Logical Miscellanea

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June 10, 2015

Abstract

This document comprises miscellaneous notes on various logical topics, particularly concerned with decision problems.

Contents

1 The Hintikka-Fraïssé Game Normal Form

We give an account of the game normal form largely following [3]. $\mathcal{L}$ is some first-order language with a finite signature, i.e., $\mathcal{L}$ has only finitely many constant, function and predicate symbols. We take $\bot$, $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$ as the logical primitives and assume that implication and bi-implication are defined in the standard way. Define the quantifier rank $\mathsf{qr}(\psi)$ of a formula ψ by induction on the structure of ψ as follows:

$$
\begin{array}{rclcl}
\mathsf{qr}(\psi) & = & 0 & & \text{if } \psi \text{ is atomic or } \bot \\
\mathsf{qr}(\neg\psi) & = & \mathsf{qr}(\psi) & & \\
\mathsf{qr}(\psi_1 \wedge \psi_2) & = & \mathsf{qr}(\psi_1 \vee \psi_2) & = & \mathsf{max}\{\mathsf{qr}(\psi_1), \mathsf{qr}(\psi_2)\} \\
\mathsf{qr}(\exists x \cdot \psi) & = & \mathsf{qr}(\forall x \cdot \psi) & = & \mathsf{qr}(\psi) + 1
\end{array}
$$

A formula ϕ of $\mathcal{L}$ is said to be *unnested* if function symbols only appear in subterms of the form $y = f(x_1, \ldots, x_n)$ and constant symbols only appear in subterms of the form $y = c$ where f is a function symbol, c is a constant symbol and y and the x_i are variables. At the expense of introducing additional quantifiers, any formula may be converted into a logically equivalent unnested formula with the same free variables.

We write $v_1, v_2, \ldots$ for some fixed enumeration of the variables of $\mathcal{L}$. Writing $\mathsf{frees}(\phi)$ for the set of free variables of a formula ϕ, define for $n \in \mathbb{N}$:

$$
\Phi(n) \quad = \quad \{\phi \in \mathcal{L} \mid \phi \text{ is atomic} \wedge \phi \text{ is unnested} \wedge \mathsf{frees}(\phi) \subseteq \{v_1, \ldots, v_n\}\}.
$$

Then define index sets $X^{n,r}$ for $n, r \in \mathbb{N}$:

$$
\begin{array}{rcl}
X^{n,0} & = & \mathbb{P}(\Phi(n)); \\
X^{n,r+1} & = & \mathbb{P}(X^{n,r}).
\end{array}
$$

Since $\mathcal{L}$ has a finite signature, the sets $X^{n,r}$ are all finite. We assume given an effective encoding of $\mathcal{L}$ as natural numbers and an effective total ordering of $\mathcal{L}$. If ψ_i is any set of formulas indexed by some non-empty finite set I, $\bigvee_{i \in I} \psi_i$ and $\bigwedge_{i \in I} \psi_i$ denote the formulas $\psi_{i_1} \vee \ldots \vee \psi_{i_k}$ and $\psi_{i_1} \wedge \ldots \wedge \psi_{i_k}$ respectively where the i_j enumerate the elements of I so that the ψ_{i_j} are listed in order without repetition. If I is empty, $\bigvee_{i \in I} \psi_i$ and $\bigwedge_{i \in I} \psi_i$ denote $\bot$ and $\neg\bot$ respectively.

Define functions $\gamma^{n,r}_{\_} : X^{n,r} \to \mathcal{L}$ for $n, r \in \mathbb{N}$ (with the argument written as a subscript) such that for $i \in X^{n,0} = \Phi(n)$ and $j \in X^{n,r+1}$:

$$
\gamma^{n,0}_i \quad = \quad \bigwedge_{\phi \in i} \phi \wedge \bigwedge_{\phi \in \Phi(n) \setminus i} \neg\phi;
$$

$$\gamma_j^{n,r+1} \quad = \quad (\bigwedge_{J \in j} \exists v_{n+1} \cdot \gamma_J^{n+1,r}) \wedge (\forall v_{n+1} \cdot \bigvee_{J \in j} \gamma_J^{n+1,r}).$$

So $\mathsf{frees}(\gamma_i^{n,r}) \subseteq \{v_1, \ldots, v_n\}$ and $\mathsf{qr}(\gamma_i^{n,r}) = r$. We call these formulas $\gamma_i^{n,r}$ *game normal atoms*. If K is a structure for $\mathcal{L}$ and $i \in X^{n,r}$, write $K_i^{n,r}$ for the subset of K^n defined by the formula $\gamma_i^{n,r}$. The following lemma says that, for given n and r, the non-empty $K_i^{n,r}$ form a partition of K^n.

