrm{Su}(\mathbb{P})^k, d\big)$. To see this, let $\mathbf{A}, \mathbf{B} \in \mathrm{Su}(\mathbb{P})^k$, and let $n \in \mathbb{N}$ be such that $\mathbf{A}|_{\leq n} = \mathbf{B}|_{\leq n}$, that is, $A_j|_{\leq n} = B_j|_{\leq n}$ for $1 \leq j \leq k$. Then, for $\mathbf{u} \in \mathbb{N}^k$ and $1 \leq j \leq k$,

$$\begin{aligned}\Big(G_{j,\mathbf{u}} + \mathbf{u} \circledast \mathbf{A}\Big)\Big|_{\leq n+1} \;&=\; \Big(G_{j,\mathbf{u}} + \mathbf{u} \circledast (\mathbf{A}|_{\leq n})\Big)\Big|_{\leq n+1}\\ &=\; \Big(G_{j,\mathbf{u}} + \mathbf{u} \circledast (\mathbf{B}|_{\leq n})\Big)\Big|_{\leq n+1}\\ &=\; \Big(G_{j,\mathbf{u}} + \mathbf{u} \circledast \mathbf{B}\Big)\Big|_{\leq n+1},\end{aligned}$$

since $0 \in G_{j,\mathbf{u}}$ implies $\sum_{j=1}^k u_j \geq 2$, by the elementary property of $\mathbf{G}(\mathbf{Y})$. Thus

$$\mathbf{A}|_{\leq n} = \mathbf{B}|_{\leq n} \;\Rightarrow\; \mathbf{G}(\mathbf{A})|_{\leq n+1} \;=\; \mathbf{G}(\mathbf{B})|_{\leq n+1}.$$

This implies

$$d_k\big(\mathbf{G}(\mathbf{A}), \mathbf{G}(\mathbf{B})\big) \;\leq\; \frac{1}{2} d_k(\mathbf{A}, \mathbf{B}),$$

so $\mathbf{G}$ defines a contraction mapping on $\big(\mathrm{Su}(\mathbb{P})^k, d_k\big)$. Since this is a complete metric space, $\mathbf{G}$ has a unique fixpoint $\mathbf{T}$, and $\mathbf{T} = \lim_{n\to\infty} \mathbf{G}^{(n)}(\mathbf{A})$, for any choice of $\mathbf{A}$ in $\mathrm{Su}(\mathbb{P})^k$. $\mathbf{T}$ is the unique solution to $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ in $\mathrm{Su}(\mathbb{P})^k$. With $\mathbf{A} = \boldsymbol{\emptyset}$ one also has $\mathbf{T} = \mathbf{G}^{(\infty)}(\boldsymbol{\emptyset})$, by Lemma 3.10, since the sequence $\mathbf{G}^{(n)}(\boldsymbol{\emptyset})$ is nondecreasing. This proves (a).

Item (b) follows from Proposition 3.6.

Now assume that the system is reduced elementary. Then each T_i is a nonempty set of positive integers.

For (c), first note that given i and $\mathbf{u}$ such that $G_{i,\mathbf{u}} \neq \emptyset$, there is a $q \geq 0$ (any $q \in G_{i,\mathbf{u}}$) such that

$$T_i \;\supseteq\; q + \sum_{\substack{1\leq j\leq k\\ u_j\neq 0}} u_j * T_j.$$

We can choose q to be positive if $G_{i,\mathbf{u}} \neq \{0\}$; if $G_{i,\mathbf{u}} = \{0\}$ then $\sum_j u_j \geq 2$. From this, $i \to j$ implies $T_i \supseteq p + T_j$, for some positive p; hence

$$i \to^+ j \quad \text{implies} \quad T_i \supseteq p + T_j, \quad \text{for some positive } p. \tag{3.6}$$

Now suppose $[i] \neq \emptyset$. Then $i \to^+ i$, so $T_i \supseteq p + T_i$, for some positive p, that is, T_i is periodic.

For the second part of (c), from $i \to^+ j$ follows $T_i \supseteq p_1 + T_j$, for some positive p_1, by (3.6). The hypothesis of (c) gives $T_j \supseteq p_2 + T_a + T_b$, for some a, b (possibly equal) in $[i]$, and some $p_2 \geq 0$. Finally $a \to^+ i$ and $b \to^+ i$ show that $T_a \supseteq p_3 + T_i$ and $T_b \supseteq p_4 + T_i$, for positive p_3, p_4. With $p = p_1+p_2+p_3+p_4$ one has $T_i \supseteq p + 2 * T_i$. Then Lemmas 2.13 and 2.11 (d) give the desired conclusion.

For (d), just apply Lemma 2.6.

Now to prove (e) and (f). The expression (3.4) for $\mathfrak{m}_i$ is given in Lemma 3.5. To derive the formula (3.5) for $\mathfrak{q}_i$, as well as to prove (f), from item (a) one has

$$\mathbf{T} \;=\; \mathbf{G}(\mathbf{T}) \;=\; \mathbf{G}^{(\infty)}(\boldsymbol{\emptyset}) \;:=\; \bigvee_{n\geq 0} \mathbf{G}^n(\boldsymbol{\emptyset}). \tag{3.7}$$

Let

$$\mathbf{H}(\mathbf{Y}) \;:=\; \bigvee_{\mathbf{u}\in\mathbb{N}^k} \big(\mathbf{H}_\mathbf{u} + \mathbf{u} \circledast \mathbf{Y}\big), \quad \text{where} \quad \mathbf{H}_\mathbf{u} \;:=\; \mathbf{G}_\mathbf{u} + \mathbf{u} \circledast \boldsymbol{\mathfrak{m}} - \boldsymbol{\mathfrak{m}}.$$

Then $\mathbf{H}(\mathbf{Y}) \in \mathsf{Dom}(\mathbf{Y})^k$. By Proposition 3.6, the equation $\mathbf{Y} = \mathbf{H}(\mathbf{Y})$ has a minimum solution $\mathbf{S}$; it satisfies

$$\mathbf{S} \;=\; \mathbf{H}^{(\infty)}(\boldsymbol{\emptyset}) \;=\; \mathbf{H}(\mathbf{S}) \;=\; \bigvee_{\mathbf{u}\in\mathbb{N}^k} \big(\mathbf{H}_\mathbf{u} + \mathbf{u} \circledast \mathbf{S}\big). \tag{3.8}$$

Since $\mathbf{H}(\mathbf{Y}) \;=\; \mathbf{G}(\mathbf{Y} + \boldsymbol{\mathfrak{m}}) - \boldsymbol{\mathfrak{m}}$,

$$\mathbf{H}^{(n)}(\mathbf{Y}) \;=\; \mathbf{G}^{(n)}(\mathbf{Y} + \boldsymbol{\mathfrak{m}}) - \boldsymbol{\mathfrak{m}}, \quad \text{for } n \geq 0. \tag{3.9}$$

From (3.7), (3.8) and (3.9),

$$\mathbf{S} \;=\; \bigvee_{n\geq 0} \mathbf{H}^{(n)}(\boldsymbol{\emptyset}) \;=\; \bigvee_{n\geq 0} \mathbf{G}^{(n)}(\boldsymbol{\emptyset}) - \boldsymbol{\mathfrak{m}} \;=\; \mathbf{G}^{(\infty)}(\boldsymbol{\emptyset}) - \boldsymbol{\mathfrak{m}} \;=\; \mathbf{T} - \boldsymbol{\mathfrak{m}}.$$

Thus

$$\mathbf{0} \in \mathbf{S} \;=\; \mathbf{T} - \boldsymbol{\mathfrak{m}} \;=\; \mathbf{G}(\mathbf{T}) - \boldsymbol{\mathfrak{m}} \;=\; \mathbf{H}(\mathbf{S}),$$

so, for $1 \le j \le k$,

$$S_j \;=\; \bigcup_{\mathbf{u}\in\mathbb{N}^k} \Big(H_{j,\mathbf{u}} + \mathbf{u} \circledast \mathbf{S}\Big) \;\supseteq\; \bigcup_{\mathbf{u}\in\mathbb{N}^k} H_{j,\mathbf{u}}. \tag{3.10}$$

By definition, $q_j = \gcd(S_j)$, so (3.10) implies

$$\mathfrak{q}_j \,\Big|\, \bigcup_{\mathbf{u}\in\mathbb{N}^k} H_{j,\mathbf{u}} \quad \text{for } 1 \le j \le k. \tag{3.11}$$

For $i \to j$ there is a $\mathbf{u} \in \mathbb{N}^k$ such that $G_{i,\mathbf{u}} \ne \emptyset$ and $u_j > 0$, thus

$$S_i \;\supseteq\; H_{i,\mathbf{u}} + S_j \quad \text{and } H_{i,\mathbf{u}} \ne \emptyset.$$

Then (3.11) and the fact that $q_i \,|\, S_i$ imply $\mathfrak{q}_i \,\big|\, S_j$ whenever $i \to j$. Thus

$$\mathfrak{q}_i \,\big|\, \mathfrak{q}_j \text{ whenever } i \to j, \tag{3.12}$$

which is item (f) of the theorem.

From (3.11) and (3.12),

$$i \to^* j \;\Rightarrow\; \mathfrak{q}_i \,\Big|\, \bigcup_{\mathbf{u}\in\mathbb{N}^k} H_{j,\mathbf{u}},$$

which implies

$$\mathfrak{q}_i \,\Big|\, \mathfrak{q}'_i \;:=\; \gcd\bigg(\bigcup_{j\,:\, i\to^* j}\ \bigcup_{\mathbf{u}\in\mathbb{N}^k} H_{j,\mathbf{u}}\bigg). \tag{3.13}$$

To finish the proof of (e) one needs $\mathfrak{q}'_i \,\big|\, \mathfrak{q}_i$. The key step is to show, by induction on n, that

$$i \to^* j \;\Rightarrow\; \mathfrak{q}'_i \,\Big|\, H^{(n)}_j(\boldsymbol{\emptyset}), \quad \text{for } n \ge 0. \tag{3.14}$$

Ground Case: (n=0)
Clearly $\mathfrak{q}'_i \,\big|\, \emptyset$.

Induction Step:
Assume that $\mathfrak{q}'_i \,\Big|\, H^{(n)}_j(\boldsymbol{\emptyset})$ whenever $i \to^* j$. One has

$$H^{(n+1)}_j(\boldsymbol{\emptyset}) \;=\; \bigcup_{\mathbf{u}\in\mathbb{N}^k} \Big(H_{j,\mathbf{u}} + \mathbf{u} \circledast \mathbf{H}^{(n)}(\boldsymbol{\emptyset})\Big). \tag{3.15}$$

Suppose that $i \to^* j$. Let $\mathbf{u} \in \mathbb{N}^k$. Then $\mathfrak{q}'_i \,\big|\, H_{j,\mathbf{u}}$ (by the definition of $\mathfrak{q}'_i$ in (3.13)). Clearly

$$H_{j,\mathbf{u}} \;=\; \emptyset \;\Rightarrow\; \mathfrak{q}'_i \,\Big|\, \Big(H_{j,\mathbf{u}} + \mathbf{u} \circledast \mathbf{H}^{(n)}(\boldsymbol{\emptyset})\Big) \;=\; \emptyset. \tag{3.16}$$

Suppose $H_{j,\mathbf{u}} \ne \emptyset$. If $\mathbf{u} = \mathbf{0}$, we have already noted that $\mathfrak{q}'_i \,\big|\, H_{j,\mathbf{0}}$. So suppose $u_m > 0$. Then $j \to m$; and, since $i \to^* j$, one has $i \to^* m$. By the induction hypothesis, $\mathfrak{q}'_i \,\Big|\, H^{(n)}_m(\boldsymbol{\emptyset})$. Consequently $\mathfrak{q}'_i \,\big|\, \big(\mathbf{u} \circledast \mathbf{H}^{(n)}(\boldsymbol{\emptyset})\big)$. Since, as noted above, $\mathfrak{q}'_i \,\big|\, H_{j,\mathbf{u}}$, it follows that

$$H_{j,\mathbf{u}} \;\ne\; \emptyset \;\Rightarrow\; \mathfrak{q}'_i \,\Big|\, \Big(H_{j,\mathbf{u}} + \mathbf{u} \circledast \mathbf{H}^{(n)}(\boldsymbol{\emptyset})\Big). \tag{3.17}$$

Items (3.16) and (3.17) show that for $\mathbf{u} \in \mathbb{N}^k$,

$$i \to^* j \;\Rightarrow\; \mathfrak{q}'_i \,\Big|\, \Big(H_{j,\mathbf{u}} + \mathbf{u} \circledast \mathbf{H}^{(n)}(\boldsymbol{\emptyset})\Big).$$

From this and (3.15) one then has $i \to^* j \;\Rightarrow\; \mathfrak{q}_i' \,\Big|\, H_j^{(n+1)}(\mathbf{\emptyset})$, finishing the proof of (3.14). Thus

$$i \to^* j \;\Rightarrow\; \mathfrak{q}_i' \,\Big|\, H_j^{(\infty)}(\mathbf{\emptyset}) \;=\; S_j\,.$$

Setting j equal to i gives $\mathfrak{q}_i' \,\big|\, S_i$, proving that $\mathfrak{q}_i' \,\big|\, \mathfrak{q}_i = \gcd(S_i)$. □

The following corollary will be used to provide information (see Corollary 4.7) on the spectra of irreducible systems studied in combinatorics.

COROLLARY 3.12. *Suppose* $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ *is an irreducible reduced elementary system of set equations with solution* $\mathbf{T}$*. Let* $\mathfrak{m} := \mathfrak{m}(\mathbf{T})$ *and* $\mathfrak{q} := \mathfrak{q}(\mathbf{T})$*. Then the following hold:*

(a) *All* T_i *are infinite and periodic, with the same parameter* $\mathfrak{q}_i$*, namely* $\mathfrak{q}_i = \mathfrak{q}$*, where*

$$\mathfrak{q} \;:=\; \gcd\left(\bigcup_{\substack{1\le j\le k\\ \mathbf{u}\in\mathbb{N}^k}} \Big(G_{j,\mathbf{u}} + \mathbf{u} \circledast \mathfrak{m} - \mathfrak{m}_j\Big)\right).$$

(b) *If there is a* j *such that* $G_j(\mathbf{Y})$ *has a term* $G_{j,\mathbf{u}} + \mathbf{u} \circledast \mathbf{Y}$*, with* $G_{j,\mathbf{u}} \neq \emptyset$ *and* $\sum_\ell u_\ell \ge 2$*, then* $\mathfrak{p}_i = \mathfrak{q}$*, for* $1 \le i \le k$*, and for all* i*,* T_i *is the union of a finite set and a single arithmetical progression:*

$$T_i \;=\; T_i|_{<\mathfrak{c}_i} \cup (\mathfrak{c}_i + \mathfrak{q}\cdot\mathbb{N}) \;\subseteq\; \mathfrak{m}_i + \mathfrak{q}\cdot\mathbb{N}.$$

PROOF. By Theorem 3.11(c). □

REMARK 3.13. Elementary systems $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ have unique solutions in $\mathrm{Su}(\mathbb{P})^k$, but this need not be the case in $\mathrm{Su}(\mathbb{N})^k$. Consider the single equation elementary system $Y = 2 * Y$. It has infinitely many solutions in $\mathrm{Su}(\mathbb{N})$, namely any $A \subseteq \mathbb{N}$ which satisfies the two conditions: (i) $0 \in A$, and $A + A \subseteq A$. Thus, for example, $p \cdot \mathbb{N}$ is a solution, for $p \in \mathbb{N}$. However, there is a unique solution of $Y = 2 * Y$ in $\mathrm{Su}(\mathbb{P})$, namely the empty set.

4. Elementary Power Series Systems

4.1. General background for power series systems.

Recall that $\mathbb{R}$ is the set of reals, $\mathbb{N}$ the set of nonnegative integers, and $\mathbb{P}$ the set of positive integers. We continue the upper case/lower case convention connecting a power series to its coefficients, for example, $A(x) = \sum_n a(n)x^n$.

The following table gives the notations needed for this section:

$\mathbf{z}$	$=$	$z_1, \ldots, z_k$
$\mathbf{z}^{\mathbf{u}}$	$=$	$z_1^{u_1} \cdots z_k^{u_k}$
$\mathbb{F}$	$=$	a field
$\mathbb{F}[[\mathbf{z}]]$	$=$	set of power series $A(\mathbf{z}) = \sum_{\mathbf{u}} a_{\mathbf{u}} \mathbf{z}^{\mathbf{u}}$ over $\mathbb{F}$
$\mathbb{F}[[\mathbf{z}]]^k$	$=$	$\big\{(A_1(\mathbf{z}), \ldots, A_k(\mathbf{z})) : A_i(\mathbf{z}) \in \mathbb{F}[[\mathbf{z}]]\big\}$
$\mathbb{F}[[\mathbf{z}]]_0$	$=$	$\{A(\mathbf{z}) \in \mathbb{F}[[\mathbf{z}]] : A(\mathbf{0}) = 0\}$
$\left[x^{\leq m}\right] A(x)$	$=$	$a(0) + a(1)x + \cdots + a(m)x^m$
$J_{\mathbf{G}}(x, \mathbf{y})$	$=$	the Jacobian matrix of $\mathbf{G}(x, \mathbf{y})$ with respect to $\mathbf{y}$
$\mathsf{Spec}(T(x))$	$=$	$\{n \geq 0 : t(n) \neq 0\}$, for $T(x) \in \mathbb{F}[[x]]$
$\mathsf{Spec}(\mathbf{T}(x))$	$=$	$\big(\mathsf{Spec}(T_1(x)), \ldots, \mathsf{Spec}(T_k(x))\big)$, for $\mathbf{T}(x) \in \mathbb{F}[[x]]^k$
The following items assume $\mathbb{F} = \mathbb{R}$, the field of real numbers		
$A(\mathbf{z}) \trianglerighteq B(\mathbf{z})$	says	$a_{\mathbf{u}} \geq b_{\mathbf{u}}$ for all $\mathbf{u}$
$\mathbf{A}(\mathbf{z}) \trianglerighteq \mathbf{B}(\mathbf{z})$	says	$A_i(\mathbf{z}) \trianglerighteq B_i(\mathbf{z})$ for all i
$\mathsf{Dom}[\mathbf{z}]$	$=$	$\{A(\mathbf{z}) \in \mathbb{R}[[\mathbf{z}]] : A(\mathbf{z}) \trianglerighteq 0\}$
$\mathsf{Dom}_0[\mathbf{z}]$	$=$	$\{A(\mathbf{z}) \in \mathsf{Dom}[\mathbf{z}] : A(\mathbf{0}) = 0\}$
$\mathbf{G}(x, \mathbf{y})$	$=$	a member of $\mathsf{Dom}_0[x, \mathbf{y}]^k$
$\mathsf{Dom}_{J0}[x, \mathbf{y}]$	$=$	$\big\{\mathbf{G}(x, \mathbf{y}) \in \mathsf{Dom}_0[x, \mathbf{y}]^k : J_{\mathbf{G}}(0, \mathbf{0}) = \mathbf{0}\big\}$, where $\mathbf{y} = y_1, \ldots, y_k$

For $k \geq 1$, the set $\mathbb{F}[[x]]^k$ becomes a complete metric space (see *Analytic Combinatorics* [**19**], p. 731) when equipped with the metric

$$d_k\big(\mathbf{A}(x), \mathbf{B}(x)\big) \ := \ \begin{cases} 2^{-\min \operatorname{ldegree}\left(A_i(x) - B_i(x) \,:\, 1 \leq i \leq k\right)} & \text{if } \mathbf{A}(x) \neq \mathbf{B}(x) \\ 0 & \text{if } \mathbf{A}(x) = \mathbf{B}(x), \end{cases}$$

where the *ldegree* of a power series $A(x)$ is its lowest degree, that is, $\operatorname{ldegree}(A(x)) := \min\{n \in \mathbb{N} : a(n) \neq 0\}$. One has $d_k\big(\mathbf{A}_n(x), \mathbf{B}_n(x)\big) \to 0$ as $n \to \infty$ iff for $m \geq 0$ there is an $N \geq 0$ such that $[x^{\leq m}]\mathbf{A}_n(x) = [x^{\leq m}]\mathbf{B}_n(x)$ for $n \geq N$; that is, for n sufficiently large, the corresponding coordinates of $\mathbf{A}_n(x)$ and $\mathbf{B}_n(x)$ agree on their first $m+1$ coefficients. The subset $\mathbb{F}[[x]]_0^k$ of $\mathbb{F}[[x]]^k$ is, with the same metric, also a complete metric space.

Let $k \geq 1$ be given, and let $\mathbf{y} := y_1, \ldots, y_k$. Given a k-tuple of formal power series $\mathbf{G}(x, \mathbf{y}) \in \mathbb{F}[[x, \mathbf{y}]]_0^k$, and $\mathbf{A}(x) \in \mathbb{F}[[x]]_0^k$, the composition $\mathbf{G}(x, \mathbf{A}(x))$ is a member of $\mathbb{F}[[x]]_0^k$. Such a $\mathbf{G}(x, \mathbf{y})$ can be viewed as a mapping from $\mathbb{F}[[x]]_0^k$ to itself, a mapping whose n-fold composition with itself will be expressed by $\mathbf{G}^{(n)}(x, \mathbf{y})$, also a member of $\mathbb{F}[[x, \mathbf{y}]]_0^k$. More precisely,

$$\begin{aligned} \mathbf{G}^{(0)}(x, \mathbf{y}) &= \mathbf{y}, \\ \mathbf{G}^{(n+1)}(x, \mathbf{y}) &= \mathbf{G}(x, \mathbf{G}^{(n)}(x, \mathbf{y})). \end{aligned}$$

The power series in the ith coordinate of $\mathbf{G}^{(n)}(x,\mathbf{y})$ will be denoted by $G_i^{(n)}(x,\mathbf{y})$, that is,

$$\mathbf{G}^{(n)}(x,\mathbf{y}) \quad = \quad \big(G_1^{(n)}(x,\mathbf{y}),\ldots,G_k^{(n)}(x,\mathbf{y})\big).$$

Basic results, on the existence and uniqueness of power series solutions to systems of equations, hold in the general setting of power series over a field; however, the natural setting for analyzing the generating functions of combinatorial classes is to work with power series over the real field $\mathbb{R}$.

