# Non-Definability
# in
# First Order Logic
# and
# Monadic Second Order Logic

# Ehrenfeucht-Fraïssé Games



## Tools to Show Non-Definability

- Ehrenfeucht-Fraïssé Games
- Translation Schemes and transductions
- Feferman-Vaught Theorem for sums
- $0-1$ Laws





## Proving non-definability

The class of $\tau$-structures of finite even cardinality, $EVEN(\tau)$, is *not definable* in First Order Logic, (not even in Monadic Second Order Logic):

- For $FOL$: use compactness. Every formula true in all finite even structures has an infinite model.
- For $FOL$ (restricted to finite structures): use Pebble Games (Ehrenfeucht-Fraïssé Games)
- For $MSOL$: use Pebble Games adapted to $MSOL$.

Similarly, $DisPath(n)$ is not $FOL$-definable even for $n = 1$.





## Compactness of $FOL$

Recall:
$\Sigma$ is satisfiable if there is a $\tau$-structure $\mathcal{A}$ such that $\mathcal{A} \models \Sigma$.

**Theorem:**[Gödel-Mal'cev]
Let $\Sigma$ be an infinite set of $FOL(\tau)$-sentences.
$\Sigma$ is satisfiable iff every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable.

*This theorem was stated and proved in* **Logic for CS** *for Propositional Logic.*
*This theorem was stated, but probably not proved in* **Logic for CS** *for First Order Logic.*
*The proof for $FOL$ is very similar to the one for Propositional Logic.*

## Using Compactness

Let $\phi_n$ be the sentence which says that the universe contains at least $n$ elements.

Let $\Sigma_{even}$ consist of

$$\{(\phi_{2n+1} \rightarrow \phi_{2n+2}) : n \in \mathbb{N}\}$$

All **finite** models of $\Sigma_{even}$ are of even cardinality.

Assume there is $\psi_{even}$ such that

$$\mathcal{A} \models \psi_{even} \text{ iff } \mid A \mid = 2n$$

Define

$$\Sigma_1 = \{\psi_{even}\} \cup \{\phi_n : n \in \mathbb{N}\} \cup$$

Every finite subset $\Sigma_0 \subseteq \Sigma_1$ is satisfiable
(by a finite model of even cardinality).

But $\Sigma_1$ has no model,
contradicting compactness.



## $MSOL$ is not compact

Let $\tau_{a,b} = \tau_{graph} \cup \{a, b\}$ be the vocabulary of graphs with two constants.

In $MSOL(\tau_{a,b})$ we have a formula $\phi_{conn}$ which says that the graph is connected.

Let $\psi_n(a, b)$ say that the shortest path between $a, b$ is of length $n$. This is in $FOL(\tau_{a,b})$.

Now every finite subset of

$$\Sigma = \{\phi_{conn} \cup \{\psi_n(a, b) : n \in \mathbb{N}\}$$

is satisfiable, but $\Sigma$ is not.





## Quantifier rank of a formula, I

We write a formula $\phi$ as a tree:

$$\exists X_1 \forall x_2 \, (x_2 \in X_1 \rightarrow \exists x_3 E(x_2, x_3))$$

$\exists X_1$

$\forall x_2$

$\rightarrow$

$x_2 \in X_1$ $\qquad$ $\exists x_3$

$E(x_2, x_3)$

The quantifier rank is biggest number of quantifiers one can find along a path in this tree. Here it is 3.





## Quantifier rank of a formula, II

- For formulas in prenex normal form the quantifier rank equals the number of quantifiers.
- If we reuse variables, the quantifier rank can be smaller than the number of quantifiers used in prenex normal form.

$$\forall x_1 \, (\exists x_2 E(x_1, x_2) \wedge \exists x_2 \neg E(x_1, x_2))$$

Quantifier rank 2

$$\forall x_1 \exists x_2 \exists x_3 \, (E(x_1, x_2) \wedge \neg E(x_1, x_3))$$

Quantifier rank 3

## Ehrenfeucht-Fraïssé Games, I

Given two $\tau$-structures $\mathcal{A}_0$ and $\mathcal{A}_1$ and their powersets $P(A_0)$ and $P(A_1)$.

Two players I (spoiler), II (duplicator)

$k$ numbered pebbles for each structure

Two kind of moves: Set- and point-moves

Play for $n$ moves

$i$-th move:
I chooses $\alpha \in \{0,1\}$ and put pebble on an element in $P(A_\alpha)$ (Set-move) or
in $A_\alpha$ (point move).
II puts corresponding pebble on set or point.



## Ehrenfeucht-Fraïssé Games, II

After $n$ moves we have from $\mathcal{A}_0$

$$A_1^0, a_2^0, A_3^0, \ldots, a_{n-1}^0, A_n^0$$

and from $\mathcal{A}_1$

$$A_1^1, a_2^1, A_3^1, \ldots, a_{n-1}^1, A_n^1$$

These two sequences are *locally isomorphic* if for all $j, k$

$$a_k^0 \in A_j^0 \text{ iff } a_k^1 \in A_j^1$$

and for each $m$-ary $R \in \tau$ and $j_1, j_2, \ldots j_m$

$$R^{\mathcal{A}_0}(a_{j_1}^0, a_{j_2}^0, \ldots, a_{j_m}^0) \text{ iff } R^{\mathcal{A}_1}(a_{j_1}^1, a_{j_2}^1, \ldots, a_{j_m}^1)$$





**Lemma:** Two sequences in $\mathcal{A}_0$ and $\mathcal{A}_1$ respectively

$$A_1^0, a_2^0, A_3^0, \ldots, a_{n-1}^0, A_n^0$$

and

$$A_1^1, a_2^1, A_3^1, \ldots, a_{n-1}^1, A_n^1$$

are locally isomorphic iff for all quantifierfree formulas $B$ we have

$$\mathcal{A}_0 \models B(A_1^0, a_2^0, A_3^0, \ldots, a_{n-1}^0, A_n^0)$$

iff

$$\mathcal{A}_1 \models B(A_1^1, a_2^1, A_3^1, \ldots, a_{n-1}^1, A_n^1)$$

**Proof:**
Use induction over the construction of $B$.





## Ehrenfeucht-Fraïssé Games, III

Winning the game:

II wins if the correspondence on the pebbles induces a local isomorphism (including the sets).

