

# ***Convex Hulls in Two and Three Dimensions***

**The Advanced CG Course**

# **Outline**

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# **CH with Divide-and-Conquer**

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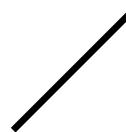
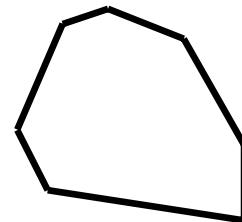
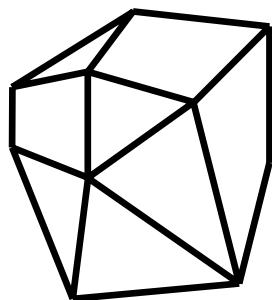
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# Polytope a reminder

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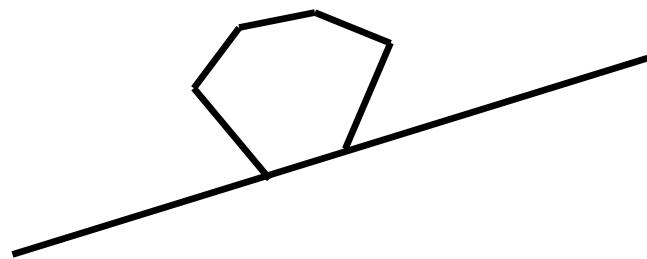
*finite*  
*polytope*      *d-polytope*



# Supporting Hyperplane

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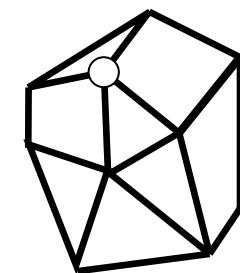
- $H$   
 $H^+$        $H^-$
- $H$  supports  $d$        $P$   
 $H \cap P$   
 $H^+$        $H^-$



# Faces of a Polytope a reminder

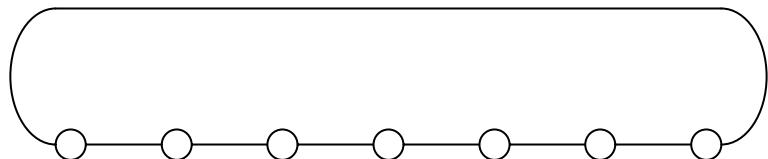
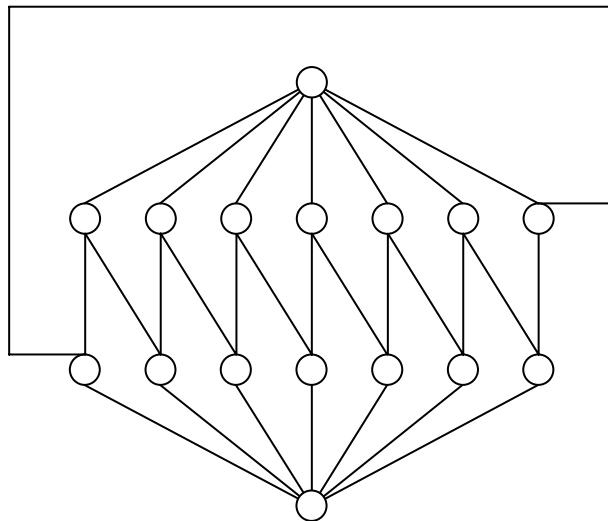
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- $P$   $d$   $H$   $P$   $face$
- $k$   $k$ 
  - vertex*
  - edge*
  - facet*



# **Representation of a 2-polytope**

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# Divide-and-Conquer Convex Hulls in Dimension 2: the Algorithm

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- $A$
- $A$
- $A$        $A_1$        $A_2$
- $A_1$        $A_2$
- $A_1$        $A_2$   
             $A$        $A_1$        $A_2$

