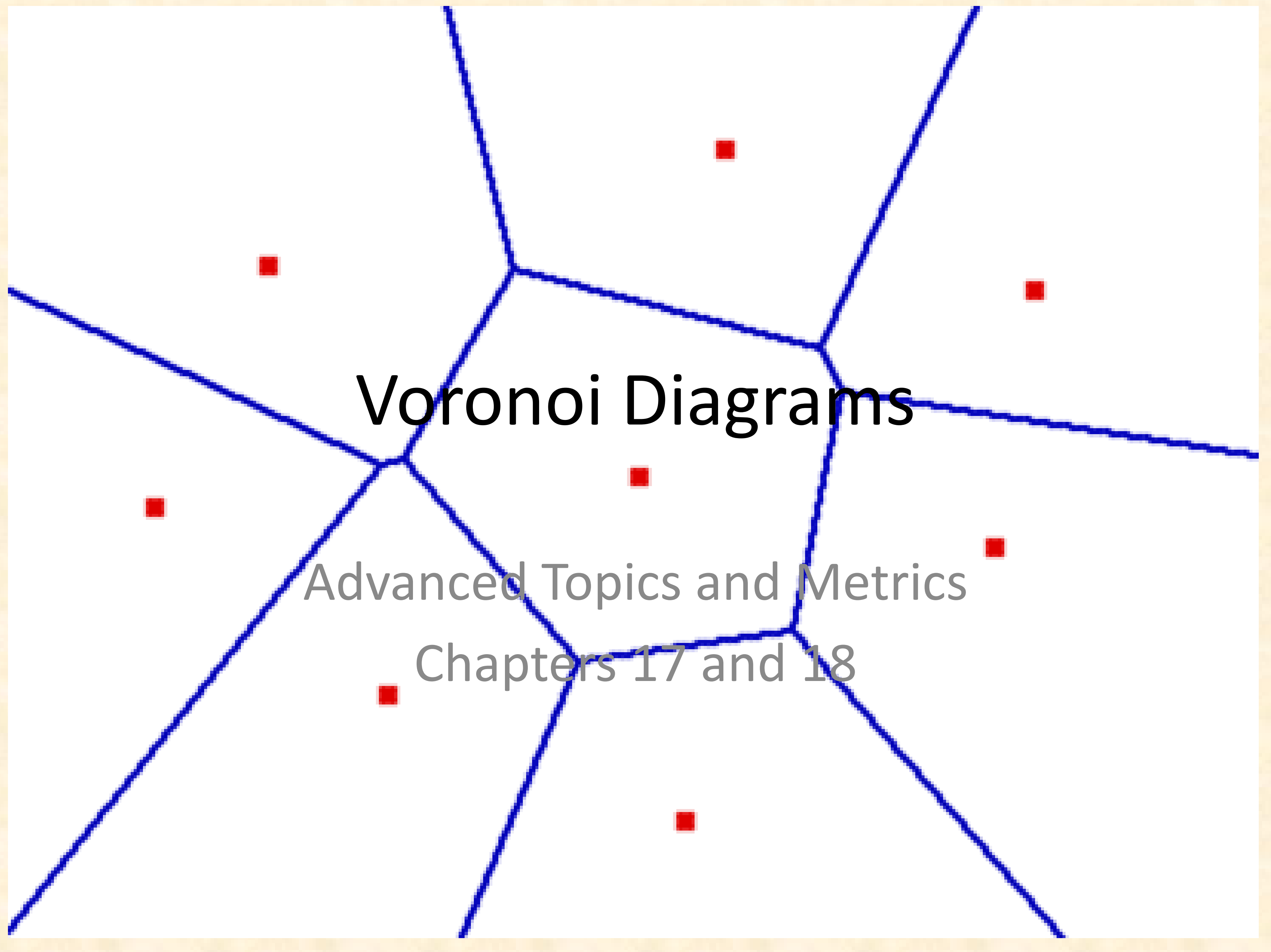


Voronoi Diagrams

Advanced Topics and Metrics

Chapters 17 and 18

Definitions

- ***M:*** a set of *n* points in $\mathbb{E}^d$, $M_1, \ldots, M_n$, called sites.

$$V(M_i) = \{X \in \mathbb{E}^d : \delta(X, M_i) \le \delta(X, M_j), j \ne i\}$$

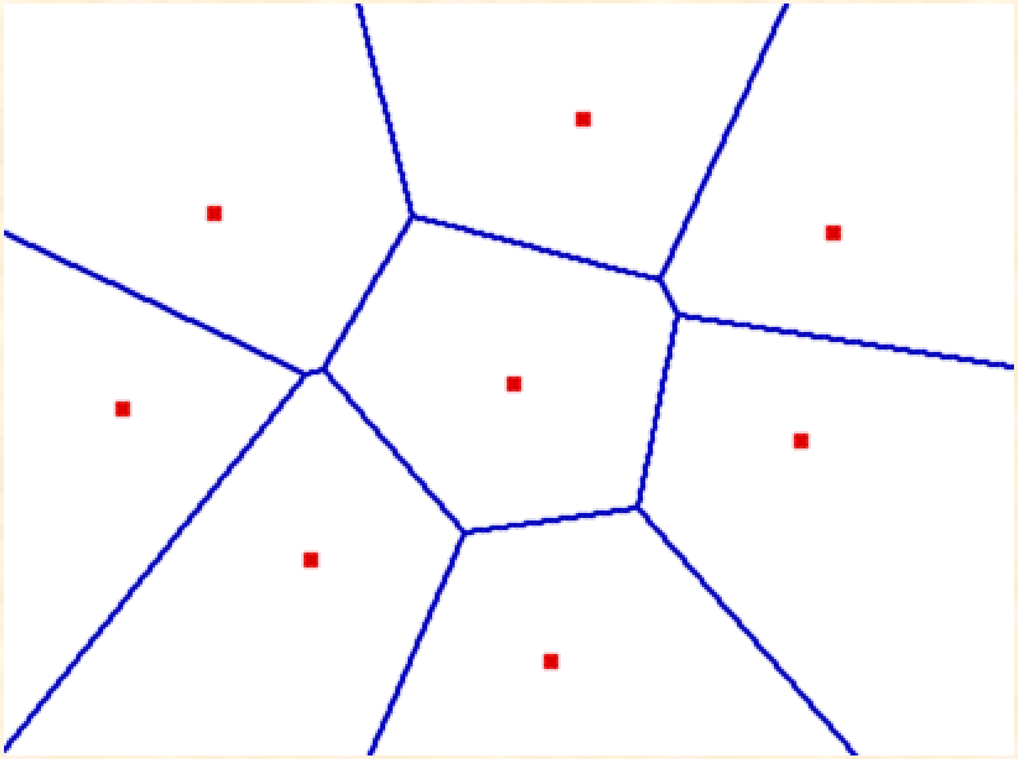
- $V(M_i)$: The intersection of a finite number of closed half-spaces, perpendicular bisectors of point pairs.
- ***Vor***(***M***): A complex of $V(M_i)$ cells.
- L_2-*general* position: no sphere contains *d+2* sites on its boundary

Diagram as regions of Sphere Centers

- $V(M_i)$ as the set of the center of spheres
  - Boundary of contains M_i, interior contains no M_j.
  - When d=2
    - Voronoi Edges: centers of spheres containing exactly 2 points on their edge
    - Voronoi Verteces: center of spheres containing 3 or more points on the edge (in L_2-*general*, exactly 3).

Voronoi Diagrams and Polytopes

- By treating points as spheres, we can describe the diagram as a polytope in *d+1* dimensions

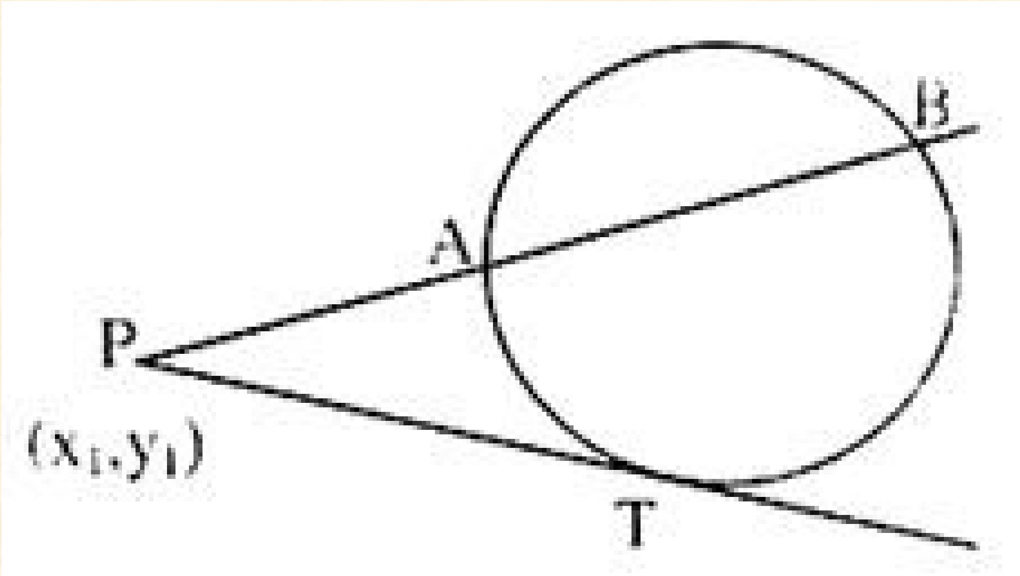


Power of a point w.r.t. a sphere

- $\mathbb{E}^d$: D dimension Euclidean space, origin *O*.
- *X*: a point in $\mathbb{E}^d$.
- Σ: a sphere in $\mathbb{E}^d$ with center *C*, radius *r*.

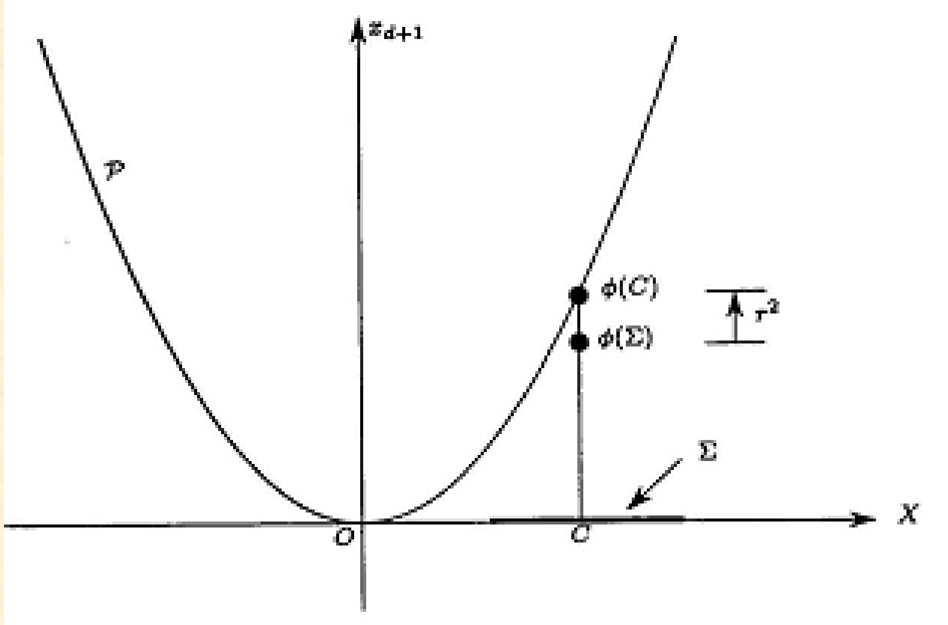
  $\Sigma(X)=0$, where: $\Sigma(X)=|XC|^2-r^2$ (17.1)
- Interior of the sphere: $\mathrm{int}(\Sigma)=\{X: \Sigma(X)<0\}$
- Exterior of the sphere: $\mathrm{ext}(\Sigma)=\{X: \Sigma(X)>0\}$

- σ : Power w.r.t. origin: $\Sigma(O) = \sigma = |C|^2 - r^2$ (17.2)
- If D is any line containing X, and intersecting Σ at A and B $\Sigma(X) = |XA| \cdot |XB|$ (17.3)
- When D is tangent at T, $\Sigma(X) = XT^2$ (17.4)



Representation of Spheres

- Let ϕ be a mapping that takes a sphere Σ in $\mathbb{E}^d$ with center *C*, power w.r.t. *O* σ to the point $\phi(\Sigma)=(C, \sigma)$ in $\mathbb{E}^{d+1}$.
- Allows us to treat spheres in $\mathbb{E}^d$ as points in $\mathbb{E}^{d+1}$



Embedding Spheres as Points

- Embed $\mathbb{E}^d$ as the hyperplane in $\mathbb{E}^{d+1}$, with $x_{d+1}=0$.
  - x_{d+1} is the vertical direction, positive is up
  - Each point in $\mathbb{E}^d$ is a sphere of radius $|C|$.
- X denotes a point in $\mathbb{E}^d$ with vector $(x_1,\ldots,x_d)$,
- $\underline{X}$ denotes a point in $\mathbb{E}^{d+1}$, with vector $(x_1,\ldots,x_{d+1})$.
- Projection means a vertical projection.

