INT. J. CONTROL, 2003, VOL. 76, NO. 5, 453–458

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On static feedback for the ℓ_1 and other optimal control problems

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Although ℓ_1-optimal linear state feedback controllers are known to be dynamic, it has been shown that suboptimal performance arbitrarily closed to optimal can be achieved by using a static non-linear feedback law. In this paper, this fact is established by using a novel approach which shows that the result is a natural consequence of elementary state-space theory. The approach is motivated by recent works in active vision systems, which have considered a state-feedback problem tightly connected with ℓ_1 optimization. This problem, which has independent interest, is discussed in some detail. The new formulation of the problem provides additional insight in ℓ_1 state-feedback. In particular, it leads naturally to some extensions which do not follow in a straightforward manner from previous works on the subject.

1. Introduction

It is well known that constant gain feedback can be used for solving optimal and suboptimal control problems with an $\mathcal{H}_2$ or an $\mathcal{H}_\infty$ objective. In other words, the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance achievable by a constant state-feedback cannot be improved upon by using a dynamic, linear or non-linear, time-varying or time-invariant controller (see, for instance, Doyle *et al.* 1989). Unfortunately, this remarkable property of constant gain feedback does not hold for other control objectives. For instance, Diaz-Bobillo and Dahleh (1992) presented an example showing that state feedback ℓ_1-optimal or slightly suboptimal controllers can be dynamic. By projecting this result one step forward, this suggested that the nice separation principle observed for $\mathcal{H}_2$ or $\mathcal{H}_\infty$, by which an optimal controller is formed by estimating in an optimal manner the states and then applying an optimal constant gain to the estimates, may not be true in the ℓ_1 setting.

The lack of optimality of constant gains for ℓ_1 control does not mean that one should abandon non-dynamic controllers when solving ℓ_1 optimization problems. Indeed, it was shown in Shamma (1993) that suboptimal performance arbitrarily close to the optimal can be achieved if one allows the feedback law to be a *non-linear, state dependent* gain, i.e. that the performance achievable by static feedback cannot be improved by including dynamics. The proof in Shamma (1993) is non-constructive, but Blanchini and Sznaier (1995 a) presented a memoryless variable structure controller achieving the desired performance, by building on some results obtained by Schweppe and others in the 70s (Schweppe 1973). An alternative algorithm more in the spirit of Schweppe (1973) was presented in Shamma (1994). More recently, Fialho and Georgiou (1997) presented two algorithms that allow the computation of ℓ_1 static feedback with a prescribed rate of exponential convergence. Although all these works provide algorithms which can in principle be implemented in practice, the resulting feedback laws are in general complicated, to the point that one could argue that they do not offer any advantage over dynamic feedback. As such, it is probably fair to say that, at least in the current status of implementation cost, the appeal of the results in Shamma (1993, 1994) and Blanchini and Sznaier (1995 a) lies more in the fact that a separation structure may exist for ℓ_1-optimal control rather than in the actual applicability.

The research presented in this paper evolves from the work of the second and third authors on two mode tracking systems (Rivlin *et al.* 1998). The problem considered there was to characterize the target set on which a dynamic controller can be initialized to work properly, whenever the plant is not assumed to be at rest for $t = 0$. This problem was formulated as a state feedback problem with non-zero initial state; the objective was to keep the ℓ_∞ norm of the output below a given bound assuming that the initial state satisfies some constraints and a disturbance input was bounded in an ℓ_∞ sense by some known constant. The memoryless variable structure controller described in Blanchini and Sznaier (1995 a) could have been used for this problem; as shown in the next section, if the states of an appropriate linear dynamic controller are properly initialized, then the desired level of performance can also be achieved using the original dynamic linear feedback law. Optimality properties of static feedback for ℓ_1 control follow from this observation in a straightforward manner, whenever the input disturbance is assumed to be bounded by a known constant. If the input is

Received 1 October 1999. Revised and accepted 1 November 2002.

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International Journal of Control ISSN 0020–7179 print/ISSN 1366–5820 online © 2003 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/0020717031000087644

assumed to be bounded by an *unknown* constant, then some additional work is required to obtain the same result.