PROPOSITION 4.1. *Let* $\mathbf{G}(x,\mathbf{y}) \in \mathbb{F}[[x,\mathbf{y}]]^k$. *If*

(a) $\mathbf{G}(0,\mathbf{0}) = \mathbf{0}$ *and*
(b) $J_{\mathbf{G}}(0,\mathbf{0}) = \mathbf{0}$

then the equational system $\mathbf{y} = \mathbf{G}(x,\mathbf{y})$

(i) *has a unique solution* $\mathbf{T}(x)$ *in* $\mathbb{F}[[x]]_0^k$, *and,*
(ii) *for any* $\mathbf{A}(x) \in \mathbb{F}[[x]]_0^k$, *one has, in the complete metric space* $\big(\mathbb{F}[[x]]_0^k, d_k\big)$,

$$\mathbf{T}(x) = \lim_{n\to\infty} \mathbf{G}^{(n)}(x,\mathbf{A}(x)).$$

If, furthermore,

(c) $\mathbb{F} = \mathbb{R}$,

then

(iii) $\mathbf{G}(x,\mathbf{y}) \trianglerighteq \mathbf{0} \ \Rightarrow\ \mathbf{T}(x) \trianglerighteq \mathbf{0}$.

PROOF. For $\mathbf{A}(x), \mathbf{B}(x) \in \mathbb{F}[[x]]_0^k$, the hypotheses guarantee that

$$[x^{\le n}]\mathbf{A}(x) = [x^{\le n}]\mathbf{B}(x) \quad\Rightarrow\quad [x^{\le n+1}]\mathbf{G}(x,\mathbf{A}(x)) = [x^{\le n+1}]\mathbf{G}(x,\mathbf{A}(x)).$$

This implies that $\mathbf{G}(x,\mathbf{y})$ is a contraction mapping on the complete metric space $\big(\mathbb{F}[[x]]_0^k, d_k\big)$; consequently (i)–(ii) follow. Item (iii) follows from (ii). □

For $T(x) \in \mathbb{F}[[x]]$, let $\mathfrak{m} := \mathfrak{m}\big(\mathsf{Spec}(T(x))\big)$ and $\mathfrak{q} := \mathfrak{q}\big(\mathsf{Spec}(T(x))\big)$, as in Definition 2.7.[11] Assuming $T(x) \neq 0$, one has the following:

(a) The largest power of x dividing $T(x)$ is $x^{\mathfrak{m}}$, since $\mathfrak{m}$ is the smallest index n such that $t(n) \neq 0$.
(b) The monomial $x^{\mathfrak{q}}$ is the largest power of x such that, for $n \ge 0$, $t(n) \neq 0$ implies $\mathfrak{q} \mid (n - \mathfrak{m})$.
(c) There is a (unique) power series $V(x) \in \mathbb{F}[[x]]$ such that $T(x) = x^{\mathfrak{m}} V(x^{\mathfrak{q}})$, and one has $\gcd(\mathsf{Spec}(V(x))) = 1$.
(d) Suppose $T(x) \in \mathsf{Dom}_0[x]$, and it has radius of convergence $\rho \in (0,\infty)$. By Pringsheim's Theorem, ρ is a singularity of $T(x)$, and thus it easily follows that $\rho\cdot\omega^j$, $j = 0,\ldots,\mathfrak{q}-1$, where ω is a primitive $\mathfrak{q}$th root of unity, are among the dominant singularities[12] of $T(x)$. If $T(x)$ is the solution to a well conditioned equation $y = G(x,y)$ then these are the only dominant singularities—this leads to $\mathfrak{q}$ having a prominent role in the expression for the asymptotics of the coefficients $t(n)$ of $T(x)$. (See §7).

[11] In *Analytic Combinatorics* [**19**], $\mathfrak{q}$ is called the *period* of $T(x)$, as well as of the sequence $t(n)$. It is only used in situations where $\mathfrak{p} = \mathfrak{q}$.

[12] Given a power series $A(x)$, the singularities on the circle of convergence are called the *dominant* singularities. A study of $A(x)$ near its dominant singularities is usually the first step, and often the main step, towards determining an asymptotic formula for the coefficients $a(n)$ of $A(x)$.

Under favorable conditions—such as those encountered in [**2**], a study of the solution $T(x)$ to $y = G(x, y)$, a non y-linear single equation system—$\mathsf{Spec}(T(x))$ is the union of a finite set and an arithmetical progression, and the coefficients $t(n)$ of $T(x)$ have 'nice' asymptotics for n on this spectrum. It would be an important achievement to show that any system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ built from standard components has a solution $\mathbf{T}(x)$ with the $T_i(x)$ exhibiting the positive features just described. A first, and very modest step in this direction, is to show that such systems have spectra of the appropriate kind, namely eventually periodic spectra.

4.2. Nonnegative power series and elementary systems.

There were two stages in our investigation [**2**] of generating functions defined by a single equation. The first considered $y = G(x, y)$, where $G(x, y)$ was a power series with nonnegative coefficients. The second looked at more complex equations $y = \Theta(y)$ involving operators like Multiset, Sequence, and Cycle. The same two stages will be followed in this study of generating functions defined by systems of equations.

Definition 4.2. A power series $A(\mathbf{z}) \in \mathbb{R}[[\mathbf{z}]]$ is *nonnegative* if $A(\mathbf{z}) \trianglerighteq 0$, that is, each coefficient $a_{\mathbf{u}}$ is nonnegative. $\mathbf{A}(\mathbf{z}) \in \mathbb{R}[[\mathbf{z}]]^k$ is nonnegative if each $A_i(\mathbf{z})$ is nonnegative. A system $\mathbf{y} = \mathbf{G}(x, \mathbf{y}) \in \mathbb{R}[[x, \mathbf{y}]]^k$ is nonnegative if $\mathbf{G}(x, \mathbf{y})$ is nonnegative.

When applied to nonnegative power series, the Spec operator acts like a homomorphism, as the next lemma shows. This is used to convert equational systems satisfied by generating functions into equational systems satisfied by spectra.

Lemma 4.3. *Let $c > 0$ and let $A(x), A_i(x), B(x) \in \mathbb{R}[[x]]$ be nonnegative power series. Then*

(a) $\mathsf{Spec}\big(c \cdot A(x)\big) = \mathsf{Spec}(A(x))$
(b) $\mathsf{Spec}\big(A(x) + B(x)\big) = \mathsf{Spec}(A(x)) \cup \mathsf{Spec}(B(x))$
(c) $\mathsf{Spec}\Big(\sum_i A_i(x)\Big) \ = \ \bigcup_i \mathsf{Spec}(A_i(x)),$ *provided* $\sum_i A_i(x) \in \mathbb{R}[[x]]$
(d) $\mathsf{Spec}\big(A(x) \cdot B(x)\big) = \mathsf{Spec}(A(x)) + \mathsf{Spec}(B(x))$
(e) $\mathsf{Spec}\big(A(x) \circ B(x)\big) = \mathsf{Spec}(A(x)) * \mathsf{Spec}(B(x))$, *provided* $B(x) \in \mathbb{R}[[x]]_0$.

Proof. The first four cases (scalar multiplication, addition, summation and Cauchy product) are straightforward, as is composition:

$$\begin{aligned}\mathsf{Spec}\big(A(x) \circ B(x)\big) \ &= \ \mathsf{Spec}\Big(\sum_{i \geq 1} a(i) B(x)^i\Big) \ = \bigcup_{i \in \mathsf{Spec}(A(x))} \mathsf{Spec}\left(B(x)^i\right) \\ &= \bigcup_{i \in \mathsf{Spec}(A(x))} i * \mathsf{Spec}(B(x)) \ = \ \mathsf{Spec}(A(x)) * \mathsf{Spec}(B(x)).\end{aligned}$$

□

One defines the *dependency digraph* for an equational system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ parallel to the way one defines it for a system of set equations $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$, namely $i \to j$ iff $G_i(x, \mathbf{y})$ depends on y_j, for $1 \leq i, j \leq k$.

Lemma 4.4 (Tests for eventual dependence). *Given a nonnegative system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$, the following are equivalent:*

(a) *One has $i \to^+ j$.*

(b) *There is an* $m \in \{1, \ldots, k\}$ *such that the* (i, j) *entry of* $J_{\mathbf{G}}(x, \mathbf{y})^m$ *is not* 0.
(c) *The* (i, j) *entry of* $\sum_{m=1}^{k} J_{\mathbf{G}}(x, \mathbf{y})^m$ *is not* 0.

In practice one only works with equational systems that have a connected dependency digraph. Otherwise the system trivially breaks up into several independent subsystems. There has been considerable interest in *irreducible* nonnegative equational systems $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$, where every y_i eventually depends on every y_j, that is, $i \rightarrow^{+} j$ holds for all i, j. One can also express this by the condition:

$$\textit{the matrix } \sum_{n=1}^{k} J_{\mathbf{G}}(x, \mathbf{y})^n \textit{ has all entries nonzero.}$$

Such systems behave, in many ways, like irreducible 1-equation systems. Clearly an equational system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ is irreducible if, for some n, no entry of $J_{\mathbf{G}}(x, \mathbf{y})^n$ is zero. In this case the matrix $J_{\mathbf{G}}(x, \mathbf{y})$ (and the system) is said to be *primitive* (or *aperiodic* in *Analytic Combinatorics* [**19**]).

However, even some nonnegative irreducible systems $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ can be easily decomposed into several independent subsystems—this will happen precisely when $J_{\mathbf{G}}(x, \mathbf{y})$ is *imprimitive*, that is, irreducible but not primitive. This case happens precisely when there is a permutation of the indices $1, \ldots, k$ transforming $J_{\mathbf{G}}(x, \mathbf{y})$ into $\widehat{J}_{\mathbf{G}}(x, \mathbf{y})$ such that, for some $n \geq 1$, $\widehat{J}_{\mathbf{G}}(x, \mathbf{y})^n$ has a block diagonal form with at least two blocks. By choosing the permutation to maximize the number of blocks obtainable, one finds that each block gives rise to an irreducible system that is primitive. Awareness of this possibility of decomposing irreducible systems is important for practical computational work.

A power series $G(x, \mathbf{y})$ can be expressed in the form

$$\sum_{\mathbf{u} \in \mathbb{N}^k} G_{\mathbf{u}}(x) \cdot \mathbf{y}^{\mathbf{u}},$$

where $\mathbf{y}^{\mathbf{u}}$ is the monomial $y_1^{u_1} \cdots y_k^{u_k}$. The associated set expression $G(\mathbf{Y})$, where $G_{\mathbf{u}} := \mathsf{Spec}(G_{\mathbf{u}}(x))$, is given by

$$G(\mathbf{Y}) \ := \ \bigcup_{\mathbf{u} \in \mathbb{N}^k} \big(G_{\mathbf{u}} + \mathbf{u} \circledast \mathbf{y}\big).$$

DEFINITION 4.5. A nonnegative power series $\mathbf{G}(x, \mathbf{y})$ is *elementary* if it satisfies the conditions of Proposition 4.1, namely:
(a) $\mathbf{G}(0, \mathbf{0}) = \mathbf{0}$ and
(b) $J_{\mathbf{G}}(0, \mathbf{0}) = \mathbf{0}$.
An equational system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ is *elementary* iff $\mathbf{G}(x, \mathbf{y})$ is elementary.

This definition is easily seen to be equivalent to requiring: for $\mathbf{u} \in \mathbb{N}^k$ and $1 \leq i \leq k$,

$$G_{i, \mathbf{u}}(0) \neq 0 \ \Rightarrow \ \sum_{j=1}^{k} u_j \geq 2.$$

Consequently a nonnegative power series system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ is elementary iff the associated system of set equations $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ is elementary.

The next result is the main theorem on power series systems.

THEOREM 4.6. *For an elementary system* $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ *the following hold:*

(a) *The system has a unique solution* $\mathbf{T}(x)$ *in* $\mathbb{R}[[x]]_0^k$.
Let $\mathfrak{m} := \mathfrak{m}(\mathsf{Spec}(\mathbf{T}(x)))$ *and* $\mathfrak{q} := \mathfrak{q}(\mathsf{Spec}(\mathbf{T}(x)))$.

(b) $\mathbf{T}(x) \trianglerighteq \mathbf{0}$, *that is, the coefficients of each* $T_i(x)$ *are nonnegative.*

(c) $\mathbf{T}(x) = \lim_{n\to\infty} \mathbf{G}^{(n)}(x, \mathbf{A}(x))$, *for any* $\mathbf{A}(x) \in \mathbb{R}[[x]]_0^k$.

(d) $\mathbf{Y} = \mathsf{Spec}(\mathbf{T}(x))$ *is the unique solution to the elementary system of set equations* $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$, *where*

$$\mathbf{G}(\mathbf{Y}) \ := \ \bigvee_{\mathbf{u}\in\mathbb{N}^k} \big(\mathbf{G}_{\mathbf{u}} + \mathbf{u} \circledast \mathbf{Y}\big).$$

(e) $\mathsf{Spec}(\mathbf{T}(x)) \ = \ \lim_{n\to\infty} \mathbf{G}^{(n)}(\mathbf{A})$, *for any* $\mathbf{A} \in \mathrm{Su}(\mathbb{P})^k$.
Thus $\mathsf{Spec}(\mathbf{T}(x)) \ = \ \mathbf{G}^{(\infty)}(\boldsymbol{\emptyset}) \ := \ \bigvee_{n\ge 1} \mathbf{G}^{(n)}(\boldsymbol{\emptyset})$.

(f) $T_i(x) = 0$ *iff* $G_i^{(k)}(x, \mathbf{0}) = 0$ *iff* $\mathsf{Spec}(T_i(x)) = \emptyset$ *iff* $G_i^{(k)}(\boldsymbol{\emptyset}) = \emptyset$ *iff* $\mathfrak{m}_i = \infty$.

Now we assume that the system has been reduced by eliminating all y_i *for which* $T_i(x) = 0$.

(g) $[i] \neq \emptyset$ *implies* $\mathsf{Spec}(T_i(x))$ *is periodic. If also there is a* $j \in [i]$ *such that, for some* $\mathbf{u} \in \mathbb{N}^k$, *one has* $G_{j,\mathbf{u}}(x) \neq 0$ *and* $\sum\{u_\ell : \ell \in [i]\} \geq 2$, *then* $\mathfrak{p}_i = \mathfrak{q}_i$ *and* $\mathsf{Spec}(T_i(x))$ *is the union of a finite set with a single arithmetical progression, namely* $\mathsf{Spec}(T_i(x)) = \mathsf{Spec}(T_i(x))|_{<\mathfrak{c}_i} \cup \big(\mathfrak{c}_i + \mathfrak{q}_i \cdot \mathbb{N}\big) \subseteq \mathfrak{m}_i + \mathfrak{q}_i \cdot \mathbb{N}$.

(h) *If* $[i] = \emptyset$ *and the* ith *set equation can be written in the form*

$$Y_i \ := \ P_i \ + \ \bigcup_{\mathbf{Q}\in\mathcal{Q}_i} \sum_{j=1}^{k} \big(Q_j * Y_j\big),$$

with P_i *[eventually] periodic, and* $\mathcal{Q}_i$ *a finite set of* k*-tuples* $\mathbf{Q} = (Q_1, \ldots, Q_k)$ *of [eventually] periodic subsets* Q_j *of* $\mathbb{N}$, *and if, for* $i \to j$, *one has* $\mathsf{Spec}(T_j(x))$ *being [eventually] periodic, then* $\mathsf{Spec}(T_i(x))$ *is [eventually] periodic.*

(i) *The periodicity parameters* $\mathfrak{m}, \mathfrak{q}$ *of* $\mathsf{Spec}(\mathbf{T}(x))$ *can be found from* $\mathbf{G}^{(k)}(\boldsymbol{\emptyset})$ *and the* $\mathbf{G}_{\mathbf{u}}$ *via the formulas*

$$(4.1)\quad \mathfrak{m}_i \ := \ \mathfrak{m}(\mathsf{Spec}(T_i(x))) \ = \ \min\Big(G_i^{(k)}(\boldsymbol{\emptyset})\Big)$$

$$(4.2)\quad \mathfrak{q}_i \ := \ \mathfrak{q}(\mathsf{Spec}(T_i(x))) \ = \ \gcd\Bigg(\bigcup_{j\,:\,i\to^* j} \ \bigcup_{\mathbf{u}\in\mathbb{N}^k} \Big(G_{j,\mathbf{u}} + \mathbf{u} \circledast \mathfrak{m} - \mathfrak{m}_j\Big)\Bigg).$$

(j) $\mathfrak{q}_i \mid \mathfrak{q}_j$ *whenever* $i \to j$.

Proof. Items (a)–(c) are immediate from Proposition 4.1. For (d) simply apply Spec to both sides of $\mathbf{T}(x) = \mathbf{G}(x, \mathbf{T}(x))$. For (e)–(j) note that the hypotheses of the theorem imply that $\mathbf{G}(\mathbf{Y})$ satisfies the hypotheses of Theorem 3.11. □

The study of irreducible systems $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ has been of interest because of parallels with work on single equation systems, with a particular interest in *non* $\mathbf{y}$*-linear systems*, that is, systems where some $G_i(x, \mathbf{y})$ has a nonvanishing Hessian matrix $\big[\partial^2 G_i(x, \mathbf{y})/\partial y_r \partial y_s\big]$. (See, for example, *Analytic Combinatorics* [**19**], VII.6.)

Corollary 4.7. *Suppose* $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ *is an irreducible reduced elementary system with solution* $\mathbf{T}(x)$. *Let* $\mathfrak{m} := \mathfrak{m}(\mathsf{Spec}(\mathbf{T}(x)))$ *and* $\mathfrak{q} := \mathfrak{q}(\mathsf{Spec}(\mathbf{T}(x)))$.

(a) *All* $\mathsf{Spec}(T_i(x))$ *are infinite and periodic, with the same parameter* $\mathfrak{q}_i$*, namely* $\mathfrak{q}_i = \mathfrak{q}$*, where*

$$\mathfrak{q} \;:=\; \gcd\Big(\bigcup_{\substack{1\le j\le k\\ \mathbf{u}\in\mathbb{N}^k}} \Big(G_{j,\mathbf{u}} + \mathbf{u} \circledast \mathfrak{m} - \mathfrak{m}_j\Big)\Big).$$

(b) *If also there is a* j *such that* $G_j(x,\mathbf{y})$ *is not* $\mathbf{y}$*-linear, then, for* $1 \le i \le k$*,* $\mathfrak{p}_i = \mathfrak{q}$ *and*

$$\mathsf{Spec}(T_i(x)) \;=\; \mathsf{Spec}(T_i(x))\big|_{<\mathfrak{c}_i} \cup (\mathfrak{c}_i + \mathfrak{q}\cdot\mathbb{N}) \;\subseteq\; \mathfrak{m}_i + \mathfrak{q}\cdot\mathbb{N}.$$

Thus each $\mathsf{Spec}(T_i(x))$ *is eventually an arithmetical progression.*

PROOF. By Corollary 3.12. □

Systems that arise in combinatorial problems are invariably reduced since the solution gives generating functions for nonempty classes of objects. However, if one should encounter a nonreduced elementary *polynomial* system $\mathbf{y} = \mathbf{G}(x,\mathbf{y})$, Theorem 4.6 (f) provides an efficient way to determine which of the solution components $T_i(x)$ will be 0, namely let $\lambda(0) = 0$, and let λ map any nonzero $A(x) \in \mathsf{Dom}_0[x]$ to its lowest degree term, setting the coefficient to 1; extend this to $\mathsf{Dom}_0[x]^k$ coordinatewise. Then

$$T_i(x) = 0 \quad \text{iff} \quad \Big((\lambda\circ\mathbf{G})^{(k)}(x,\mathbf{0})\Big)_i = 0.$$

4.3. Irreducible elementary y-linear systems.

Irreducible elementary y-linear equations $y = G_0(x) + G_1(x)y$, with solution $y = T(x)$, do not, in general, have the property that $\mathsf{Spec}(T(x))$ is eventually a single arithmetical progression. For example, let $y = T(x)$ be the power series solution to

$$y \;=\; x + x^2 + x^3 y.$$

The periodicity parameters of $\mathsf{Spec}(T(x))$ are $\mathfrak{m} = 1$, $\mathfrak{q} = 1$, $\mathfrak{p} = 3$, and $\mathfrak{c} = 1$. $\mathsf{Spec}(T(x))$ is readily seen to be

$$\{3n+1 : n \ge 0\} \cup \{3n+2 : n \ge 0\},$$

and the set of periods of $\mathsf{Spec}(T(x))$ is the same as the set of eventual periods of $\mathsf{Spec}(T(x))$, namely $3\cdot\mathbb{N}$.

Nonetheless, the periodic spectrum of the solution of an irreducible elementary y-linear equation does have a particularly simple expression.

PROPOSITION 4.8. *Given a 1-equation irreducible elementary* y*-linear system*

$$y \;=\; G(x,\mathbf{y}) \;:=\; G_0(x) + G_1(x)\cdot y,$$

the solution is

$$T(x) \;=\; \Big(\sum_{n\ge 0} G_1(x)^n\Big)\cdot G_0(x),$$

the spectral equation is

$$Y \;=\; G(Y) \;:=\; G_0 \cup (G_1 + Y),$$

and the spectrum is

$$\mathsf{Spec}(T(x)) \;=\; \Big(\bigcup_{n\ge 0}(n * G_1)\Big) + G_0 \;=\; G_0 + \mathbb{N} * G_1.$$

Let $\mathfrak{m} := \mathfrak{m}(\mathsf{Spec}(T(x)))$ *and* $\mathfrak{q} := \mathfrak{q}(\mathsf{Spec}(T(x)))$*. Then*

$$\mathfrak{m} = \min(G_0) \text{ and } \mathfrak{q} = \gcd\big((G_0 - \mathfrak{m}) \cup G_1\big).$$

The proof of the proposition is straightforward. From the form of the solution for $\mathsf{Spec}(T(x))$, one sees that every periodic subset of $\mathbb{P}$ is the spectrum of the solution to some 1-equation irreducible elementary y-linear system.

The next three examples, of $\mathbf{y}$-linear systems, are staples in the study of systems.