**Theorem:** (Ehrenfeucht-Fraïssé, 1953/61)
II has a winning strategy for the $k$-pebble $n$-moves game on $\mathcal{A}_0$ and $\mathcal{A}_1$ iff they satisfy the same $MSOL(\tau)$-sentences with $k$ variables and quantifier depth $n$.

If no set-moves are played this holds for $FOL(\tau)$.

We write $\mathcal{A}_0 \sim_{k,n}^{MSOL} \mathcal{A}_1$ iff
II has a winning strategy in the game with set moves and
$\mathcal{A}_0 \sim_{k,n}^{FOL} \mathcal{A}_1$ in the game without set moves.

## Winning strategies, I

A winning **strategy** is a function which takes a position of length $n$

$$A_1^0, a_2^0, A_3^0, \ldots, a_{n-1}^0, A_n^0$$

$$A_1^1, a_2^1, A_3^1, \ldots, a_{n-1}^1, A_n^1$$

from $\mathcal{A}_0$ and $\mathcal{A}_1$ respectively
together with a move of player I, say
$X_{n+1}^i \in \{a_{n+1}^i, A_{n+1}^i\}$ as input and returns
$X_{n+1}^{1-i} \in \{a_{n+1}^{1-i}, A_{n+1}^{1-i}\}$ as output such that

$$A_1^0, a_2^0, A_3^0, \ldots, a_{n-1}^0, A_n^0, X_{n+1}^0$$

$$A_1^1, a_2^1, A_3^1, \ldots, a_{n-1}^1, A_n^1, X_{n+1}^1$$

is a winning position
(if it exists, else it is undefined).



## Winning strategies, II

**Proposition:**

$\mathcal{A}_0 \sim_{k,n}^{FOL} \mathcal{A}_1$ and $\mathcal{A}_0 \sim_{k,n}^{MSOL} \mathcal{A}_1$

are *equivalence relations* between $\tau$-structures.
I.e., they are **symmetric, reflexive** and **transitive**.

**Proof:**
Reflexivity: Copy literally
Symmetry: The structures play exchangeable roles (but not the players)
Transitivity: Play on the intermediate board





## Winning EF-Games, I

$\tau = \emptyset$

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$\tau = \{R_2\}$, linear orders

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## Winning EF-Games, II

**Theorem:**

Let $\tau = \emptyset$.
For two sets $\mathcal{A}_0$ and $\mathcal{A}_1$
of size $m_0$ and $m_1$ respectively,
we have $\mathcal{A}_0 \sim_{k,n}^{FOL} \mathcal{A}_1$
(in the game without set moves)

iff

$m_0 = m_1$ or
$k \leq m_0$ and $k \leq m_1$

## Winning EF-Games, III

**Theorem:**

Let $\tau = \{R_2\}$.
For two cycle graphs $\mathcal{G}_0$ and $\mathcal{G}_1$ of size $v_0$ and $v_1$ respectively, we have $\mathcal{G}_0 \sim^{FOL}_{k,n} \mathcal{G}_1$ (in the game without set moves)

provided

$v_0 = v_1$ or
$2^k \leq v_0$ and $2^k \leq v_1$

Does the converse hold ?



## Ehrenfeucht-Fraïssé Games, IV

**Theorem:** (Feferman, Vaught, 1956)

If $\mathcal{A}_0 \sim^{MSOL}_{k,n} \mathcal{B}_0$ and $\mathcal{A}_1 \sim^{MSOL}_{k,n} \mathcal{B}_1$
then $\mathcal{A}_0 \sqcup \mathcal{A}_1 \sim^{MSOL}_{k,n} \mathcal{B}_0 \sqcup \mathcal{B}_1$

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**Theorem:** (Feferman, Vaught, 1956)

If $\mathcal{A}_0 \sim^{FOL}_{k,n} \mathcal{B}_0$ and $\mathcal{A}_1 \sim^{FOL}_{k,n} \mathcal{B}_1$
then $\mathcal{A}_0 \times \mathcal{A}_1 \sim^{FOL}_{k,n} \mathcal{B}_0 \times \mathcal{B}_1$

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The same holds for "gluing" operations.





## Winning EF-Games, IV

**Theorem:**

Let $\tau = \{R_2\}$.
Let $\mathcal{G}_0$ consist of one cycle of size $2^k$ and $\mathcal{G}_1$ consist of two cycles of size $2^k$.

Then we have $\mathcal{G}_0 \sim^{FOL}_{k,n} \mathcal{G}_1$
(in the game without set moves)

**Corollary:**
Connectivity is not $FOL$-definable in the language of graphs.





## Winning EF-Games, V

Now we play the game for $MSOL$ for $\tau = \emptyset$.

$2^n$ elements $\qquad$ $2^n - 1$ elements

How many moves does player I need to win?

$2^n$ elements $\qquad$ $2^{n-1}$ elements

How many moves does player I need to win?

## Winning EF-Games, VI

The rôle of the pebbles.

How long can we play (without set moves) with **two** pebbles?

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How long can we play with **three** pebbles?