## **D&C in 2D Algorithm (cont )**

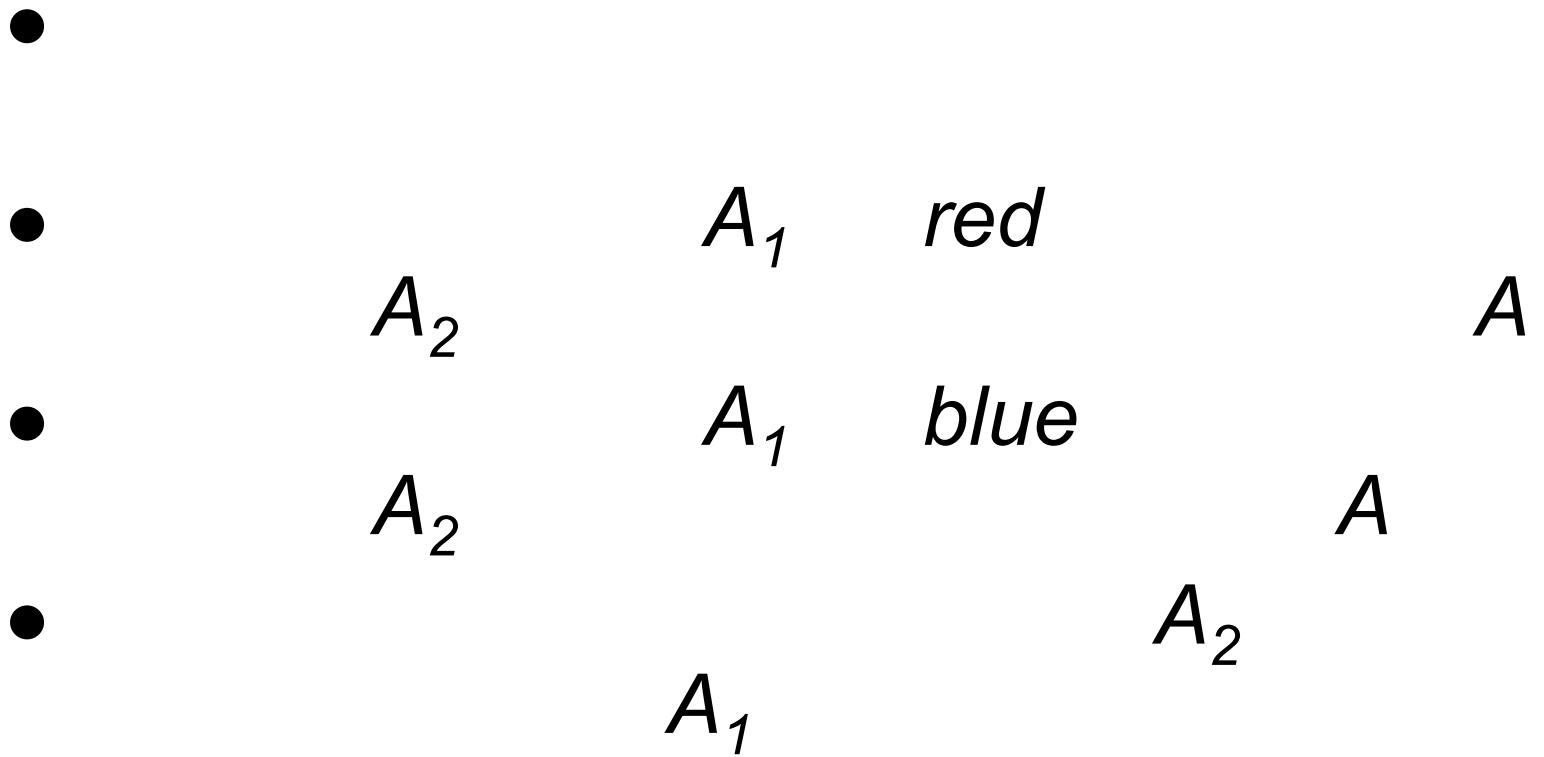
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# Merging in 2D: Coloring Edges

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# Coloring Edges (cont )

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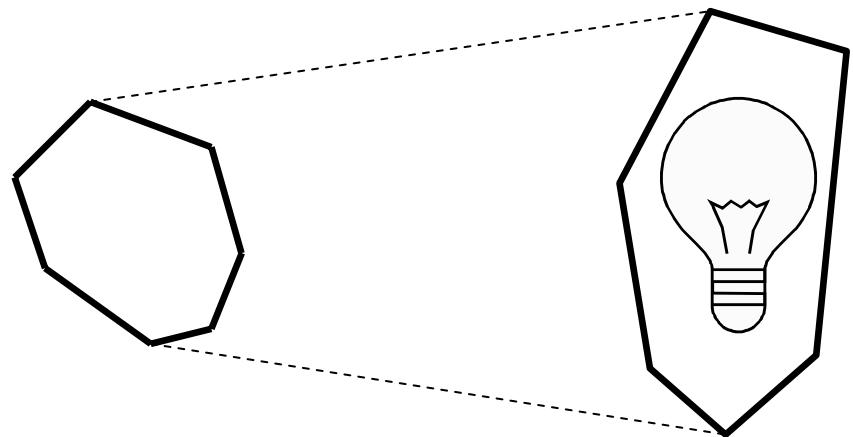
- 