EXAMPLE 4.9 (Postage Spectra). Given that one has stamps in denominations $d_1, \ldots, d_r$, the associated *postage spectrum* is the set S of amounts of postage one can put on a package using only these sizes of stamps. With $D = \{d_1, \ldots, d_r\}$ being the set of denominations of the stamps, let $D(x) = \sum_{i=1}^r n_i x^{d_i}$, where n_i is the number of distinct stamps of denomination d_i. Let $S(x)$ be the generating function with $s(n)$ giving the number of *ordered* ways to realize the postage amount n, that is, using a *sequence* of such stamps. Then $S(x)$ is the solution to the irreducible elementary y-linear equation

$$y \;=\; D(x) + D(x) \cdot y.$$

The spectrum $\mathsf{Spec}(S(x))$ is the solution to the elementary set equation

$$Y = D \cup (D + Y),$$

which, by Proposition 4.8, is $\mathsf{Spec}(S(x)) = \mathbb{P} * D$. (Of course, one easily sees that $\mathbb{P} * D$ must be the spectrum, without the help of Proposition 4.8.)

We need a quick lemma, using results from Section 2.

LEMMA 4.10. *Suppose* $B \subseteq \mathbb{N}$ *and* $B \cap \mathbb{P} \neq \emptyset$*. Let* $A = \mathbb{P} * B$*. Then* A *is a periodic set. Let* $\mathfrak{c} := \mathfrak{c}(A)$*,* $\mathfrak{p} := \mathfrak{p}(A)$*,* $\mathfrak{q} := \mathfrak{q}(A)$*. Then*

(a) $\mathfrak{p} = \mathfrak{q} = \gcd(A) = \gcd(B)$*, and*
(b) $A = A\big|_{<\mathfrak{c}} \cup \big(\mathfrak{c} + \mathfrak{p} \cdot \mathbb{N}\big) \;\subseteq\; \mathfrak{p} \cdot \mathbb{N}$.

PROOF. Note that $A \cap \mathbb{P} \supseteq B \cap \mathbb{P} \neq \emptyset$, and

$$A + A \;=\; (\mathbb{P} + \mathbb{P}) * B \;\subseteq\; \mathbb{P} * B \;=\; A,$$

thus Lemma 2.13 applies with $(r, s) = (0, 2)$. Since $\gcd(A) = \gcd(B)$, one has the desired conclusions. □

Returning to the postage spectra, by Lemma 4.10, $\mathsf{Spec}(S(x))$ is periodic, $\mathfrak{q} = \mathfrak{p} = \gcd(S) = \gcd(D)$, and $\mathsf{Spec}(S(x)) = \mathsf{Spec}(S(x))\big|_{<\mathfrak{c}} \cup (\mathfrak{c} + \mathfrak{q} \cdot \mathbb{N})$, where $\mathfrak{c} := \mathfrak{c}(\mathsf{Spec}(S(x)))$, $\mathfrak{q} := \mathfrak{q}(\mathsf{Spec}(S(x)))$, $\mathfrak{p} := \mathfrak{p}(\mathsf{Spec}(S(x)))$.[13]

EXAMPLE 4.11 (Paths in Labelled Digraphs). The objective in this example is to find a set equation system that determines the set of lengths of the paths going from vertex 1 to vertex 4 in the labelled digraph in Fig. 1.

[13]The number $\gamma(D) := \mathfrak{c}(\mathsf{Spec}(S(x)))$ is called the *conductor* of D by Wilf (see [**32**], §3.15.). $\gamma(D) - 1$ is called the *Frobenius number*, and the problem of finding it is called the Frobenius Problem (or Coin Problem). The problem can easily be reduced to the case that $\gcd(D) = 1$, in which case every number $\geq \gamma(D)$ is in $\mathbb{N} * D$, but $\gamma(D) - 1 \notin \mathbb{N} * D$. For D a finite set of positive integers, considerable effort has been devoted to finding a formula for $\gamma(D)$, for D with few elements. The only known closed forms are for D with 1 or 2 elements. For $D = \{b_1, b_2\}$, with 2 coprime elements, both > 1, the solution is $\gamma(D) = (b_1 - 1)(b_2 - 1)$, found by Sylvester in 1884. Finding $\gamma(D)$ is known to be NP-hard. (See Ramírez-Alfonsín [**26**]; Shallit [**27**].)

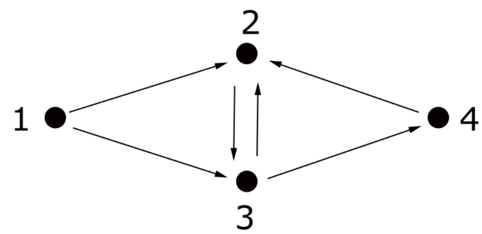

FIGURE 1. A Labelled Digraph

For $1 \le i \le 4$, let $L_i(x) = \sum_{n\ge1} \ell_i(n)x^n$ be the generating function for the lengths of paths going from vertex i to vertex 4, that is, $\ell_i(n)$ counts the number of paths of length n from vertex i to vertex 4. Then $\mathbf{y} = \mathbf{L}(x)$ satisfies the following system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$:

$$\begin{array}{rclcrcl} y_1 &=& x\cdot\big(y_2 + y_3\big) && y_2 &=& x\cdot y_3 \\ y_3 &=& x\cdot\big(y_2 + 1 + y_4\big) && y_4 &=& x\cdot y_2. \end{array}$$

One has $\mathbf{G}(x, \mathbf{y}) \trianglerighteq 0$ and $\mathbf{G}(0, \mathbf{0}) = J_{\mathbf{G}}(0, \mathbf{0}) = \mathbf{0}$, so the system is elementary. The associated elementary spectral system $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ is:

$$\begin{array}{rclcrcl} Y_1 &=& 1 + (Y_2 \cup Y_3) && Y_2 &=& 1 + Y_3 \\ Y_3 &=& \{1\} \cup (1 + (Y_2 \cup Y_4)) && Y_4 &=& 1 + Y_2. \end{array}$$

To calculate the $\mathfrak{m}_i$ and $\mathfrak{q}_i$ for this system, first

$$\mathbf{G}(\boldsymbol{\varnothing}) = \begin{bmatrix} \varnothing \\ \varnothing \\ \{1\} \\ \varnothing \end{bmatrix} \qquad \mathbf{G}^{(2)}(\boldsymbol{\varnothing}) = \begin{bmatrix} \{2\} \\ \{2\} \\ \{1\} \\ \varnothing \end{bmatrix}$$

$$\mathbf{G}^{(3)}(\boldsymbol{\varnothing}) = \begin{bmatrix} \{2,3\} \\ \{2\} \\ \{1,3\} \\ \{3\} \end{bmatrix} \qquad \mathbf{G}^{(4)}(\boldsymbol{\varnothing}) = \begin{bmatrix} \{2,3,4\} \\ \{2,4\} \\ \{1,3,4\} \\ \{3\} \end{bmatrix},$$

and thus, by (4.1), $\mathfrak{m} = (2, 2, 1, 3)$. For such a simple example, one easily finds the $\mathfrak{m}_i$ by inspection—$\mathfrak{m}_i$ is the length of the shortest path in Fig. 1 from vertex i to vertex 4.

To calculate the $\mathfrak{q}_i$ let

$$S_j := \bigcup_{\mathbf{u}} \big(G_{j,\mathbf{u}} + \mathfrak{m} * \mathbf{u} - \mathfrak{m}_j\big), \quad \text{for } 1 \le j \le 4.$$

Then $S_1 = \{0, 1\}$, $S_2 = \{0\}$, $S_3 = \{0, 2, 3\}$, and $S_4 = \{0\}$. The digraph in Fig. 1 is, conveniently, also the dependency digraph of the system, and $\{2, 3, 4\}$ is a strong component. From (4.2), $\mathfrak{q}_i = \gcd\Big(\bigcup_{i\to^* j} S_j\Big)$, so $\mathfrak{q}_1 = \gcd\big(S_1 \cup S_2 \cup S_3 \cup S_4\big) = \gcd\{0, 1, 2, 3\} = 1$, and $\mathfrak{q}_2 = \mathfrak{q}_3 = \mathfrak{q}_4 = \gcd\big(S_2 \cup S_3 \cup S_4\big) = \gcd\{0, 2, 3\} = 1$.

EXAMPLE 4.12 (Regular Languages). A set $\mathcal{R}$ of words over an m-letter alphabet is a *regular language* if it is precisely the set of words accepted by some finite-state deterministic automaton. A word is accepted by such an automaton if, starting at state 0, one can follow a path to a final state with the labels on the successive edges of the path spelling out the word. Let the states of the automaton be $0, \ldots, k$. We write $i \to j$ if there is an edge in the automaton going from i to j. If $i \to j$, let a_{ij} denote the letter from the alphabet that labels the edge $i \to j$. For each $i \in \{0, \ldots, k\}$ let $\mathcal{U}_i$ be the set of letters a_{ij} for which j is a final state, and let $\mathcal{R}_i$ be the set of words traversed when going from a state i to a final state.

$\mathcal{R}_i$ is the union of $\mathcal{U}_i$ with the classes $a_{ij}\mathcal{R}_j$ for which $i \to j$ is an edge in the automaton. Thus the specification of the $\mathcal{R}_i$ is given by the system of equations

$$\mathcal{R}_i \; = \; \mathcal{U}_i \cup \bigcup_{j \,:\, i \to j} a_{ij}\mathcal{R}_j, \quad 0 \le i \le k.$$

This leads to a system of $\mathbf{y}$-linear equations of a particularly simple form for the generating functions, and for the spectra, namely for $0 \le i \le k$, with $c_i = |\mathcal{U}_i|$,

$$\begin{aligned} R_i(x) &= x \cdot \big(c_i + \sum_{j \,:\, i \to j} R_j(x)\big) \\ \mathsf{Spec}(\mathcal{R}_i) &= A_i \cup \big(1 + \bigcup_{j \,:\, i \to j} \mathsf{Spec}(\mathcal{R}_j)\big), \end{aligned}$$

where $A_i = \emptyset$ if $c_i = 0$, and $A_i = \{1\}$ otherwise.

The theory of the generating functions for regular languages was worked out by Berstel [**5**], 1971 (his results were augmented by Soittola [**30**], 1976). Given a regular language $\mathcal{R}$, one can, as noted above, partition it into classes $\mathcal{R}_i$ such that the generating functions $R_i(x)$ satisfy a system of linear equations $\mathbf{y} = x(C + M \cdot \mathbf{y})$, where C is a column matrix with entries from $\mathbb{N}$, and M is a 0,1-square matrix. The $\mathsf{Spec}(\mathcal{R}_i)$ are eventually periodic, and by Cramer's rule, the generating functions $R_i(x)$ are rational functions; also they are given by $\mathbf{R}(x) = x\big(I - xM\big)^{-1}C$. Berstel showed that each $R_i(x)$ is the sum of a finite number of $R_{ij}(x)$, with each $\mathsf{Spec}(R_{ij}(x))$ being either finite or eventually an arithmetical progression. For those $\mathsf{Spec}(R_{ij}(x))$ which are not finite, there is a polynomial $P_{ij}(x) \neq 0$ and finitely many polynomials $P_{ij\ell}(x)$, a real β_{ij}, and numbers $\beta_{ij\ell}$, with $\beta_{ij} > \max\{|\beta_{ij\ell}|\}$, such that, on the set $\mathsf{Spec}(R_{ij}(x))$, one has the coefficients $r_{ij}(n)$ having an exact polynomial exponential form, and polynomial exponential asymptotics, given by (see *Analytic Combinatorics* [**19**], p. 302):

$$\begin{aligned} r_{ij}(n) &= P_{ij}(n)\beta_{ij}^n + \sum_{\ell} P_{ij\ell}(n)\beta_{ij\ell}^n \quad \text{for } n \in \mathsf{Spec}(R_{ij}(x)) \\ &\sim P_{ij}(n)\beta_{ij}^n \quad \text{on the set } \mathsf{Spec}(R_{ij}(x)). \end{aligned}$$

4.4. Relaxing the conditions on $\mathbf{G}(x, \mathbf{y})$. Recall that a power series system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ is elementary if (i) $\mathbf{G}(x, \mathbf{y}) \trianglerighteq \mathbf{0}$, (ii) $\mathbf{G}(0, \mathbf{0}) = \mathbf{0}$, and (iii) $J_{\mathbf{G}}(0, \mathbf{0}) = \mathbf{0}$. The 'elementary system' requirement of Theorem 4.6 is usually true for irreducible power series systems $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ arising in combinatorics—see, for example, *Analytic Combinatorics* [**19**], where most of the irreducible examples are such that x is a factor of $\mathbf{G}(x, \mathbf{y})$, a property of $\mathbf{G}(x, \mathbf{y})$ which immediately guarantees that the second and third of the three conditions hold.

Dropping the first requirement, that $\mathbf{G}(x, \mathbf{y}) \trianglerighteq \mathbf{0}$, leads to a difficult area of research, where little is known, even with a single equation $y = G(x, y)$—see the final sections of [**2**] for several remarks on the difficulties mixed signs in $G(x, y)$ pose when trying to determine the asymptotics of the coefficients $t(n)$ of a solution

$y = T(x)$. Such mixed sign situations can arise naturally, for example when dealing with the construction Set, which forms subsets of a given set of objects. The method developed here, for studying the spectra of the solutions $T_i(x)$ of a system $\mathbf{y} = \mathbf{G}(x,\mathbf{y})$, very much depends on $\mathbf{G}(x,\mathbf{y}) \trianglerighteq \mathbf{0}$, in particular, being able to claim that $\mathsf{Spec}\big(\mathbf{G_u}(x)\cdot\mathbf{T}(x)^{\mathbf{u}}\big)$ is equal to $\mathbf{G_u} + \mathbf{u}\circledast\mathbf{T}$. This equality can fail with mixed signs, for example, the spectrum of $G(x)\cdot T(x) = (1-x)\cdot(x+x^2)$ is not the same as $\mathsf{Spec}(1-x) + \mathsf{Spec}(x+x^2)$.

The second condition, $\mathbf{G}(0,\mathbf{0}) = \mathbf{0}$, is essential if the solution $\mathbf{T}(x)$ provides generating functions $T_i(x)$ for combinatorial classes $\mathcal{T}_i$ since, in these cases, $\mathsf{Spec}(T_i) \subseteq \mathbb{P}$, so $0 \notin \mathsf{Spec}(T_i)$, for any i. Thus the discussion will be limited to dropping the third requirement, that $J_{\mathbf{G}}(0,\mathbf{0}) = \mathbf{0}$. This simply means that $\mathbf{y}$-linear terms with constant coefficients are permitted to appear in the $G_i(x,\mathbf{y})$, in which case a number of new possibilities can arise when classifying the solutions of such systems:

(a) There may be no (formal power series) solution, for example, $y = x + y$.
(b) There may be a solution, but not $\trianglerighteq \mathbf{0}$, for example, $y = x + 2y$.
(c) There may be infinitely many solutions, for example, $y_1 = y_2$, $y_2 = y_1$.

One can express the system $\mathbf{y} = \mathbf{G}(x,\mathbf{y})$ as

$$\mathbf{y} \;=\; \mathbf{G}(x,\mathbf{0}) + J_{\mathbf{G}}(0,\mathbf{0})\cdot\mathbf{y} + \mathbf{H}(x,\mathbf{y}),$$

where

$$\mathbf{H}(x,\mathbf{y}) \;=\; \sum_{i=1}^{k} y_i\cdot\mathbf{H}_i(x,\mathbf{y}),$$

with each $\mathbf{H}_i(x,\mathbf{y}) \in \mathsf{Dom}_0[x,\mathbf{y}]^k$. The obvious approach to such a system, with $J_{\mathbf{G}}(0,\mathbf{0}) \neq \mathbf{0}$, is to write it in the form

$$\big(I - J_{\mathbf{G}}(0,\mathbf{0})\big)\cdot\mathbf{y} \;=\; \mathbf{G}(x,0) + \mathbf{H}(x,\mathbf{y})$$

and solve for $\mathbf{y}$.

Definition 4.13 (of $\widehat{\mathbf{G}}(x,\mathbf{y})$). Given $\mathbf{G}(x,\mathbf{y}) \trianglerighteq \mathbf{0}$, with $\mathbf{G}(0,\mathbf{0}) = \mathbf{0}$, if the matrix $I - J_{\mathbf{G}}(0,\mathbf{0})$ has an inverse that is nonnegative then let

$$\widehat{\mathbf{G}}(x,\mathbf{y}) \;:=\; \Big(I - J_{\mathbf{G}}(0,\mathbf{0})\Big)^{-1}\cdot\Big(\mathbf{G}(x,\mathbf{0}) + \mathbf{H}(x,\mathbf{y})\Big).$$

Given a nonnegative square matrix M, let $\Lambda(M)$ denote the largest real eigenvalue of M. (Note: From the Perron-Frobenius theory we know that a nonnegative square matrix M has a nonnegative real eigenvalue; hence there is indeed a largest real eigenvalue $\Lambda(M)$, and it is ≥ 0. The Perron-Frobenius theory also tells us that $\Lambda(M)$ has a nonnegative eigenvector.)

Theorem 4.14. *Let $\mathbf{G}(x,\mathbf{y}) \in \mathbb{R}[[x,\mathbf{y}]]^k$ satisfy the two conditions*

$$\mathbf{G}(x,\mathbf{y}) \trianglerighteq \mathbf{0}, \quad \textit{and } \mathbf{G}(0,\mathbf{0}) = \mathbf{0},$$

that is, $\mathbf{G}(x,\mathbf{y}) \in \mathsf{Dom}_0(x,\mathbf{y})^k$.

(a) *Suppose $I - J_{\mathbf{G}}(0,\mathbf{0})$ has a nonnegative inverse. Then the following hold:*

(i) *The system $\mathbf{y} = \widehat{\mathbf{G}}(x,\mathbf{y})$ is equivalent to the system $\mathbf{y} = \mathbf{G}(x,\mathbf{y})$, that is, they have the same solutions (but not necessarily the same dependency digraph).*

(ii) *$\widehat{\mathbf{G}}(x,\mathbf{y})$ is elementary.*

(iii) *Consequently* $\mathbf{T}(x) := \widehat{\mathbf{G}}^{(\infty)}(x, \mathbf{0})$ *is the unique solution in* $\mathbb{R}[[x]]_0$ *of* $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ *as well as of* $\mathbf{y} = \widehat{\mathbf{G}}(x, \mathbf{y})$*. The periodicity properties of* $\mathbf{T}(x)$ *are as stated in Theorem 4.6, with* $\mathbf{G}$ *replaced by* $\widehat{\mathbf{G}}$*.*

(b) *Suppose that* $\mathbf{G}^{(k)}(x, \mathbf{0})$ *has all entries nonzero, that is, the associated system* $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ *of set equations is reduced. Then the following are equivalent:*
(i) $I - J_{\mathbf{G}}(0, \mathbf{0})$ *has a nonnegative inverse.*
(ii) *The equation* $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ *has a solution* $\mathbf{T}(x) \in \mathsf{Dom}_0[x]^k$*.*
(iii) $\Lambda\big(J_{\mathbf{G}}(0, \mathbf{0})\big) < 1$.

Proof. (a): Given that $I - J_{\mathbf{G}}(0, \mathbf{0})$ has a nonnegative inverse, one can transform either of $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ and $\mathbf{y} = \widehat{\mathbf{G}}(x, \mathbf{y})$ into the other by simple operations that preserve solutions. It is routine to check that $\widehat{\mathbf{G}}(x, \mathbf{y})$ is elementary; then use Theorem 4.6.

(b): Assume $\mathbf{G}^{(k)}(x, \mathbf{0})$ has all entries nonzero. (i) $\Rightarrow$ (ii) follows from (a). If (ii) holds then

$$\mathbf{T}(x) \ = \ \mathbf{G}^{(k)}\big(x, \mathbf{T}(x)\big) \ \trianglerighteq \ \mathbf{G}^{(k)}(x, \mathbf{0}).$$

Let $\mathbf{v} \geq 0$ be a left eigenvector of $\Lambda\big(J_{\mathbf{G}^{(k)}}(0, \mathbf{0})\big)$. From

$$\mathbf{T}(x) \ = \ \mathbf{G}^{(k)}(x, \mathbf{0}) \ + \ J_{\mathbf{G}^{(k)}}(0, \mathbf{0}) \cdot \mathbf{T}(x) \ + \ \widetilde{\mathbf{H}}\big(x, \mathbf{T}(x)\big),$$

one has

$$\mathbf{v} \cdot \mathbf{T}(x) \ = \ \mathbf{v} \cdot \mathbf{G}^{(k)}(x, \mathbf{0}) \ + \ \Lambda\big(J_{\mathbf{G}^{(k)}}(0, \mathbf{0})\big) \cdot \mathbf{v} \cdot \mathbf{T}(x) \ + \ \mathbf{v} \cdot \widetilde{\mathbf{H}}\big(x, \mathbf{T}(x)\big). \tag{4.3}$$

Since all $T_i(x)$ are nonnegative and nonzero, one has $\mathbf{v} \cdot \mathbf{T}(x)$ and $\mathbf{v} \cdot \mathbf{G}^{(k)}(x, \mathbf{0}) + \mathbf{v} \cdot \widetilde{\mathbf{H}}\big(x, \mathbf{T}(x)\big)$ are nonzero power series with nonnegative coefficients; consequently (4.3) implies $\Lambda\big(J_{\mathbf{G}^{(k)}}(0, \mathbf{0})\big) < 1$. From $J_{\mathbf{G}^{(k)}}(0, \mathbf{0}) = J_{\mathbf{G}}(0, \mathbf{0})^k$ it follows that $\Big(\Lambda\big(J_{\mathbf{G}}(0, \mathbf{0})\big)\Big)^k$ is an eigenvalue of $J_{\mathbf{G}^{(k)}}(0, \mathbf{0})$, and thus also < 1. This clearly implies $\Lambda\big(J_{\mathbf{G}}(0, \mathbf{0})\big) < 1$, so (ii) $\Rightarrow$ (iii).

If (iii) holds, then by Neumann's expansion theorem (see [**22**], p. 201), one knows that $I - J_{\mathbf{G}}(0, \mathbf{0})$ has an inverse, and $\big(I - J_{\mathbf{G}}(0, \mathbf{0})\big)^{-1} = \sum_{n \geq 0} J_{\mathbf{G}}(0, \mathbf{0})^n$, a nonnegative matrix. Thus (iii) $\Rightarrow$ (i). □

The condition that $\mathbf{G}^{(k)}(x, \mathbf{0})$ has all entries nonzero is usual for power series systems in combinatorics since the $T_i(x)$ in the solution of $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ are generating functions for nonempty classes $\mathcal{T}_i$.