Lemma 1 *For any structure K for $\mathcal{L}$ and any $n, r \in \mathbb{N}$, we have*

$$\textit{(a)} \quad K \quad \models \quad \bigwedge_{\substack{i,j \in X^{n,r} \\ i \neq j}} \forall v_1 \ldots v_n \cdot \neg(\gamma_i^{n,r} \wedge \gamma_j^{n,r})$$

$$\textit{(b)} \quad K \quad \models \quad \forall v_1 \ldots v_n \cdot \bigvee_{i \in X^{n,r}} \gamma_i^{n,r}$$

Proof: Induction on r: when $r = 0$ the matrix of the universally quantified formula in both (a) and (b) is an instance of a propositional tautology. So assume $r > 0$. For (a), it suffices to consider the case where $i \setminus j \neq \{\}$ so that there is an $I \in i$ with $I \notin j$; by the inductive hypothesis, under any given interpretation of $v_1, \ldots, v_n$ in K, $\exists v_{n+1} \cdot \gamma_I^{n+1,r}$ and $\forall v_{n+1} \cdot \bigvee_{J \in j} \gamma_J^{n+1,r}$ cannot both be satisfied, but $\gamma_i^{n,r}$ implies the former and $\gamma_j^{n,r}$ implies the latter so $\gamma_i^{n,r} \wedge \gamma_j^{n,r}$ cannot hold. For (b), given an interpretation of $v_1, \ldots, v_n$ in K we have to find an i such that $\gamma_i^{n,r}$ holds; let i be the set of $I \in X^{n+1,r}$ such that $\exists v_{n+1} \cdot \gamma_I^{n+1,r}$ holds; then evidently $\bigwedge_{I \in i} \exists v_{n+1} \cdot \gamma_I^{n+1,r}$ holds while $\forall v_{n+1} \cdot \bigvee_{I \in i} \gamma_I^{n+1,r}$ follows from the inductive hypothesis; but the conjunction of these two formulas is $\gamma_i^{n,r}$. ■.

We say a formula is *game normal* or in *game normal form* if it has the form $\bigvee_{i \in I} \gamma_i^{n,r}$ for some $n, r \in \mathbb{N}$ and $I \subseteq X^{n,r}$. We refer to n and r as the *arity* and *rank* of the game normal formula respectively. We assume given an effective encoding of the elements of the index sets $X^{n,r}$ as natural numbers.

The following theorem says that every formula is logically equivalent to one in game normal form.

Theorem 2 *(Game Normal Form) There is primitive recursive algorithm which, given an unnested formula ψ of $\mathcal{L}$ with $\mathsf{frees}(\psi) \subseteq \{v_1, \ldots, v_n\}$ and $\mathsf{qr}(\psi) = r$ finds $n' \leq n$, $r' \leq r$ and a subset I of $X^{n',r'}$ such that in any structure K for $\mathcal{L}$ one has:*

$$K \models \psi \Leftrightarrow \bigvee_{i \in I} \gamma_i^{n',r'}.$$

Proof: Let us call a formula *tiered* if each subformula of the form $\exists v_p \cdot \chi$ or $\forall v_p \cdot \chi$ has the property that $\mathsf{frees}(\chi) \subseteq \{v_1, \ldots, v_p\}$. Consider the primitive recursive function $\mathcal{V} : \mathbb{N} \times \mathcal{L} \to \mathcal{L}$ defined as follows:

$$\begin{array}{rcll}
\mathcal{V}(p, \chi) & = & \chi & \text{if } \chi \text{ is atomic} \\
\mathcal{V}(p, \neg\chi) & = & \neg\mathcal{V}(p, \chi) & \\
\mathcal{V}(p, \chi \vee \rho) & = & \mathcal{V}(p, \chi) \vee \mathcal{V}(p, \rho) & \\
\mathcal{V}(p, \chi \wedge \rho) & = & \mathcal{V}(p, \chi) \wedge \mathcal{V}(p, \rho) & \\
\mathcal{V}(p, \exists v_q \cdot \chi) & = & \exists v_r \cdot (\mathcal{V}(r+1, \chi)[v_r/v_q]) & \text{where } r = \mathsf{max}\{p, q\} \\
\mathcal{V}(p, \forall v_q \cdot \chi) & = & \forall v_r \cdot (\mathcal{V}(r+1, \chi)[v_r/v_q]) & \text{where } r = \mathsf{max}\{p, q\}
\end{array}$$

Note that no renaming of bound variables is required when the substitution is performed in calculating $\mathcal{V}(r+1, \chi)[v_r/v_q]$. Thus if ψ is the input formula and $m = \mathsf{max}\{q \mid q = 0 \vee v_q \in \mathsf{frees}(\psi)\}$, then $\mathcal{V}(m+1, \psi)$ is a tiered formula that is logically equivalent to ψ. Our first step is to replace ψ by $\mathcal{V}(m+1, \psi)$ and from then on we will work exclusively with tiered formulas. We next replace all subformulas of the form $\forall x \cdot \chi$ by $\neg\exists x \cdot \neg\chi$.

From the definition of the $\gamma_K^{m,0}$, the following bi-implication, where $\phi \in \Phi(m)$ and $m \in \mathbb{N}$, is a propositional tautology and hence holds in any structure.

$$\phi \;\Leftrightarrow\; \bigvee_{\substack{K \in X^{m,0} \\ \phi \in K}} \gamma_K^{m,0} \tag{1}$$

We now use (1) to replace each atomic subformula of ψ by an equivalent formula in game normal form taking m as small as possible in each case. Note that the game normal atoms are tiered so the resulting formula will be tiered.