Homogeneous Coordinates and Matrix Notation

- $\boldsymbol{X}=(x_1,\ldots,x_d,t)$, $\underline{\boldsymbol{X}}=(x_1,\ldots,x_{d+1},t)$
- We can then rewrite the equation of the sphere, Σ, as $\boldsymbol{X}\boldsymbol{\Sigma}\underline{\boldsymbol{X}}^t = 0$, with

$$\boldsymbol{\Sigma} = \begin{pmatrix} I_d & -C^t \\ -C & \sigma \end{pmatrix}$$

Paraboloid *P*

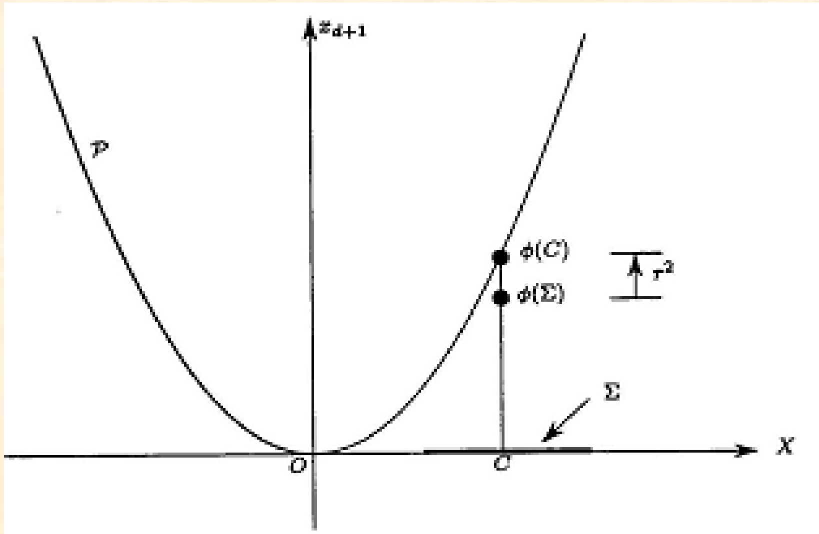
- ϕ maps points in $\mathbb{E}^d$, as spheres of radius 0, to a paraboloid of revolution, *P,* with vertical axis and equation:

$$x_{d+1} = \sum_{i=1}^{d} x_i^2 = X \cdot X$$

- In homogenous coordinates:

$$\underline{\boldsymbol{X}}\Delta_P \underline{\boldsymbol{X}}^t = 0$$

$$\Delta_P = \begin{pmatrix} I_d & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}$$



Paraboloid *P*

- Real spheres: non-negative radius; on or below this paraboloid
- Imaginary spheres: negative radius; above this paraboloid.

Polarity

- Any hyperplane ***H*** in a projective space has a homogeneous equation of the form

$$\boldsymbol{H} = \{\boldsymbol{X} : \sum_{i=1}^{d+1} h_i x_i = 0\}$$

- Let *S* be the matrix $S = \begin{pmatrix} I_d & 0 \\ 0 & -1 \end{pmatrix}$
- We let ***H**** be the point $(h_1, \ldots, h_d, -h_{d+1})$.

$$\boldsymbol{H} = \{\boldsymbol{X} : \boldsymbol{H} * S\boldsymbol{X}^t = 0\}$$

Polarity

- Chapter 7.3
- The *polar set, A**: Given a point *A*,

  $A^* = \{X : \langle X, A\rangle = 1\}$

  - A^* the polar hyperplane of A
  - A is the pole of A^*
- Polarity is the duality that connects points and planes.

Polarity w.r.t. *Q*

- Consider the quadric Q in $\mathbb{E}^{d+1}$, defined by the homogeneous equation: $Q(\underline{\boldsymbol{X}}) = \underline{\boldsymbol{X}}\Delta_Q\underline{\boldsymbol{X}}^t = 0$
- Points *X* and *Y* are conjugates w.r.t. *Q* if
  $Q(\underline{\boldsymbol{X}},\underline{\boldsymbol{Y}}) = \underline{\boldsymbol{X}}\Delta_Q\underline{\boldsymbol{Y}}^t = 0$
- $A^* = \{\, \underline{X} : \underline{\boldsymbol{A}}\Delta_Q\underline{\boldsymbol{X}}^t = 0 \,\}$,
- Polarity w.r.t. *Q* maps hyperplane $\underline{H}$ to the pole $\underline{H}^*$

Polarity w.r.t. *P*

- $A^* = \{\underline{X} : \underline{\boldsymbol{A}}\Delta_Q \underline{\boldsymbol{X}}^t = 0\}$
- If $Q=P$, and $\phi(\Sigma)=(C,\sigma)$, $\phi(\Sigma)^*$ can be rewritten as $x_{d+1} = 2C \cdot X - \sigma$
- Since polarity preserves incidences: for point $\underline{X}$ and hyperplane $\underline{H}$,

$$\underline{X} \in \underline{H} \text{ iff } \underline{H}^* \in \underline{X}^*$$

  - This also preserves the side of the parabola

Orthogonal Spheres

- Σ_1, Σ_2 (centers and radii C_1, C_2, r_1, r_2) are *orthogonal* if

$$\Sigma_1(C_2) = r_2^2$$
$$\Sigma_2(C_1) = r_1^2$$

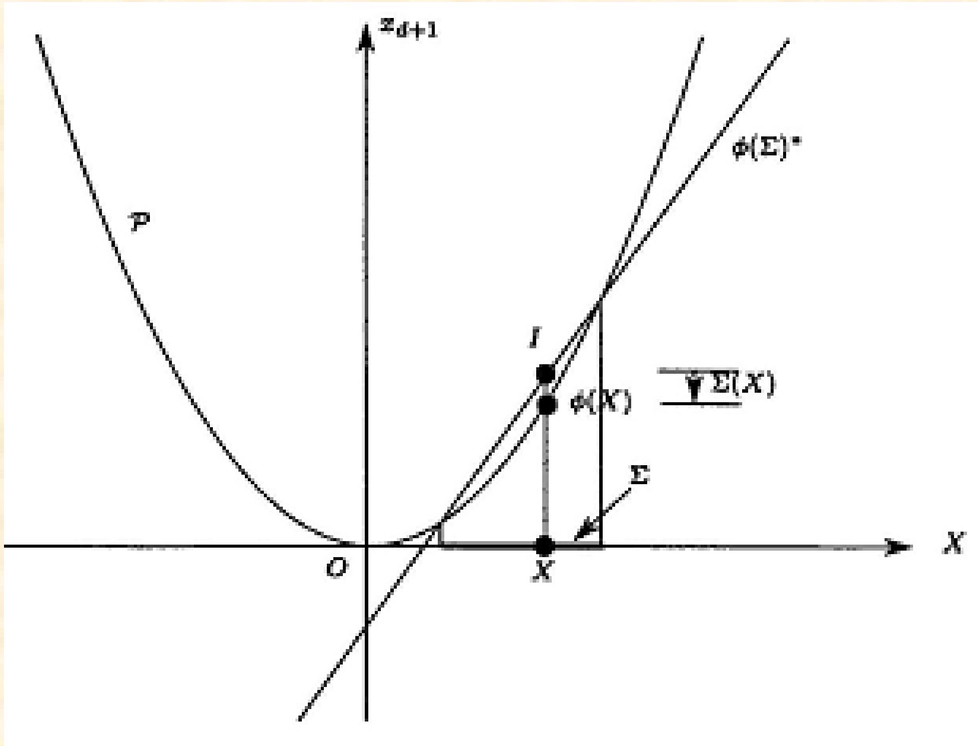
- We can rewrite these equivalent equations as

$$C_1 \cdot C_2 - \frac{1}{2}(\sigma_1 + \sigma_2) = 0$$

- This shows that 2 spheres are orthogonal if their mappings are conjugate w.r.t. *P*.

Lemma 17.2.1

- The set of spheres in $\mathbb{E}^d$ that are orthogonal to a given sphere is mapped by φ to the polar hyperplane φ(Σ)*

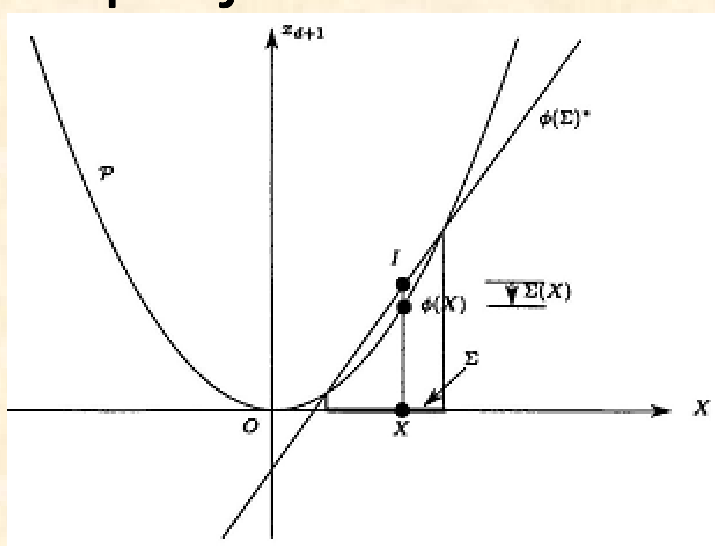


Paraboloid and Hyperplane

- The set of spheres that passing through X = the set of spheres that intersect sphere with $r=0$, centered at X.
- Image of these spheres is $\phi(X)^*$
- $\phi(X)^*$ is tangent to P, since X is only sphere with $r=0$ orthogonal to X

Lemma 17.2.2

- Let Σ be a sphere in $\mathbb{E}^d$.
- The intersection of ϕ(Σ)* with *P* is the image under ϕ of the set of spheres with radius 0 that are orthogonal to Σ, namely Σ itself
- ϕ(Σ)* ∩ *P* in $\mathbb{E}^{d+1}$ projects onto Σ in $\mathbb{E}^d$.



Lemma 17.2.2

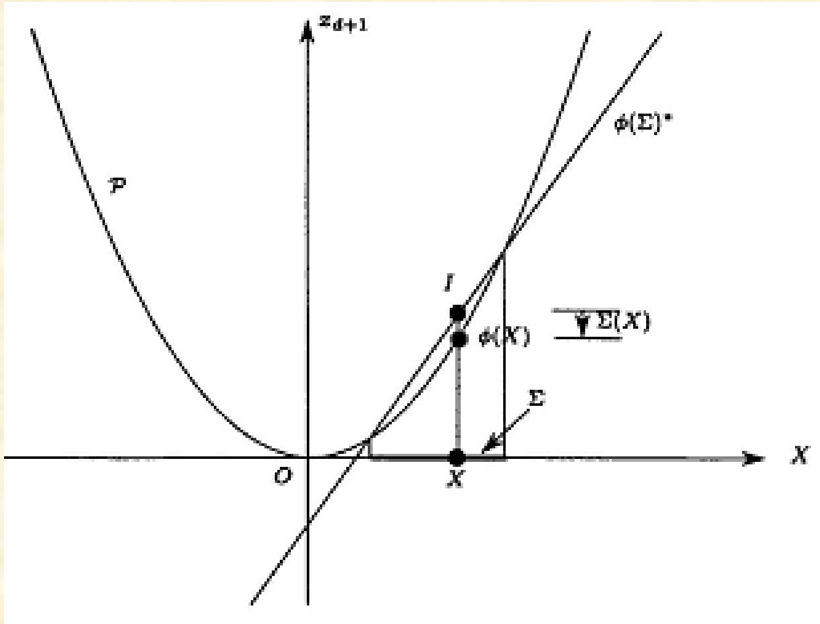
- $P \cap \underline{H}$ projects onto $\mathbb{E}^d$ as a sphere $\phi^{-1}(\underline{H}^*)$, centered at the vertical projection of $\underline{H}^*$
- The points of sphere Σ lifted onto P in $\mathbb{E}^{d+1}$ belong to a unique hyperplane that intersects P exactly at these points. This hyperplane is $\phi(\Sigma)^*$

Lemma 17.2.3

- From 17.2.2, the power of *X* w.r.t. Σ equals square of the radius of the sphere Σ_X
  - Orthogonal to Σ, centered at *X*
- This can be computed in $\mathbb{E}^{d+1}$
- Σ_X is mapped to a point $\underline{I}$ in $\mathbb{E}^{d+1}$, the intersection of the vertical line passing through *X* with φ(Σ)*
- The x_{d+1} coordinates of φ(*X*) and $\underline{I}$ are: X^2 and $\Sigma_X(O)=X^2 - \Sigma_X(X)$, because $r_{\Sigma X} = \Sigma(X)$

Lemma 17.2.3

- The power of *X* with respect to a sphere Σ equals the signed vertical distance from the point ϕ(*X*) to the hyperplane ϕ(Σ)*



Lemma 17.2.4

- Let X and Σ be respectively a point and a sphere in $\mathbb{E}^d$. If $\underline{H}$ is a hyperplane in $\mathbb{E}^{d+1}$, we denote by $\underline{H}^-$ the half-space lying below $\underline{H}$. Then:

$$X \in \Sigma \Leftrightarrow \phi(X) \in \phi(\Sigma)^* \Leftrightarrow \phi(\Sigma) \in \phi(X)^*$$

$$X \in int(\Sigma) \Leftrightarrow \phi(X) \in \phi(\Sigma)^{*-} \Leftrightarrow \phi(\Sigma) \in \phi(X)^{*-}$$

$$X \in ext(\Sigma) \Leftrightarrow \phi(X) \in \phi(\Sigma)^{*+} \Leftrightarrow \phi(\Sigma) \in \phi(X)^{*+}$$

$$X \in \Sigma \Leftrightarrow \phi(X) \in \phi(\Sigma)^* \Leftrightarrow \phi(\Sigma) \in \phi(X)^*$$

$$X \in int(\Sigma) \Leftrightarrow \phi(X) \in \phi(\Sigma)^{*-} \Leftrightarrow \phi(\Sigma) \in \phi(X)^{*-}$$

$$X \in ext(\Sigma) \Leftrightarrow \phi(X) \in \phi(\Sigma)^{*+} \Leftrightarrow \phi(\Sigma) \in \phi(X)^{*+}$$

- Any point lying below $\phi(X)^*$ is the image of a sphere whose interior contains X. Any point above is the image of a sphere whose exterior contains X.