A somewhat tedious calculation shows that one can use the formulas (4.1) and (4.2) of Theorem 4.6 to calculate the $\mathfrak{m}_i$ and $\mathfrak{q}_i$ with the original system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ as well as with the derived system $\mathbf{y} = \widehat{\mathbf{G}}(x, \mathbf{y})$. It can be useful to note that if the two hypotheses of Theorem 4.14 hold, then the condition that $\mathbf{G}^{(k)}(x, \mathbf{0})$ has all entries nonzero is equivalent to requiring that $\mathbf{G}^{(j)}(x, \mathbf{0})$ has all entries nonzero, for some j, $1 \leq j \leq k$.

Remark 4.15. The uniqueness of the solution $\mathbf{T}(x)$ in $\mathbb{R}[[x]]_0^k$, for a power series system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ satisfying the two hypotheses of Theorem 4.14 and the hypothesis of part (a), does not in general extend to the associated spectral system $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ when $J_{\mathbf{G}}(0, \mathbf{0}) \neq \mathbf{0}$. For example, consider the consistent single equation system $y = G(x, y)$, where $G(x, y) = x^2 + (1/2)y + xy$. The spectral system $Y = G(Y)$ is $Y = \{2\} \cup Y \cup (1 + Y)$, which has three solutions: $\mathbb{N}$, $1 + \mathbb{N}$, and $2+\mathbb{N}$. On the other hand, the elementary system $y = \widehat{G}(x,y)$ is $y = 2x^2+2xy$; its spectral system is $Y = \{2\} \cup (1+Y)$, which has the unique solution $2+\mathbb{N}$.

4.5. An example of the tools.

The following simple example uses all the tools developed so far.

EXAMPLE 4.16. Consider the class $\mathcal{T}$ of planar trees with blue and red nodes, defined by the conditions:

(i) Every blue node that is not a leaf has exactly three nodes immediately below it, but not all of the same color.
(ii) Every red node that is not a leaf has exactly two nodes immediately below it.

Let $\mathcal{B}$ be the collection of trees in $\mathcal{T}$ with blue roots, and $\mathcal{R}$ the collection of those with red roots. Then, letting $\bullet_B$ be a blue node and $\bullet_R$ a red node, one has the equational specification (see §5)

$$\begin{aligned}
\mathcal{B} &= \{\bullet_B\} \cup \frac{\bullet_B}{\mathcal{B}+\mathcal{R}+\mathcal{R}} \cup \frac{\bullet_B}{\mathcal{R}+\mathcal{B}+\mathcal{R}} \cup \frac{\bullet_B}{\mathcal{R}+\mathcal{R}+\mathcal{B}} \cup \\
&\quad \frac{\bullet_B}{\mathcal{B}+\mathcal{B}+\mathcal{R}} \cup \frac{\bullet_B}{\mathcal{B}+\mathcal{R}+\mathcal{B}} \cup \frac{\bullet_B}{\mathcal{R}+\mathcal{B}+\mathcal{B}} \\
\mathcal{R} &= \{\bullet_R\} \cup \frac{\bullet_R}{\mathcal{T}+\mathcal{T}} \\
\mathcal{T} &= \mathcal{B} \cup \mathcal{R}.
\end{aligned}$$

The three generating functions, $B(x)$ for $\mathcal{B}$, $R(x)$ for $\mathcal{R}$, and $T(x)$ for $\mathcal{T}$, are related by the system of equations:

$$\begin{aligned}
B(x) &= x + 3x \cdot B(x) \cdot R(x)^2 + 3x \cdot B(x)^2 \cdot R(x) \\
R(x) &= x + x \cdot T(x)^2 \\
T(x) &= B(x) + R(x).
\end{aligned}$$

Thus $(B(x), R(x), T(x))$ gives a solution (y_1, y_2, y_3) for the system of polynomial equations:

$$\begin{aligned}
y_1 &= x + 3x \cdot y_1 \cdot y_2^2 + 3x \cdot y_1^2 \cdot y_2 \\
y_2 &= x + x \cdot y_3^2 \\
y_3 &= y_1 + y_2.
\end{aligned}$$

The spectra $\mathsf{Spec}(\mathcal{B}), \mathsf{Spec}(\mathcal{R}), \mathsf{Spec}(\mathcal{T})$ are related by the set equations

$$\begin{aligned}
\mathsf{Spec}(\mathcal{B}) &= \{1\} \cup (1+\mathsf{Spec}(\mathcal{B})+2*\mathsf{Spec}(\mathcal{R})) \cup (1+2*\mathsf{Spec}(\mathcal{B})+\mathsf{Spec}(\mathcal{R})) \\
\mathsf{Spec}(\mathcal{R}) &= \{1\} \cup (1+2*\mathsf{Spec}(\mathcal{T})) \\
\mathsf{Spec}(\mathcal{T}) &= \mathsf{Spec}(\mathcal{B}) \cup \mathsf{Spec}(\mathcal{R}),
\end{aligned}$$

so $(\mathsf{Spec}(\mathcal{B}), \mathsf{Spec}(\mathcal{R}), \mathsf{Spec}(\mathcal{T}))$ is a solution to the system of set equations

$$\begin{aligned}
Y_1 &= \{1\} \cup (1+Y_1+2*Y_2) \cup (1+2*Y_1+Y_2) \\
Y_2 &= \{1\} \cup (1+2*Y_3) \\
Y_3 &= Y_1 \cup Y_2.
\end{aligned}$$

Next,

$$\mathbf{G}(x,y_1,y_2,y_3) = \begin{bmatrix} x + 3x \cdot y_1 \cdot y_2^2 + 3x \cdot y_1^2 \cdot y_2 \\ x + x \cdot y_3^2 \\ y_1 + y_2 \end{bmatrix},$$

so

$$\mathbf{G}^{(2)}(x,0,0,0) \;=\; \begin{bmatrix} 6x^4 + x \\ x \\ 2x \end{bmatrix}$$

has all entries nonzero—this implies $\mathbf{G}^{(3)}(x,\mathbf{0})$ has all entries nonzero.

The Jacobian matrix $J_{\mathbf{G}}(x,\mathbf{y})$ is

$$J_{\mathbf{G}}(x,y_1,y_2,y_3) \;=\; \begin{bmatrix} 3xy_2^2 + 6xy_1y_2 & 6xy_1y_2 + 3xy_1^2 & 0 \\ 0 & 0 & 2xy_3 \\ 1 & 1 & 0 \end{bmatrix}$$

so

$$J_{\mathbf{G}}(x,0,0,0) \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of $J_{\mathbf{G}}(0,0,0,0)$ are the roots of $\det\big(\lambda I - J_{\mathbf{G}}(0,0,0,0)\big) = 0$, that is, $\lambda^3 = 0$. Thus $\Lambda\big(J_{\mathbf{G}}(0,0,0,0)\big) = 0 < 1$, so the system $\mathbf{y} = \mathbf{G}(x,\mathbf{y})$ has a solution $\mathbf{T}(x) \in \mathbb{R}[[x]]_0^3$, and the solution has all entries nonzero. The inverse of $I - J_{\mathbf{G}}(0,0,0,0)$ is a nonnegative matrix:

$$\big(I - J_{\mathbf{G}}(0,0,0,0)\big)^{-1} \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Thus

$$\widehat{\mathbf{G}}(x,\mathbf{y}) \;=\; \begin{bmatrix} x + 3xy_1y_2^2 + 3xy_1^2y_2 \\ x + xy_3^2 \\ 2x + 3xy_1y_2^2 + 3xy_1^2y_2 + xy_3^2 \end{bmatrix},$$

so $\mathbf{y} = \widehat{\mathbf{G}}(x,\mathbf{y})$ is an irreducible reduced non $\mathbf{y}$-linear elementary system. Thus Corollary 4.7 applies.

The spectral system $\mathbf{Y} = \widehat{\mathbf{G}}(\mathbf{Y})$ is

$$\begin{aligned} Y_1 &= \{1\} \cup \big(1 + Y_1 + 2 * Y_2\big) \cup \big(1 + 2 * Y_1 + Y_2\big) \\ Y_2 &= \{1\} \cup \big(1 + 2 * Y_3\big) \\ Y_3 &= \{1\} \cup \big(1 + Y_1 + 2 * Y_2\big) \cup \big(1 + 2 * Y_1 + Y_2\big) \cup \big(1 + 2 * Y_3\big). \end{aligned}$$

One readily calculates that $\mathfrak{m}_i = \mathfrak{p}_i = \mathfrak{q}_i = 1$, for $1 \le i \le 3$.

5. Constructions, Operators and Equational Specifications

The general theory of setting up equational specifications $\boldsymbol{\mathcal{Y}} = \boldsymbol{\Gamma}(\boldsymbol{\mathcal{Y}})$ for combinatorial classes $\mathcal{A}_1,\ldots,\mathcal{A}_k$ is developed by forming the $\Gamma_i(\boldsymbol{\mathcal{Y}})$ from compositions of basic constructions, and then translating the specifications into systems of equations

(a) $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$, for the spectra $\mathsf{Spec}(\mathcal{A}_1),\ldots,\mathsf{Spec}(\mathcal{A}_k)$, and
(b) $\mathbf{y} = \mathbf{G}(x,\mathbf{y})$, for the generating functions $A_1(x),\ldots,A_k(x)$,

There are two important approaches to constructions: one is to have constructions that may be influenced by the nature of the objects in combinatorial classes, and may have a limited range of application; the other is to only have constructions that apply to all combinatorial classes. §5.1–§5.2 look at the first approach, based on Compton's papers [**9**], [**10**], from the late 1980s. The second approach, following *Analytic Combinatorics* [**19**], is sketched in §5.3.

5.1. Constructions for relational classes: $\cup, +, \mathsf{MSet}$.

Many combinatorial classes of interest are classes of relational structures such as digraphs, functional digraphs, graphs, posets, forests or planar forests.[14] The count function for a class of relational structures counts up to isomorphism. A class $\mathcal{A}$ of relational structures will always be assumed to belong to a fixed finite relational language, for example, $\mathcal{A}$ could be a class of binary relational structures (also known as a class of digraphs).

For $\mathcal{A}$ a class of relational structures, a *k-ary construction* Θ *for* $\mathcal{A}$ is a mapping $\Theta : \mathrm{Su}(\mathcal{A})^k \to \mathrm{Su}(\mathcal{A})$ from k-tuples of subclasses of $\mathcal{A}$ to subclasses of $\mathcal{A}$. The following gives three constructions that apply to any class of relational structures that belong to a given relational language. Recall that $A(x) \trianglelefteq B(x)$ means $a(n) \le b(n)$ for all n.

(a) *Union* ($\cup$)

Given two classes $\mathcal{A}, \mathcal{B}$ of relational structures, the *union* construction is exactly as in set theory. One has:

$$\begin{aligned} (\mathcal{A} \cup \mathcal{B})(x) &\trianglelefteq A(x) + B(x) \\ \mathsf{Spec}(\mathcal{A} \cup \mathcal{B}) &= \mathsf{Spec}(\mathcal{A}) \cup \mathsf{Spec}(\mathcal{B}). \end{aligned} \tag{5.1}$$

One has equality in (5.1) iff $\mathcal{A}$ and $\mathcal{B}$ are (up to isomorphism) disjoint.

(b) *Relational Sum* ($+$)

$\mathfrak{a} + \mathfrak{b}$ is defined for a pair of relational structures $\mathfrak{a}, \mathfrak{b}$ by first replacing $\mathfrak{a}$ and $\mathfrak{b}$ by isomorphic structures $\mathfrak{a}'$ and $\mathfrak{b}'$ whose universes are disjoint; then define $\mathfrak{a} + \mathfrak{b}$ to be $\mathfrak{a}' + \mathfrak{b}'$, the relational structure whose universe is the union of the universes of $\mathfrak{a}'$ and $\mathfrak{b}'$, and for each relational symbol R, one has $R_{\mathfrak{a}'+\mathfrak{b}'} := R_{\mathfrak{a}'} \cup R_{\mathfrak{b}'}$. Clearly $\|\mathfrak{a} + \mathfrak{b}\| = \|\mathfrak{a}\| + \|\mathfrak{b}\|$.

A useful defining property of $\mathfrak{a} + \mathfrak{b}$ is as follows. For any connected structure $\mathfrak{c}$, let $\nu_{\mathfrak{c}}(\mathfrak{a})$ be the number of components of $\mathfrak{a}$ that are isomorphic to $\mathfrak{c}$. Then $\nu_{\mathfrak{c}}(\mathfrak{a} + \mathfrak{b}) = \nu_{\mathfrak{c}}(\mathfrak{a}) + \nu_{\mathfrak{c}}(\mathfrak{b})$, for all $\mathfrak{c}$.

One has the following for the generating function and spectrum:

$$\begin{aligned} (\mathcal{A} + \mathcal{B})(x) &\trianglelefteq A(x) \cdot B(x) \\ \mathsf{Spec}(\mathcal{A} + \mathcal{B}) &= \mathsf{Spec}(\mathcal{A}) + \mathsf{Spec}(\mathcal{B}). \end{aligned} \tag{5.2}$$

One has equality in (5.2) iff

(*) given any $\mathfrak{c} \in \mathcal{A}+\mathcal{B}$, there are, up to isomorphism, unique $\mathfrak{a} \in \mathcal{A}$, $\mathfrak{b} \in \mathcal{B}$ such that $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$.

(c) *Relational Multiset* (MSet)

For $\mathcal{A}$ a class of relational structures let

$$\mathsf{MSet}_M(\mathcal{A}) := \bigcup_{m \in M} \underbrace{\mathcal{A} + \cdots + \mathcal{A}}_{m}.$$

Then

$$\begin{aligned} \mathsf{MSet}_M(\mathcal{A})(x) &\trianglelefteq \mathsf{MSet}_M(A(x)) \\ &:= \sum_{m \in M} \sigma_m\big(\mathfrak{s}_m, A(x), \ldots, A(x^m)\big) \\ \mathsf{Spec}\big(\mathsf{MSet}_M(\mathcal{A})\big) &= M * \mathsf{Spec}(\mathcal{A}), \end{aligned} \tag{5.3}$$

[14] Each of these classes, except planar forests, is closed under sum and the extraction of components, and thus can be viewed as an *additive number system* (see [**7**]).

where $\sigma_m(\mathfrak{s}_m, y_1, \ldots, y_m)$ is the Pólya cycle index polynomial on m variables, for the symmetric group $\mathfrak{s}_m$.

If $\mathcal{A}$ is a class of connected structures then one has equality in (5.3).

REMARK 5.1. MSet_M is written simply as MSet when $M = \mathbb{P}$. Let Θ be a unary construction that acts on classes $\mathcal{A}$ by performing constructions on finitely many objects from $\mathcal{A}$. Given a nonempty set M of positive integers, let $\Theta_M(\mathcal{A})$ be the class of all objects that one can construct by applying Θ to m-many objects from $\mathcal{A}$, for $m \in M$. (Repeats are allowed, and they are counted with multiplicity when determining if the count is in M). Θ_M is called a *restriction* of Θ.

5.2. Specifications for m-colored forests: $\bullet_i/, \cup, +, \mathsf{MSet}$.

For an illustration of how one derives systems of equations for spectra from equational specifications of classes of relational structures, we choose the setting of m-colored forests.

Let $\bullet_i$ be the 1-element forest with color i, for $1 \le i \le m$. Then

$$\begin{aligned} \{\bullet_i\}(x) &= x \\ \mathsf{Spec}(\{\bullet_i\}) &= \{1\}. \end{aligned}$$

In addition to the three constructions in § 5.1, there is the construction $\bullet_i/$, which adds a root with color i to a forest. Clearly $\| \bullet_i / F\| = 1 + \|F\|$. Extend this construction to classes $\mathcal{F}$ of m-colored forests by $\bullet_i/\mathcal{F} = \{\bullet/F : F \in \mathcal{F}\}$. Then

$$\begin{aligned} \big(\bullet_i /\mathcal{F}\big)(x) &= x \cdot F(x) \\ \mathsf{Spec}\big(\bullet_i /\mathcal{F}\big) &= 1 + \mathsf{Spec}(\mathcal{F}). \end{aligned}$$

Given a k-tuple $\boldsymbol{\mathcal{F}}$ of classes of m-colored forests, a *specification* for $\boldsymbol{\mathcal{F}}$ is a system of equations $\boldsymbol{\mathcal{Y}} = \boldsymbol{\Gamma}(\boldsymbol{\mathcal{Y}})$, where each $\Gamma_i(\boldsymbol{\mathcal{Y}})$ is a composition of the basic constructions for m-colored forests, and $\boldsymbol{\mathcal{F}}$ is the minimum solution $\boldsymbol{\Gamma}^{(\infty)}(\boldsymbol{\emptyset})$ of this system. There is a routine method to translate the specification into a system $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ of equations for the spectra.

5.3. The standard constructions of *Analytic Combinatorics*. The set calculus that has been developed in this article can also be applied to specifications $\boldsymbol{\mathcal{Y}} = \boldsymbol{\Gamma}(\boldsymbol{\mathcal{Y}})$, where the Γ_i are composed from the class $\mathcal{Z}$, consisting of a single object of size 1, and the following *standard constructions* found in *Analytic Combinatorics* [**19**]:[15]

(a) *Disjoint Union* $(\sqcup)$[16] Given any two combinatorial classes $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A}', \mathcal{B}'$ be isomorphic copies that are disjoint. Let

$$\mathcal{A} \sqcup \mathcal{B} \ := \ \mathcal{A}' \cup \mathcal{B}'.$$

(b) *Combinatorial Product* $(\times)$ Given any two combinatorial classes $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A} \times \mathcal{B}$ be the Cartesian product $\mathcal{A} \times \mathcal{B}$, with $\|(\mathfrak{a}, \mathfrak{b})\| = \|\mathfrak{a}\| + \|\mathfrak{b}\|$.

(c) *Combinatorial Multiset* (MSET) Given any combinatorial class $\mathcal{A}$ and nonempty $M \subseteq \mathbb{P}$, let $\mathrm{MSET}_M(\mathcal{A})$ be the class of formal unordered sums $\mathfrak{a}_1 + \cdots + \mathfrak{a}_m$ of m objects from $\mathcal{A}$, for $m \in M$, with size equal to $\|\mathfrak{a}_1\| + \cdots + \|\mathfrak{a}_m\|$.

[15] The constructions of the previous sections have standard analogs: the analog of $\{\bullet\}$ is $\mathcal{Z}$, of $\cup$ is $\sqcup$, of $+$ is $\times$, and of MSet is MSET.

[16] The *disjoint union* construction in [**19**] uses *sum* for its primary name, and the symbol used is $+$ instead of $\sqcup$.

(d) *Combinatorial Sequence* (SEQ) Given any combinatorial class $\mathcal{A}$ and nonempty $M \subseteq \mathbb{P}$, let $\mathrm{SEQ}_M(\mathcal{A})$ be the class of formal ordered sums $\mathfrak{a}_1 + \cdots + \mathfrak{a}_m$ of m objects from $\mathcal{A}$, for $m \in M$, with size equal to $\|\mathfrak{a}_1\| + \cdots + \|\mathfrak{a}_m\|$.

(e) *Combinatorial Cycle* (CYC) Given a combinatorial class $\mathcal{A}$ and a nonempty set $M \subseteq \mathbb{P}$, $\mathrm{CYC}_M(\mathcal{A})$ is the class of formal cyclic arrangements of m members of $\mathcal{A}$, where $m \in M$, with the size of the cycle being the sum of the sizes of the members.

(f) *Combinatorial Directed Cycle* (DCYC) Given a combinatorial class $\mathcal{A}$ and a nonempty set $M \subseteq \mathbb{P}$, $\mathrm{DCYC}_M(\mathcal{A})$ is the class of formal directed cyclic arrangements of m members of $\mathcal{A}$, where $m \in M$, with the size of the directed cycle being the sum of the sizes of the members.

These constructions are not relational constructions (as defined in §5.1). For example, letting $\mathcal{D}$ be the class of digraphs one has $(\mathcal{D} \sqcup \mathcal{D})(x) = 2D(x)$, so $\mathcal{D} \sqcup \mathcal{D}$ has twice as many objects of each size as $\mathcal{D}$. Thus $\mathcal{D} \sqcup \mathcal{D}$ cannot be viewed as a class of digraphs.

Nonetheless, given a specification $\boldsymbol{\mathcal{Y}} = \mathbf{\Gamma}(\boldsymbol{\mathcal{Y}})$, where $\mathbf{\Gamma}(\boldsymbol{\mathcal{Y}})$ is composed of relational constructions as in the previous sections, if one lets $\widehat{\mathbf{\Gamma}}(\boldsymbol{\mathcal{Y}})$ be the expression obtained by replacing the relational constructions in $\mathbf{\Gamma}(\boldsymbol{\mathcal{Y}})$ by their standard analogs (see Footnote 15), then one usually finds that *the combinatorial classes specified by* $\boldsymbol{\mathcal{Y}} = \mathbf{\Gamma}(\boldsymbol{\mathcal{Y}})$ *have the same generating functions and spectra as that of the classes specified by* $\boldsymbol{\mathcal{Y}} = \widehat{\mathbf{\Gamma}}(\boldsymbol{\mathcal{Y}})$.[17]

The standard constructions have two important properties which make the analysis of classes defined by specifications proceed smoothly in *Analytic Combinatorics*:

(1) Standard constructions $\Theta(\mathbf{Y})$ are *total*, that is, they are defined for all arguments $\boldsymbol{\mathcal{A}}$ of combinatorial classes.