We now have a formula that is constructed from game normal atoms using the propositional connectives and the existential quantifier. We reduce this to game normal form using a system of rules each based on one or more bi-implications defined in primitive recursive schemata. These are used as left-to-right rewrite rules possibly conditional on a primitive recursive test on the subformula being rewritten. For each rule there is a primitive recursive numeric measure of the formula that decreases when the rule is applied and that is not increased by any other rule. This means that the overall process is a primitive recursive function of the original formula.

We now list the rules and the measures according to the form of the subformula type rule deals with. The bi-implications used in each rule are easily seen to be valid in any structure for $\mathcal{L}$ using lemma 1 and the definitions of the game normal atoms as necessary. Note that there is no rule for universal quantification, because the only universal quantifiers in the formula are inside subformulas that are already in game normal form. Once a subformula is in game normal form, its subformulas are never rewritten, but a rule may eliminate or replace it or an enclosing formula.

¬-gnf: The bi-implication is the following where $m, q \in \mathbb{N}$ and $I \subseteq X^{m,q}$:

$$\neg \bigvee_{a \in I} \gamma_a^{m,q} \quad \Leftrightarrow \quad \bigvee_{a \in X^{m,q} \setminus I} \gamma_a^{m,q} \tag{2}$$

The measure is the number of negations appearing outside a game normal subformula.

gnf-∨-gnf: For a disjunction between formulas in game normal form with the same arity and rank, the bi-implication is the following, where $m, q \in \mathbb{N}$ and $I, J \subseteq X^{m,q}$:

$$\bigvee_{a \in I} \gamma_a^{m,q} \vee \bigvee_{b \in J} \gamma_b^{m,q} \quad \Leftrightarrow \quad \bigvee_{a \in I \cup J} \gamma_a^{m,q} \tag{3}$$

To avoid an infinite regress, this is not to be used if the left-hand side is already in game normal form. For a disjunction between formulas in game normal form where the arities or ranks do not agree we use the following bi-implications, where $m, q \in \mathbb{N}$ and $j \in X^{m,q}$:

$$\gamma_j^{m,q} \quad \Leftrightarrow \quad \bigvee_{a \in X^{m+1,q} \setminus X^{m,q}} \gamma_{a \cup j}^{m+1,q} \tag{4}$$

$$\gamma_j^{m,q} \quad \Leftrightarrow \quad \gamma_{\{j\}}^{m,q+1} \tag{5}$$

These are used only as needed to make the arities and ranks agree.

The measure is the number of disjunctions appearing outside a game normal subformula.

gnf-∧-gnf: For a conjunction between formulas in game normal form with the same arity and rank, the bi-implication is the following, where $m, q \in \mathbb{N}$ and $I, J \subseteq X^{m,q}$:

$$\bigvee_{a\in I}\gamma_a^{m,q}\wedge\bigvee_{b\in J}\gamma_b^{m,q} \quad\Leftrightarrow\quad \bigvee_{a\in I\cap J}\gamma_a^{m,q} \tag{6}$$

As in the case of disjunction, (4) and (5) are used as necessary to make the arities and ranks agree.

The measure is the number of conjunctions appearing outside a game normal subformula.

$\exists$-$\vee$: For an existentially quantified disjunction, the bi-implication is the following:

$$(\exists x\cdot\, \chi\vee\rho) \quad\Leftrightarrow\quad (\exists x\cdot\, \chi)\vee(\exists x\cdot\, \rho) \tag{7}$$

The measure is the sum over all existentially quantified subformulas of the number of disjunctions in the matrix that appear outside a game normal subformula.

$\exists$-gnf: For an existentially quantified formula with bound variable v_m for some $m\in\mathbb{N}$ and a matrix that is a game normal atom $\gamma_i^{m',q}$ for some $m',q\in\mathbb{N}$ and $i\in X^{m+1,q}$. If $v_m\notin \mathsf{frees}(\gamma_i^{m',q})$ we use the following bi-implication, valid whenever $x\notin\mathsf{frees}(\chi)$:

$$(\exists x\cdot\, \chi) \quad\Leftrightarrow\quad \chi \tag{8}$$

If $v_m\in\mathsf{frees}(\gamma_i^{m',q})$, then, as the formula is tiered, we must have $m'=m>0$ and then we use the following bi-implication:

$$(\exists v_m\cdot\, \gamma_i^{m,q}) \quad\Leftrightarrow\quad \bigvee_{\substack{K\in X^{m-1,q+1}\\ i\in K}}\gamma_K^{m-1,q+1} \tag{9}$$

The measure is the number of existential quantifiers appearing outside a game normal subformula.

That completes the list of rules and measures. Clearly all of these rules transform a tiered formula constructed from game normal atoms using the propositional connectives and the existential quantifier into a formula of the same kind.