The new approach discussed in this paper provides further insight into the ℓ_1 static problem, by showing that the extremum property of static non-linear feedback can actually be interpreted in terms of basic state-space theory. As a by-product of the new insight into the problem, the simple derivation can be extended to show the optimality of static feedback for the *continuous* and the *sampled-data* cases. Although the former was previously considered in Blanchini and Sznaier (1995 a), the result for the latter is new and appears to be challenging for other approaches (Blanchini and Sznaier 1995 b). Moreover, it is possible to show that static feedback controllers achieve the same performance as more complex linear or non-linear controllers; the main conclusion is that these optimality properties are a natural consequence of basic state-space theory.

The notation used in this paper is fairly standard; for a vector $x \in \mathbb{R}^m$, the notation $|x| \le 1$ should be understood entry by entry. For a vector $x \in \mathbb{R}^n$, $\|x\|_\infty = \max_i |x_i|$, $\|x\|_1 = \sum_i |x_i|$. Given a sequence $x = \{x(t)\}$, the ℓ_1 and ℓ_∞ norms are

$$\|x\|_\infty = \sup_t \|x(t)\|_\infty$$
$$\|x\|_1 = \sum_t \|x(t)\|_1$$

Also, the projection operator $\mathcal{P}$ is defined as

$$(\mathcal{P}_t(x))(\tau) \doteq \begin{cases} x(\tau) & 0 \le \tau \le t \\ 0 & t < \tau \end{cases}$$

To simplify the notation, no special symbol is used for sequences, vector and the associated norms. The meaning can be clarified from the context.

The approach taken here is elementary, inspired by some of the driving ideas in Blanchini and Sznaier (1995 a), from where the following definition is taken.

Definition 1 (reachable set)**:** Consider a system of the form

$$x(t+1) = f(x(t), w(t))$$

Given a sequence $w = \{w(0), w(1), \ldots, \}$, denote by $\phi(t, x_0, w(\cdot))$ the corresponding trajectory. The origin-reachable state set $\mathcal{R}$ is defined as

$$\mathcal{R} \doteq \{x : x = \phi(t, 0, w), t < \infty,\ |w(\tau)| \le 1 \text{ for every } 0 \le \tau \le t\}$$

2. Problem formulation

Consider the discrete-time state-space system

$$\left.\begin{aligned} x(t+1) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= Cx(t) + D_{11} w(t) + D_{12} u(t) \end{aligned}\right\} \qquad (1)$$

with $x(0) = 0$. The state $x(\cdot) \in \mathbb{R}^n$, while $w(\cdot) \in \mathbb{R}^{m_w}$, $u(\cdot) \in \mathbb{R}^{m_u}$. All signals are defined for $t \ge 0$ and are assumed to be 0 for negative t. Later on this restriction will be relaxed, and signals for negative (but finite) t's will be considered.

Two related optimal-control problems are of interest. In the first, the disturbance signal is known *a priori* to be norm-bounded.

Problem 1 (bounded input)**:** Given $\gamma > 0$, compute if possible a feedback control law $u = \mathcal{F}(x)$ such that the closed-loop system be stable and

$$\sup_{\|w\|_\infty \le 1} \|z\|_\infty < \gamma \qquad (2)$$

It is worth stressing that the control law is *static*, in the sense that the control action $u(\tau)$ at time τ is a function of $x(\tau)$ at time τ only. Problem 1 is motivated by the work of the second and third authors in the control of an active vision system (Rivlin and Rotstein 2000), but appears to be of interest well beyond its original application. This is so because controllers are ultimately applied in systems which behave linearly only for relatively small disturbances.

The second problem of interest is obtained when the bound on the norm of the disturbance is relaxed.