Radical Hyperplanes

- Given Σ_1, Σ_2, the set of points in $\mathbb{E}^d$ with the same power with respect to Σ_1 and Σ_2 is the *radical hyperplane* H_{12}:

$$H_{12} : \Sigma_1(X) - \Sigma_2(X) = 0$$

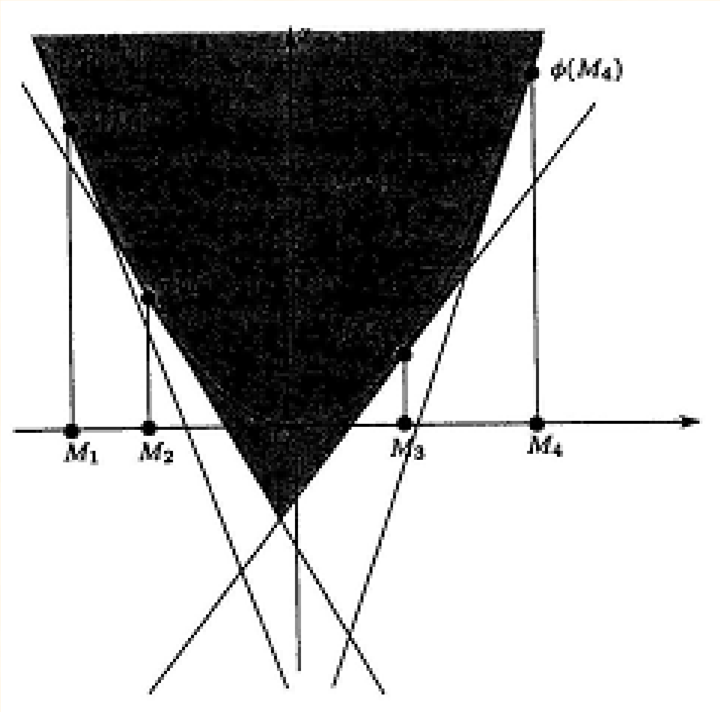
- The points that are the centers of spheres orthogonal to both spheres
- These points are mapped to $\phi(\Sigma_1)^* \cap \phi(\Sigma_2)^*$

Application to Voronoi Diagrams

- By 17.2.1, $\phi(M_i)^*$ denotes the polar hyperplane tangent to P at $\phi(M_i)$
- The set of spheres containing NO site M_i is mapped to the intersection of the half-spaces lying above the hyperplanes

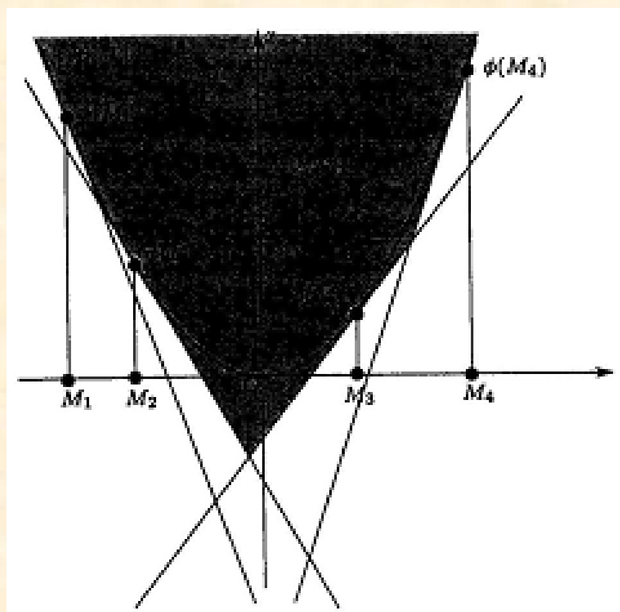
Voronoi Polytope

- The aforementioned intersection is an unbounded polytope that contains *P*.
- This is the *voronoi polytope*, ***V***(***M***)



Theorem 17.2.5

- The Voronoi Diagram of ***M***, ***Vor***(***M***), is a cell complex of dimension *d* in $\mathbb{E}^d$ whose faces are obtained by projecting onto $\mathbb{E}^d$ the proper faces of the Voronoi Polytope ***V***(***M***)



Proof of 17.2.5

- The boundary of ***V***(***M***) is a pure cell complex of dimension *d*, hence so is ***Vor***(***M***).
- Let *A* be a point on a facet of ***V***(***M***), that is contained in $\phi(M_i)^*$.
- *A* is the image of a sphere centered on the projection of *A* that passes through M_i, and contains no other site.
- Therefore, *A* must belong to $V(M_i)$

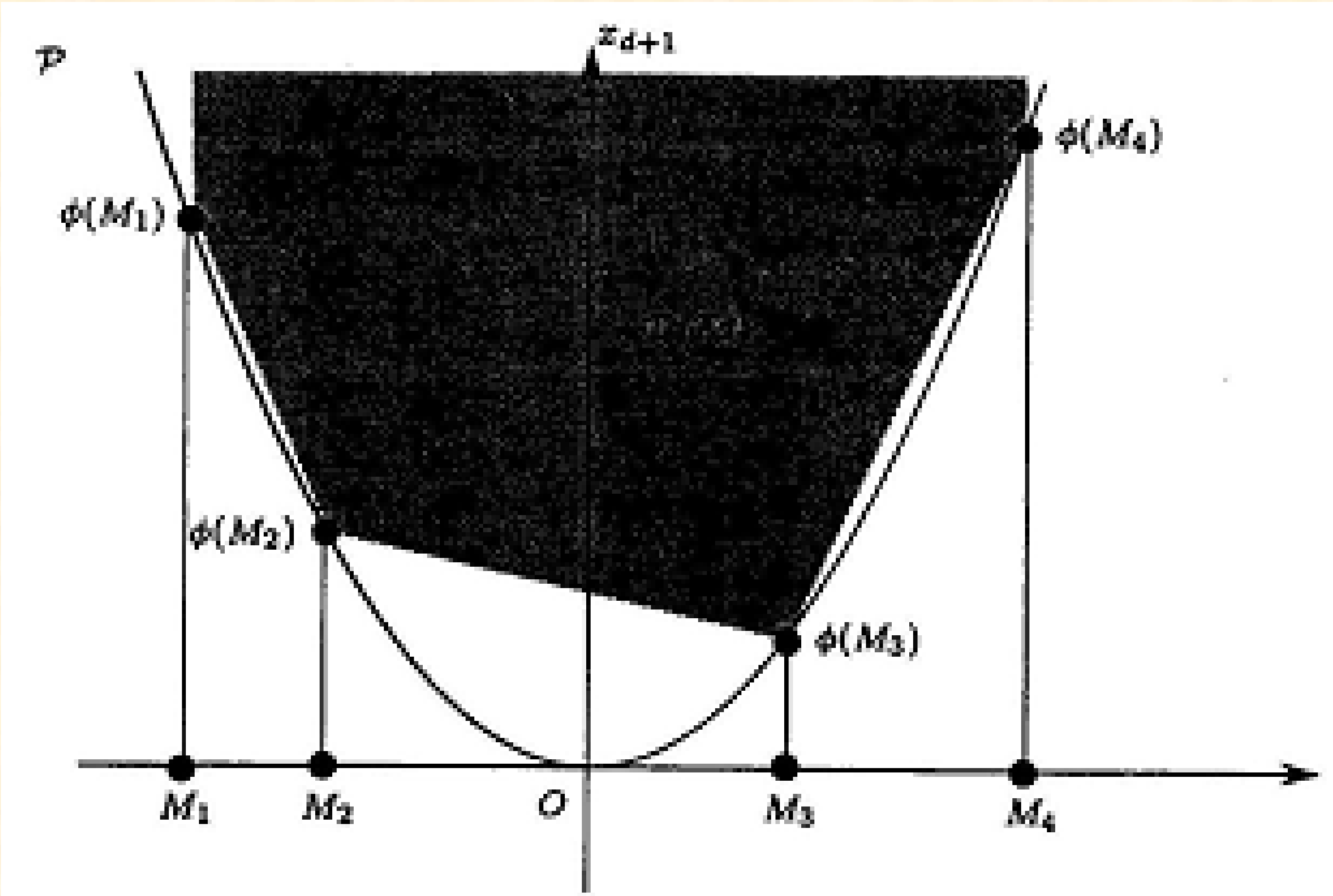
Implications

- Combinatorial properties of VDs follow directly from those of polytopes
  - If points are in general position, ***V***(***M***) is a simple (d+1)-polytope
    - Each vertex is incident to d+1 hyperplanes
    - This is the L_2-*general* assumption
- Computing the VD of *n* points ≈ computing the intersection of *n* half-spaces in 1 higher dimension

Corollary 17.2.6

- The complexity (# of faces) of Voronoi Diagrams of n points is $\Theta(n^{\text{ceil}(d/2)})$
- This can be computed in $O(n\log n + n^{\text{ceil}(d/2)})$ time, which is worst case optimal

Delaunay Complexes



Delaunay Complexes

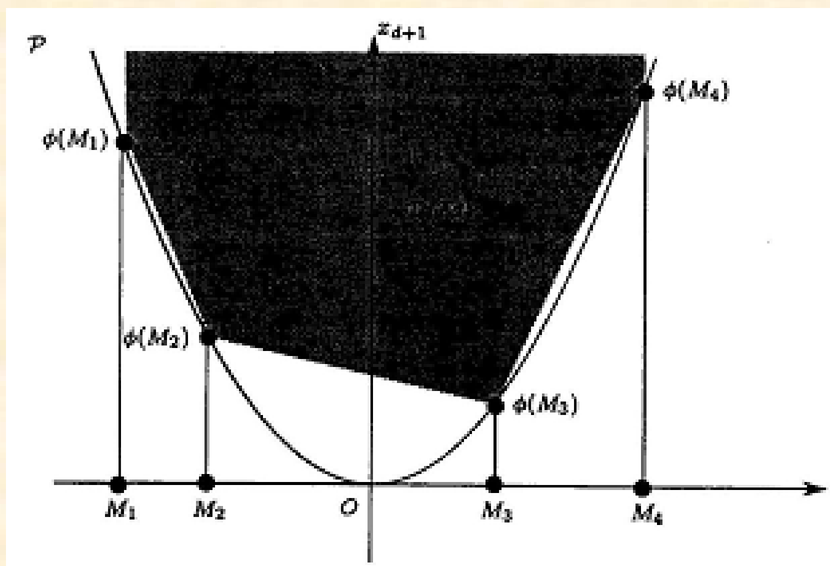
- Define ***M*** as before, and the mapping onto the paraboloid *P* as well.
- Denote by ***D***(***M***) the convex hull of the mappings of the points M_i as well as some 'high' point *O'*.
- *K*-face: a *k*-dimensional face:
  - 0-face is vertex, 1-face is a line, d-1 is a facet

D(***M***)

- ***D***(***M***) forms the convex hull of ϕ(***M***) and some point *O'*, a high point on x_{d+1} axis, such that the convex hull is stable as *O*' rises to infinity
- The faces of this convex hull that do not contain *O*' form the lower envelope of *conv*(ϕ(***M***))
- **Del**(***M***) is the projection of the convex hull.

Del(***M***)

- A vertical projection of the polytope ***D***(***M***) $\mathbb{E}^d$.
- The *k*-faces of ***Del***(***M***)are in 1-to-1 correspondence with the *k*-faces of ***D***(***M***) that do not contain *O*'



Vor(***M***) and ***Del***(***M***)

- Exists a bijection between the faces of ***V***(***M***) and the faces of ***D***(***M***) that do not contain *O'*
  - Maps the facet of ***V***(***M***) containing $\phi(M_i)^*$ to the point $\phi(M_i)$.
- The bijection exists between the *k*-faces of ***Vor***(***M***) and the (*d*-*k*)-faces of ***Del***(***M***) that reverses inclusion relationships
- ***Del***(***M***) is the dual of ***Vor***(***M***).

Theorem 17.3.1

- The Delaunay complex of ***M*** is the dual to the Voronoi Diagram.
- Its faces are obtained by projecting the faces of the lower envelope of the convex hull of ϕ(***M***), obtained by lifting the ***M*** onto *P*.
- Computing the Delaunay complex is equivalent to computing the Convex hull

Corollary 17.3.2

- The Delaunay complex of n sites in $\mathbb{E}^d$ can be computed in time $O(n\log n + n^{\text{ceil}(d/2)})$ time, which is worst case optimal

Delaunay Triangulations

- In L_2*-general,* ***V***(***M***) is a simple polytope, ***D***(***M***) is a simplicial polytope, and ***Del***(***M***) is a simplicial complex.
- If not L_2*-general*, some of the faces are not simple (triangles).
  - There could be many valid triangulations. All of them are considered Delaunay triangulations.

Characteristic Properties

- Theorem 17.3.3
  - Any *d*-face in the complex can be circumscribed by a sphere that passes through all its vertices $M_{i0},\ldots,M_{il}$, and whose interior contains no site of $\boldsymbol{M}$

Proof of 17.3.3

- $\boldsymbol{M}_k$: a subset of ***M*** containing *k* sites
- Pick a *d*-face, *T*, of ***Del***(***M***), with vertices $\boldsymbol{M}_k$. *T* is the convex hull of $\boldsymbol{M}_k$. The CH of $\phi(\boldsymbol{M}_k)$ form a *d*-face *F* of the total CH, by thm 17.3.1.
- $\underline{H}_F$: the hyperplane that supports *F*
- $\underline{H}_F \cap P$ projects onto $\mathbb{E}^d$ as a sphere Σ circumscribed to $conv(M_{i0},\ldots,M_{il})$, centered on the projection of the pole, $\underline{H}_F{}^*$.

Proof, Continued

- $\underline{H}_F$ * is the intersection of the polar hyperplanes $\phi(\boldsymbol{M}_k)$, and it is projected onto $\mathbb{E}^d$ at C
- C is the vertex of the VD that is incident to the cells that correspond to the sites $\boldsymbol{M}_k$, and none of the interiors can contain any other site.

Theorem 17.3.4

- Let $\boldsymbol{M}_k$ be a subset of k sites in $\boldsymbol{M}$. The CH of $\boldsymbol{M}_k$ is a face of the Delaunay complex iff there exists a (d-1) sphere passing through the vertexes $\boldsymbol{M}_k$, and such that no point in $\boldsymbol{M}$ is on the interior of this sphere.

Proof of 17.3.4

- Necessary
  - Result of 17.3.3, and sphere circumscribed to a face also circumscribed to a subface.
- Assume exists a (d-1)-sphere Σ that passes through the points of M_k, and has no interior sites. Let <u>*H*</u> be the hyperplane, $\phi(\Sigma)^*$ containing the projections of the points

Proof of 17.3.4

- The halfspace lying below $\underline{H}$ does not contain any points in $\phi(\boldsymbol{M})$, according to 17.2.4
- Thus, $\underline{H}$ is a hyperplane supporting $\boldsymbol{D}(\boldsymbol{M})$ along the convex hull of the k sites, and so $conv(\phi(M_{i0}) \ldots. \phi(M_{ik}))= \underline{H} \cap \boldsymbol{D}(\boldsymbol{M})$ is a face of $\boldsymbol{D}(\boldsymbol{M})$
- Therefore, from 17.3.1, $\boldsymbol{M}_k$ makes a face of $\boldsymbol{Del}(\boldsymbol{M})$

Corollary 17.3.5

- Any Delaunay Triangulation of a set of ***M*** sites is such that the sphere circumscribed to any *d*-simplex in the triangulation contains no point of M in its interior.
- Conversely, any triangulation satisfying this property is a Delaunay Triangulation

Characteristic of DT

- Consider any DT $\boldsymbol{T}(\boldsymbol{M})$
- Let $S_1=M_1\ldots M_d M_{d+1}$ and $S_1=M_1\ldots M_d M_{d+2}$ be a pair of adjacent d-simplices in $\boldsymbol{T}(\boldsymbol{M})$ (circumscribed to Σ_1 Σ_2). that share a common face $F=M_1\ldots M_d$
- (S_1,S_2) is regular if M_{d+1} is not in $int(\Sigma_2)$

Characteristics

- If Σ_1 differs from Σ_2, regularity is equivalent to M_{d+2} not being in $int(\Sigma_1)$.
- M_{d+1} does not belong to $int(\Sigma_2)$ iff $\Sigma_2(M_{d+1})>0$. But the hyperplane $\underline{H}_F$ that supports F is the radical hyperplane of Σ_1 and Σ_2

Theorem 17.3.6

- Since $\Sigma_1(M_{d+1})=0$, the half-space bounded by $\underline{H}_F$ that contains M_{d+1} (or resp. M_{d+2}) consists of the points whose power w.r.t. Σ_1 is smaller (resp. greater) than their power w.r.t. Σ_2.