It is easy to see by induction on the construction of such a formula that each measure is primitive recursive and is decreased by application of the corresponding rule and is not increased by application of any of the other rules. Moreover if none of the of the rules is applicable, as will be the case when the algorithm terminates, the result will be in game normal form with arity $n' \leq n$ and rank $r' \leq r$ and I, n' and r' may be read off from it. ■

2 Skolem's Decision Procedure for Algebras of Sets

$\mathcal{S}$ is the language of Skolem's Klassenkalkül [4]. $\mathcal{S}$ has a constant symbol $\emptyset$, a unary function symbol $\sim$, binary function symbols $\cup$ and $\cap$ and a binary relation symbol $\subseteq$. Given any set X, let $\mathcal{P}(X)$ be the structure for $\mathcal{S}$ in which the domain is the power set $\mathbb{P}X$, in which $\emptyset$, $\cup$, $\cap$ and $\sim$ are interpreted as the usual set-theoretic operations ($\sim$ being complementation relative to X, $\sim a = X \setminus a$) and in which $\subseteq$ is interpreted as the subset relation. We call X the *universe* of the structure and we write $\mathbb{U}$ for $\sim\emptyset$, so that $\mathbb{U}$ denotes the universe. We will give a quantifier elimination procedure for an expansion $\mathcal{S}'$ of $\mathcal{S}$ that adds for each $n \in \mathbb{N}$, a unary relation symbol L_n intended to be interpreted so that $\mathsf{L}_n(a)$ holds iff a has at least n elements. $\mathcal{S}'$ is a definitional expansion of $\mathcal{S}$ since we have:

$$\begin{array}{rcl} \mathsf{L}_0(a) & \Leftrightarrow & \emptyset = \emptyset \\ \mathsf{L}_{n+1}(a) & \Leftrightarrow & \exists b \cdot \emptyset \neq b \subseteq a \wedge \mathsf{L}_n(a \cap \sim b) \end{array}$$

Note that in the structure $\mathcal{P}(X)$, $\mathsf{L}_0(a)$ is always true: L_0 is provided only to make the notation uniform.

If $a_1, \ldots a_k$ is any list of pairwise distinct variables and $I \subseteq \{1, \ldots, k\}$, define the term a_I of $\mathcal{S}'$ as follows:

$$a_I = \Big(\bigcap_{i \in I} a_i \Big) \cap \Big(\bigcap_{i \notin I} \sim a_i \Big) .$$

In any structure $\mathcal{P}(X)$, $a_I \cap a_J = \emptyset$ holds unless $I = J$. Using de Morgan's laws one has that if t is a term of $\mathcal{S}'$ with $\mathsf{frees}(t) = \{a_1, \ldots, a_k\}$, there is a unique index set $U \subseteq \mathbb{P}\{1, \ldots, k\}$ such that the following equation holds in $\mathcal{P}(X)$ for any X:

$$t = \bigcup_{I \in U} a_I$$

(where we ensure that each summand contains all the free variables using $a = (a \cap b) \cup (a \cap \sim b)$ as necessary). Here, by convention, $\bigcup_{I \in \emptyset} a_I$ means $\emptyset$. Also when $U = \mathbb{P}\{1, \ldots, k\}$, we can further simplify to get $t = \mathbb{U}$. We call the result the *disjunctive normal form* of t; it can clearly be effectively calculated from t.

It will be convenient to define $\#(X)$ to be $|X|$ if X is finite and to be the symbol ∞ otherwise. We order and add elements of $\mathbb{N} \cup \{\infty\}$ in the usual way.

Theorem 3 *There is an effective quantifier elimination procedure for $\mathcal{S}'$, i.e., an effective procedure that calculates for each formula ϕ a quantifier-free formula ϕ_{qf} with the same free variables such that $\phi \Leftrightarrow \phi_{qf}$ holds in every structure $\mathcal{P}(X)$.*

Proof: By standard arguments for a quantifier elimination process, it is enough to show that if ψ is a quantifier-free formula with $\mathsf{frees}(\psi) = \{a_1, \ldots, a_k\}$, then $\exists a_k \cdot\ \psi$ is equivalent to a quantifier-free formula. The following hold for any $a, b, c \in \mathbb{P}X$:

$$\begin{array}{rcl} a = b & \Leftrightarrow & a \subseteq b \wedge b \subseteq a \\ a \subseteq b & \Leftrightarrow & \neg \mathsf{L}_1(a \cap \sim b) \\ \mathsf{L}_n((a \cap c) \cup (b \cap \sim c)) & \Leftrightarrow & \displaystyle\bigvee_{m=0}^{n} (\mathsf{L}_m(a \cap c) \wedge \mathsf{L}_{n-m}(b \cap \sim c)) \end{array}$$

After putting all terms in ψ in disjunctive normal form, we may use the above equivalences to reduce to the case where the atomic predicates in ψ are all of the form $\mathsf{L}_n(\emptyset)$, $\mathsf{L}_n(\mathbb{U})$ or $\mathsf{L}_n(a_I)$ for $I \subseteq \{1, \ldots, k\}$. After putting ψ in disjunctive normal form, moving quantifiers through disjunctions and moving out conjuncts with no occurrence of a_k, we may further assume that ψ is a conjunction of literals $\mathsf{L}_n(a_I)$ and $\neg \mathsf{L}_n(a_I)$.

There are now two cases to consider.