Problem 2: Given $\gamma > 0$, compute if possible a feedback control law $u = \mathcal{F}(x)$ such that the closed-loop system be stable and

$$\sup_{\|w\|_\infty < \infty} \frac{\|z\|_\infty}{\|w\|_\infty} < \gamma \qquad (3)$$

Consider next a finite dimensional linear time-invariant controller solving the γ-suboptimal state-feedback ℓ_1 problem. Suppose that the controller has a state-space representation of the form

$$\left.\begin{aligned} \hat{x}(t+1) &= A_K \hat{x}(t) + B_K x(t) \\ u(t) &= C_K \hat{x}(t) + D_K x(t) \end{aligned}\right\} \qquad (4)$$

with $\hat{x} \in \mathbb{R}^{n_k}$, $\hat{x}(0) = 0$. The closed loop system can then be written as

$$\left.\begin{aligned} x_{cl}(t+1) &= A_c x_{cl}(t) + B_c w(t) \qquad (5) \\ z(t) &= C_c x_{cl}(t) + D_c w(t) \qquad (6) \end{aligned}\right\}$$

where $x_{cl}(t) = [x'(t)\ \ \hat{x}'(t)]'$ and

$$A_c = \begin{bmatrix} A + B_2 D_K & B_2 C_K \\ B_K & A_K \end{bmatrix}, \qquad B_c = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

$$C_c = [C + D_{12} D_K \quad 0], \qquad D_c = D_{11}$$

The fact that the controller is ℓ_1 sub-optimal, implies that the closed-loop system is internally stable and $\sup(\|z\|_\infty/\|w\|_\infty) < \gamma$ (see Dahleh and Diaz-Bobillo 1994 for details). In the remainder of this paper, the sup-optimality level will be set to $\gamma = 1$; this is without loss of generality, after appropriately scaling the objective function.

The controller (4) is in general *dynamic*; it will be shown next that a *non-linear, static* controller exists achieving the same performance.

3. Main result

For each $s > 0$, let $\mathcal{C}_s$ denote the matrix

$$\mathcal{C}_s \triangleq [B_c \quad A_c B_c \quad \cdots \quad A_c^{s-1} B_c] \tag{7}$$

and $J = [I_n \ 0]$, $\hat{J} = [0 \ I_{n_k}]$. Then, assuming $x(0) = \hat{x}(0) = 0$, the state x at time $t = s$ is given by

$$x(s) = J\mathcal{C}_s \begin{bmatrix} w(s-1) \\ w(s-2) \\ \vdots \\ w(0) \end{bmatrix}$$

Let $\mathcal{R} \subset \mathbb{R}^{n+n_k}$ denote the origin-reachable state for the closed-loop interconnection of the plant (1) with the controller (4), and $\mathcal{R}_1 \subset \mathbb{R}^n$ denotes the projection over the states of the plant, namely $\mathcal{R}_1 = J\mathcal{R}$.

Assume now that $x(0) = x_0 \neq 0$, with $x_0 \in \mathcal{R}_1$. Then the controller (4) initialized with $\hat{x}_K(0) = 0$, can in general no longer guarantee $\|z\|_\infty \leq 1$. As claimed next, it is nevertheless possible to select $\hat{x}(0) = \hat{x}_0$ in such a way that the interconnection of (1) and (4) with initial state

$$\begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix}$$

still solves Problems 1 and 2. To establish this fact, two auxiliary optimization problems are required. With respect to Problem 1, fix $s > 0$ and consider the solution to the quadratic programming (QP) problem

$$\left.\begin{aligned} &\min_{\text{s.t.}} (w^\nu)^{\mathrm{T}} P (w^\nu) \\ &\qquad J\mathcal{C}_s w^\nu = x_0 \\ &\qquad |w^\nu(i)| \leq 1 \quad i = 0, \ldots, s-1 \end{aligned}\right\} \tag{8}$$

for some positive definite P.

The fact that $x_0 \in \mathcal{R}_1$ implies that (8) has a solution for some $\bar{s} < \infty$. Therefore, if the problem has no feasible solution for a given s, then one must replace $s \to s+1$ and attempt to solve again. For Problem 2, consider

$$\left.\begin{aligned} &\min_{\text{s.t.}} \varepsilon \\ &\qquad J\mathcal{C}_s w^\nu = x_0 \\ &\qquad |w^\nu(i)| \leq \varepsilon, \quad i = 0, \ldots, s-1 \end{aligned}\right\} \tag{9}$$

The solution to (9) gives a worst-case disturbance from an ℓ_1 perspective.