  $\Sigma_1(M_{d+2}) > \Sigma_2(M_{d+2})=0$
- This proves M_{d+2} does not belong to the interior of Σ_1

Theorem 17.3.6

- Consider a triangulation ***T***(***M***)
- Then, ***T***(***M***) is a DT iff all pairs of adjacent *d*-simplices in ***T***(***M***) are regular

Proof of 17.3.6

- Necessary as a consequence of 17.3.3
- $\phi(S)$ = the k-simplex whose vertices are the images of the vertices of a k-simplex S.
- C = the union of the $\phi(S)$'s for all the faces S of the DT $\boldsymbol{T}(\boldsymbol{M})$.
- The sufficiency proof is to show that C is the graph of a real valued convex function over *conv*(***M***)

Proof, Cont.

- We consider S_1 and S_2 that share the common face F, with circumscribing spheres Σ_1 and Σ_2
- By 17.2.4, the regularity condition is equivalent to $\phi(M_{d+1})$ being in $\phi(\Sigma_2)^{*+}$, and vice versa.
- If (S_1, S_2) regular, $\phi(F)$ is locally convex (there is a hyperplane containing $\phi(F)$ such that $\phi(S_1)$ and $\phi(S_2)$ belong to the half-space above this hyperplane)

Proof Cont

- This is true for any (*d-1*)-face of *C* incident to 2 *d*-faces, and so *C* is locally convex at any point
- *C* is defined over a convex subset, the *conv*(***M***).
- Therefore, *C* is convex and is the lower envelope of the polytope ***D***(***M***) which proves that ***T***(***M***) is a DT of ***M***.

Optimality of a DT

- Because we can triangulate in many ways, we have many ways to define optimality
  - Compactness
  - Equiangularity

Compactness

- The smallest enclosing sphere of each simplex *S*
- ***T***(***M***) corresponds to a function $\Sigma_T(\boldsymbol{M})$, defined over *conv*(***M***) as the power of a point *X* w.r.t. the sphere Σ circumscribing any *d*-simplex containing *X*

Lemma 17.3.7

- Let ***Det***(***M***) be a Delaunay triangulation of ***M***, and ***T***(***M***) be another triangulation. Then
  - For all X in $conv(\boldsymbol{M})$, $\Sigma_{Det}(X) \geq \Sigma_T(X)$

Proof of 17.3.7

- Consider a d-simplex T in $\boldsymbol{T}(\boldsymbol{M})$ containing X, Σ the cirumscribed sphere, and ϕ(T) the d-simplex projection
- By 17.2.3, $\Sigma_T(X)$ is the (negative) vertical distance from ϕ(Σ)* to ϕ(X).
- ϕ(Σ)* is the affine hull of ϕ(T)
- For a given X, this signed vertical distance is maximized when ϕ(T) is a face of the convex hull of ϕ($\boldsymbol{M}$); when T is a simplex of a DT

Lemma 17.3.8

- If T is a d-simplex circumscribed to Σ_T, then

$$\min_{X \in T} \Sigma_T(X) = \Sigma_T(C'_T) = -r'^2_T$$

- where C'_T and r'_T are the center and radius of the smallest enclosing sphere

Proof

- Let Σ_T be the sphere circumscribed to T, centered at C_T with radius r_T.
  - $\Sigma_T(X) = XC_\mathrm{T}^2 - r_T^2$ is minimized when $X = C_T$, and so $\Sigma_T(X)$ is greater than $-r_T^2$.
- If C_T contained in T, Σ_T is the smallest enclosing sphere, and so $r'_T = r_T$

Proof, Cont.

- If C_T not contained in T, the smallest containing sphere is centered on a k-face (k<d) F, the face such that the orthogonal project of C_T onto the plane that supports F falls inside F.
- r'_T of this sphere is that of the (k-1)-sphere circumscribed to F
- C'_T minimizes $XC_T . XC_T$ when X is in T
- $C_TC'_T{}^2 + r'_T{}^2 = r_T{}^2$

Most Compact Triangulation

- The *maximum min-containment radius* of $\boldsymbol{T}(\boldsymbol{M})$: $C(\boldsymbol{T}(\boldsymbol{M})) = \max_{T \in \boldsymbol{T}(\boldsymbol{M})} r'_T$
- The most compact triangulation minimizes $C(\boldsymbol{T}(\boldsymbol{M}))$.

Theorem 17.3.9

- Delaunay Triangulations are the most compact among all triangulations
  - Other triangulations might also be the most compact, even if not Delaunay

Proof of 17.3.9

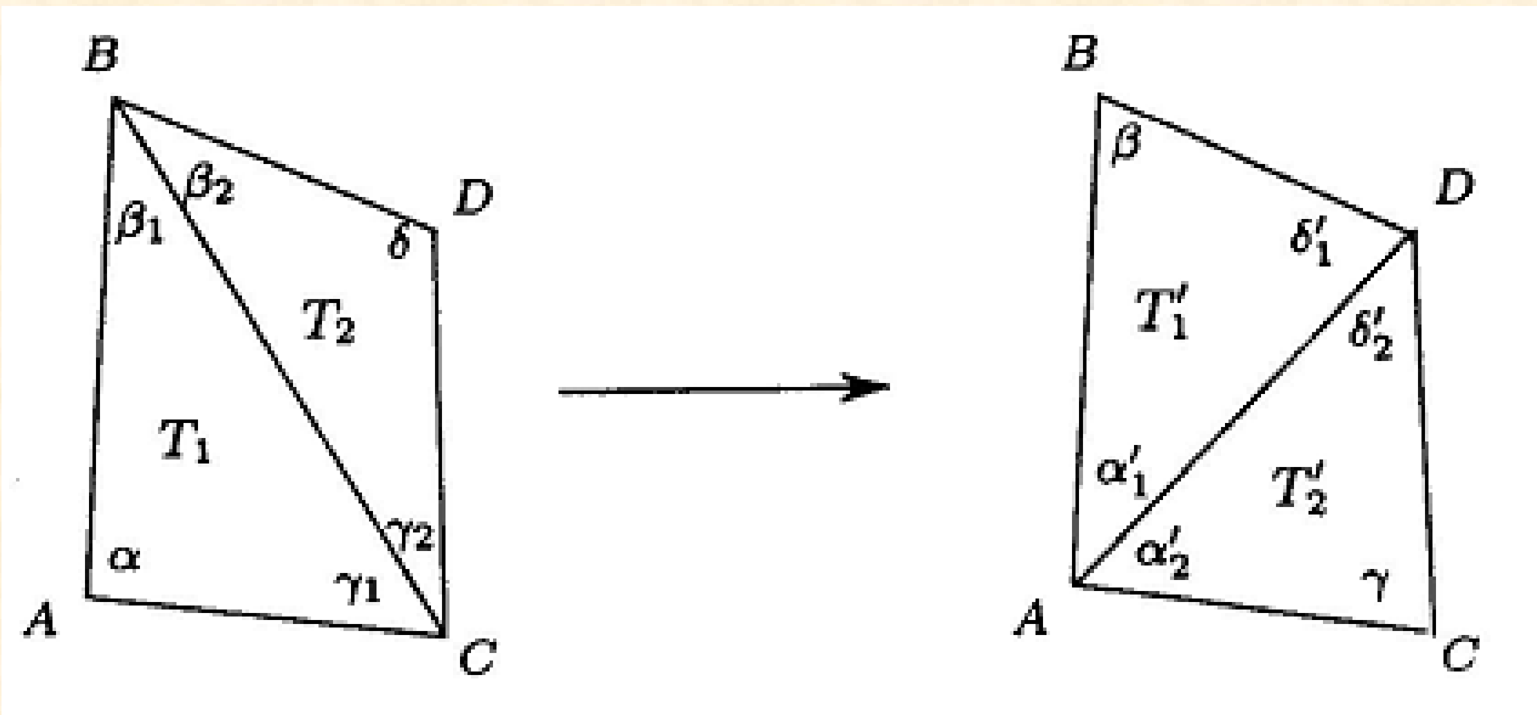
- Define $\boldsymbol{T}(\boldsymbol{M})$, $\boldsymbol{Det}(\boldsymbol{M})$
- X_T is the point minimizing $\Sigma_T(X)$, and similarly X_{Det}.
- X_T is the center of the smallest sphere circumscribed to the simplex containing X_T (by 17.3.8)
- We denote the radii as r'_T and r'_{Det}.
- $C(\boldsymbol{T}(\boldsymbol{M}))= r'_T$, $C(\boldsymbol{Det}(\boldsymbol{M}))=r'_{Det}$
- By 17.3.7 and 8:
- $\Sigma_T(X_T) =-r'^2_T \leq \Sigma_T(X_{det}) \leq \Sigma_{Det}(X_{Det})=-r'^2_{Det}$

Equiangularity (d=2)

- Given a triangulation $\boldsymbol{T}(\boldsymbol{M})$, the angle vector is $Q(\boldsymbol{T}(\boldsymbol{M})) = (a_1,\ldots,a_{3t})$, where each a is an angle of the t triangles, sorted by increasing value.
- We know that the sum of the angles = $t\pi$
- A triangulation that maximizes the angle vector for the lexicographic ordering also maximizes the smallest ordering
- This is a *globally equiangular* triangulation

Theorem 17.3.10

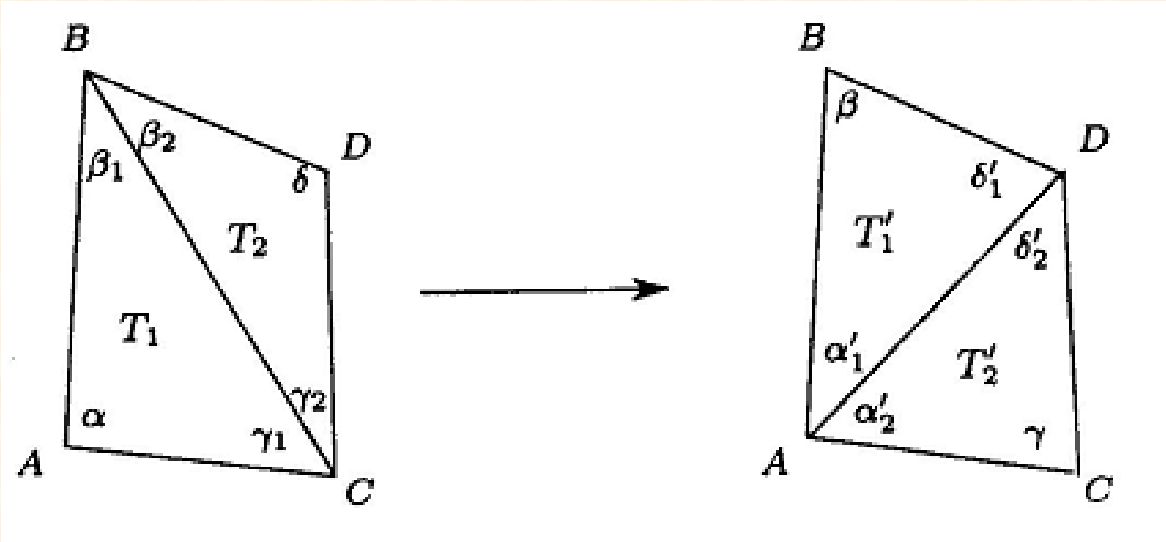
- A globally equiangular triangulation of a set of *M* sites in the plane is always a DT