(2) Standard constructions $\Theta(\mathbf{Y})$ are *admissible*, that is, they are total, and furthermore, given combinatorial classes $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ with $\boldsymbol{\mathcal{A}}(x) = \boldsymbol{\mathcal{B}}(x)$, then

$$\Theta(\boldsymbol{\mathcal{A}})(x) \;=\; \Theta(\boldsymbol{\mathcal{B}})(x).$$

To this list we can add the following:

(3) Standard constructions $\Theta(\mathbf{Y})$ are *spectrally admissible*, that is, given combinatorial classes $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ with $\mathsf{Spec}(\boldsymbol{\mathcal{A}}) = \mathsf{Spec}(\boldsymbol{\mathcal{B}})$, then

$$\mathsf{Spec}(\Theta(\boldsymbol{\mathcal{A}})) \;=\; \mathsf{Spec}(\Theta(\boldsymbol{\mathcal{B}})).$$

Each admissible k-ary construction Θ induces a k-ary operator on $\mathbb{N}[[x]]_0$, which we also call Θ, defined as follows: For $\mathbf{A}(x) \in \mathbb{N}[[x]]_0{}^k$, let $\Theta(\mathbf{A}(x))$ be $\Theta(\boldsymbol{\mathcal{A}})(x)$, where $\boldsymbol{\mathcal{A}}$ is any combinatorial class with $\boldsymbol{\mathcal{A}}(x) = \mathbf{A}(x)$. If Θ is a standard construction then the induced operator is called a *standard* operator. For the standard operators discussed in this section, one finds the following formulas in [**19**]:

LEMMA 5.2. *For combinatorial classes $\mathcal{A}$, $\mathcal{B}$ one has:*

(a) Disjoint Union ($\sqcup$)

$$A(x) \sqcup B(x) \;=\; A(x) + B(x),$$

thus $\mathsf{Spec}(\mathcal{A} \sqcup \mathcal{B}) = \mathsf{Spec}(\mathcal{A}) \cup \mathsf{Spec}(\mathcal{B})$.

[17]The conditions, under which these two specifications lead to the same generating functions and spectra, do not seem to be detailed in the literature. For simple examples, like the specification for trees, or planar trees, the reader will readily see that the italicized claim holds.

(b) Product ($\times$)

$$A(x) \times B(x) \;=\; A(x) \cdot B(x),$$

thus $\mathsf{Spec}(\mathcal{A} \times \mathcal{B}) = \mathsf{Spec}(\mathcal{A}) + \mathsf{Spec}(\mathcal{B})$.

(c) Combinatorial Multiset (MSET)

$$\text{MSET}_M(A(x)) \;=\; \sum_{m \in M} \sigma_m(\mathfrak{s}_m, A(x), \ldots, A(x^m)),$$

where $\sigma_m(\mathfrak{s}_m, y_1, \ldots, y_m)$ *is the Pólya cycle index polynomial on* m *variables for the symmetric group* $\mathfrak{s}_m$. *Thus* $\mathsf{Spec}\big(\text{MSET}_M(\mathcal{A})\big) = M * \mathsf{Spec}(\mathcal{A})$.

(d) Combinatorial Sequence (SEQ)

$$\text{SEQ}_M(A(x)) \;=\; \sum_{m \in M} A(x)^m,$$

thus $\mathsf{Spec}\big(\text{SEQ}_M(\mathcal{A})\big) = M * \mathsf{Spec}(\mathcal{A})$.

(e) Combinatorial Cycle (CYC)

$$\text{CYC}_M(A(x)) \;=\; \sum_{m \in M} \sigma_m(\mathfrak{d}_m, A(x), \ldots, A(x^m)),$$

where $\sigma_m(\mathfrak{d}_m, y_1, \ldots, y_m)$ *is the Pólya cycle index polynomial on* m *variables for the dihedral group* $\mathfrak{d}_m$ *of order* $2m$. *Thus* $\mathsf{Spec}\big(\text{CYC}_M(\mathcal{A})\big) = M * \mathsf{Spec}(\mathcal{A})$.

(f) Combinatorial Directed Cycle (DCYC)

$$\text{DCYC}_M(\mathcal{A}(x)) \;=\; \sum_{m \in M} \sigma_m(\mathfrak{c}_m, A(x), \ldots, A(x^m)),$$

where $\sigma_m(\mathfrak{c}_m, y_1, \ldots, y_m)$ *is the Pólya cycle index polynomial on* m *variables for the cyclic group* $\mathfrak{c}_m$ *of order* m. *Thus* $\mathsf{Spec}\big(\text{DCYC}_M(\mathcal{A})\big) = M * \mathsf{Spec}(\mathcal{A})$.

Recall that

$$\begin{aligned}
\mathsf{Dom}_0[x] &= \big\{A(x) \in \mathbb{R}[[x]] : A(0) = 0, A(x) \trianglerighteq 0\big\} \\
\mathsf{Dom}_0[x, \mathbf{y}] &= \big\{G(x, \mathbf{y}) \in \mathbb{R}[[x, \mathbf{y}]] : G(0, \mathbf{0}) = 0, G(x, \mathbf{y}) \trianglerighteq 0\big\} \\
\mathsf{Dom}_{J0}[x, \mathbf{y}] &= \{\mathbf{G}(x, \mathbf{y}) \in \mathsf{Dom}_0[x, \mathbf{y}]^k : J_{\mathbf{G}}(0, \mathbf{0}) = \mathbf{0}\}.
\end{aligned}$$

The $A(x), B(x)$ in Lemma 5.2 range over the power series in $\mathbb{N}[[x]]_0$. After noting that the right sides of the equations in (a)–(f) of Lemma 5.2 are defined for all $A(x), B(x) \in \mathsf{Dom}_0[x]$, we will assume that these formulas define the standard operators on $\mathsf{Dom}_0[x]$.

DEFINITION 5.3. A k-ary operator Θ on $\mathsf{Dom}_0[x]$ is *spectrally admissible* if for $\mathbf{A}(x), \mathbf{B}(x) \in \mathsf{Dom}_0[x]^k$ one has

$$\mathsf{Spec}(\mathbf{A}(x)) = \mathsf{Spec}(\mathbf{B}(x)) \;\Rightarrow\; \mathsf{Spec}(\Theta(\mathbf{A}(x))) = \mathsf{Spec}(\Theta(\mathbf{B}(x))).$$

An operator $\mathbf{\Theta}$ on $\mathsf{Dom}_0[x]^k$ is *spectrally admissible* if each Θ_i is spectrally admissible.

LEMMA 5.4. *Each* $\mathbf{G}(x, \mathbf{y}) \in \mathsf{Dom}_{J0}[x, \mathbf{y}]^k$ *defines an operator on* $\mathsf{Dom}_0[x]^k$, *namely* $\mathbf{A}(x) \mapsto \mathbf{G}(x, \mathbf{A}(x))$, *that is spectrally admissible. Such operators are called* elementary operators.[18] *As a spectrally admissible operator,* $\mathbf{G}(x, \mathbf{y})$ *induces a set*

[18] Many natural operators are not elementary operators. For example, the operator Θ : $A(x) \mapsto A(x^2)$ is not elementary. There is no power series $G(x, y) \in \mathsf{Dom}[x, y]$ such that $A(x^2) = \sum_n G_n(x) A(x)^n$ for $A(x) \in \mathsf{Dom}_0[x]$. Likewise, the operator $\Theta : A(x) \mapsto A^{(2)}(x)$ is not elementary.

operator on $\mathrm{Su}(\mathbb{P})^k$*, namely*

$$\mathbf{G} : \mathbf{A} \mapsto \bigvee_{\mathbf{u}\in\mathbb{N}^k} \big(\mathbf{G_u} + \mathbf{u} \circledast \mathbf{A}\big).$$

THEOREM 5.5 (Systems based on Spectrally Admissible Operators).

(a) *Elementary operators* $\mathbf{G}(x, \mathbf{y})$ *and standard operators* Θ *are spectrally admissible.*

(b) *Each standard unary operator* Θ_M *is spectrally equivalent to the elementary operator* $\sum_{j\in M} y^j$*, and* $\mathsf{Spec}\big(\Theta_M\big(A(x)\big)\big) = M * \mathsf{Spec}(A(x))$.

(c) *The sum* $\Theta_1 + \Theta_2$*, product* $\Theta_1 \cdot \Theta_2$ *and composition* $\Theta_1 \circ \Theta_2$ *of spectrally admissible operators are spectrally admissible.*

(d) *Any combination of elementary operators and standard operators—using the operations of sum, product and composition—yields an operator that is spectrally admissible and spectrally equivalent to an elementary operator.*

(e) *Suppose* $\mathbf{\Theta}$ *and* $\mathbf{\Theta}'$ *have components* Θ_i, Θ'_i *that are combinations as described in* (d)*. If* $\mathbf{\Theta}$ *and* $\mathbf{\Theta}'$ *are spectrally equivalent then*

$$\mathsf{Spec}\big(\mathbf{\Theta}^{(\infty)}(\mathbf{0})\big) \;=\; \mathsf{Spec}\big(\mathbf{\Theta}'^{(\infty)}(\mathbf{0})\big).$$

(f) *Let* $\mathbf{y} = \mathbf{\Theta}(\mathbf{y})$ *be a system with solution* $\mathbf{T}(x) \in \mathsf{Dom}_0[x]^k$*, where the operators* Θ_i *are combinations as described in item* (d)*. By* (d)*,* $\mathbf{\Theta}$ *is spectrally equivalent to an elementary operator* $\mathbf{G}(x, \mathbf{y})$*. Let* $\mathbf{U}(x)$ *be the unique solution to* $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ *guaranteed by Theorem 4.6. Then* $\mathsf{Spec}(\mathbf{T}(x)) = \mathsf{Spec}(\mathbf{U}(x))$.

Thus periodicity properties for the $T_i(x)$ *can be deduced by applying Theorem 4.6 to* $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$.

PROOF. Items (a) through (e) are straightforward. For item (f), one has

$$\mathsf{Spec}(\mathbf{T}(x)) \;=\; \mathsf{Spec}\big(\mathbf{\Theta}\big(\mathbf{T}(x)\big)\big) \;=\; \mathsf{Spec}\big(\mathbf{G}\big(x, \mathbf{T}(x)\big)\big) \;=\; \mathbf{G}(\mathsf{Spec}(\mathbf{T}(x))),$$

where $\mathbf{G}$ is the set operator corresponding to $\mathbf{G}(x, \mathbf{y})$. So $\mathsf{Spec}(\mathbf{T}(x))$ is a solution of $\mathbf{Y} = \mathbf{G}\big(\mathbf{Y}\big)$. Now $\mathbf{U}(x) = \mathbf{G}(\mathbf{U}(x))$ implies that $\mathsf{Spec}(\mathbf{U}(x))$ is also a solution of $\mathbf{Y} = \mathbf{G}\big(\mathbf{Y}\big)$. Theorem 3.11 says that the elementary system $\mathbf{Y} = \mathbf{G}(\mathbf{Y})$ has a unique solution, so $\mathsf{Spec}(\mathbf{U}(x)) = \mathsf{Spec}(\mathbf{T}(x))$. Consequently the periodicity properties of $\mathsf{Spec}(\mathbf{T}(x))$ are those of $\mathsf{Spec}(\mathbf{U}(x))$, and thus Theorem 4.6 can be used to analyze $\mathsf{Spec}(\mathbf{T}(x))$. □

6. Monadic Second Order Classes

When working with relational structures like graphs and trees, logicians have been able to strengthen some first order logic results to *monadic second order logic* (MSO logic) results.[19] A primary reason for the success with MSO logic is the powerful connection between Ehrenfeucht-Fraïssé games and sentences of quantifier depth at most q.[20] These games, although very combinatorial in nature, have not

[19]This is first order logic augmented with unary predicates U as variables— this means that one can quantify over subsets as well as individual elements, and say that an element belongs to a subset. The fact that the U are predicates and not domain elements make the logic *second order*, and the fact that these predicates have only one argument (for example, $U(x)$) makes the logic *monadic*.

[20]The connection with Ehrenfeucht-Fraïssé games fails if one has quantification over more general relations, like binary relations.

been widely used in the combinatorics community. (For the following discussion, one can find the background material needed on MSO logic and Ehrenfeucht-Fraïssé games in Chapter VI of [**7**].)

6.1. MSO classes of m-colored chains.

One of the early success stories in the study of MSO classes of structures was Büchi's analysis of MSO classes of m-colored chains, connecting them to the regular languages discussed in Example 4.12.

THEOREM 6.1 (Büchi [**6**], 1960). *MSO classes of colored chains are precisely the regular languages.*

For more detail on this result, the precursor to Compton's Specification Theorem in §6.3, see Appendix B. Combining this with the results stated in Example 4.12 for regular languages, one immediately sees that MSO classes of m-colored chains are eventually periodic, their generating functions are determined by systems of particularly simple $\mathbf{y}$-linear equations, and there is Berstel's penetrating analysis of the nature of the coefficients of the generating functions.

6.2. Trees and forests.

Compton extended parts of Büchi's analysis of m-colored chains to the setting of m-colored trees, to show that MSO classes of m-colored trees have an equational specification.

When speaking of structures, in particular the models of a sentence φ, it will be understood that *only finite structures are being considered.* All results stated for trees and forests can easily be extended to trees and forests with finitely many unary predicates, or, if one prefers, to finitely colored trees and forests.

A *tree* $\mathfrak{t} = (T, <)$ is a poset such that: (i) there is a unique maximal element r called the *root* of the tree, and (ii) every interval $[t, r]$ is linear. A *forest* $\mathfrak{f} = (F, <)$ is a poset whose components are trees. Let TREES be the class of (finite) trees, and let $\mathsf{FORESTS}$ be the class of (finite) forests.

The class $\mathsf{FORESTS}$ is defined by the following first order sentence φ_F of quantifier depth 3:

$$\begin{aligned}
&(\forall x)[\neg(x < x)]\\
&\ \wedge (\forall x)(\forall y)[(x < y) \rightarrow \neg(y < x)]\\
&\ \wedge (\forall x)(\forall y)(\forall z)[(x < y)\ \wedge\ (y < z)\ \rightarrow\ (x < z)]\\
&\ \wedge (\forall x)(\forall y)(\forall z)[(x < y)\ \wedge\ (x < z)\ \rightarrow\ (y < z) \vee (y = z) \vee (z < y)]
\end{aligned}$$

The class TREES is defined by the following first order sentence φ_T, also of quantifier depth 3:

$$\varphi_F \wedge (\exists x)(\forall y)[(y = x) \vee (y < x)].$$

A forest $\mathfrak{f}$ is determined (up to isomorphism) by the number of each (isomorphism type of) tree appearing in it, thus by its component counting function $\nu_{\mathfrak{f}} : \mathsf{TREES} \rightarrow \mathbb{N}$. One can combine two forests $\mathfrak{f}_1$ and $\mathfrak{f}_2$ into a single forest $\mathfrak{f}_1 + \mathfrak{f}_2$ which is determined up to isomorphism by $\nu_{\mathfrak{f}} = \nu_{\mathfrak{f}_1} + \nu_{\mathfrak{f}_2}$. Extend this operation to classes $\mathcal{F}$ of forests by $\mathcal{F}_1 + \mathcal{F}_2 = \{\mathfrak{f}_1 + \mathfrak{f}_2 : \mathfrak{f}_i \in \mathcal{F}_i\}$. The *ideal class* $\mathcal{O}$ of forests is introduced with the properties $\mathcal{O} \cup \mathcal{F} = \mathcal{O} + \mathcal{F} = \mathcal{F}$ (it is introduced solely as a notational device to smooth out the presentation).

Define the operation $*$ between nonempty subsets A of $\mathbb{N}$ and nonempty classes $\mathcal{F}$ of forests by

$$\begin{aligned} n * \mathcal{F} &= \begin{cases} \mathcal{O} & \text{if } n = 0 \\ \underbrace{\mathcal{F} + \cdots + \mathcal{F}}_{n-\text{fold}} & \text{if } n \geq 1 \end{cases} \\ A * \mathcal{F} &= \bigcup_{a \in A} (a * \mathcal{F}). \end{aligned}$$

$A * \mathcal{F}$ is just a minor extension of the construction $\mathsf{MSet}_A(\mathcal{F})$.

6.3. Compton's specification of MSO classes of trees.

Definition 6.2. Given a positive integer q, a MSO_q sentence φ is a MSO sentence of quantifier depth q, and a MSO_q class is a class defined by a MSO_q sentence.[21] $\mathfrak{f} \equiv_q^{\mathrm{MSO}} \mathfrak{f}'$ means that $\mathfrak{f}$ and $\mathfrak{f}'$ satisfy the same MSO_q sentences.

The binary relation $\equiv_q^{\mathrm{MSO}}$ is an equivalence relation of finite index on $\mathsf{FORESTS}$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ be the $\equiv_q^{\mathrm{MSO}}$-classes of $\mathsf{FORESTS}$, the equivalence classes of forests under the relation $\equiv_q^{\mathrm{MSO}}$. These classes are MSO_q classes—the $\mathcal{F}_i$ are the minimal nonempty MSO_q classes of forests—and every MSO_q class $\mathcal{F}$ of forests is a union of some of the $\mathcal{F}_i$, namely the $\mathcal{F}_i \subseteq \mathcal{F}$. Two forests satisfy the same MSO_q sentences iff they are in the same $\mathcal{F}_i$.

In the following, when given a MSO_q class $\mathcal{F}$ of forests, it will be assumed that $q \geq 3$, so that one can use the MSO_q sentence φ_F, resp. φ_T, to express "is a forest", resp. "is a tree". Consequently, if one member of an $\mathcal{F}_i$ is a tree, then so is every member of $\mathcal{F}_i$. Let $\mathcal{T}_1, \ldots, \mathcal{T}_k$ be the $\mathcal{F}_i$ whose members are trees. Each $\mathcal{T}_i$ is a MSO_q class of trees—the $\mathcal{T}_i$ are the minimal nonempty MSO_q classes of trees—and every MSO_q class $\mathcal{T}$ of trees is a union of some of the $\mathcal{T}_i$, namely the $\mathcal{T}_i \subseteq \mathcal{T}$.

If $\mathcal{T}_i$ has a 1-element tree in it then, up to isomorphism, no other tree is in $\mathcal{T}_i$ (since every member of $\mathcal{T}_i$ will satisfy $(\forall x)(\forall y)(x = y)$). Let $\bullet$ denote the 1-element tree, and assume $\mathcal{T}_1 = \{\bullet\}$. This is the only $\mathcal{T}_i$ with a 1-element member.

Given a tree $\mathfrak{t}$ with more than one element, let $\partial\mathfrak{t}$ be the forest that results from removing the root from $\mathfrak{t}$; and for $\mathcal{F}$ a class of forests let

$$\partial\mathcal{F} \quad := \quad \big\{\partial\mathfrak{t} : \mathfrak{t} \in \mathcal{F}, \|\mathfrak{t}\| \geq 2\big\}.$$

Recall from §5.2 that, given any forest $\mathfrak{f}$, $\bullet/\mathfrak{f}$ is the tree that results by adding a root to the forest; and for $\mathcal{F}$ a class of forests, $\bullet/\mathcal{F} := \{\bullet/\mathfrak{f} : \mathfrak{f} \in \mathcal{F}\}$.

Lemma 6.3. *Let q be a positive integer ≥ 3.*

(a) *In the class $\mathsf{FORESTS}$, the operations of addition and adding a root preserve $\equiv_q^{\mathsf{MSO}}$, that is,*

$$\mathfrak{f}_i \equiv_q^{\mathsf{MSO}} \mathfrak{f}_i' \;\Rightarrow\; \sum_i \mathfrak{f}_i \equiv_q^{\mathsf{MSO}} \sum_i \mathfrak{f}_i', \text{ and}$$

$$\mathfrak{f} \equiv_q^{\mathsf{MSO}} \mathfrak{f}' \;\Rightarrow\; \bullet/\mathfrak{f} \equiv_q^{\mathsf{MSO}} \bullet/\mathfrak{f}'.$$

(b) *There is a constant C_q such that, for all trees $\mathfrak{t}$ and all $n \geq C_q$, one has $n * \mathfrak{t} \equiv_q^{\mathsf{MSO}} C_q * \mathfrak{t}$.*

(c) *The relation $\equiv_q^{\mathsf{MSO}}$ is decidable.*

---

[21] Since every MSO sentence of quantifier depth $\leq q$ is logically equivalent to one of quantifier depth $\mathfrak{q}$, any class defined by a MSO sentence of quantifier depth $\leq q$ is a MSO_q class.

PROOF. One can find a discussion of the first item of (a) in [**7**], based on E-F games. Use E-F games for (b) and (c) as well. (a)–(c) are basic tools of Gurevich and Shelah ([**20**], 2003). □

DEFINITION 6.4. With C_q as in Lemma 6.3(b), let $\mathcal{S}_q$ be the collection of k-tuples $\mathbf{S} := (S_1, \ldots, S_k)$ such that each S_j is either a singleton $\{n_j\}$, with $0 \le n_j < C_q$, or the cofinite set $\mathbb{N}|_{\ge C_q}$. For $\mathbf{S} \in \mathcal{S}_q$, define the class of forests $\mathcal{F}_{\mathbf{S}}$ by

$$\mathcal{F}_{\mathbf{S}} \;:=\; \sum_{j=1}^{k} \big(S_j * \mathcal{T}_j\big).$$

Clearly there are $(1 + C_q)^k$ distinct choices for $\mathbf{S} \in \mathcal{S}_q$, every forest is in some $\mathcal{F}_{\mathbf{S}}$, and given two distinct members $\mathbf{S}$ and $\mathbf{S}'$ of $\mathcal{S}_q$, the classes $\mathcal{F}_{\mathbf{S}}$ and $\mathcal{F}_{\mathbf{S}'}$ are disjoint.

The next lemma gives the crucial structure result for MSO_q classes of forests.

LEMMA 6.5.

(a) *For $\mathbf{S} \in \mathcal{S}_q$ there is an $r \in \{1, \ldots, \ell\}$ such that $\mathcal{F}_{\mathbf{S}} \subseteq \mathcal{F}_r$, that is, all forests in $\mathcal{F}_{\mathbf{S}}$ are equivalent modulo $\equiv_q^{\mathrm{MSO}}$.*

(b) *Let $\mathcal{F}$ be a MSO_q class of forests. Then there is an $\mathbb{S} \subseteq \mathcal{S}_q$ such that*

$$\mathcal{F} \;=\; \bigcup_{\mathbf{S} \in \mathbb{S}} \mathcal{F}_{\mathbf{S}}.$$

PROOF. For (a), let $\mathbf{n}$ and $\mathbf{n}'$ be two k-tuples of nonnegative integers satisfying the condition $n_j \ne n'_j$ implies $n_j, n'_j \ge C_q$, for $1 \le j \le k$. Then Lemma 6.3 shows that every forest in $\sum_{j=1}^{k} \big(n_j * \mathcal{T}_j\big)$ is equivalent modulo $\equiv_q^{\mathrm{MSO}}$ to every forest in $\sum_{j=1}^{k} \big(n'_j * \mathcal{T}_j\big)$. Thus, given $\mathbf{S} \in \mathcal{S}_q$, all members of $\mathcal{F}_{\mathbf{S}}$ are equivalent modulo $\equiv_q^{\mathrm{MSO}}$, and thus they are in some $\equiv_q^{\mathrm{MSO}}$ class $\mathcal{F}_r$.