Note that w^ν solving (9) will generically depend on s, and so will ε, i.e. $\varepsilon = \varepsilon_s$, with $\varepsilon_{s+1} \leq \varepsilon_s$. To circumvent this difficulty, the following assumption can be made.

Assumption 1: *The closed-loop transfer matrix has a finite impulse response of length $\bar{s}$.*†

Let $\mathbf{w}_s^\nu = [w_s^\nu(s-1)^{\mathrm{T}} \ w_s^\nu(s-2)^{\mathrm{T}} \ \cdots \ w_s^\nu(0)^{\mathrm{T}}]^{\mathrm{T}}$ denote a solution to (8) or (9). In the case of Problem 1, the positive definiteness of P implies that the solution is unique. For Problem 2 the solution may not be unique, but one could pick a solution uniquely by choosing an ordering criteria for the solutions. In any of these two cases, $\mathbf{w}_s^\nu$ can be thought of as a function of x_0 *alone*, i.e. $\mathbf{w}_s^\nu = \mathcal{N}[x_0]$.

Consider the initial state for the controller

$$\begin{aligned} \hat{x}_0 &= \hat{J}\mathcal{C}_s \mathbf{w}_s^\nu \\ &= \hat{J}\mathcal{C}_s \mathcal{N}[x_0] = \hat{\mathcal{K}}[x_0] \end{aligned} \tag{10}$$

which is a non-linear function of x_0. The following lemma shows that $\hat{\mathcal{K}}[x_0]$ is an adequate initial state for the dynamic controller.

Lemma 1: *Consider the interconnection of the plant* (1) *with the controller* (4), *such that if $x(0) = \hat{x}(0) = 0$ then $\|w\|_\infty \leq 1$ yields $\|z\|_\infty < 1$. Consider, also, a vector $x_0 \in \mathcal{R}_1$. Then the same interconnection for $x(0) = x_0$ and $\hat{x}(0) = \hat{\mathcal{K}}[x_0]$ is such that $\|w\|_\infty \leq 1$ yields $\|z\|_\infty < 1$.*

Proof: The idea of the proof is to assume that the plant has been driven to the state x_0 by a disturbance active from $t = -s$, and finding the corresponding state of the controller. More specifically, augment the disturbance sequence by defining

$$\begin{bmatrix} w(-s) \\ w(-s+1) \\ \vdots \\ w(-1) \end{bmatrix} = \mathbf{w}_s^\nu$$

where $\mathbf{w}_s^\nu$ solves Problems 1 or 2 above. Now drive the closed-loop interconnection from time $t = -s$, assuming $x(-s) = \hat{x}(-s) = 0$. By the assumption on the closed-loop interconnection, the resulting output $z(\cdot)$ will be such that (with a small abuse of notation) $\|z\|_\infty < 1$ for $\|w\| \leq 1$. Moreover, $x(0) = x_0$ and $\hat{x}(0) = \hat{x}_0$. Since the behaviour of the closed-loop system for $t \geq 0$ is determined by the state-space model and the initial states $x(0)$, $\hat{x}(0)$, this concludes the proof. □

† Note that by McDonald and Pearson (1991, Theorem 9), this assumption is without loss of generality.

The problem solved in Lemma 1 constitutes the original motivation for the present research. Indeed, in the active vision problem considered in Rivlin and Rotstein (2000) and Rivlin *et al.* (1998), a control mechanism is presented composed of two control laws which are switched on or off according to the current status of the system. Lemma 1 shows how to choose the initial state of one of the controllers after each switching, so as to guarantee the success of the overall control law.

The optimality proof of ℓ_1 static feedback can now be obtained by recursing on the previous lemma.