Case *(i)* $k = 1$: if $a_k = a_1$ is its only free variable, ψ is a conjunction of literals of the form $\mathsf{L}_n(a)$, $\neg \mathsf{L}_n(a)$, $\mathsf{L}_n(\sim a)$ and $\neg \mathsf{L}_n(\sim a)$, i.e., ψ comprises sets of lower and upper bounds on the sizes of the sets a_1 and $\sim a_1$. E.g., if ψ contains the conjunct $\neg \mathsf{L}_4(\sim a_1)$, then ψ imposes the upper bound $\#(\sim a_1) \leq 3$. Let L_0 and U_0 (resp. L_1 and U_1) be the sets of lower and upper bounds that ψ imposes on a (resp. $\sim a$). Let $l_p = \mathsf{max}(\{0\} \cup L_p)$, $u_p = \mathsf{min}(\{\infty\} \cup U_p)$, $p = 0, 1$. ψ then holds iff the following constraints on the sizes of a_1 and $\sim a_1$ are satisfied:

$$\begin{array}{c} l_0 \leq \#(a_1) \leq u_0 \\ l_1 \leq \#(\sim a_1) \leq u_1. \end{array}$$

Let $l = l_0 + l_1$ and $u = u_0 + u_1$. If, either $l_0 > u_0$ or $l_1 > u_1$, then the constraints are unsatisfiable and $\exists a_1 \cdot\ \psi$ is equivalent to $\neg \mathsf{L}_0(\emptyset)$. If $l_0 \leq u_0$, $l_1 \leq u_1$ and $u = \infty$, then the constraints are satisfiable providing the universe has at least l elements and so $\exists a_1 \cdot\ \psi$ is equivalent to $\mathsf{L}_l(\mathbb{U})$. Finally, if $l_0 \leq u_0$, $l_1 \leq u_1$ and $u < \infty$, then the constraints are satisfiable providing the universe has at least l elements and at most u elements and so $\exists a_1 \cdot \psi$ is equivalent to $\mathsf{L}_l(\mathbb{U}) \wedge \neg \mathsf{L}_{u+1}(\mathbb{U})$.

Case *(ii)* $k > 1$: In this case, by grouping together the literals $L_n(a_I)$ and $\neg L_n(a_I)$ where $J \subseteq \{1, \ldots, k-1\}$ and $I = J$ or $I = J \cup \{k\}$, we find that there are sets $J_{pq} \subseteq \mathbb{N}$, $p, q = 0, 1$, such that we have the following logical equivalence:

$$\exists a_k \cdot \psi \Leftrightarrow \exists a_k \cdot \bigwedge_{J \subseteq \{1,\ldots,k-1\}} \left(\begin{array}{ll} & \bigwedge_{n \in J_{00}} \ L_n(a_J \cap a_k) \\ \wedge & \bigwedge_{n \in J_{10}} \neg L_n(a_J \cap a_k) \\ \wedge & \bigwedge_{n \in J_{01}} \ L_n(a_J \cap \sim a_k) \\ \wedge & \bigwedge_{n \in J_{11}} \neg L_n(a_J \cap \sim a_k) \end{array} \right)$$

Let χ be the formula on the right-hand side of the above formula then, I claim that in $\mathcal{P}(X)$ we have

$$\chi \Leftrightarrow \bigwedge_{J \subseteq \{1,\ldots,k-1\}} \exists a_k \cdot \left(\begin{array}{ll} & \bigwedge_{n \in J_{00}} \ L_n(a_J \cap a_k) \\ \wedge & \bigwedge_{n \in J_{10}} \neg L_n(a_J \cap a_k) \\ \wedge & \bigwedge_{n \in J_{01}} \ L_n(a_J \cap \sim a_k) \\ \wedge & \bigwedge_{n \in J_{11}} \neg L_n(a_J \cap \sim a_k) \end{array} \right).$$

For the left-to-right direction is trivial, while, if x_J is a witness for the existential formula forming the conjunct indexed by J on the right-hand side, then so is $a_J \cap x_J$, and then, as the a_J are pairwise disjoint, $\bigcup_J (a_J \cap x_J)$ is a witness for χ.

So we may assume that for given $J \subseteq \{1, \ldots, k-1\}$, ψ is a conjunction of literals $L_n(a_J \cap a_k)$, $\neg L_n(a_J \cap a_k)$, $L_n(a_J \cap \sim a_k)$ and $\neg L_n(a_J \cap \sim a_k)$. I.e., ψ comprises sets of lower and upper bounds on the sizes of the sets $a_J \cap a_k$ and $a_J \cap \sim a_k$. As in case *(i)*, we can now compute $l_0, u_0, l_1, u_1 \in \mathbb{N} \cup \{\infty\}$ such that ψ is equivalent to the following constraints on the sizes of these sets:

$$\begin{array}{c} l_0 \leq \#(a_J \cap a_k) \leq u_0 \\ l_1 \leq \#(a_J \cap \sim a_k) \leq u_1. \end{array}$$

And then, writing $l = l_0 + l_1$ and $u = u_0 + u_1$, we see as in case *(i)* that if either $l_0 > u_0$ or $l_1 > u_1$ then $\exists a_k \cdot \psi$ is equivalent to $\neg \mathsf{L}_0(\emptyset)$, while otherwise it is equivalent to $\mathsf{L}_l(a_J)$ if $u = \infty$ or to $\mathsf{L}_l(a_J) \wedge \neg \mathsf{L}_{u+1}(a_J)$ if $u < \infty$. ∎

If C is a set whose elements are cardinals or the symbol ∞, we write S^C for the set of sentences of $\mathcal{S}$ that are valid in $\mathcal{P}(X)$ whenever the universe X has either $|X| \in C$ or $\#(X) \in C$. We write S^∞ for $\mathsf{S}^{\{\infty\}}$, the set of sentences valid over all infinite universes. Part *(ii)* of the following corollary implies that if a sentence ϕ fails to hold over some universe, then it fails to hold over some finite universe.