For (b), since $\mathcal{F}$ is an $\equiv_q^{\mathrm{MSO}}$ class of forests, for each $\mathbf{S} \in \mathcal{S}_q$ one has either $\mathcal{F}_{\mathbf{S}} \subseteq \mathcal{F}$ or $\mathcal{F}_{\mathbf{S}} \cap \mathcal{F} = \emptyset$. Consequently $\mathcal{F}$ is a union of some of the $\mathcal{F}_{\mathbf{S}}$. □

LEMMA 6.6. *For $2 \le i \le k$, the class of forests $\partial\mathcal{T}_i$ is definable by a MSO_q sentence.*

PROOF. Suppose $\mathfrak{f} \in \partial\mathcal{T}_i$ and $\mathfrak{f} \equiv_q^{\mathrm{MSO}} \mathfrak{f}'$. By Lemma 6.3(a), $\bullet/\mathfrak{f} \equiv_q^{\mathrm{MSO}} \bullet/\mathfrak{f}'$. Since $\bullet/\mathfrak{f} \in \mathcal{T}_i$, and $\mathcal{T}_i$ is a $\equiv_q^{\mathrm{MSO}}$ class, it follows that $\bullet/\mathfrak{f}' \in \mathcal{T}_i$, so $\partial\mathcal{T}_i$ is closed under $\equiv_q^{\mathrm{MSO}}$, proving that $\partial\mathcal{T}_i$ is a MSO_q class. □

THEOREM 6.7 (Compton, see [**33**]). *Let $\mathcal{T}$ be a class of trees defined by a MSO_q sentence. Then:*

(a) *$\mathcal{T}$ is a union of some of the $\mathcal{T}_i$, and*

(b) *the $\mathcal{T}_i$ satisfy a system of equations*

$$\Sigma_q : \begin{cases} \mathcal{T}_1 &=& \Phi_1\big(\mathcal{T}_1, \ldots, \mathcal{T}_k\big) \\ &\vdots& \\ \mathcal{T}_k &=& \Phi_k\big(\mathcal{T}_1, \ldots, \mathcal{T}_k\big), \end{cases}$$

where $\Phi_1\big(\mathcal{T}_1, \ldots, \mathcal{T}_k\big)$ is $\{\bullet\}$, and, for $2 \le i \le k$, there is an $\mathbb{S}_i \subseteq \mathcal{S}_q$ such that

$$\Phi_i\big(\mathcal{T}_1, \ldots, \mathcal{T}_k\big) \;:=\; \bullet\Big/ \bigcup_{\mathbf{S} \in \mathbb{S}_i} \mathcal{F}_{\mathbf{S}} \;=\; \bullet\Big/ \bigcup_{\mathbf{S} \in \mathbb{S}_i} \sum_{j=1}^{k} \big(S_j * \mathcal{T}_j\big). \tag{6.1}$$

PROOF. (a) is obviously true. For (b) note that for $2 \le i \le k$, $\mathcal{T}_i = \bullet/\partial\mathcal{T}_i$. Lemma 6.6 says $\partial\mathcal{T}_i$ is definable by a MSO_q sentence. Then Lemma 6.5(b) shows that $\partial\mathcal{T}_i$ can be expressed as $\bigcup_{\mathbf{S}\in\mathbb{S}_i}\mathcal{F}_{\mathbf{S}}$, for a suitable $\mathbb{S}_i \subseteq \mathcal{S}_q$. One only needs to attach the root $\bullet$ to have (6.1). $\square$

Compton's specification leads to a system of equations that define generating functions.

COROLLARY 6.8. *For $\mathcal{T}$ as in Compton's Theorem, one has:*

$$T(x) \;=\; \sum_{\mathcal{T}_i\subseteq\mathcal{T}} T_i(x) \tag{6.2}$$

$$T_i(x) \;=\; \begin{cases} x & \text{for } i=1\\ x\cdot \displaystyle\sum_{\mathbf{S}\in\mathbb{S}_i}\sum_{u_1\in S_1}\cdots\sum_{u_k\in S_k}\mathbf{T}(x)^{\mathbf{u}} & \text{for } 2\le i\le k, \end{cases} \tag{6.3}$$

where $\mathbf{T}(x)^{\mathbf{u}} := T_1(x)^{u_1}\cdots T_k(x)^{u_k}$.

Applying Spec to Σ_q gives a system of set equations for the spectra.

COROLLARY 6.9. *For $\mathcal{T}$ as in Compton's Theorem,*

$$\mathsf{Spec}(\mathcal{T}) \;=\; \bigcup_{\mathcal{T}_i\subseteq\mathcal{T}} \mathsf{Spec}(\mathcal{T}_i) \tag{6.4}$$

$$\mathsf{Spec}(\mathcal{T}_i) \;=\; \begin{cases} \{1\} & \text{for } i=1\\ 1+\displaystyle\bigcup_{\mathbf{S}\in\mathbb{S}_i}\sum_{j=1}^{k}\big(S_j * \mathsf{Spec}(\mathcal{T}_j)\big) & \text{for } 2\le i\le k. \end{cases} \tag{6.5}$$

REMARK 6.10. Compton [**11**] described his equational specification for the minimal classes $\mathcal{T}_1,\dots,\mathcal{T}_k$ of trees defined by MSO_q sentences to Alan Woods, during a visit to Yale in 1986; at the time Woods was a PostDoc at Yale. A careful analysis of the Jacobian of the system (6.3) by Woods [**33**] (1997) led to a proof that the class of (m-colored) trees has a MSO limit law. (It is not a 0–1 law.)

REMARK 6.11. To handle m-colored trees and forests, let $\bullet_i$ be a 1-element tree with the color i, for $1 \le i \le m$. Let $\mathcal{T}_1,\dots,\mathcal{T}_k$ be the $\equiv_q^{\mathrm{MSO}}$ classes of m-colored trees, where $\mathcal{T}_i = \{\bullet_i\}$, for $1\le i\le m$. Note that the roots of members of any class $\mathcal{T}_i$, $1\le i\le k$, all have the same color, say i'. Thus for $m < i \le k$, $\mathcal{T}_i = \bullet_{i'}/\partial\mathcal{T}_i$. With this, the development of Compton's results for m-colored trees and forests proceeds exactly as before.

6.4. The spectrum of a MSO class of trees is eventually periodic.

The dependency digraph D_q for Σ_q is defined parallel to the definition for systems of set equations. D_q has vertices $1,\dots,k$ and, referring to (6.5), directed edges given by $i \to j$ iff there is a $\mathbf{S}\in\mathbb{S}_i$ such that $S_j \ne \{0\}$. One defines a *height function* on D_q by setting $h(1)=0$, and then, for $2\le i\le k$, use the inductive definition $h(i) = 1+\max\{h(j) : i \to^+ j,\ \text{but not } j\to^+ i\}$.

COROLLARY 6.12. *The spectrum of a MSO class $\mathcal{T}$ of trees is eventually periodic.*

PROOF. It suffices to prove this result for the $\equiv_q^{\mathrm{MSO}}$ classes $\mathcal{T}_i$, $1\le i\le k$, in view of Lemma 2.6 (which guarantees that eventual periodicity is preserved by finite union). $\mathsf{Spec}(\mathcal{T}_1) = \{1\}$ is eventually periodic. So suppose $2 \le i \le k$. Note that $i \to j$ implies $\mathsf{Spec}(\mathcal{T}_i) \supseteq p_{ij} + \mathsf{Spec}(\mathcal{T}_j)$ for some positive integer p_{ij}, by (6.5). Thus $i \to^+ j$ implies the same conclusion.

If $[i] \neq \emptyset$ then $i \to^+ i$, so $\mathsf{Spec}(\mathcal{T}_i) \supseteq p + \mathsf{Spec}(\mathcal{T}_i)$, for some $p \in \mathbb{P}$. Consequently $\mathsf{Spec}(\mathcal{T}_i)$ is actually periodic.

If $[i] = \emptyset$ then one argues, by induction on the height $h(i)$, that $\mathsf{Spec}(\mathcal{T}_i)$ is eventually periodic. The ground case, $h(i) = 0$, holds precisely for $i = 1$, a case discussed above. Now suppose the result holds for all $\mathcal{T}_i$ with $h(i) \le n$. If $h(i) = n + 1$ then $2 \le i \le k$, and one has

$$\mathsf{Spec}(\mathcal{T}_i) \quad = \quad 1 + \bigcup_{\mathbf{S} \in \mathbb{S}_i} \sum_{j=1}^{k} \big(S_j * \mathsf{Spec}(\mathcal{T}_j)\big).$$

For the j such that there is an $\mathbf{S}$ with $S_j \neq \{0\}$ (there is at least one such j since $i > 1$) one has $i \to j$, so $h(j) < h(i)$, implying $\mathsf{Spec}(\mathcal{T}_j)$ is eventually periodic (by the induction hypothesis). The S_j are either singleton or cofinite subsets of $\mathbb{N}$, and therefore eventually periodic. Then Lemma 2.6 shows $\mathsf{Spec}(\mathcal{T}_i)$ is eventually periodic, since being eventually periodic is preserved by finite unions, (finite) sums, and $*$, proving the result. We have the additional information that those $\mathsf{Spec}(\mathcal{T}_i)$ with i belonging to a strong component of the dependency digraph are actually periodic. □

Corollary 6.13. *The spectrum of a MSO class $\mathcal{F}$ of forests is eventually periodic.*

Proof. Since $\bullet/\mathcal{F}$ is a MSO class of trees, one has $\{1\} + \mathsf{Spec}(\mathcal{F})$ eventually periodic; hence so is $\mathsf{Spec}(\mathcal{F})$. □

One can view unary functions as multisets of directed cycles of trees. Since forests are posets, they carry the structure of a digraph with up directed edges; hence they are partial unary functions. In order to complete such digraphs to unary functions, one only needs to add to each tree in the forest an edge directed from the root of the tree to a member of the tree. Recall that a monounary algebra is an algebra (A, f), where f is a unary function on A.

Theorem 6.14 (Gurevich and Shelah [**20**], 2003). *Let $\mathcal{M}$ be a MSO class of monounary algebras. Then the spectrum $\mathsf{Spec}(\mathcal{M})$ is eventually periodic.*

Proof. It suffices to show that one can find a MSO class of forests with the same spectrum. Let $\mathcal{F}$ be the class of forests defined as follows:

> for each forest in the class there exists a subset V of the forest, with exactly one element from each tree in the forest, such that if one adds a directed edge from the root of each tree in the forest to the unique node of the tree in V, then one has a functional digraph which satisfies a sentence defining $\mathcal{M}$.

Clearly this condition can be expressed by a MSO sentence, so $\mathsf{Spec}(\mathcal{F})$ is eventually periodic; hence so is $\mathsf{Spec}(\mathcal{M})$. □

This theorem is almost best possible for MSO classes—for example, one cannot replace 'monounary algebra' with 'digraph' or 'graph' as one can easily find classes of such structures where the theorem fails to hold. For example, the class $\mathcal{G}$ of finite graphs that look like rectangular grids is a MSO class, and since $\mathsf{Spec}(\mathcal{G})$ is the set of composite numbers, by Remark 2.5 we see that this is not an eventually periodic set.

The converse, that every eventually periodic set $S \subseteq \mathbb{P}$ can be realized as the spectrum of a MSO sentence for monounary algebras, is easy to prove. Although the proof presented here of the Gurevich and Shelah Theorem comes after considerable development of the theory of spectra defined by equations, actually all that is needed for this proof, beyond Compton's Specification Theorem, is Lemma 2.6.

In a related direction, one has the following:

Corollary 6.15. *A MSO class of graphs with bounded defect has an eventually periodic spectrum.*

Proof. A connected graph has defect d if $d+1$ is the minimum number of edges that need to be removed in order to have an acyclic graph. Thus trees have defect $= -1$. A graph has defect d if the maximum defect of its components is d. For graphs of defect at most d, introduce $d+2$ colors, one to mark a choice of a root in each component, and the others to mark the endpoints of edges which, when removed, convert the graph into a forest. For a MSO class of m-colored graphs of defect at at most d, carrying out this additional coloring in all possible ways gives a MSO class of $m+d+2$ colored graphs. Then removing the marked edges from each graph converts this into a MSO class of $m+d+2$ colored forests with the same spectrum. □

This can be easily generalized further to MSO classes of digraphs with bounded defect, giving a slight generalization of the Gurevich-Shelah result (since trees have defect -1, functional digraphs have defect 0). These examples suffice to indicate the value of knowing that MSO classes of trees have eventually periodic spectra. The method of showing that MSO spectra are eventually periodic, by reducing them to the spectra of MSO classes of trees, has been successfully pursued by Fischer and Makowsky in [**18**] (2004)—they prove that a MSO class that is contained in a class of bounded patch-width has an eventually periodic spectrum. In the same year Shelah [**29**] proved that MSO classes having a certain recursive constructibility property have eventually periodic spectra, and in 2007 Doron and Shelah [**13**] showed that the bounded patch-width result is a consequence of the constructibility property.

6.5. Effective tree procedures.

What follows is a program, given q, to effectively find a value for C_q and representatives of the $\equiv_q^{\mathsf{MSO}}$ classes of TREES and of $\mathsf{FORESTS}$, with applications to the decidability results mentioned in Gurevich and Shelah ([**20**], 2003), and an effective procedure to construct Compton's system of equations for trees. The particular classes of trees constructed in the WHILE loop of this program are similar to the classes $\mathcal{T}_k^m$ used in 1990 by Compton and Henson (see [**12**], p. 38) to prove lower bounds on computational complexity.

Program Steps	Comments
$\mathsf{FindReps}:= \mathrm{PROC}(q)$	q is the quantifier depth
$\widehat{\mathcal{F}}_0 := \emptyset$	Initialize collection of forests
$\widehat{\mathcal{T}}_0 := \{\bullet\}$	Initialize collection of trees
$t(0) := 1$	cardinality of $\widehat{\mathcal{T}}_0/\equiv_q^{\mathsf{MSO}}$
$d(0) := 1$	initialize $d(n)$
$n := 0$	initialize n

WHILE $t(n-1) < t(n)$ OR $d(n) > 0$ DO
$n := n + 1$ — augment the value of n
$\widehat{\mathcal{F}}_n := \Big\{ \sum_{\mathfrak{t} \in \widehat{\mathcal{T}}_{n-1}} m_{\mathfrak{t}} * \mathfrak{t} : m_{\mathfrak{t}} \le n \Big\}$ — make forests using at most n copies of each tree in $\widehat{\mathcal{T}}_{n-1}$
$\widehat{\mathcal{T}}_n := \widehat{\mathcal{T}}_{n-1} \cup \bullet/\widehat{\mathcal{F}}_n$
$t(n) := \big|\widehat{\mathcal{T}}_n/\equiv_q^{\mathsf{MSO}}\big|$, — # of $\equiv_q^{\mathsf{MSO}}$ classes represented by $\widehat{\mathcal{T}}_n$
$d(n) := \big|\big\{\mathfrak{t} \in \widehat{\mathcal{T}}_{n-1} : (n+1) * \mathfrak{t} \not\equiv_q n * \mathfrak{t}\big\}\big|$ — # of failures of $(n+1) * \mathfrak{t} \equiv_q^{\mathsf{MSO}} n * \mathfrak{t}$, for $\mathfrak{t} \in \widehat{\mathcal{T}}_{n-1}$
END WHILE
Define $N := n$.
Choose a maximal set $\mathsf{REP}_{\mathsf{TREES}} := \{\mathfrak{t}_1, \ldots, \mathfrak{t}_k\}$ of $\equiv_q^{\mathsf{MSO}}$ distinct trees $\mathfrak{t}_i$ from $\widehat{\mathcal{T}}_{N-1}$.
Choose a maximal set $\mathsf{REP}_{\mathsf{FORESTS}} := \{\mathfrak{f}_1, \ldots, \mathfrak{f}_\ell\}$ of $\equiv_q^{\mathsf{MSO}}$ distinct forests $\mathfrak{f}_j$ from $\widehat{\mathcal{F}}_{N-1}$.
RETURN $(N, \mathsf{REP}_{\mathsf{TREES}}, \mathsf{REP}_{\mathsf{FORESTS}})$
END PROC

THEOREM 6.16. *The procedure* $\mathsf{FindReps}(q)$ *halts for all* $q \in \mathbb{N}$*, giving an effective procedure to find a set* $\mathsf{REP}_{\mathsf{TREES}}$ *of representatives for the* $\equiv_q^{\mathsf{MSO}}$ *equivalence classes of finite trees, a set* $\mathsf{REP}_{\mathsf{FORESTS}}$ *of representatives for the* $\equiv_q^{\mathsf{MSO}}$ *equivalence classes of finite forests, and a number* N *such that, for any finite tree* $\mathfrak{t}$ *and* $n \ge N$*, one has* $n * \mathfrak{t} \equiv_q^{\mathsf{MSO}} (N-1) * \mathfrak{t}$.

PROOF. Define $\widehat{\mathcal{F}}_n$ and $\widehat{\mathcal{T}}_n$ recursively for all $n \in \mathbb{N}$ by changing the WHILE loop in the program into an unconditional DO loop. Then the classes $\widehat{\mathcal{F}}_n$ and $\widehat{\mathcal{T}_n}$ are finite and monotone nondecreasing; furthermore every finite forest is in some $\widehat{\mathcal{F}}_n$, and every finite tree is in some $\widehat{\mathcal{T}}_n$.

From Lemma 6.3(b), one knows that eventually $d(n) = 0$ (indeed, for $n \ge C_q$). Also, since $\equiv_q^{\mathrm{MSO}}$ has finite index, and the $\mathcal{T}_n$ are monotone nondecreasing, for n sufficiently large, one has $t(n-1) = t(n)$. Thus the program will eventually exit the WHILE loop, returning the triple $(N, \mathsf{REP}_{\mathsf{TREES}}, \mathsf{REP}_{\mathsf{FORESTS}})$.

From the definition of N, one has $t(N-1) = t(N)$ and $d(N) = 0$. Thus

$$\Big(\widehat{\mathcal{T}}_{N-1}/\equiv_q^{\mathsf{MSO}}\Big) = \Big(\widehat{\mathcal{T}}_N/\equiv_q^{\mathsf{MSO}}\Big)$$
$$(N+1) * \mathfrak{t} \equiv_q^{\mathsf{MSO}} N * \mathfrak{t} \quad \text{for } \mathfrak{t} \in \widehat{\mathcal{T}}_{N-1}.$$

We will prove, by induction, for $n \ge N$,

$$t(n-1) = t(n),\ d(n) = 0 \text{ and } \Big(\widehat{\mathcal{F}}_{n-1}/\equiv_q^{\mathsf{MSO}}\Big) = \Big(\widehat{\mathcal{F}}_n/\equiv_q^{\mathsf{MSO}}\Big). \tag{6.6}$$

We have already noted that it is true for $n = N$. Suppose this is true for some $n \ge N$. Then

$$\Big(\widehat{\mathcal{T}}_{n-1}/\equiv_q^{\mathsf{MSO}}\Big) = \Big(\widehat{\mathcal{T}}_n/\equiv_q^{\mathsf{MSO}}\Big) \tag{6.7}$$

and

$$(n+1) * \mathfrak{t} \equiv_q^{\mathsf{MSO}} n * \mathfrak{t} \quad \text{for } \mathfrak{t} \in \widehat{\mathcal{T}}_{n-1}. \tag{6.8}$$

Then

$$\begin{aligned}
\widehat{\mathcal{F}}_{n+1}/\equiv_q^{\mathsf{MSO}} &:= \Big\{\sum_{\mathfrak{t}\in\widehat{\mathcal{T}}_n} \big(m_{\mathfrak{t}} * \mathfrak{t}\big) : m_{\mathfrak{t}} \le n+1\Big\}\Big/ \equiv_q \\
&\subseteq \Big\{\sum_{\mathfrak{t}\in\widehat{\mathcal{T}}_{n-1}} \big(m_{\mathfrak{t}} * \mathfrak{t}\big) : m_{\mathfrak{t}} \in \mathbb{N}\Big\}\Big/ \equiv_q \quad \text{by (6.7)} \\
&= \Big\{\sum_{\mathfrak{t}\in\widehat{\mathcal{T}}_{n-1}} \big(m_{\mathfrak{t}} * \mathfrak{t}\big) : m_{\mathfrak{t}} \le n\Big\}\Big/ \equiv_q \quad \text{by (6.8)} \\
&= \widehat{\mathcal{F}}_n/\equiv_q^{\mathsf{MSO}},
\end{aligned}$$

so $\big(\widehat{\mathcal{F}}_{n+1}/\equiv_q^{\mathsf{MSO}}\big) = \big(\widehat{\mathcal{F}}_n/\equiv_q^{\mathsf{MSO}}\big)$.

Next,

$$\begin{aligned}
\widehat{\mathcal{T}}_{n+1}/\equiv_q^{\mathsf{MSO}} &= \big(\bullet/\widehat{\mathcal{F}}_{n+1}\big)\Big/\equiv_q^{\mathsf{MSO}} \\
&= \big(\bullet/\widehat{\mathcal{F}}_{n}\big)\Big/\equiv_q^{\mathsf{MSO}} \quad \text{by Lemma 6.3(a)} \\
&= \widehat{\mathcal{T}}_n/\equiv_q^{\mathsf{MSO}},
\end{aligned}$$

so $t(n) = t(n+1)$.

Given $\mathfrak{t} \in \widehat{\mathcal{T}}_{n+1}$, choose $\mathfrak{t}' \in \widehat{\mathcal{T}}_n$ such that $\mathfrak{t} \equiv_q^{\mathsf{MSO}} \mathfrak{t}'$. Then

$$(n+2)*\mathfrak{t} \ \equiv_q^{\mathsf{MSO}}\ (n+2)*\mathfrak{t}' \ \equiv_q^{\mathsf{MSO}}\ (n+1)*\mathfrak{t}' \ \equiv_q^{\mathsf{MSO}}\ (n+1)*\mathfrak{t},$$

so $d(n+1) = 0$. This finishes the proof of (6.6).