Theorem 1: *There exists a dynamic internally stabilizing controller as* (4) *solving Problem 1 or 2 only if there exists a static feedback control law achieving the same performance.*

Proof: Enforcing Assumption 1, take $s = \bar{s}$. Suppose first that the controller (4) is such that

$$\|w\|_\infty \leq 1 \quad \rightarrow \quad \|z\|_\infty < 1 \tag{11}$$

The claim is that the static feedback control defined by

$$u_s(t) \doteq C_K \hat{\mathcal{K}}[x(t)] + D_K x(t) = \mathcal{K}[x(t)] \tag{12}$$

with $\mathcal{K}$ as defined in (10), also achieves this performance level. To show this, it suffices to prove that if $x(t) \in \mathcal{R}_1$, then also $x(t+1) \in \mathcal{R}_1$. From Lemma 1, $x(t) \in \mathcal{R}_1$ implies that

$$\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \in \mathcal{R}$$

From (5), the sub-optimality of controller (4) and the assumption $\|w\|_\infty \leq 1$, the state

$$x(t+1) = (A + B_2 D_K)x(t) + B_2 C_K \hat{x}(t) + B_1 w(t) \tag{13}$$

is such that $x(t+1) \in \mathcal{R}_1$. Note that (13) can be written as

$$\begin{aligned} x(t+1) &= Ax(t) + B_2(C_K \hat{x}(t) + D_K x(t)) + B_1 w(t) \\ &= Ax(t) + B_2 u_s(t) + B_1 w(t) \end{aligned}$$

where $u_s(t)$ is as in (12). This shows that the control law is suboptimal for Problem 1.

Suppose now that the control law is not optimal for Problem 2. Then there exists a w with $\|w\|_\infty = \delta$ such that when using the control law u_s, $\|z\|_\infty \geq \delta$. By linearity of the plant, and the definition of u_s, this implies that there exists a w_n with $\|w_n\|_\infty = 1$ such that $\|z\|_\infty \geq 1$. But this contradicts the conclusion of the previous paragraph, thus concluding the proof. □

Remark 1: The assumption that the closed-loop system has finite impulse response of length $\bar{s}$ can be relaxed; the resulting state-feedback control law is in general time-varying, since it depends of the current time t. Now, for each finite t, the cost of computing the control action $u_s(t)$ is essentially given by the cost of solving the corresponding finite optimization program, this cost may become too expensive for a large t, and hence the approach may become non-implementable independently of the sampling interval.

Remark 2: It is clear that $\mathcal{K}[0] = 0$, since the 0-vector solves the linear programming problems. Moreover, continuity of the solution to perturbed linear programming problems (Franklin 1980, p. 69) implies that the static controller is uniformly continuous.

The fact that the closed-loop system with controller $\mathcal{K}(\cdot)$ is ℓ_∞ stable does not imply that it is asymptotically stable. The following lemma shows that, under a mild assumption, exponential stability can be actually guaranteed.

Lemma 2: *Let*

$$w = \{w(0),\ w(1),\ \ldots,\ w(t-1),\ 0,\ 0,\ \ldots\}$$

and assume that B_1 is square and invertible. Then the state trajectory $\phi(t, 0, w)$ converges exponentially to zero as $t \to \infty$.

Proof: It is assumed again that the closed-loop system has finite impulse response

$$\begin{aligned} J\mathcal{C} &= [B_1 \,|\, JA_c B_c \quad \cdots \quad JA_c^{\bar{s}-1} B_c] \\ &= [B_1 \,|\, J\mathcal{C}_2] \end{aligned}$$

For a given t and a state $x(t)$ of the plant, the solution $\mathbf{w}_t^\nu$ to the optimization (9) verifies

$$x(t) = J\mathcal{C} \begin{bmatrix} w_t^\nu(t-1) \\ w_t^\nu(t-2) \\ \vdots \\ w_t^\nu(t-1) \end{bmatrix}$$

with $\max_i |w_t^\nu(i)| = \varepsilon$. Since $w(t) = 0$, it is straightforward to verify that

$$x(t+1) = J\mathcal{C} \begin{bmatrix} 0 \\ w_t^\nu(t-1) \\ \vdots \\ w_t^\nu(t-l-1) \end{bmatrix} \tag{14}$$

so that $\max_i |w_{t+1}^\nu(i)| \leq \varepsilon$, with w_{t+1}^ν denoting the entries of the optimal solution $\mathbf{w}_{t+1}^\nu$. Write $x(t+1) = x_1(t+1) + x_2(t+1)$, where for some $\delta \in (0, 1)$

$$x_1(t+1) = (1-\delta)x(k+1) = J\mathcal{C} \begin{bmatrix} 0 \\ w_k^\nu(k-1) \\ \vdots \\ w_k^\nu(k-l-1) \end{bmatrix} (1-\delta)$$