Corollary 4 (i) *If* $C \subseteq \mathbb{N} \cup \{\infty\}$ *is recursive and* $\#(C)$ *is given,* S^C *is decidable.* (ii) *If* κ *is any infinite cardinal,* $\mathsf{S}^\infty = \mathsf{S}^{\{\kappa\}} = \bigcup_{n \in \mathbb{N}} \mathsf{S}^{\{m:\mathbb{N} | m \geq n\}}$.

Proof: Given a sentence $\phi \in \mathcal{S}$, the theorem gives us a quantifier-free formula $\phi_{qf} \in \mathcal{S}'$ equivalent to ϕ in $\mathcal{P}(X)$ for any X. As in the proof of the theorem, we can take ϕ_{qf} to be a conjunction of literals of the form $\mathsf{L}_n(\emptyset)$, $\neg\mathsf{L}_n(\emptyset)$, $\mathsf{L}_n(\mathbb{U})$ and $\neg\mathsf{L}_n(\mathbb{U})$, i.e. in $\mathcal{P}(X)$, ϕ_{qf} comprises sets of lower and upper bounds on $\#(\emptyset)$ and $\#(X)$. We can therefore find $l, u \in \mathbb{N} \cup \{\infty\}$ such that ϕ_{qf} holds iff $l \leq \#(X) \leq u$, since, if the bounds on $\#(\emptyset) = 0$ are consistent, then they can be ignored, while if not, then we can take $l > u$. As ϕ_{qf} and ϕ are equivalent, and ϕ_{qf} holds iff the universe X has $l \leq \#(X) \leq u$, we have $\phi \in \mathsf{S}^C$ iff $C \subseteq \{l, \ldots, u\}$. Given $\#(C)$, it is an easy exercise to give an algorithm to decide this, completing the proof of *(i)*.

For *(ii)*, note that each of $\phi_{qf} \in \bigcup_{n\in\mathbb{N}} \mathsf{S}^{\{m:\mathbb{N}|m\geq n\}}$, $\phi_{qf} \in \mathsf{S}^{\{\kappa\}}$ and $\phi_{qf} \in \mathsf{S}^{\infty}$ is equivalent to $u = \infty$. ∎

The proof of the theorem and part *(i)* of the corollary go through almost unchanged if we add to $\mathcal{S}'$ a unary predicate symbol L_∞ whose intended interpretation is such that $\mathsf{L}_\infty(a)$ holds iff a is infinite.

For later use, we define the following shorthand:

$$\mathsf{Part}(a_1, \ldots, a_n) := \bigcup_{i=1}^{n} a_i = \mathbb{U} \wedge \bigwedge_{i=1}^{n-1} \bigwedge_{j=i+1}^{n} a_i \cap a_j = \emptyset$$

so that $\mathsf{Part}(a_1, \ldots, a_n)$ holds iff the a_i comprise a partition of the universe into n pairwise disjoint sets.

3 The Feferman-Vaught Theorem

Let K_i be a family of structures for a first order language $\mathcal{L}$ with i ranging over some index set I. The *direct product* $\prod_{i \in I} K_i$ is a structure for $\mathcal{L}$ whose domain is the set-theoretic product $\prod_{i \in I} \mathsf{dom}\, K_i$, so that an element in the direct product is a function $X : I \to \bigcup_{i \in I} \mathsf{dom}\, K_i$, such that $X(i) \in \mathsf{dom}\, K_i$ for all i in I. The function symbols of $\mathcal{L}$ are interpreted pointwise: if f is an n-place function symbol, $f(X_1, \ldots, X_n)(i) = f(X_1(i), \ldots, X_n(i))$ (where, by abuse of notation, the first f is the interpretation of f in the product and the second is its interpretation in the i-th factor); similarly, if R is an n-place relation symbol, $R(X_1, \ldots, X_n)$ holds in the direct product iff $R(X_1(i), \ldots, X_n(i))$ holds in K_i for all $i \in I$; if c is a constant symbol c is interpreted in the direct product so that its i-th component $c(i)$ is the interpretation of c in K_i.