Proof of 17.3.10

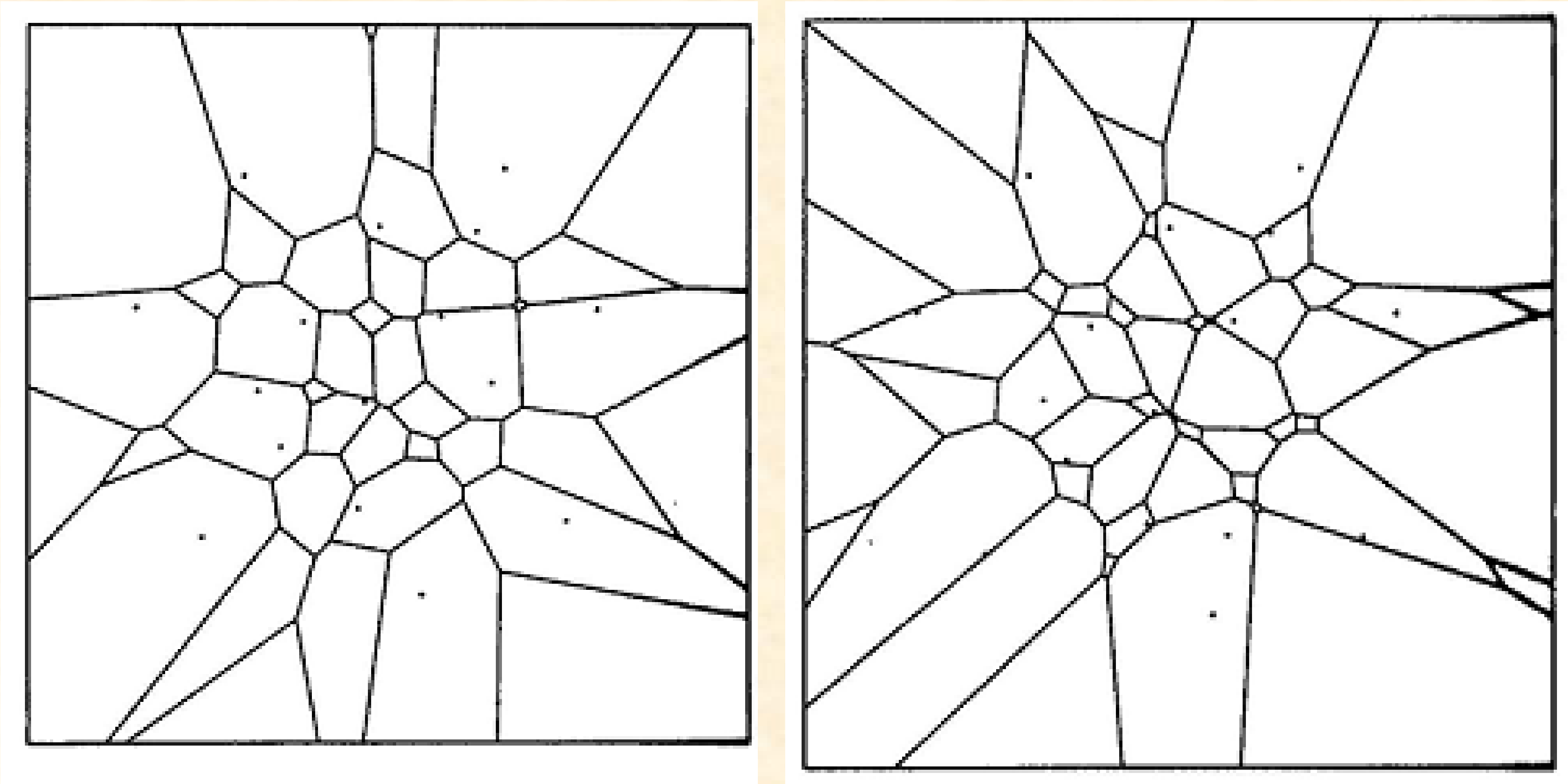
- Consider 2 triangles $T_1=ABC$, and $T_2=BCD$, where *ABCD* is a strictly convex quadrilateral (all vertices of the convex hull)
- We can flip diagonals to increase equiangularity
  – This is a regularization rule, since it changes a pair of adjacent triangles into a regular pair of triangles

- Let Σ_1 and Σ_2 be the circumscribing circles.
  - AD is flipped only if Σ_1 contains D
- Let *abcd* be the angles at the vertices ABCD, and *b* and *c* are split into b_1, b_2, c_1 and c_2
  - We also split *a* and *d* into a'_1, a'_2, b'_1, b'_2 for the potential split



- If *a* is the smallest angle, then we don't flip, but $d=\pi-b_2-c_2 < \pi - a$, so that $a+d< \pi$, so A is not inside Σ_2
- If *b1* is the smallest, then we flip only if *d'2* is greater, which only happens when *D* in Σ_1
- Parallels for all other smallest angles
- After we flip, $Q(\boldsymbol{T_1}(\boldsymbol{M}))>Q(\boldsymbol{T}(\boldsymbol{M}))$
- Progressive flippings increases the angle vector. Since there are a finite number of triangulations, we must reach a maximum, with only regular pairs of adjacent triangles, and is a DT by 17.3.6

Higher Order Voronoi Diagrams



Higher Order Voronoi Diagrams

- Level-*k*
  - A point is at Level *k* of an arrangement ***A*** if it belongs to exactly *k* open half-spaces, $\phi(M_i)^{*-}$, such that each half-space does not the reference point (the origin).

Voronoi Diagram of Order k

- Let $\boldsymbol{M}_k$ be a subset of $\boldsymbol{M}$ of size k.
- $V_k(\boldsymbol{M}_k)$: the region or points that are closer to $\boldsymbol{M}_k$ than any other sites

$$V_k(M_k) = \left\{X : \forall M_i \in M_k, \forall M_j \in M \setminus M_k, \left\|XM_i\right\| \leq \left\|XM_j\right\|\right\}$$

- The union of V_k forms $\boldsymbol{Vor}_{\mathrm{k}}(\boldsymbol{M})$
- $\boldsymbol{Vor}_1(\boldsymbol{M}) = \boldsymbol{Vor}(\boldsymbol{M})$.

Theorem 17.4.1

- ***Vor***$_k$(***M***) is a cell complex of dimension d in $\mathbb{E}^d$. The cells of this complex correspond to the cells at level k in the arrangement ***A*** of the hyperplanes induced by the projections of the points. A cell in the diagram is obtained by projecting the corresponding cell in the arrangement. The l-faces of ***Vor***$_k$(***M***) are obtained by projecting the l-faces common to cells at level k in arrangement ***A***.

Proof of 17.4.1

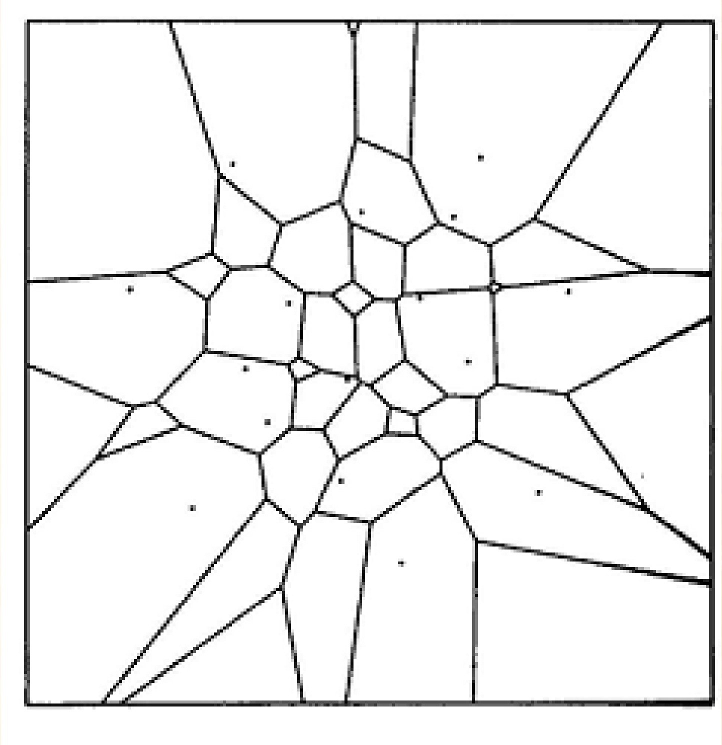
- The proof relies on lemma 17.2.4
- A sphere whose interior contains k points is mapped by ϕ to a point at level k in the arrangement $\boldsymbol{A}$ of the hyperplanes.
- This is easily verifiable.

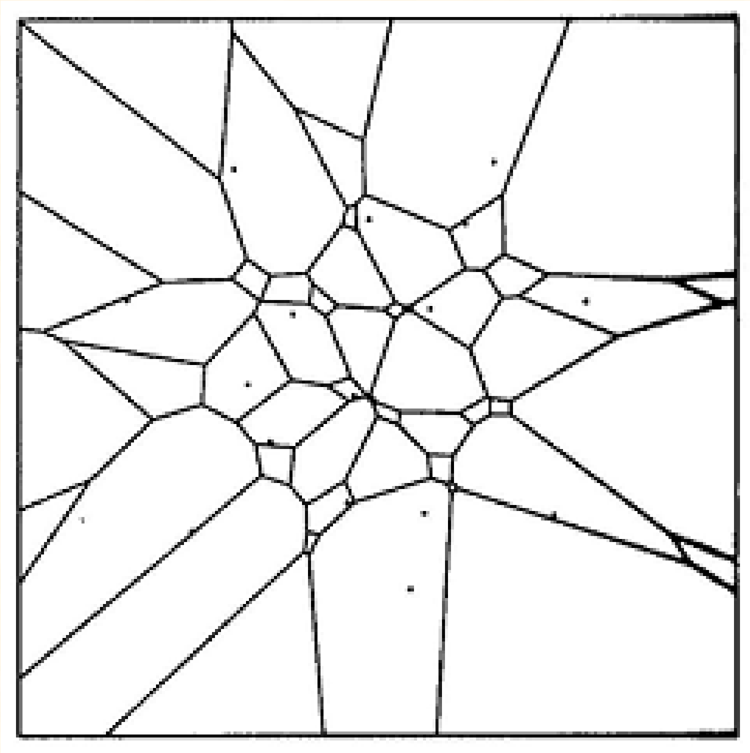
$Vor_k(M)$

- Once constructed, easy to find k-nearest neighbors, by finding the region of the arrangement containing the point.
- Theorem 17.4.2
  - The overall complexity of the first k voronoi diagrams of a set of n points is $O(n^{\text{ceil}((d+1)/2)}k^{\text{ceil}((d+1)/2)})$. The k diagrams can be computed in time $O(n^{\text{ceil}((d+1)/2)}k^{\text{ceil}((d+1)/2)})$ for d≥3, or $O(nk^2\log(n/k))$ for d=2

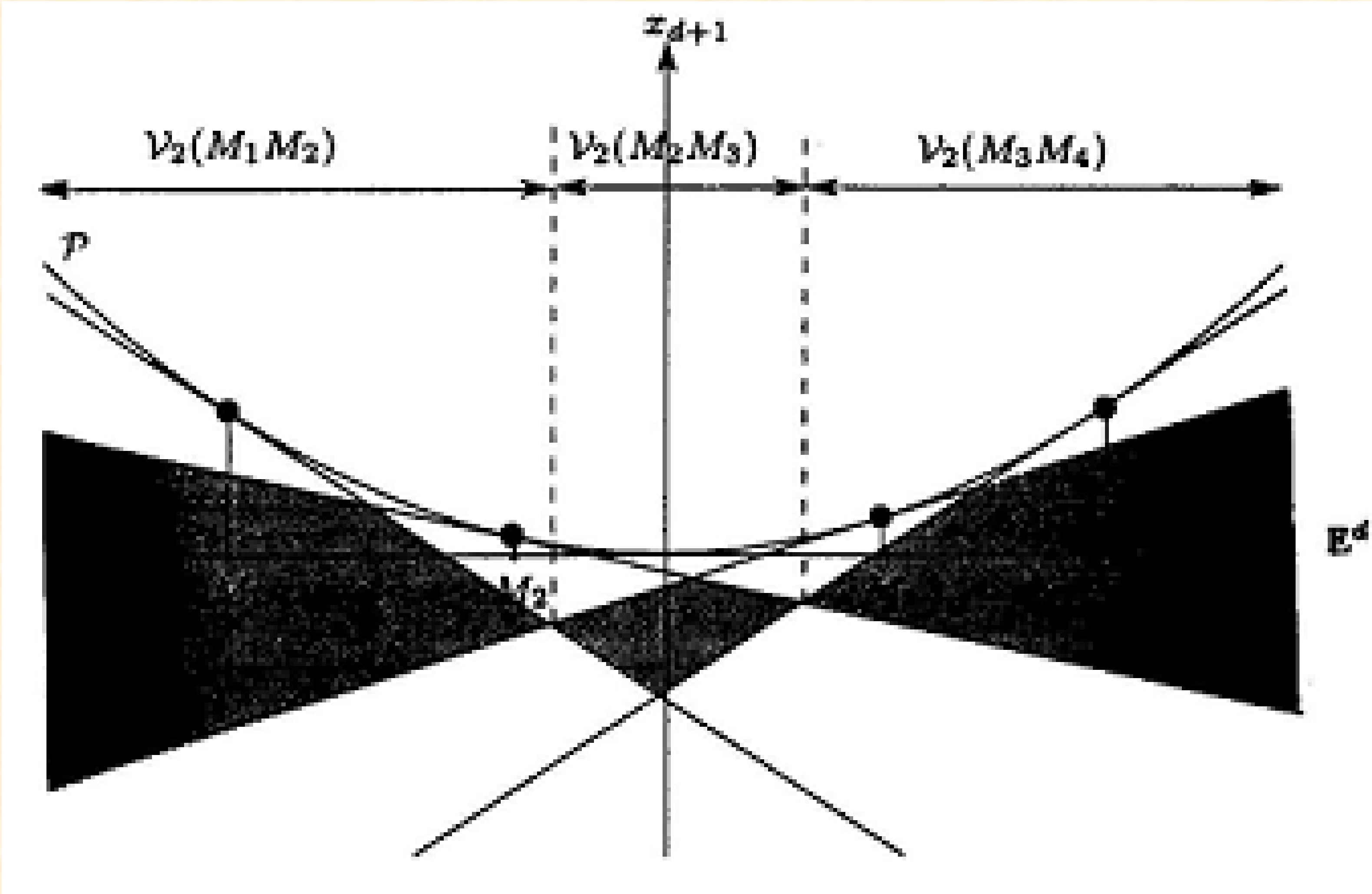
Examples

- ***Vor***$_{n-1}$(***M***) is the furthest-point Voronoi Diagram.
- ***Vor***$_2$, ***Vor***$_3$





Projection of Vork(M)



Non-Euclidean Metrics

Power Diagrams

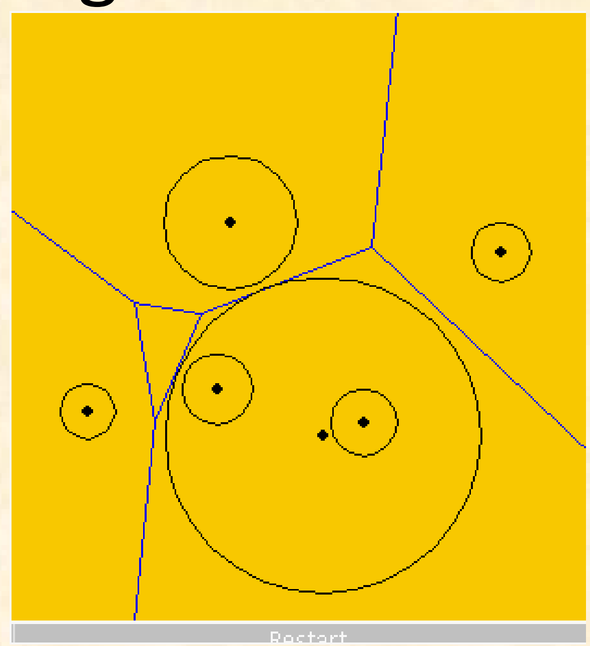
- Let $S=\{\Sigma_1,\ldots, \Sigma_n\}$ be a set of spheres in $\mathbb{E}^d$.
- $P(\Sigma_i) \rightarrow$ all points whose power w.r.t. Σ_i is smaller than power w.r.t. any other sphere.

$$P(\Sigma_i) = \left\{X \in E^d : \forall j \neq i, \Sigma_i(X) \leq \Sigma_j(X)\right\}$$

- The region $P(\Sigma_i)$ is a convex polytope, the intersection of a finite set of half-spaces bounded by radical hyperplanes.
- The complex is called the Power Diagram of S, *Pow*(*S*).