$$x_2(t+1) = \delta x(k+1) = B_1 z$$

From (14), the norm of $z = \delta B_1^{-1} x(k+1)$ can be bounded by

$$\|z\|_\infty \le \delta \|B_1^{-1} J\mathcal{C}\|_1 \varepsilon = \delta\lambda\varepsilon$$

with $\lambda \doteq \|B_1^{-1} J\mathcal{C}\|_1$. Then

$$\left\| \begin{bmatrix} z \\ w_t^\nu(t-1) \\ \vdots \\ w_t^\nu(t-l-1) \end{bmatrix} \right\|_\infty \le \max\{\delta\lambda\varepsilon,\ (1-\delta)\varepsilon\}$$

To find the minimum upper bound, take $\delta\lambda\varepsilon = (1-\delta)\varepsilon$, which gives $\delta = 1/(1+\lambda)$ and implies

$$\|\mathbf{w}_{t+1}^\nu\|_\infty \le \frac{\lambda}{1+\lambda} \|\mathbf{w}_t^\nu\|_\infty$$

Repeating this argument for increasing t concludes the proof. □

The assumption about the invertibility of B_1 is in principle without loss of generality as explained in Shamma (1993), which involves enlarging and perturbing B_1 so that it becomes invertible, without increasing the induced norm by much. Unfortunately, this results in a very slow guaranteed exponential decay rate; a similar drawback also holds true for the technique in Blanchini and Sznaier (1995 a). For Problem 1, one could choose $P = \mathcal{C}_s^{\mathrm{T}} R \mathcal{C}_s$, where $R > 0$ denotes the unique solution to the discrete-time Lyapunov equation

$$R - A_c^{\mathrm{T}} R A_c = Q$$

for some $Q > 0$. Then the closed-loop system will be exponentially stable, with a guaranteed exponential decay rate of

$$\left(1 - \frac{\lambda_{\min}(Q)^2}{\lambda_{\max}(R)^2}\right)^{1/2}$$

which depends only on the parameters of the linear closed-loop system. Simulation results suggest that $\mathcal{K}(\cdot)$ provides exponential stability also for Problem 2, and establishing this fact is the subject of current research.

Remark 3: For Problem 1, similar arguments may be used if (1) and (4) are a continuous-time plant and controller respectively. Searching for a feasibility solution for the optimization (8) can be reformulated as a time-optimal control problem (Lee and Markus 1967). Unfortunately, this procedure cannot be applied in practice, since it involves solving an optimal control problem continuously in time.

Remark 4: Perhaps worthier of note, sub-optimality of static feedback is also true if the plant (1) is continuous and the controller (4) is discrete and is connected to the plant via sampling and hold devices. Moreover, since the performance of a continuous time dynamic controller can be approximated arbitrarily close by a sampled-data scheme (Dullerud and Francis 1992), this provides an approximate solution to the continuous-time case. As opposed to the approach on the previous remark or in Blanchini and Sznaier (1995 a), the resulting controller may be implementable, if the corresponding optimization problems can be solved fast enough.

Remark 5: In the proof of Theorem 1 it is assumed that the dynamic controller is linear, time-invariant and finite dimensional. However, a careful inspection of the proof in Lemma 1 shows that the same is true if the dynamic controller is non-linear, time varying or infinite dimensional, since the main point is that one should be able to produce the current state of the plant by means of a fictitious disturbance, and this is independent of the feedback interconnection (i.e. the class to which the controller belongs).

Corollary 1: *There exists a possibly non-linear, time-varying, infinite dimensional dynamic stabilizing controller as* (4) *such that* $\|z\|_\infty \le 1$ *for* $\|w\|_\infty \le 1$ *only if there exists a static feedback control law achieving the same performance.*

Remark 6: If instead of an induced ℓ_∞/ℓ_∞ the performance objective is an induced ℓ_∞/l_q or ℓ_p/ℓ_∞ norm, the same construction will hold after reformulating problems (8) and (9).

In the setup outlined in the previous remark, Theorem 1 can be re-stated as follows.