If $\theta(x_1, \ldots, x_n)$ is a formula of $\mathcal{L}$ and $X_1, \ldots, X_n$ are elements of the direct product $\prod_{i \in I} K_i$, the *boolean value* of $\theta(X_1, \ldots, X_n)$ is written $||\theta(X_1, \ldots, X_n)||$ and is defined by $||\theta(X_1, \ldots, X_n)|| = \{i \in I \mid K_i \models \theta(X_1(i), \ldots, X_n(i))\}$. Thus $\theta(X_1, \ldots, X_n)$ holds in the direct product iff $||\theta(X_1, \ldots, X_n)|| = I$. We can now state the Feferman-Vaught theorem:

Theorem 5 *Let $\mathcal{L}$ be any first-order language and let $\mathcal{S}$ be the language of the algebra of sets. There is primitive recursive function that associates with each formula $\zeta(x_1, \ldots, x_k) \in \mathcal{L}$ a formula $\Phi(a_1, \ldots, a_p)$ of $\mathcal{S}$ and a list of formulas $\zeta_1(x_1, \ldots, x_k)$, $\ldots$, $\zeta_p(x_1, \ldots, x_k)$ such that for any list $\overline{X} = (X_1, \ldots, X_k)$ of elements of the direct product $\prod_{i \in I} K_i$ the following are equivalent:*

$$\begin{array}{ll} \text{(i)} & \prod_{i \in I} K_i \models \zeta(\overline{X}) \\ \text{(ii)} & \mathcal{P}(I) \models \Phi(\big|\big|\zeta_1(\overline{X})\big|\big|, \ldots, \big|\big|\theta_p(\overline{X})\big|\big|) \end{array}$$

Proof: Table 1 on page 14 illustrates the algorithm described in the following proof when ζ the sentence $\exists x\, y \cdot x \not\leq y \wedge y \not\leq x$ (with a slight optimisation that we will discuss after the proof).

Let us call a sequence $(\Phi, \zeta_1, \ldots, \zeta_p)$ with the stated property a *determining sequence* for ζ. We first note that if $(\Psi, \eta_1, \ldots, \eta_q)$ is a determining sequence for η, then we can always arrange for the η_i to be exhaustive and mutually exclusive. To do this, let $r = 2^q$, enumerate the subsets of $\{1, \ldots, q\}$ as $A_1, \ldots, A_r$, let $B_i := \{j \mid i \in A_j\}$, $i = 1, \ldots, q$, and define:

$$\begin{array}{lcl} \Omega & := & \Psi(\bigcup_{i \in B_1} a_i, \ldots, \bigcup_{i \in B_q} a_i) \\ \theta_i & := & \bigwedge_{i \in A_i} \eta_i \wedge \bigwedge_{i \in \{1, \ldots, q\} \setminus A_i} \neg \eta_i \quad i = 1, \ldots, r. \end{array}$$

Then $(\Omega, \theta_1, \ldots, \theta_r)$ is also a determining sequence for η and, since $\bigvee_{i=1}^{r} \theta_i$ and $\bigwedge_{i=1}^{r-1} \bigwedge_{j=i+1}^{r} \neg(\theta_i \wedge \theta_j)$ are both propositional tautologies, the θ_i are exhaustive and mutually exclusive.

If ζ is atomic then it is immediate from the definitions that $(a_1 = \mathbb{U}, \zeta)$ is a determining sequence for ζ. It is also easily verified that if ζ is $\neg\eta$ and $(\Psi, \eta_1, \ldots, \eta_q)$ is a determining sequence for η, then $(\neg\Psi, \eta_1, \ldots, \eta_q)$ is a determining sequence for ζ and that if ζ is $\eta \circ \theta$ where $\circ$ is any binary propositional connective and if $(\Psi, \eta_1, \ldots, \eta_q)$ and $(\Omega, \theta_1, \ldots, \theta_r)$ are determining sequences for η and θ then $(\Psi \circ \Omega(a_{q+1}, \ldots, a_{q+r}), \eta_1, \ldots, \eta_q, \theta_1, \ldots, \theta_r)$ is a determining sequence for ζ. If ζ has the form $\forall x \cdot \eta$, we treat it in the same way as $\neg\exists x \cdot \neg\eta$.

It remains to consider the case when ζ has the form $\exists x \cdot \eta$. In this case, we can assume that it has the form $\exists x_{k+1} \cdot \eta(x_1, \ldots, x_{k+1})$ and, by induction and the remarks above, that $(\Psi, \eta_1, \ldots, \eta_q)$ is a determining sequence for η, where the η_i are exhaustive and mutually exclusive. I claim that, if we define:

$$\begin{array}{rcl} \Phi & := & \exists b_1 \ldots b_q \cdot \bigwedge_{i=1}^{q} b_i \subseteq a_i \wedge \mathsf{Part}(b_1, \ldots, b_q) \wedge \Psi(b_1, \ldots, b_q) \\ \zeta_i & := & \exists x_{k+1} \cdot \theta_i \quad i = 1, \ldots, q \end{array}$$

then $(\Phi, \zeta_1, \ldots, \zeta_q)$ is a determining sequence for ζ. By induction, if $X_1, \ldots, X_{k+1}$ are elements of $\prod_{i \in I} K_i$ and $\overline{X} = (X_1, \ldots, X_k)$ then $\prod_{i \in I} K_i \models \eta(\overline{X}, X_{k+1})$ iff $\mathcal{P}(I) \models \Psi(\left|\left|\eta_1(\overline{X}, X_{k+1})\right|\right|, \ldots, \left|\left|\eta_p(\overline{X}, X_{k+1})\right|\right|)$. To complete the proof we must show that $\prod_{i \in I} K_i \models \zeta(\overline{X})$ iff $\mathcal{P}(I) \models \Phi(\left|\left|\zeta_1(\overline{X})\right|\right|, \ldots, \left|\left|\zeta_p(\overline{X})\right|\right|)$.