Power Diagrams

- As before, we map the spheres to a point in $\mathbb{E}^{d+1}$. However, these points are not on the paraboloid. The faces of the Power diagram are obtained by projecting the intersections of the polar hyperplanes.



Order-*k* Power Diagrams

- A similar parallel to the previous.

Affine Diagrams

- Defined for sites and for a distance such that the set of points equidistant from 2 objects is a hyperplane.
- We can extend the idea of VDs to more general sites, non-Euclidean distances.
- VD's and Power Diagrams are *Affine Diagrams,* and any affine diagram is a power diagram.
- We can also derive many nonaffine diagrams from affine diagrams

Affine Diagrams

- Cells of AD's are convex polytopes.
- To any AD of n objects corresponds a set of $\binom{n}{2}$ perpendicular bisectors, H_{ij}, $1 \leq i < j \leq n$
- These hyperplanes must satisfy

$$H_{ij} \cap H_{jk} = H_{ij} \cap H_{ik} = H_{ik} \cap H_{jk} \stackrel{def}{=} I_{ijk}$$

- For any $1 \leq i < j < k \leq n$
- The diagram is simple if the I_{ijk} are disjoint and nonempty

Theorem 18.2.1

- Any simple AD in $\mathbb{E}^d$ is the power diagram of a set of spheres in $\mathbb{E}^d$.
- Proof:
  - We embed $\mathbb{E}^d$ in $\mathbb{E}^{d+1}$ as the hyperplane of $x_{d+1}=0$.
  - We construct a set of *n* hyperplanes in $\mathbb{E}^{d+1}$ such that the projection of the intersections is exactly the hyperplanes H_{ij}

Proof of 18.2.1

- We construct *n* hyperplanes, $P_1,\ldots,P_n$, in $\mathbb{E}^{d+1}$
- We ensure that the vertical projections of $P_i \cap P_j$ for any i<j is exactly H_{ij}
- Each P_i is the polar hyperplane of a sphere $\Sigma_i=\phi^{-1}(P_i^*)$, which is the projection of $P_i \cap P$.
- So H_{ij} is the radical hyperplane of Σ_i and Σ_j. So the affine diagram is the power diagram

Constructing P_i

- Denote h_{ij} the vertical projection of H_{ij} onto P_i.
- Take P_1 and P_2 as non-vertical hyperplanes that intersect P_1 along h_{12}. For any k>2, we must take P_k that make the appropriate intersections with P_1 and P_2. If I_{12k} exists, then P_k must exist.

Proof Cont.

- We show that the projections of the intersections are H_{ij}.
- We have constructed H_{12}, H_{1k}, and H_{2k} for one k.
- If the other cells I_{ijk} exist, then their intersections much exist

Theorem 18.2.2

- The affine diagram with Hyperplanes H_{ij} and equations of hyperplanes

  $-2(C_i - C_j)\cdot X + \sigma_i - \sigma_j = 0$

  is the power diagram of the spheres Σ_i.
- Proof: We can rewrite the equation of the hyperplane as $\Sigma_i(X) - \Sigma_j(X) = 0$

Diagrams for General Quadratic Distance

- Consider 2 points *X* and *A* in $\mathbb{E}^d$.
- The General Quadratic Distance is defined as:

$$\delta_Q(X, A) = (X - A)\Delta(X - A)^t + p(A)$$

- Where Δ is a real symmetric *d x d* matrix and where *p*(*A*) is a real number

Diagrams

- VD is when Δ is the identity matrix and $p(X)=0$
- Furthest point diagrams (VD of order $n\text{-}1$) are when Δ is the negative identity matrix, and $p(X)=0$.
- Power Diagrams are when Δ is the identity matrix and $p(X)$ does not equal 0.

General Quadratic Distance

- If *A*, *B* are points, then the formula for the hyperplane can be rewritten as
  $H_{AB} : 2(B - A)\Delta X^t + A\Delta A^t - B\Delta B^t + p(A) - p(B) = 0$
- Therefore, the VD for a general quadratic distance is an affine diagram

Weighted Diagrams

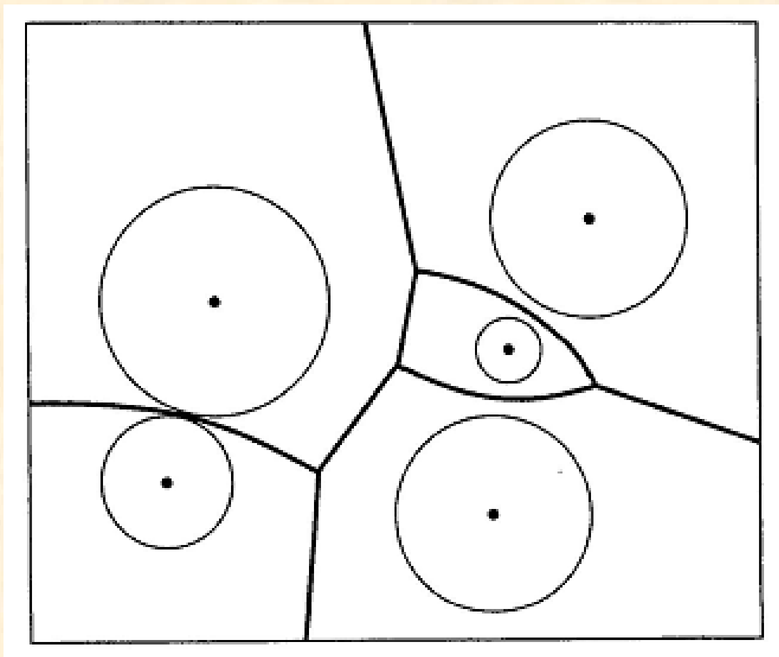
- Non-affine.
- Defined over finite sets of points and a weighted Euclidean distance

Additively Weighted Diagrams: *Vor₊*(*M*)

- Additive weighted distance formula

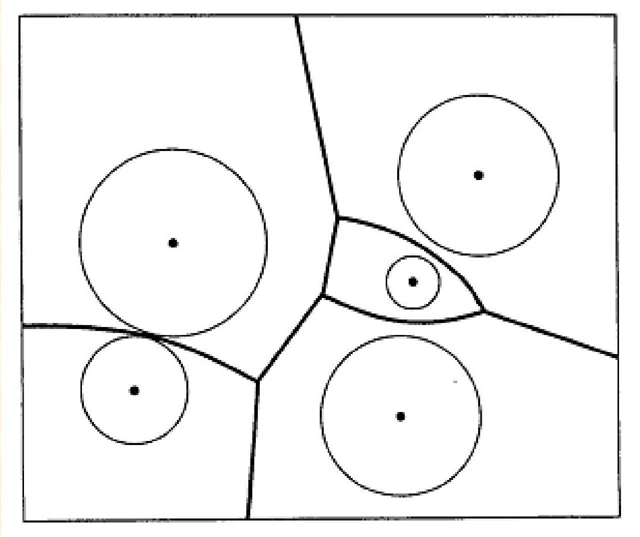
$$\delta_+(X, M_i) = \|XM_i\| - r_i$$

- We assume that all r_i values, the weights, are non-negative.



$Vor_+(M)$

- Consider Σ_i in $\mathbb{E}^d$, centered at M_i with radius r_i, and let ψ be the bijection that maps Σ_i to the point $\psi(\Sigma_i)=(M_i,r_i)$ in $\mathbb{E}^{d+1}$
- The spheres of radius 0 correspond to the hyperplane $x_{d+1}=0$ in $\mathbb{E}^{d+1}$.



Projection ψ

- Points at additive distance r from M_i are centers of spheres tangent to Σ_i with radius r, inside or outside Σ_i.
- Ψ generates a cone of revolution

  $C(\Sigma) : x_{d+1}=|XC|-r$

  - Apex of C is (C,-r),
  - symmetrical with respect to $x_{d+1}=0$,
  - has an aperture angle π/4

ψ

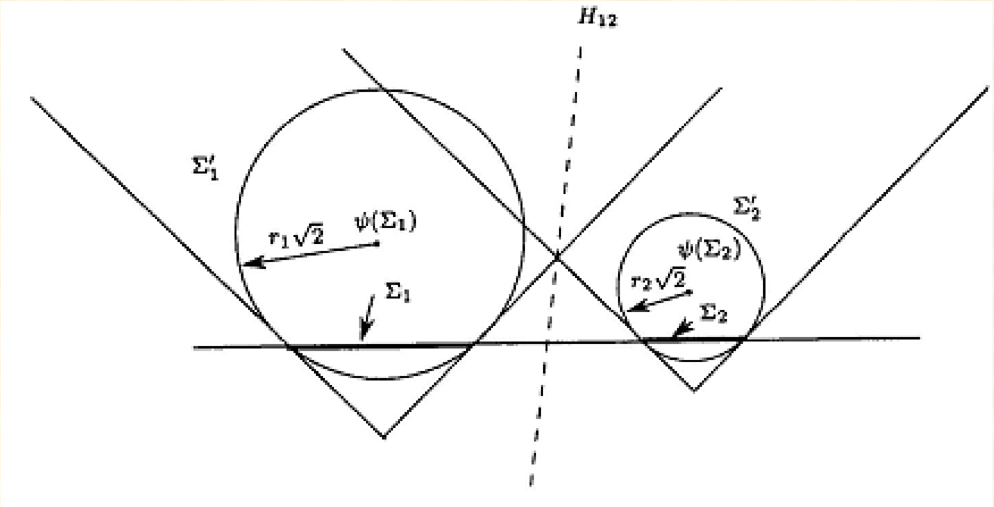
- The projection I_X of a point X on the cone $C(\Sigma)$ is the projection under ψ of the sphere centered at X and tangent to Σ
- Signed vertical distance from X to I_X equals the additive distance from X to C weighted by r
- Each sphere, Σ_i, corresponds to a cone $C(\Sigma_i)$, also denoted C_i.
- The projection of the lower envelope of the cones is exactly $\boldsymbol{Vor_+(M)}$

Equidistance w.r.t. additive distance

- The set of points in $\mathbb{E}^d$ equidistant (w.r.t. additive distance) from 2 points == the projection of intersection of cones
- $C_1 : (x_{d+1}+r_1)^2 = XM_1^2,\ x_{d+1}+r_1>0$
- $C_2 : (x_{d+1}+r_2)^2 = XM_2^2,\ x_{d+1}+r_2>0$
- $H_{12} : -2(M_1-M_2)\cdot X-2(r_1-r_2)x_{d+1}+M_1^2-r_1^2-M_2^2+r_2^2$

Vor$_+$ and Power Diagram

- There exists a correspondence between ***Vor***$_+$ and a power diagram
- Take spheres Σ_i' in $\mathbb{E}^{d+1}$, centered at $\psi(\Sigma_i)$ with radius $r_i\sqrt{2}$



$\boldsymbol{Vor_+}$ and PD

- The cell of $\boldsymbol{Vor_+}(\boldsymbol{M})$ that corresponds to M_i $V_+(M_i)$, is the projection of intersection of the cone C_i with the cell of the PD corresponding to the sphere Σ_i'
- X is in $V_+(M_i)$ iff the projection X_i of X onto C_i has a smaller x_{d+1} coordinate than of X_j $j \neq i$

  (X, x_{d+1}) of X_i must obey:

  $(x_{d+1}+r_i)^2 = XM_i^2$, $(x_{d+1}+r_j)^2 \leq XM_j^2$, $j \neq i$

Computation of $\boldsymbol{Vor_+}$

- $\Sigma_i'(X_i) \leq \Sigma_j'(X_i)$ for any $j \neq i$
- $\boldsymbol{Vor_+}$ can be computed by
  - Compute Σ_i'
  - Compute the power diagram of the Σ_i's
  - For all i, project onto $\mathbb{E}^d$ the intersection of C_i with the cell of the PD that corresponds to Σ_i'.

