Exercise 2 – Due 15.12.2005

Purpose

See limitations of the soundness gap amplification technique, using a counter example due to Andrej Bogdanov.

Definitions

For $G = (V, E)$ a d-regular graph, and $S \subset V$, let $e(S, \bar{S})$ be the number of edges crossing from S to $\bar{S} = V \setminus S$. Formally,

$$e(S, \bar{S}) = |(S \times \bar{S}) \cap E|.$$

G is called a (d, λ)-edge expander if it is d-regular and for all $S \subset V$ with density $\theta = |S|/n$ we have

$$\left|e(S, \bar{S}) - \theta(1-\theta)dn\right| \leq \lambda\sqrt{\theta(1-\theta)} \cdot n.$$

A *cycle* of length r in G is a sequence of vertices $v_0, \ldots, v_{r-1}$ such that $(v_i, v_{i+1 \bmod (r)}) \in E$. Let the *girth* of G be the minimal length of a cycle in G.

We use the following well Theorem about expander graphs (due to Lubotsky, Phillips and Sarnak).

Theorem *For infinitely many $d > 2$ and infinitely many n, there exists a $(d, 2\sqrt{d})$-edge expander on n vertices with girth $\geq \frac{2}{3}\log_d n$.*

Questions

Let G be a $(d, 2\sqrt{d})$-edge expander with girth $\geq \frac{2}{3}\log_d n$, as in the above mentioned Theorem. Let $\mathcal{G}$ be the constraint graph over graph G, with alphabet $\Sigma = \{0, 1\}$ and each edge constraint is an inequality constraint, i.e. for every edge $e = (u, v) \in E$ and any assignment $A : V \to \Sigma$, we have $C_e(A(u), A(v)) = \mathsf{accept}$ iff $A(u) \neq A(v)$.

Prove the following:

1. The initial soundness is large: $s(\mathcal{G}) \geq 1/2 - O(1/\sqrt{d})$.
2. Powering does not increase it significantly: If $n > d^{32a}$ then $s(\mathcal{G}^a) \leq 1/2$.

   (Hint: *(i)* Start with a random assignment to vertices. *(ii)* Extend it to a locally consistent assignment to balls of radius a, using the large girth of G. *(iii)* Measure the expected number of violated constraints in $\mathcal{G}^a$. *(iv)* Use linearity of expectation.)