The expected time to construct a binary search tree.

Let $T(n)$ denote the expected time to construct a tree with n nodes. Then $T(0) = 0$. If the root of the tree is the jth smallest node, then the left tree has $j-1$ nodes and the right one $n-j$ nodes. Also, when constructing the tree every node other than j is compared to j and inserted either to the left tree or to the right one. Thus

$$\begin{aligned} T(n) &= \frac{1}{n}\sum_{j=1}^{n}[(n-1)+T(j-1)+T(n-j)] \\ &= (n-1)+\frac{2}{n}\sum_{j=1}^{n-1}T(j) \\ nT(n) &= n(n-1)+2\sum_{j=1}^{n-1}T(j) \end{aligned}$$

$$\begin{aligned} (n-1)T(n-1) &= (n-1)(n-2)+2\sum_{j=1}^{n-2}T(j) \\ nT(n)-(n-1)T(n-1) &= n(n-1)-(n-1)(n-2)+2T(n-1) \\ nT(n) &= 2n-2+(n+1)T(n-1) \end{aligned}$$

Divide by $n(n+1)$:

$$\begin{aligned} \frac{T(n)}{n+1} &= \frac{2}{n+1}-\frac{2}{n(n+1)}+\frac{T(n-1)}{n} \\ &< \frac{2}{n+1}+\frac{T(n-1)}{n} \\ \frac{T(n)}{n+1} &< 2\sum_{j=1}^{n}\frac{1}{j+1}+\frac{T(0)}{1} \\ &= 2(H_n-H_1)+0 \le 2H_n, \end{aligned}$$

where $H_n = \sum_{i=1}^{n} 1/i$ is the *Harmonic Series*. Its asymptotic behavior is

$$H_n = \ln n + 0.57721... + O(\frac{1}{n}).$$

we get

$$\begin{aligned} \frac{T(n)}{n+1} &\le 2H_{n+1} = 2\ln(n+1)+O(1) \\ T(n) &\le 2(n+1)\ln(n+1)+O(n+1) = 2n\ln n + O(n) \\ &\approx 2.8854 n \log_2 n + O(n). \end{aligned}$$