Computation of ***Vor***$_+$

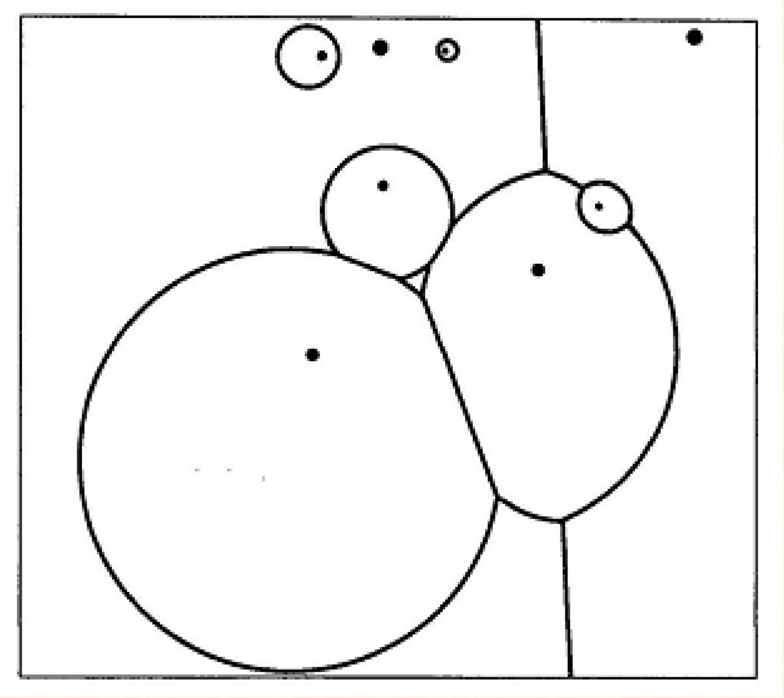
- The PD of Σ_i' can be computed in time $O(n^{(\text{floor}(d/2)+1)})$
- The intersections can be computed in $O(n^{(\text{floor}(d/2)+1)})$
- Theorem 18.3.1
  - ***Vor***$_+$ has complexity $O(n^{(\text{floor}(d/2)+1)})$
  - Can be computed in time $O(n^{(\text{floor}(d/2)+1)})$

Optimality of Computation

- This result is optimal in odd dimensions, but is not optimal in d=2, and possibly not for any even dimension (can be computed in optimal time O($n$log$n$)

Multiplicatively Weighted Diagrams $\boldsymbol{Vor_*(M)}$

- Similar to $\boldsymbol{Vor_+}$, but with multiplicative distance $\delta_*(X, M_i) = p(M_i)\|XM_i\|$
- Where $p(M_i)$ is a positive real number (from now on, p_i)



Vor_*

- Set of points at equal multiplicative distance from 2 sites M_i, M_j is a sphere of the equation $p_i(X-M_i)^2 = p_j(X-M_j)^2$
- In normalized form

$$X^2 - 2\frac{p_i M_i - p_j M_j}{p_i - p_j} \cdot X + \frac{p_i M_i^2 - p_j M_j^2}{p_i - p_j} = 0$$

- And in $\mathbb{E}^{d+1}$, the sphere can be represented as

$$\phi(\Sigma_{ij}) = \left(\frac{p_i M_i - p_j M_j}{p_i - p_j}, \frac{p_i M_i^2 - p_j M_i^2}{p_i - p_j} \right)$$

Point $\phi(\Sigma_{ij})$

- We have mapped the sphere intersection to a point in $\mathbb{E}^{d+1}$.
- The polar hyperplane H_{ij}, w.r.t. the paraboloid P has equation

  $H_{ij}(X,x_{d+1})=$

  $(p_i-p_j)x_{d+1}-2p_iM_i\cdot X+2p_jM_j\cdot X+p_iM_i^2-p_jM_j^2$

  $= 0$

$\phi(\Sigma_{ij})$

- H_{ij}'s are the radical hyperplanes of spheres Σ_i in $\mathbb{E}^{d+1}$
- Σ_i is centered at $(p_i M_i, -\pi/2)$, with $\sigma = p_i M_i^2$.
- We now have a correspondence between ***Vor***$_*$ and the PD of Σ_i's

Vor*∗* and PD

- X is a point in $\mathbb{E}^d$, projected onto *P*

$$X \in V_*(M_i) \Leftrightarrow p_i(X - M_i)^2 \leq p_j(X - M_j)^2$$

$$\Leftrightarrow H_{ij}(X, X^2) \leq 0$$

$$\Leftrightarrow \Sigma_i(\phi(X)) \leq \Sigma_j(\phi(X))$$

$$\Leftrightarrow \phi(X) \in P(\Sigma_i)$$

Computing *Vor*$_*$

- Compute Σ_i
- Compute PD of the Σ_i's
- For each I, project the intersection of the Σ_i in the PD with the paraboloid *P*

Complexity of computation

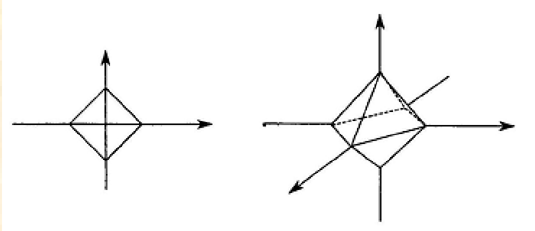
- Theorem 18.3.2
  - ***Vor***$_*$ has complexity $O(n^{(\text{floor}(d/2)+1)})$ and can be computed in time $O(n^{(\text{floor}(d/2)+1)})$
- This is optimal

L_1 metric

- Reminder, L_1 distance between X and point M

$$\delta_1(X,M)=\sum_{i=1}^{d}\left|x_i-m_i\right|$$

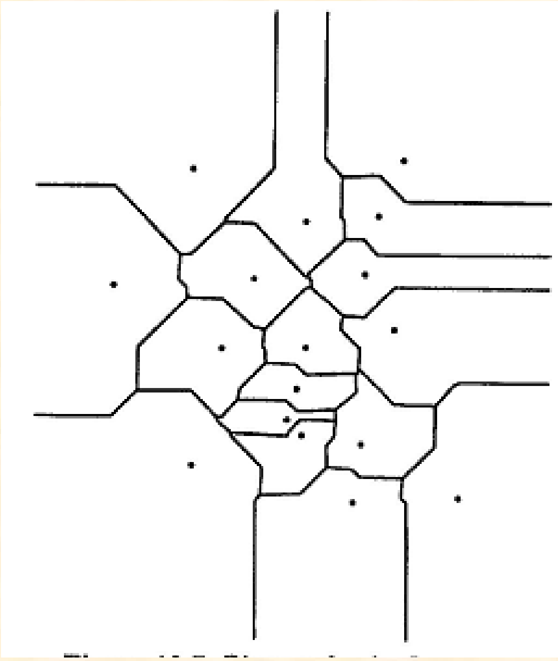
- Points at distance r from M are a polytope w/ vertices at coordinates $x_i=m_i\pm r$, and $x_j=m_j$ if $i\neq j$
  - In 2d- a tilted square
  - In 3d- a regular octahedron



- These are dual to the cube, a co-cube

$\boldsymbol{Vor}_{L1}$

- Voronoi Diagram for L_1 distance
- We create a similar mapping to $\boldsymbol{Vor}_{+}$, but mapped to a pyramid instead of a cone, with $x_{d+1}=\delta_1(X,M_i)$

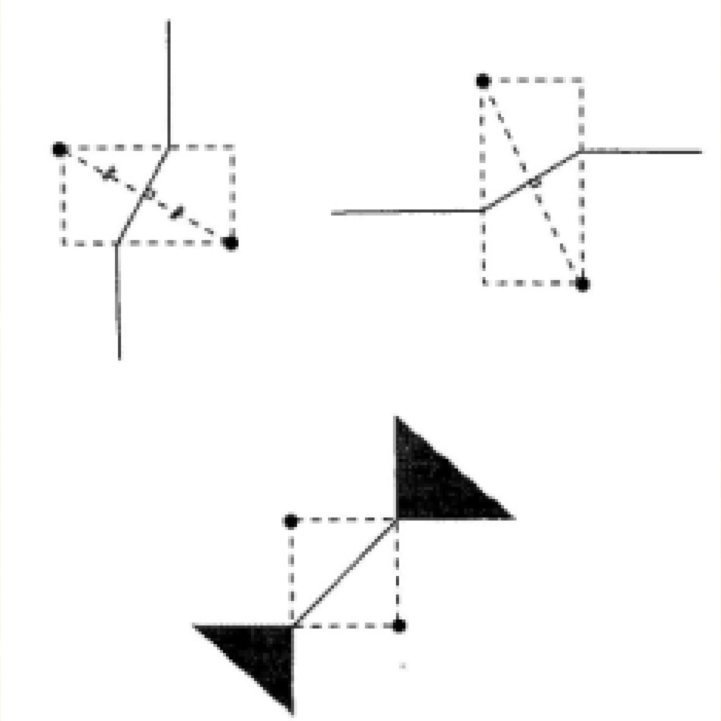


Vor$_{L1}$

- Lower envelope of the Pi pyramids
  - The graph of the function $\min_{1 \le i \le n} \delta_1(X, Mi)$
- Each portion from a distinct pyramid projects onto $\mathbb{E}^d$ as a facet, a cell of ***Vor***$_{L1}$ (M)
- The complexity is O(n^da(n)).

Region Bisectors in $\boldsymbol{Vor}_{L1}$

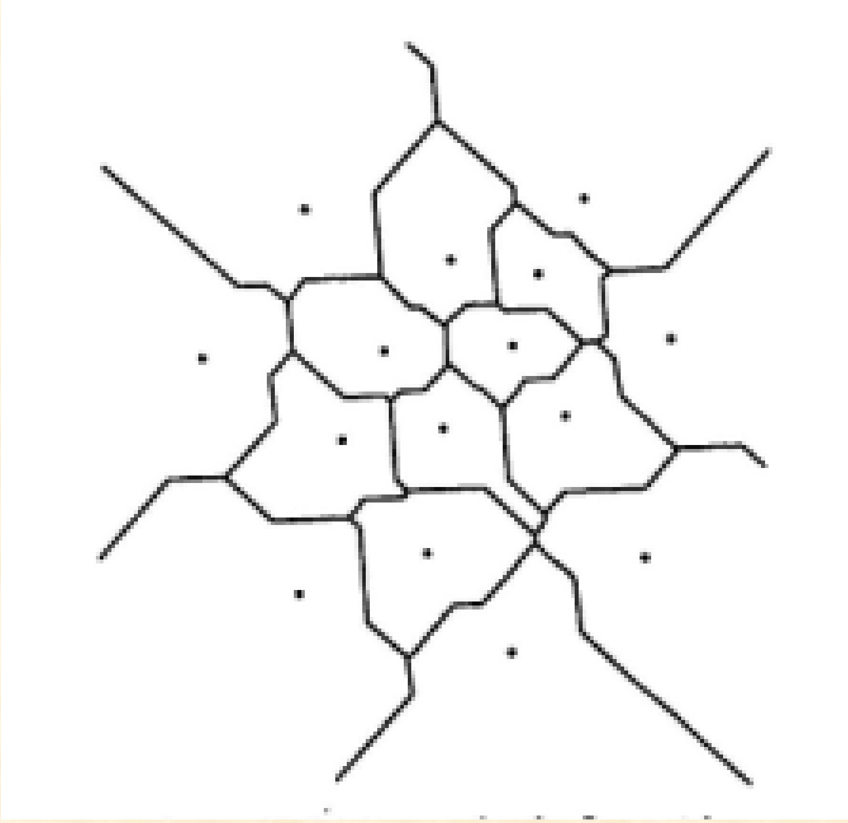
- When d=2, bisectors are generally 3 piece polylines. However, can be 1 linear segment connecting 2 regions of dimension 2



L_∞ Metric

- Using the distance metric

  $\delta_\infty(X,M)=\max|x_i-m_i|$



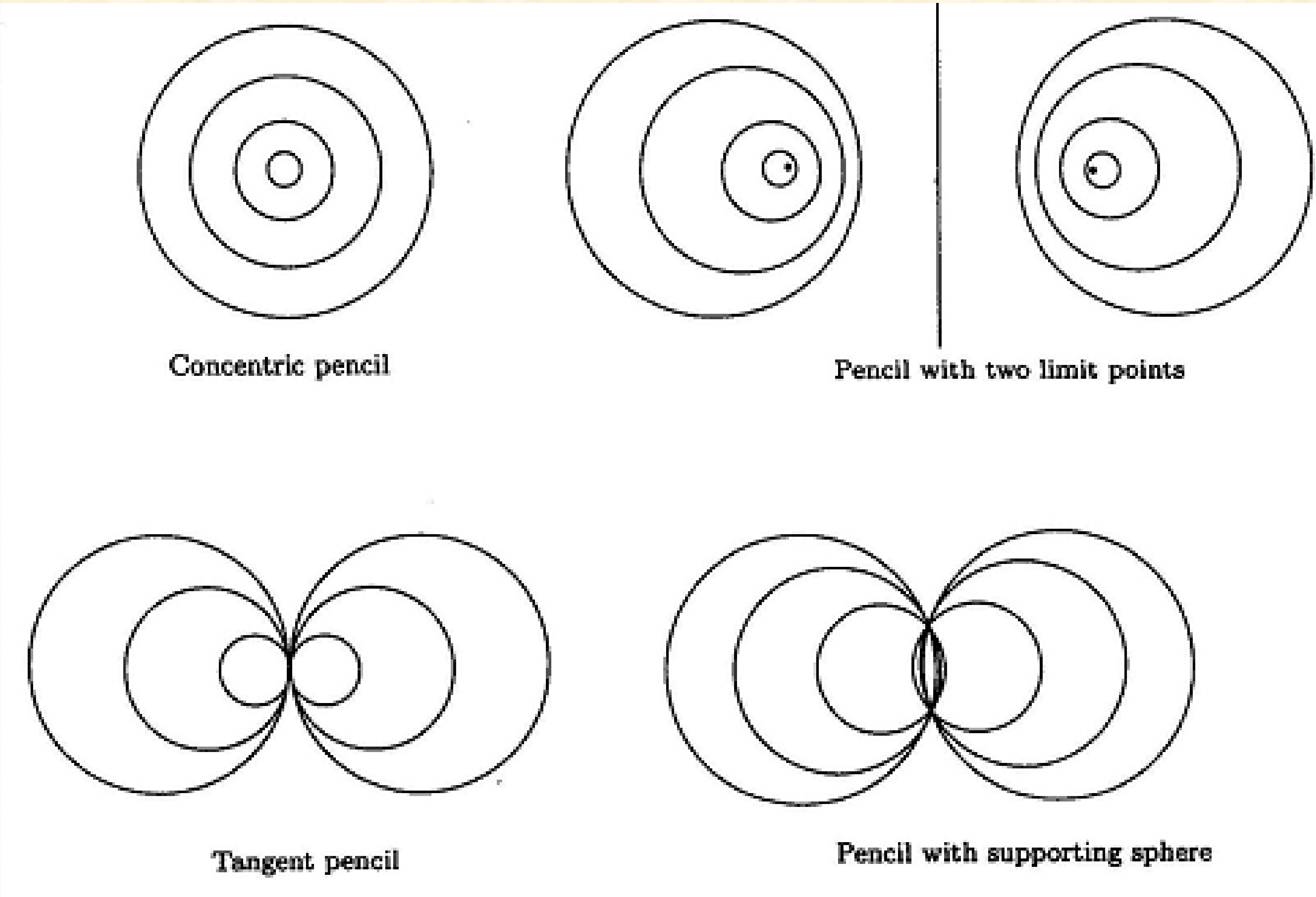
Voronoi Diagrams in Hyperbolic Spaces

Pencils of spheres

- A set of spheres, *S*, that are affine combinations of 2 given spheres, Σ_1 and Σ_2
  - F={Σ : exists real λ, s.t. for all X in $\mathbb{E}^d$

    $\Sigma(X)= \lambda\, \Sigma_1(X)+(1-\lambda)\, \Sigma_2(X)\}$
- When mapping *F* by ϕ, the image of *F* is the line $\phi(F)$ that connects $\phi(\Sigma_1)$ to $\phi(\Sigma_2)$

Types of Pencils



Types of pencils of spheres

- If $\phi(F)$ intersects P at 1 point, F contains a single sphere of radius 0, F is a pencil of concentric spheres
- If 2 intersection points, 2 limit points of radius 0
- If 1 tangent point, as if 2 limit points touching, a tangent pencil