Thus for $n \ge N$,

$$\begin{aligned}
\Big(\widehat{\mathcal{T}}_n/\equiv_q^{\mathsf{MSO}}\Big) &= \Big(\widehat{\mathcal{T}}_{N-1}/\equiv_q^{\mathsf{MSO}}\Big) \\
n*\mathfrak{t} &\equiv_q^{\mathsf{MSO}} (N-1)*\mathfrak{t}, \text{ for } \mathfrak{t}\in\widehat{\mathcal{T}}_n \\
\Big(\widehat{\mathcal{F}}_n/\equiv_q^{\mathsf{MSO}}\Big) &= \Big(\widehat{\mathcal{F}}_{N-1}/\equiv_q^{\mathsf{MSO}}\Big).
\end{aligned}$$

Consequently $\widehat{\mathcal{T}}_{N-1}$ has representatives for all $\equiv_q^{\mathsf{MSO}}$ classes of finite trees, $n*\mathfrak{t} \equiv_q^{\mathsf{MSO}} (N-1)*\mathfrak{t}$, for any finite tree $\mathfrak{t}$, and $\widehat{\mathcal{F}}_{N-1}$ has representatives for all $\equiv_q^{\mathsf{MSO}}$ classes of finite forests. Thus one can choose $C_q = N-1$.

The procedures for constructing the classes $\widehat{\mathcal{F}}_n$ and $\widehat{\mathcal{T}}_n$ are effective, as are the calculations of the functions $t(n)$ and $d(n)$. □

Further Conclusions (all can be extended to m-colorings, or adding m unary predicates to the language):

(a) The trees in $\widehat{\mathcal{T}}_n$ are all of height $\le n$.
(b) One can effectively find MSO_q sentences φ_i, $1 \le i \le k$, such that φ_i defines $\mathfrak{t}_i/\equiv_q^{\mathrm{MSO}}$, the $\equiv_q^{\mathsf{MSO}}$ equivalence class of finite trees with the representative $\mathfrak{t}_i$ in it.

[For $1 \le i \le k$, start enumerating the MSO_q sentences φ and test each one in turn until one finds one such that $\mathfrak{t}_j \models \varphi$ iff $j = i$. Then let $\varphi_i := \varphi$.]

(c) Likewise for $1 \le j \le \ell$ one can effectively find MSO_q sentences ψ_j defining $\mathfrak{f}_j/\equiv_q^{\mathrm{MSO}}$, the $\equiv_q^{\mathsf{MSO}}$ equivalence class of finite forests with $\mathfrak{f}_j$ in it.

(d) The MSO theory of FORESTS is decidable.[22] (Given ψ of quantifier depth q, it will be true of all finite forests iff it is true of each $\mathfrak{f}_j$ in $\mathsf{REP}_{\mathsf{FORESTS}}$.)
(e) Finite satisfiability for the MSO theory of monounary algebras is decidable. (This can be proved directly, by interpretation into FORESTS.)
(f) One can effectively find the Compton Equations Σ_q for the $\equiv_q^{\mathsf{MSO}}$ equivalence classes of finite trees, namely, with $[\mathfrak{t}_i]_q := \mathfrak{t}_i / \equiv_q^{\mathsf{MSO}}$, one has

$$[\mathfrak{t}_i]_q \quad = \quad \{\bullet\} \quad \text{if } \mathfrak{t}_i = \{\bullet\}; \text{ otherwise}$$

$$[\mathfrak{t}_i]_q \quad = \quad \bigcup \Big\{ \bullet \Big/ \sum_{j=1}^{k} \big(G_j * [\mathfrak{t}_j]_q\big) \ : \ G_j \in \big\{\{0\},\{1\},\ldots,\{N-2\},\mathbb{N}|_{\geq N-1}\big\} \text{ and}$$
$$\mathfrak{t}_i \equiv_q^{\mathsf{MSO}} \bullet \Big/ \sum_{j=1}^{k} \big(G_j * \mathfrak{t}_j\big) \Big\}.$$

To test the last condition (concerning $\equiv_q^{\mathsf{MSO}}$) one replaces any $G_j = \mathbb{N}|_{\geq N-1}$ by $N-1$, so one is deciding $\equiv_q^{\mathsf{MSO}}$ between two finite trees.
(g) One can effectively find the dependency digraph of Σ_q (immediate from the previous step).
(h) One can effectively find the periodicity parameters $\mathfrak{m}$, $\mathfrak{q}$ for the spectra of the $[\mathfrak{t}_i]_q$ (using Theorem 3.11(e)).

Question 1. One question stands out concerning §6.5, namely can one find an explicit bound (in terms of known functions, like exponentiation) for the value $n = N$ for which the WHILE loop halts? This would give the maximum height of the trees in the set of representatives $\mathsf{REP}_{\mathsf{TREES}}$ of the $\equiv_q^{\mathsf{MSO}}$ classes of trees.

Using Compton's equations, we have carried out a detailed study [**4**] of MSO classes $\mathcal{T}$ of finite trees whose generating functions $T(x)$ have radius of convergence $\rho = 1$. One conclusion obtained is that if the class of forests $\partial\mathcal{T}$ is closed under addition, and under extraction of trees (thus forming an additive number system as described in [**7**]), then $\mathcal{T}$ has a MSO 0–1 law.

7. Well Conditioned Systems

In the following, a method for determining the asymptotics of generating functions defined by well conditioned systems of equations is described, as an application of the formula (4.2) for $\mathfrak{q}$. Details for how to apply the method to well conditioned 2-equation systems follows the general discussion. First some terminology from [**3**] is needed:

---

[22]This item, as well as the next one, are easy corollaries of results in Rabin's 1969 paper [**25**] on the decidability of the MSO theory of two successors. Idziak and Idziak [**21**] use Rabin's result to prove that the MSO theory of finite trees is decidable. Essentially the same proof works for m-colored finite forests as well.

Monounary algebras can be interpreted into finite forests, or one can use Rabin's Theorem 2.4 and the subsequent paragraph from [**25**], which show that the MSO theory of countable monounary algebras is decidable, even when the language is augmented by variables for finite subsets. By relativizing quantifiers to a finite subset closed under the unary function, one has the fact that the MSO theory of finite monounary algebras is decidable. A more elementary proof, not depending on the Rabin result, was given by Gurevich and Shelah [**20**]—the proof given here seems to be yet more elementary.

DEFINITION 7.1. A system of equations

$$\begin{array}{rcl} y_1 & = & G_1(x, y_1, \ldots, y_k) \\ & \vdots & \\ y_k & = & G_k(x, y_1, \ldots, y_k), \end{array}$$

abbreviated $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$, is *well conditioned* if the following hold:[23]

(a) The $G_i(x, \mathbf{y})$ are power series in $\mathsf{Dom}_0[x, \mathbf{y}]$.
(b) $\mathbf{G}(x, \mathbf{y})$ is holomorphic in a neighborhood of the origin.
(c) $\mathbf{G}(0, \mathbf{y}) = \mathbf{0}$.
(d) For all i, $G_i(x, \mathbf{0}) \neq 0$.
(e) The system is irreducible (as defined in §4.2).
(f) For some i, j, k, $\dfrac{\partial^2 G_i(x, \mathbf{y})}{\partial y_j \partial y_k} \neq 0$.

The *characteristic system* of $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ is

$$\left\{ \begin{array}{rcl} \mathbf{y} & = & \mathbf{G}(x, \mathbf{y}) \\ 0 & = & \det\big(I - J_{\mathbf{G}}(x, \mathbf{y})\big), \end{array} \right.$$

where $J_{\mathbf{G}}(x, \mathbf{y})$ is the Jacobian matrix $\dfrac{\partial \mathbf{G}}{\partial \mathbf{y}}$. The positive solutions of the characteristic system, that is, the solutions $(a, \mathbf{b})$ with $a, b_1, \ldots, b_k > 0$, are called *characteristic points*. A characteristic point $(a, \mathbf{b})$ is an *eigenpoint* if the largest real eigenvalue of the matrix $J_{\mathbf{G}}(a, \mathbf{b})$ is 1.

PROPOSITION 7.2. *For a well conditioned system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$, one has the following:*

(a) *The system $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ is a reduced elementary system.*
(b) *There is a unique solution $\mathbf{y} = \mathbf{T}(x)$ in $\mathsf{Dom}_0[x]^k$.*
(c) *The $T_i(x)$ have the same radius of convergence $\rho \in (0, \infty)$, and they converge at ρ. Let $\boldsymbol{\tau} = \mathbf{T}(\rho)$.*
(d) *If $(a, \mathbf{b})$ is an eigenpoint of $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ then it is equal to $(\rho, \boldsymbol{\tau})$.*
(e) *If $(\rho, \boldsymbol{\tau})$ is a characteristic point then, among the characteristic points $(a, \mathbf{b})$, it is the one with the largest value of a.*
(f) *The $\mathsf{Spec}(T_i(x))$ are infinite and periodic, with the same parameter $\mathfrak{q}_i$, namely $\mathfrak{q}_i = \mathfrak{q}$, where*

$$\begin{array}{rcl} \mathfrak{q} & = & \gcd\Bigg(\bigcup_{\substack{1 \le j \le k \\ \mathbf{u} \in \mathbb{N}^k}} \Big(G_{j,\mathbf{u}} + \mathbf{u} \circledast \mathfrak{m} - \mathfrak{m}_j \Big) \Bigg) \\ \mathfrak{m}_i & = & \min\Big(G_i^{(k)}(\mathbf{\emptyset}) \Big). \end{array}$$

(g) *For all i, the minimum eventual period $\mathfrak{p}_i$ of $\mathsf{Spec}(T_i(x))$ is equal to $\mathfrak{q}$, so $\mathsf{Spec}(T_i(x))$ is eventually an arithmetical progression, in particular*

$$\mathsf{Spec}(T_i(x)) \;=\; \mathsf{Spec}(T_i(x))|_{<\mathfrak{c}_i} \cup (\mathfrak{c}_i + \mathfrak{q} \cdot \mathbb{N}) \;\subseteq\; \mathfrak{m}_i + \mathfrak{q} \cdot \mathbb{N}.$$

(h) *Each $T_i(x)$ can be written in the form $x^{\mathfrak{m}_i} V_i(x^{\mathfrak{q}})$, where $V_i(x) \trianglerighteq 0$, $V_i(0) \neq 0$ and $\mathsf{Spec}(V_i(x))$ is a cofinal subset of $\mathbb{N}$.*

---

[23]The condition (e) from the definition of *well conditioned* in §2 of [**3**] is redundant; hence it is omitted here.

(i) *The dominant singularities of* $T_i(x)$ *are* $\rho, \rho\cdot\omega, \ldots, \rho\cdot\omega^{\mathfrak{q}-1}$, *where* $\omega = \exp(2\pi i/\mathfrak{q})$, *a primitive* $\mathfrak{q}$*th root of unity.*

Proof. (a) follows from Definition 7.1 (b),(c),(d). Use Proposition 4.1 for (b). Item (c) is Proposition 3 (iv) of [**3**]. (d) follows from Theorem 21 (d),(e) of [**3**]. (e) is Theorem 14 (a) of [**3**]. (f) and (g) follow from Theorem 4.6 and Corollary 4.7. (h) follows from (g) and the definition of $\mathfrak{m}_i$. (i) follows from $\mathbf{T}(x) = \mathbf{G}(x, \mathbf{T}(x))$ and (h). □

Theorem 7.3. *Suppose* $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$ *is well conditioned, with an eigenpoint in the interior of the domain of* $\mathbf{G}(x, \mathbf{y})$. *Let* $\mathbf{T}(x)$, $(\rho, \boldsymbol{\tau})$ *and* $\mathfrak{q}$ *be as in Proposition 7.2. Then one has the following:*

(a) $\mathfrak{c}_i + \mathfrak{q}\cdot\mathbb{N} \subseteq \mathsf{Spec}(T_i(x)) \subseteq \mathfrak{m}_i + \mathfrak{q}\cdot\mathbb{N}$.

(b) *There are well conditioned single equation systems* $y_i = \widehat{G}_i(x, y_i)$, *with the unique solution in* $\mathsf{Dom}_0[x]$ *being* $y_i = T_i(x)$, *and with* (ρ, τ_i) *in the interior of the domain of* $\widehat{G}_i(x, y_i)$.

(c) *The coefficient sequence* $t_i(n)$ *of each* $T_i(x)$ *has the asymptotics*

$$t_i(n) \sim C_i \rho^{-n} n^{-3/2} \quad \text{on } \mathsf{Spec}(T_i(x)),$$

where (letting $\widehat{G}_{ix}(x, y_i)$ *denote* $\partial\widehat{G}_i(x, y_i)/\partial x$, *etc.),*

$$C_i \;=\; \mathfrak{q}\cdot\sqrt{\frac{\rho}{2\pi}}\cdot\sqrt{\frac{\widehat{G}_{ix}(\rho, \tau_i)}{\widehat{G}_{iy_iy_i}(\rho, \tau_i)}}.$$

(d) *The quotients* $\widehat{G}_{ix}(x, y_i)/\widehat{G}_{iy_iy_i}(x, y_i)$ *are rational functions.*

Proof. (a) is from item (g) of Proposition 7.2. (b) is from Drmota's Theorem, as presented in [**3**] (where it is Theorem 22). (c) is from Theorem 28 of [**2**] (the factor of $\mathfrak{q}$ in C_i is a consequence of (h) and (i) in Proposition 7.2). The method to prove item (d) is detailed in the case of two equations in §7.1 below. □

Next, this theorem is applied to find the asymptotics for the $t_i(n)$ in the case $k = 2$.

7.1. Systems of two equations.

Assume the system[24]

$$\begin{aligned} y &= E(x, y, z) \\ z &= F(x, y, z) \end{aligned} \tag{7.1}$$

is well conditioned, and there is an eigenpoint in the interior of the domain of $(E(x, y, z), F(x, y, z))$. In view of Proposition 7.2, let the solution in $\mathsf{Dom}_0[x]^2$ be $(y, z) = (S(x), T(x))$, let $\rho \in (0, \infty)$ be the radius of convergence of both $S(x)$ and $T(x)$, and let $(\tau_1, \tau_2) = (S(\rho), T(\rho))$.

As in the first step of the proof of Drmota's Theorem (see Theorem 22 in [**3**]), solve the first equation for y as a function of x, z, say

$$y \;=\; Y(x, z) \in \mathsf{Dom}_0(x, z).$$

[24] Instead of the usual $\mathbf{y} = \mathbf{G}(x, \mathbf{y})$, with subscripted variables and functions, this discussion of a 2-equation system will use single letters for different variables and functions—it is more reader friendly when using subscripted variables for partial derivatives. We replace y_1, y_2 by y, z, etc.

Then (again from the proof of Drmota's Theorem), (ρ, τ_2) is in the interior of the domain of $Y(x,z)$, and

$$\begin{aligned} Y(x,z) &\trianglerighteq 0 \\ Y(x,z) &= E(x, Y(x,z), z) \\ Y(\rho, \tau_2) &= \tau_1. \end{aligned} \tag{7.2}$$

Differentiating (7.2) gives:

$$\begin{aligned} Y_x(x,z) &= E_x(x, Y(x,z), z) + E_y(x, Y(x,z), z) \cdot Y_x(x,z) \\ Y_z(x,z) &= E_z(x, Y(x,z), z) + E_y(x, Y(x,z), z) \cdot Y_z(x,z) \\ Y_{zz}(x,z) &= E_{zz}(x, Y(x,z), z) + 2E_{yz}(x, Y(x,z), z) \cdot Y_z(x,z) \\ &\quad + E_{yy}(x, Y(x,z), z) \cdot Y_z(x,z)^2 + E_y(x, Y(x,z), z) \cdot Y_{zz}(x,z). \end{aligned}$$

Solving these equations for the partial derivatives of $Y(x,z)$, and and evaluating at $(x,z) = (\rho, \tau_2)$ gives

$$\begin{aligned} Y_x(\rho, \tau_2) &= \frac{E_x(\rho, \boldsymbol{\tau})}{1 - E_y(\rho, \boldsymbol{\tau})} \\ Y_z(\rho, \tau_2) &= \frac{E_z(\rho, \boldsymbol{\tau})}{1 - E_y(\rho, \boldsymbol{\tau})} \\ Y_{zz}(\rho, \tau_2) &= \frac{E_{zz}(\rho, \boldsymbol{\tau}) + 2E_{yz}(\rho, \boldsymbol{\tau}) \cdot Y_z(\rho, \tau_2) + E_{yy}(\rho, \boldsymbol{\tau}) \cdot Y_z(\rho, \tau_2)^2}{1 - E_y(\rho, \boldsymbol{\tau})}. \end{aligned}$$

Substituting $Y(x,z)$ for y in (7.1) gives a well conditioned 1-equation system solved by $z = T(x)$, namely:

$$z \ = \ \widehat{F}(x,z) \ := \ F(x, Y(x,z), z).$$

From Theorem 7.3 (c),

$$t(n) \ \sim \ C_T \rho^{-n} n^{-3/2},$$

where

$$C_T \ = \ \mathfrak{q} \cdot \sqrt{\frac{\rho}{2\pi}} \cdot \sqrt{\frac{\widehat{F}_x(\rho, \tau_2)}{\widehat{F}_{zz}(\rho, \tau_2)}}.$$

We have

$$\begin{aligned} \widehat{F}_x(x,z) &= F_x(x, Y(x,z), z) + F_y(x, Y(x,z), z) \cdot Y_x(x,z) \\ \widehat{F}_z(x,z) &= F_z(x, Y(x,z), z) + F_y(x, Y(x,z), z) \cdot Y_z(x,z) \\ \widehat{F}_{zz}(x,z) &= F_{zz}(x, Y(x,z), z) + 2F_{yz}(x, Y(x,z), z) \cdot Y_z(x,z) \\ &\quad + F_{yy}(x, Y(x,z), z) \cdot Y_z(x,z)^2 + F_y(x, Y(x,z), z) \cdot Y_{zz}(x,z). \end{aligned}$$

Evaluating these at (ρ, τ_2) gives

$$\begin{aligned} \widehat{F}_x(\rho, \tau_2) &= F_x(\rho, \boldsymbol{\tau}) + F_y(\rho, \boldsymbol{\tau}) \cdot Y_x(\rho, \tau_2) \\ \widehat{F}_{zz}(\rho, \tau_2) &= F_{zz}(\rho, \boldsymbol{\tau}) + 2F_{yz}(\rho, \boldsymbol{\tau}) \cdot Y_z(\rho, \tau_2) \\ &\quad + F_{yy}(\rho, \boldsymbol{\tau}) \cdot Y_z(\rho, \tau_2)^2 \ + \ F_y(\rho, \boldsymbol{\tau}) \cdot Y_{zz}(\rho, \tau_2). \end{aligned}$$

This information suffices to determine C_T. A similar procedure gives C_S.

For well conditioned *polynomial* systems, $(\rho, \boldsymbol{\tau})$ is a characteristic point in the interior of the domain of $\mathbf{G}(x, \mathbf{y})$, thus the method just described determines the constants C_i for such systems.

EXAMPLE 7.4. Consider the polynomial system

$$\begin{aligned} y &= E(x,y,z) := x\cdot(x+x^5y^5+x^3z^5) \\ z &= F(x,y,z) := x\cdot(1+y^3z^8). \end{aligned}$$

We solve for C_T, where $(S(x),T(x))$ is the solution.

By Proposition 7.2 (f), $\mathfrak{m}=(2,1)$ and $\mathfrak{q}=(7,7)$. There are two characteristic points:

$$x=.4275279509\ldots,y=3.5297999379\ldots,z=.4886125984\ldots$$
$$x=.7580667215\ldots,y=.7485529361\ldots,z=.8289799201\ldots$$

The second point has the largest first coordinate; by Proposition 7.2 (e) it must be $(\rho,\boldsymbol{\tau})$. From the formulas above, one has

$$\begin{aligned} Y_x(\rho,\tau_2) &= 3.6339912586\ldots \\ Y_z(\rho,\tau_2) &= 1.1106860072\ldots \\ Y_{zz}(\rho,\tau_2) &= 8.1565981501\ldots \\ \widehat{F}_x(\rho,\tau_2) &= 2.1263292470\ldots \\ \widehat{F}_{zz}(\rho,\tau_2) &= 15.1259723598\ldots \end{aligned}$$

Then

$$C_T = 0.9116233215\ldots,$$

so

$$t(n) \sim (0.9116233215\ldots)\cdot(0.7580667215\ldots)^{-n}n^{-3/2}.$$

Likewise one finds C_S and the asymptotics for $s(n)$.