Assume $\prod_{i \in I} K_i \models \zeta(\overline{X})$, i.e., $\prod_{i \in I} K_i \models \exists x_{k+1} \cdot \eta(\overline{X})$. Then there is an X_{k+1} such that $\prod_{i \in I} K_i \models \eta(X_1, \ldots, X_{k+1})$ and hence, by the inductive hypothesis $\mathcal{P}(I) \models \Psi(B_1, \ldots, B_p)$ where $B_i = \left|\left|\eta_i(\overline{X}, X_{k+1})\right|\right|$, for $i = 1, \ldots, p$. Also, $\left|\left|\zeta_i(\overline{X})\right|\right| \supseteq B_i$, since $X_{k+1}(j)$ provides a witness for $\exists x_{k+1} \cdot \theta_i(\overline{X}, x_{k+1})$ for each $j \in B_i$. As the θ_i are exhaustive and mutually exclusive the B_i partition I and so we see that $\mathcal{P}(I) \models \Phi(\left|\left|\zeta_1(\overline{X})\right|\right|, \ldots, \left|\left|\zeta_p(\overline{X})\right|\right|)$ by taking B_i as the witness for b_i, $i = 1, \ldots, p$.

Conversely, write A_i for $\left|\left|\zeta_i(\overline{X})\right|\right|$ and assume $\mathcal{P}(I) \models \Phi(A_1, \ldots, A_p)$. Then there are sets $B_i \subseteq A_i$ that partition I so that for each $j \in I$, there is a unique i such that $j \in B_i$. But $j \in B_i$ implies that $j \in S_i$ and hence that $K_i \models \exists x_{k+1} \cdot \theta_i(X_1(i), \ldots, X_k(i))$. So we may choose an element X_{k+1} of $\prod_{i \in I} K_i$ such that $K_i \models \theta(X_1(i), \ldots, X_{k+1}(i))$ for each $i \in I$, i.e., $\prod_{i \in I} K_i \models \theta(\overline{X}, X_{k+1})$. But then as $\zeta(\overline{X})$ is $\exists x_{k+1} \cdot \theta(\overline{X}, x_{k+1})$, we have $\prod_{i \in I} K_i \models \zeta(\overline{X})$. ■

We can optimize the procedure of the theorem a little by handling blocks of bound variables in one step. An example of the optimized procedure applied to the sentence $\exists x\, y \cdot x \not\leq y \wedge y \not\leq x$ is illustrated in table 1 on page 14. This shows how

θ	Φ	$\theta_1, \ldots, \theta_p$
$x \leq y$	$a_1 = \mathbb{U}$	$x \leq y$
$x \not\leq y$	$a_1 \neq \mathbb{U}$	$x \leq y$
$y \leq x$	$a_1 = \mathbb{U}$	$y \leq x$
$y \not\leq x$	$a_1 \neq \mathbb{U}$	$y \leq x$
$x \not\leq y \wedge y \not\leq x$	$a_1 \neq \mathbb{U}$ $\wedge \quad a_2 \neq \mathbb{U}$	$x \leq y,$ $y \leq x$
$x \not\leq y \wedge y \not\leq x$	$a_1 \cup a_2 \neq \mathbb{U}$ $\wedge \quad a_1 \cup a_3 \neq \mathbb{U}$	$x \leq y \wedge y \leq x,$ $x \leq y \wedge y \not\leq x,$ $x \not\leq y \wedge y \leq x,$ $x \not\leq y \wedge y \not\leq x$
$\exists x\, y \cdot x \not\leq y \wedge y \not\leq x$	$\exists b_1\, b_2\, b_3\, b_4 \cdot\ b_1 \subseteq a_1$ $\wedge \quad b_2 \subseteq a_2$ $\wedge \quad b_3 \subseteq a_3$ $\wedge \quad b_4 \subseteq a_4$ $\wedge\ \mathsf{Part}(b_1, b_2, b_3, b_4)$ $\wedge \quad b_1 \cup b_2 \neq \mathbb{U}$ $\wedge \quad b_2 \cup b_3 \neq \mathbb{U}$	$\exists x\, y \cdot x \leq y \wedge y \leq x,$ $\exists x\, y \cdot x \leq y \wedge y \not\leq x,$ $\exists x\, y \cdot x \not\leq y \wedge y \leq x,$ $\exists x\, y \cdot x \not\leq y \wedge y \not\leq x$

Table 1: An example of the Feferman-Vaught procedure

the theorem can give us decision procedures: if the sentence is to be interpreted in a cartesian product of non-trivial dense total orders (say), we can apply a decision procedure to calculate $||\theta_1|| = ||\theta_2|| = ||\theta_3|| = \mathbb{U}$ and $||\theta_4|| = \emptyset$ and infer that θ holds in the direct product iff $\Phi(\mathbb{U}, \mathbb{U}, \mathbb{U}, \emptyset)$ holds in I. Ad hoc reasoning or the quantifier elimination procedure of theorem 3 then show that this is the case iff I has more than one element (as expected: a product of non-trivial total orders is not total unless there is only one factor).

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