Types of pencils

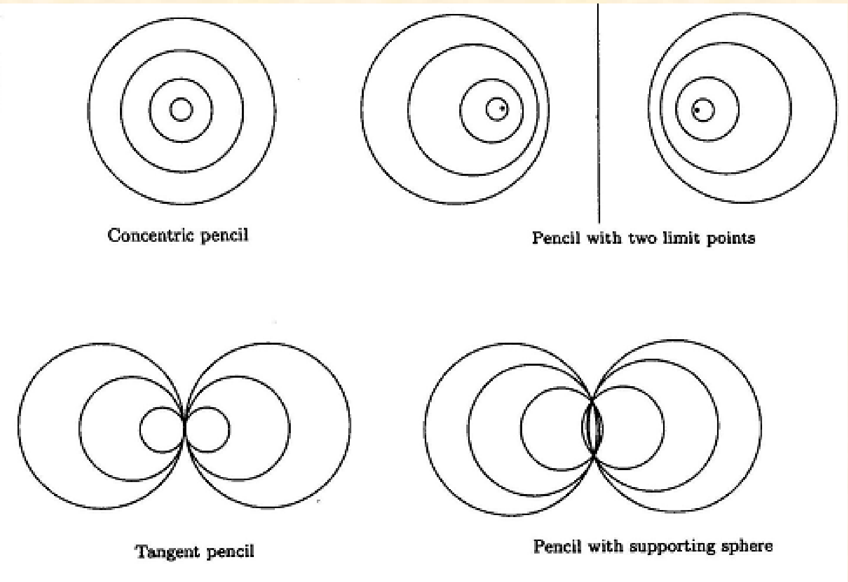
- If $\phi(F)$ does not intersect *P*, exists some family of hyperplanes tangent to *P* containing *F*.
- $\phi(\Sigma_F)$ is the set of tangent points
- Σ_F is the set of points that belong to all spheres in *F*
- All *d*-spheres in *F* intersect on the (*d-1*)-sphere Σ_1 intersect Σ_2
- Σ_F is the supporting sphere of the pencil *F*

Radical Hyperplane

- Any point in H_{12} of Σ_1 and Σ_2 has same power with respect to any particular sphere
- Therefore, H_{12} is the radical hyperplane of a pencil of spheres, or of any 2 spheres in the pencil

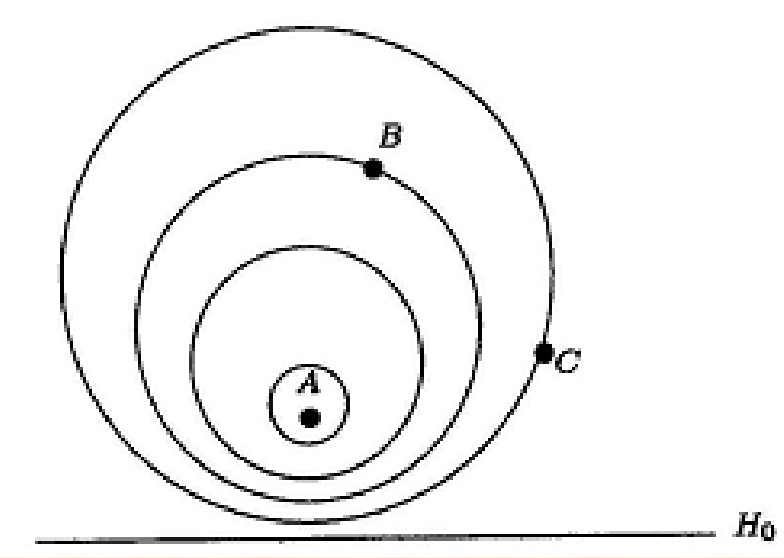
Radical Hyperplane of types of pencils

- Concentric pencil- does not exist
- Pencil with 2 limit points- the bisector
- Tangent pencil- hyperplane bisector
- Supporting Sphere- the affine hull of the supporting sphere



VD's in Hyperbolic Spaces

- The Poincare model of hyperbolic space:
  - $\mathbb{H}^d$ = {X in $\mathbb{E}^d$: xd > 0}
  - A half-space
- Hyperbolic distance: sufficient to decide whether B or C closer to A



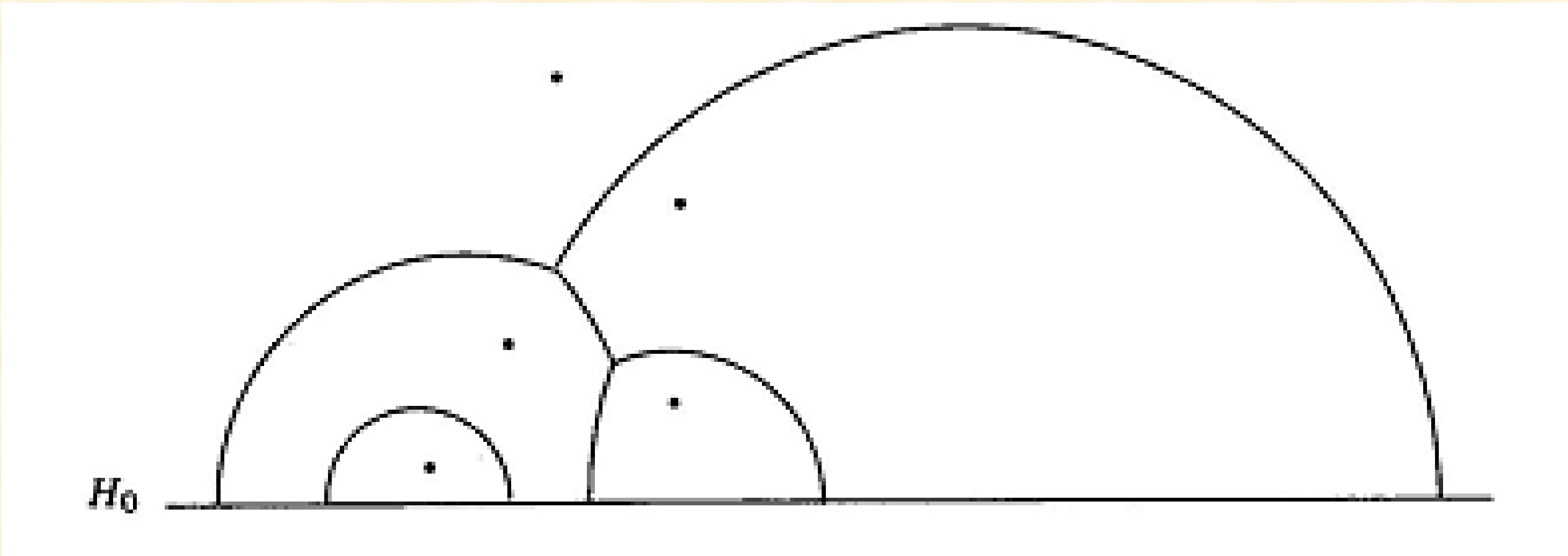
Hyperbolic Distance

- Consider a pencil F_A, with limit points A and A', where A' denotes the symmetric of A w.r.t. hyperplane H_0 of equation x_d=0, the radical hyperplane of F_A
- Distance is the radius of sphere of F_A passing through the point.

Hyperbolic VD

- Given n sites in the poincare half-space $\mathbb{H}^d$, corresponds a region

$V_h(M_i)$= {X in $\mathbb{H}^d$, $\delta_h(X,M_i) \leq \delta_h(X,M_j)$ for any $j \neq i$}



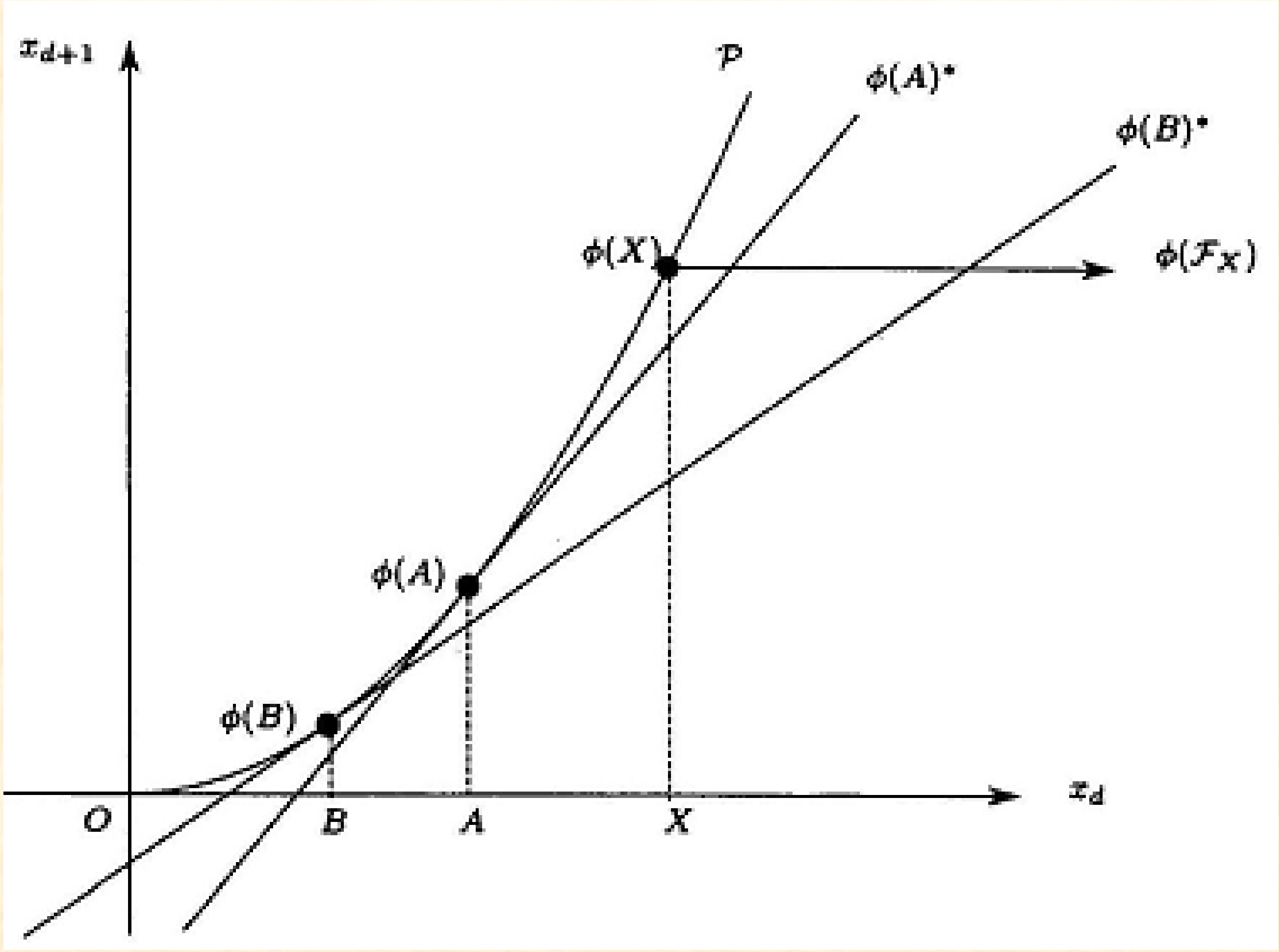
Projecting the Hyperbolic Diagram

- Identify with half-hyperplane x_{d+1}=0, x_d > 0. The hyperplane H_0 is now identified with the subspace {x_{d+1}=x_d=0}.
- F_X is mapped to a line in $\mathbb{E}^{d+1}$ parallel to the x_d axis.
- *X*, *X'* symmetric w.r.t. H_0, F_X has limit points at *X*, *X'* mapped to ϕ(*X*), ϕ(*X'*), symmetric to hyperplane x_d=0 in $\mathbb{E}^{d+1}$

Hyperbolic Diagram in $\mathbb{E}^{d+1}$

- A point X belongs to $V_h(M_i)$ iff the ray parallel to the x_d axis in $\mathbb{E}^{d+1}$ originating at $\phi(X)$ (entirely contained in P), directed toward $x_d>0$ intersects the hyperplane $\phi(M_i)^*$ before any other $\phi(M_j)^*$

Hyperbolic Diagram in $\mathbb{E}^{d+1}$



Consequences of the Projection

- Bisecting surface of 2 points for hyperbolic distance is a half-sphere
  - X is only equidistant between A and B if F_X contains a sphere passing through A and B
  - $\phi(F_X)$ intersects the intersection of $\phi(A)^*$ and $\phi(B)^*$
  - We say that $\phi(X)$ intersects Γ, the projection of the intersection of $\phi(A)^*$ and $\phi(B)^*$ parallel to x_d axis onto P

Bisecting surface

- Γ, the projection of the intersection of $\phi(A)^*$ and $\phi(B)^*$ parallel to x_d axis onto P
- Also the intersection of a hyperplane parallel to x_d and P, which projects to $\mathbb{E}^d$ as a sphere Σ_{AB}, belong to pencil with limit points A,B
- Spheres on that pencil are mapped to $\phi(\Sigma)=\lambda\phi(A)+(1-\lambda)\,\phi(B)$, with corresponding polar hyperplanes, which all contain the intersection of $\phi(A)^*$ and $\phi(B)^*$

Bisecting Surface

- H is therefore a hyperplane polar to a sphere in F_{AB}, limit points AB.
- H has a hyperplane polar to $\phi(\Sigma_{\mathrm{AB}})$, by 17.2.2, so Σ_{AB} belongs to F_{AB}
- Therefore, Σ_{AB} is the sphere in F_{AB} centered on H_0 (unique)

Consequences (2)

- X is equidistant from d+1 points, $A_0,\ldots A_d$, iff $\phi(X)$ is the projection of the intersection of their projected polar halfplanes (intersection $\phi(A_i)^*$) parallel to x_d axis, onto P
- The point at equal hyperbolic distance from d+1 points is the limit point of the pencil containing the sphere circumscribed to the d+1 points of radical hyperplane H_0

Consequences (3)

- HD can be obtained by projecting V(M)=intersection ϕ(Ai)*⁺ parallel to x_d axis onto *P*, then vertically onto x_{d+1}=0
- This double projection creates injection between VD and HD
- Can do in 1 projection
- Complexity is $O(n^{\text{ceil}(d/2)})$, computed in $O(n\log n+n^{\text{ceil}(d/2)})$ time