Appendix A. Routine proofs of preliminary material

PROOF OF THE $\mathfrak{q}$ FROM PROPOSITION 2.9. Let $a\in\mathbb{N}$, $U,V\subseteq\mathbb{N}$. Then

$$\begin{aligned} 0\in U+V &\Rightarrow \gcd(U+V)=\gcd(\gcd(U),\gcd(V)) \\ a>0\in V &\Rightarrow \gcd(a*V)=\gcd(V). \end{aligned}$$

For $A:=A_1\cup A_2$: Let $\mathfrak{m}:=\mathfrak{m}(A)$, $\mathfrak{q}:=\mathfrak{q}(A)$. Then $\mathfrak{m}=\mathfrak{m}_1$, so

$$\begin{aligned} \mathfrak{q} &= \gcd\Big((A_1-\mathfrak{m}_1)\cup(A_2-\mathfrak{m}_1)\Big) \\ &= \gcd\Big((A_1-\mathfrak{m}_1)\cup\big((A_2-\mathfrak{m}_2)+(\mathfrak{m}_2-\mathfrak{m}_1)\big)\Big) \\ &= \gcd\big(q_1,q_2,\mathfrak{m}_2-\mathfrak{m}_1\big). \end{aligned}$$

For $A:=A_1+A_2$: Let $\mathfrak{m}:=\mathfrak{m}(A)$, $\mathfrak{q}:=\mathfrak{q}(A)$. Then

$$\begin{aligned} \mathfrak{q} &:= \gcd\big(A-\mathfrak{m}\big) = \gcd\Big((A_1-\mathfrak{m}_1)+(A_2-\mathfrak{m}_2)\Big) \\ &= \gcd\big(q_1,q_2\big). \end{aligned}$$

For $A:=A_1*A_2$: Let $\mathfrak{m}:=\mathfrak{m}(A)$, $\mathfrak{q}:=\mathfrak{q}(A)$. If $A_1=\{0\}$ then $A=\{0\}$, so $\mathfrak{q}=0$. Now suppose $A_1\neq\{0\}$. Then

$$\begin{aligned} \mathfrak{q} &:= \gcd\big(A-\mathfrak{m}\big) = \gcd\big(A_1*A_2-\mathfrak{m}_1\mathfrak{m}_2\big) = \gcd\Big(\bigcup_{a_1\in A_1}\big(a_1*A_2-\mathfrak{m}_1\mathfrak{m}_2\big)\Big) \\ &= \gcd\Big(\bigcup_{a_1\in A_1}\big(a_1*(A_2-\mathfrak{m}_2)+(a_1-\mathfrak{m}_1)\mathfrak{m}_2\big)\Big) \end{aligned}$$

$$
\begin{aligned}
&= \gcd\Big\{\gcd\Big(\gcd\big(a_1 * (A_2 - \mathfrak{m}_2)\big),\ (a_1 - \mathfrak{m}_1)\mathfrak{m}_2\Big) : a_1 \in A_1\Big\}\\
&= \gcd\Big\{\gcd\Big(\gcd\big(a_1 * (A_2 - \mathfrak{m}_2)\big),\ (a_1 - \mathfrak{m}_1)\mathfrak{m}_2\Big) : a_1 \in A_1, a_1 \neq 0\Big\}\\
&= \gcd\Big\{\gcd\big(\mathfrak{q}_2,\ (a_1 - \mathfrak{m}_1)\mathfrak{m}_2\big) : a_1 \in A_1, a_1 \neq 0\Big\}\\
&= \gcd\Big(\mathfrak{q}_2,\ \gcd\big((A_1 - \mathfrak{m}_1)\mathfrak{m}_2\big)\Big) \ = \ \gcd\big(\mathfrak{q}_2,\ \mathfrak{q}_1\mathfrak{m}_2\big).
\end{aligned}
$$

□

Proof of (b)–(d) of Lemma 2.11. (b): Since $\mathfrak{c}, \mathfrak{c}+\mathfrak{p} \in A$, by Remark 2.8 we have $\mathfrak{q}|\mathfrak{p}$. Clearly $\mathfrak{p} = \mathfrak{q}$ implies $\mathfrak{p} \,\big|\, (A - \mathfrak{m})$. Conversely, if $\mathfrak{p} \,\big|\, (A - \mathfrak{m})$ then $\mathfrak{p} \mid \mathfrak{q} = \gcd(A - \mathfrak{m})$; and since $\mathfrak{q} \mid \mathfrak{p}$, one has $\mathfrak{p} = \mathfrak{q}$.

(c): (ii)⇒(i) is obvious. Assume (i) holds. Clearly $p = b$, so $A = A_0 \cup (a + p\cdot\mathbb{N})$. Let $x \in A|_{\ge\mathfrak{c}}$. For n sufficiently large, $x + pn \in A \smallsetminus A_0$, so $x + pn \in a + p\cdot\mathbb{N}$. From this we have $x \in A|_{\ge\mathfrak{c}} \Rightarrow p|(x-a)$, so $x \in A|_{\ge\mathfrak{c}} \Rightarrow p|(x-\mathfrak{c})$. Thus $A|_{\ge\mathfrak{c}} = \mathfrak{c} + \mathfrak{p}\cdot\mathbb{N}$, proving (i)⇒(ii).

(ii)⇒(iii) is obvious. Assume (iii) holds. For $x \in A|_{\ge\mathfrak{c}}$ one has $\mathfrak{q}\big(A|_{\ge\mathfrak{c}}\big) \,\big|\, (x-c)$, thus $\mathfrak{p} \,\big|\, (x-c)$. As before, $A|_{\ge\mathfrak{c}} = \mathfrak{c} + \mathfrak{p}\cdot\mathbb{N}$, proving (i)⇒(ii).

(d): Assume (i) holds. Then $A \subseteq \mathfrak{m} + \mathfrak{p}\cdot\mathbb{N} \Rightarrow \mathfrak{p} \,\big|\, (A - \mathfrak{m}) \Rightarrow \mathfrak{p} = \mathfrak{q}$, proving (iii).

Next assume (iii) holds. Then $\mathfrak{q}\big(A|_{\ge\mathfrak{c}}\big) \,\big|\, \mathfrak{p}\big(A|_{\ge\mathfrak{c}}\big) = \mathfrak{p} = \mathfrak{q} \,\big|\, \mathfrak{q}\big(A|_{\ge\mathfrak{c}}\big)$, so $\mathfrak{p} = \mathfrak{q}\big(A|_{\ge\mathfrak{c}}\big)$. By (b), $A = A\big|_{<\mathfrak{c}} \cup (\mathfrak{c} + \mathfrak{p}\cdot\mathbb{N})$. Also $\mathfrak{p} = \mathfrak{q} \,\big|\, (A - \mathfrak{m}) \Rightarrow A \subseteq \mathfrak{m} + \mathfrak{p}\cdot\mathbb{N}$. Thus (iii)⇒(ii). Finally, (ii)⇒(i) is obvious. □

Proof of Lemma 3.5. From

$$\mathbf{G}(\mathbf{Y}) \ := \ \bigvee_{\mathbf{u}\in\mathbb{N}^k} \big(\mathbf{G}_{\mathbf{u}} + \mathbf{u} \circledast \mathbf{Y}\big),$$

one has, by Lemma 3.3 (a), for $1 \le i \le k$,

$$G_i^{(n+1)}(\mathbf{A}) \ = \ \bigcup_{\mathbf{u}\in\mathbb{N}^k} \Big(G_{i,\mathbf{u}} + \sum_{j\,:\,u_j>0} u_j * G_j^{(n)}(\mathbf{A})\Big).$$

Let

$$\mathbf{b}_{\mathbf{u}} \ := \ \min \mathbf{G}_{\mathbf{u}} \quad \text{and} \quad \mathbf{b}^{(n)} \ := \ \min \mathbf{G}^{(n)}(\mathbf{A}),$$

that is, for $1 \le i \le k$,

$$b_{i,\mathbf{u}} \ = \ \min G_{i,\mathbf{u}} \quad \text{and} \quad b_i^{(n)} \ := \ \min G_i^{(n)}(\mathbf{A}).$$

Then, for $n \ge 0$,

$$\mathbf{b}^{(n+1)} \le \mathbf{b}^{(n)}, \tag{A.1}$$

since $\mathbf{A} \le \mathbf{G}(\mathbf{A})$ implies $\mathbf{G}^{(n)}(\mathbf{A}) \le \mathbf{G}^{(n+1)}(\mathbf{A})$, by repeated application of Lemma 3.4 (a).

From the above,

$$
\begin{aligned}
b_i^{(n+1)} &:= \ \min G_i^{(n+1)}(\mathbf{A})\\
&= \ \min \bigcup_{\mathbf{u}\in\mathbb{N}^k} \Big(G_{i,\mathbf{u}} + \sum_{j\,:\,u_j>0} u_j * G_j^{(n)}(\mathbf{A})\Big) \quad \text{by (3.1)},
\end{aligned}
$$

so

$$b_i^{(n+1)} = \min\Big\{b_{i,\mathbf{u}} + \sum_{j\,:\,u_j>0} u_j b_j^{(n)} : \mathbf{u} \in \mathbb{N}^k\Big\}. \tag{A.2}$$

For $n \geq 1$ let

$$I_n := \Big\{j : b_j^{(n)} < b_j^{(n-1)}\Big\}.$$

Then

$$(\forall n \geq 1)(\forall i \in I_{n+1})(\exists r \in I_n)\Big(b_i^{(n+1)} \geq b_r^{(n)}\Big), \tag{A.3}$$

which says that if some b_i decreases in round $n+1$, then it is because it depends on some b_r which decreased in round n; hence $b_i \geq b_r$. In more detail, suppose $n \geq 1$ and $i \in I_{n+1}$, that is,

$$b_i^{(n+1)} < b_i^{(n)}.$$

From (A.2), let $\mathbf{u} \in \mathbb{N}^k$ be such that

$$b_i^{(n+1)} = b_{i,\mathbf{u}} + \sum_{j\,:\,u_j>0} u_j b_j^{(n)}. \tag{A.4}$$

Let $r \in \{1, \ldots, k\}$ be such that $u_r > 0$ and $r \in I_n$. Such an r must exist, for otherwise $u_j > 0$ would imply $j \notin I_n$, that is, $b_j^{(n)} = b_j^{(n-1)}$; then, from (A.4), and from (A.2) with $n-1$ substituted for n,

$$b_i^{(n+1)} = b_{i,\mathbf{u}} + \sum_{j\,:\,u_j>0} u_j b_j^{(n-1)} \geq b_i^{(n)},$$

contradicting the assumption that $i \in I_n$, that is, $b_i^{(n+1)} < b_i^{(n)}$. For this choice of $\mathbf{u}$ and r, (A.4) implies

$$b_i^{(n+1)} \geq b_r^{(n)},$$

establishing (A.3).

Now suppose $I_n \neq \emptyset$ for some $n \geq k+1$. Then (A.3) says one can choose a sequence $i_n, \ldots, i_{n-k}$ of indices from $\{1, \ldots, k\}$ such that

$$b_{i_n}^{(n)} \geq b_{i_{n-1}}^{(n-1)} \geq \cdots \geq b_{i_{n-k}}^{(n-k)}, \tag{A.5}$$

and $i_j \in I_j$ for $n-k \leq j \leq n$. By the pigeonhole principle, there are two j such that the indices i_j are the same, say $\ell = i_p = i_q$, where $n-k \leq p < q \leq n$. Then $b_\ell^{(q)} \geq b_\ell^{(p)}$ by (A.5). But from $\ell \in I_q$ and (A.1) one has $b_\ell^{(q)} < b_\ell^{(q-1)} \leq \cdots \leq b_\ell^{(p)}$, giving a contradiction. Thus $I_n = \emptyset$ for $n > k$, completing the proof of the lemma. □

Appendix B. Büchi's Theorem

Given a finite alphabet $A = \{a_1, \ldots, a_m\}$, a *word* $w = a_{i_1} \cdots a_{i_\ell}$ over the alphabet is a string of letters from the alphabet. One can associate with w a structure $\mathfrak{c}(w) := (\{1, \ldots, \ell\}, <, U_1, \ldots, U_m)$, called an *m-colored chain*. The U_n are unary predicates, called the colors, and $<$ is a linear order on the universe of $\mathfrak{c}(w)$, namely one has $1 < \cdots < \ell$. Define $U_n(j)$ to hold in $\mathfrak{c}(w)$ (that is, the element j of the chain $\mathfrak{c}(w)$ has the color U_n) iff $n = i_j$, that is, the jth letter of the word w is a_n. The mapping $\mu : w \mapsto \mathfrak{c}(w)$ is a bijection between words over the alphabet A and m-colored chains, with the property that the length ℓ of the word w is the size of the m-colored chain $\mathfrak{c}(w)$. Thus if $\mathcal{L}$ is a set of words over the alphabet A

then $\mathcal{L}$ has the same generating function as the collection $\mu(\mathcal{L})$ of m-colored chains. Büchi proved that a set $\mathcal{L}$ of words over A is a regular language iff $\mu(\mathcal{L})$ is (up to isomorphism) a MSO class of m-colored chains, the result which we stated briefly in Theorem 6.1: *MSO classes of colored chains are precisely the regular languages.*

An important step in Büchi's proof was, given a MSO class of m-colored chains $\mathcal{C}$, to show how to find a regular language $\mathcal{R}$ such that $\mu(\mathcal{R}) = \mathcal{C}$. Knowledge of this procedure would be the starting point for Compton's discovery that every MSO class of m-colored trees has an equational specification (see §6.2). One finds $\mathcal{R}$ as follows.

Let $\mathcal{C}$ be defined by a MSO sentence φ of quantifier depth $\le q$, where $q \ge 3$. There are finitely many MSO classes of m-colored chains defined by MSO sentences of quantifier depth at most q, and they are closed under union, intersection and complement; so they form a finite Boolean algebra. Among these, let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be the minimal ones, the atoms of the Boolean algebra. Then the $\mathcal{C}_i$ are pairwise disjoint, and their union is the class of all m-colored chains. Every class of m-colored chains defined by a MSO sentence of quantifier depth at most q is a union of some of the $\mathcal{C}_i$.

For $1 \le j \le m$, let $\bullet_j$ be the 1-element chain of color U_j. Then the class $\{\bullet_j\}$ is (when closed under isomorphism) one of the $\mathcal{C}_n$, since the property that 'the size of a chain is 1 and the color of the single node is U_j' can be expressed by a MSO sentence of quantifier depth 3. We can assume $\mathcal{C}_1 = \{\bullet_1\}, \ldots, \mathcal{C}_m = \{\bullet_m\}$. Then all m-colored chains in the minimal classes $\mathcal{C}_{m+1}, \ldots, \mathcal{C}_k$ have size ≥ 2.

In each of the classes $\mathcal{C}_n$, $1 \le n \le k$, all chains have first elements of the same color. This is because the property that the first element of a chain has a given color U_i can be expressed by a MSO sentence φ_i of quantifier depth 3. Likewise, all chains in each $\mathcal{C}_n$ have the last element of the same color. Let the first element of members of $\mathcal{C}_n$ have the color $U_{\alpha(n)}$, and the last element of members of $\mathcal{C}_n$ have the color $U_{\omega(n)}$.

For an m-colored chain $\mathfrak{c}$ of size ≥ 2, let $\partial\mathfrak{c}$ be the chain that results from removing the last element from $\mathfrak{c}$; and, for any m-chain $\mathfrak{c}$, let $\mathfrak{c}\bullet_i$ be the result of adding a new last element, of color U_i, to $\mathfrak{c}$. For $j > m$ define $\partial\mathcal{C}_j := \{\partial\mathfrak{c} : \mathfrak{c} \in \mathcal{C}_j\}$; and for $j \ge 1$ define $\mathcal{C}_j\bullet_i := \{\mathfrak{c}\bullet_i : \mathfrak{c} \in \mathcal{C}_j\}$. For $j > m$ one then has $\mathcal{C}_j = \big(\partial\mathcal{C}_j\big)\bullet_{\omega(j)}$, that is, by removing and then adding back the last element, with the correct color, in each member of $\mathcal{C}_j$, one has the original class $\mathcal{C}_j$.

For $j > m$, $\partial\mathcal{C}_j$ is clearly a collection of m-colored chains. Using Ehrenfeucht-Fraïssé games, one can prove that $\partial\mathcal{C}_j$ is actually a union of some of the minimal classes $\mathcal{C}_i$, that is,

$$\partial\mathcal{C}_j \;=\; \bigcup_{\mathcal{C}_i \subseteq \partial\mathcal{C}_j} \mathcal{C}_i,$$

and thus

$$\mathcal{C}_j \;=\; \bigcup_{\mathcal{C}_i \subseteq \partial\mathcal{C}_j} \mathcal{C}_i \bullet_{\omega(j)}\,. \tag{B.1}$$

Now, for $1 \le n \le k$, define a finite-state automaton $\mathfrak{A}_n$ that accepts a regular language $\mathcal{R}_n$, with $\mu(\mathcal{R}_n) = \mathcal{C}_n$, as follows. The states of $\mathfrak{A}_n$ are $0, 1, \ldots, k$, the initial state is 0, and the unique final state is n. There is an edge from 0 to $\alpha(n)$, labelled with the letter $a_{\alpha(n)}$. For $1 \le i, j \le k$, there is an edge from i to j iff $\mathcal{C}_i\bullet_{\omega(j)} \subseteq \mathcal{C}_j$, in which case the label on the edge is $a_{\omega(j)}$.

Let $\mathcal{R}_n$ be the regular language accepted by $\mathfrak{A}_n$. It is not difficult to see that $\mu(\mathcal{R}_n) = \mathcal{C}_n$. Since a union of regular languages is a regular language, a union of some of the $\mathcal{C}_i$ also corresponds to a regular language. This finishes the sketch of how to prove, for each MSO class $\mathcal{C}$ of m-colored chains, there is a regular language $\mathcal{R}$ with $\mu(\mathcal{R}) = \mathcal{C}$.

Büchi's Theorem shows that Berstel's detailed analysis of the generating functions for regular languages (see Example 4.12) applies to the generating functions of MSO classes of m-colored chains. This is the *Berstel Paradigm* that we would like to see paralleled in the study of all MSO classes of m-colored trees. In particular, can one show that the generating functions $T(x)$ of such classes decompose into a polynomial and finitely many "nice" functions $T_i(x)$, with each spectrum $\mathsf{Spec}(T_i(x))$ being an arithmetical progression?

References

1. Jason P. Bell, Stanley N. Burris, *Compton's method for proving logical limit laws.* This volume.
2. ——, ——, and Karen A. Yeats, *Counting rooted trees: The universal law* $t(n) \sim C \cdot \rho^{-n} \cdot n^{-3/2}$. Electron. J. Combin. **13** (2006), #R63 [64pp.]
3. ——, ——, ——, *Characteristic points of recursive systems.* Electron. J. Combin. **17** (2010), #R121 [34pp.]
4. ——, ——, ——, *Monadic second-order classes of trees of radius 1.* (Submitted.)
5. Jean Berstel, *Sur les pôles et le quotient de Hadamard de séries n-rationnelles.* C.R. Acad. Sci. Paris, **272**, Sér. A-B (1971), 1079–1081.
6. J. Richard Büchi, *Weak second-order arithmetic and finite automata.* Z. Math. Logik Grundlagen Math. **6** 1960, 66–92.
7. Stanley N. Burris, *Logical Limit Laws and Number Theoretic Density.* Mathematical Surveys and Monographs, Vol. **86**, Amer. Math. Soc., 2001.
8. A. Cayley, *On the theory of the analytical forms called trees.* Phil. Magazine **13** (1857), 172–176.
9. Kevin J. Compton, *A logical approach to asymptotic combinatorics I: First-order properties.* Adv. Math. **65** (1987), 65–96.
10. ——, *A logical approach to asymptotic combinatorics. II. Monadic second-order properties.* J. Combin. Theory, Ser. A **50** (1989), 110–131.
11. ——, *Private communication*, July, 2009.
12. —— and C. Ward Henson, *A uniform method for proving lower bounds on the computational complexity of logical theories.* Ann. Pure Appl. Logic **48** (1990), 1–79.
13. Mor Doron and Saharon Shelah, *Relational structures constructible by quantifier free definable operations.* J. Symbolic Logic, **72** (2007), 1283–1298.
14. Michael Drmota, *Systems of functional equations.* Random Structures and Algorithms **10** (1997), 103–124.
15. Michael Drmota, *Random Trees.* Springer, 2009.
16. Arnaud Durand, Ronald Fagin and Bernd Loescher, *Spectra with only unary function symbols.* Proceedings of the 1997 Annual Conference of the European Association for Computer Science Logic (CSL97). [The manuscript can be found at http://www.almaden.ibm.com/cs/people/fagin/]
17. A. Durand, N.D. Jones, J.A. Makowsky, and M. More, *Fifty years of the spectrum problem.* (Preprint, July, 2009).
18. E. Fischer and J.A. Makowsky, *On spectra of sentences of monadic second order logic with counting.* J. Symbolic Logic **69** (2004), no. 3, 617–640.
19. Philippe Flajolet and Robert Sedgewick, *Analytic Combinatorics.* Cambridge University Press, 2009.
20. Yuri Gurevich and Saharon Shelah, *Spectra of monadic second-order formulas with one unary function.* 18th Annual IEEE Symposium on Logic in Computer Science, June 22–25, 2003, Ottawa, Canada.
21. Katarzyna Idziak and Pawel M. Idziak, *Decidability problem for finite Heyting Algebras.* J. Symbolic Logic **53** (1988), 729–735.

22. James M. Ortega, *Matrix Theory. A Second Course.* Plenum Press, 1987.
23. G. Pólya and R.C. Read, *Combinatorial Enumeration of Groups, Graphs and Chemical Compounds.* Springer Verlag, New York, 1987.
24. Charles Chapman Pugh, *Real Mathematical Analysis.* Undergraduate Texts in Mathematics. Springer-Verlag, 2002.
25. Michael O. Rabin, *Decidability of second-order theories and automata on infinite trees.* Trans. Amer. Math. Soc. **141** (1969), 1–35.
26. J.L. Ramírez-Alfonsín, *The Diophantine Frobenius Problem.* Oxford Lecture Series in Mathematics and its Applications, **30** (2005), Oxford University Press.
27. Jeffrey Shallit, *The Frobenius Problem and its generalizations.* Developments in Language Theory, 72–83, Lecture Notes in Comput. Sci., **5257** (2008), Springer-Verlag.
28. Heinrich Scholz, *Ein ungelöstes Problem in der Symbolischen Logik.* J. Symbolic Logic, **17**, No. 2 (1952), p. 160.
29. Saharon Shelah, *Spectra of monadic second order sentences.* Sci. Math. Jap., **59**, No. 2, (2004), 351–355.
30. Matti Soittola, *Positive rational sequences.* Theoret. Comput. Sci. **2** (1976), 317-322.
31. Larry Stockmeyer, *Classifying the computational complexity of problems.* J. Symbolic Logic, **52**, No. 1 (1987), 1–43.
32. Herbert S. Wilf, *generatingfunctionology.* 2nd ed., Academic Press, 1994.
33. Alan R. Woods, *Coloring rules for finite trees, probabilities of monadic second order sentences.* Random Structures Algorithms **10** (1997), 453–485.

Department of Mathematics, Simon Fraser University, 8888 University Dr., Burnaby, BC,V5A 1S6, Canada
E-mail address: jpb@math.sfu.ca

Department of Pure Mathematics, University of Waterloo, Waterloo, ON, N2L 3G1, Canada
E-mail address: snburris@math.uwaterloo.ca

Department of Mathematics, Simon Fraser University, 8888 University Dr., Burnaby, BC,V5A 1S6, Canada
E-mail address: karen.yeats@math.sfu.ca