Theorem 2: *There exists a dynamic internally stabilizing controller as* (4) *such that* $\|z\|_q \le 1$ *for* $\|w\|_\infty \le 1$ *(respectively* $\|z\|_\infty \le 1$ *for* $\|w\|_p \le 1$*) only if there exists a static feedback control law achieving the same performance.*

Theorem 2 states that the fact that static state feedback suffices to achieve the optimal performance does not originate from the particular case of ℓ_1-optimal control but rather is a property verified by a large class of optimal control problems.

4. Example

Consider Example 2 in Diaz-Bobillo and Dahleh (1992), where the plant

$$P(z) = \frac{2 - 2.5z^{-1} + z^{-2}}{(z^{-1} - 0.2)(23 - 2.5z^{-1} + z^{-2})}$$

is considered. It is assumed that the states are available for feedback. The controller for this plant is a dynamic

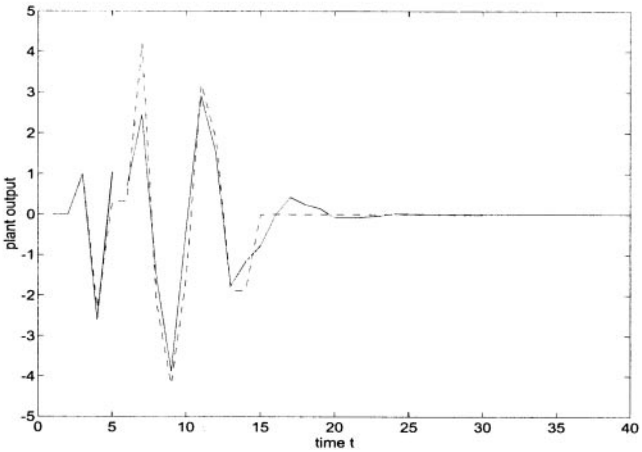

Figure 1. Simulation for the linear system worst-case input.

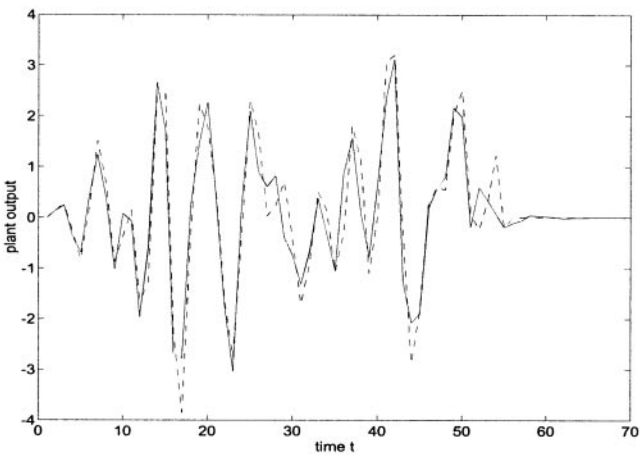

Figure 2. Simulation for a random, low-passed input of finite length.

transfer matrix with two states, three inputs and one output and yields an ℓ_1-norm of slightly more than 4.2. Figures 1 and 2 show the output of the system for two different signals: the linear controller worst case signal and a random low-pass filtered signal respectively. The output corresponding to the linear controller is shown with dashed line while the one of the non-linear static is shown with full line. Note that the static controller provides in both cases a better performance (8% better in the case of the worst case input).

5. Conclusions

In this paper, the sub-optimality of ℓ_1 static controllers for state feedback has been discussed in terms of basic state-space theory. First, the problem of computing initial conditions for a dynamic controller achieving an ℓ_1 objective was discussed. It was shown that an initial state can be computed by solving a linear or a quadratic programming problem. Then, it was shown that sub-optimality of static controllers follows naturally from the initial state construction, by applying a recursive argument. Finally, several extensions of the main result were outlined.

From the constructions in this paper one could conjecture that sub-optimality of static state feedback does not depend on the specifics of the ℓ_1 problem, but actually holds for a much larger class of performance objectives. Consequently, if a separation-like structure exists for ℓ_1 control, it probably also holds for more general ℓ_p/ℓ_p optimization problems.

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