*red*

$A_2$

*blue*

$A_1$



# Merging in 2D: Coloring Vertices

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- 

*Red*

*Blue*

*Purple*

$A_1$

$A_2$

$A$

$A$

$A$

- 

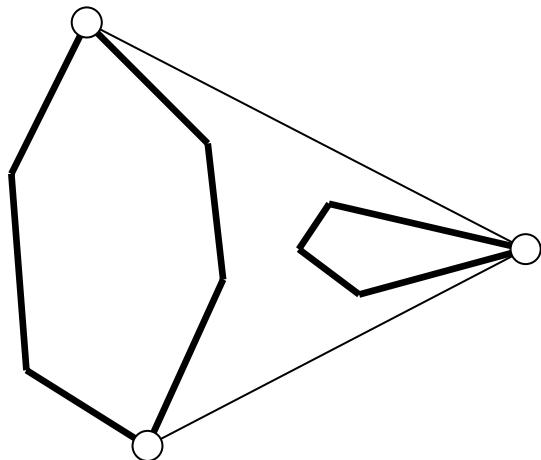
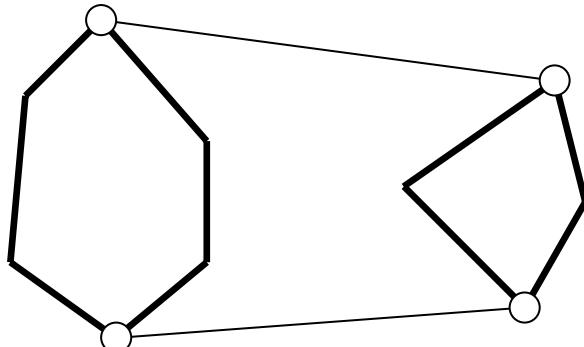
$A_1$

$A_2$

# Coloring Vertices (cont )

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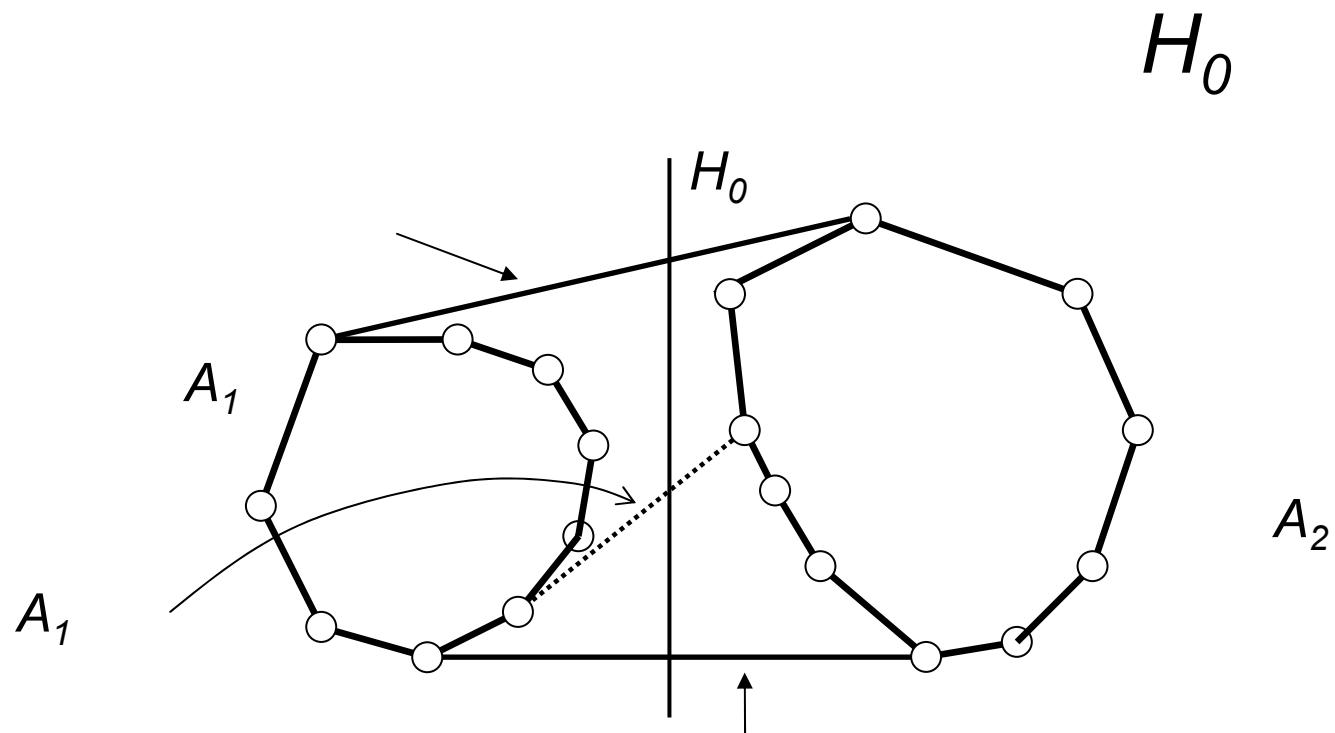
- 



# Merging Convex Hulls

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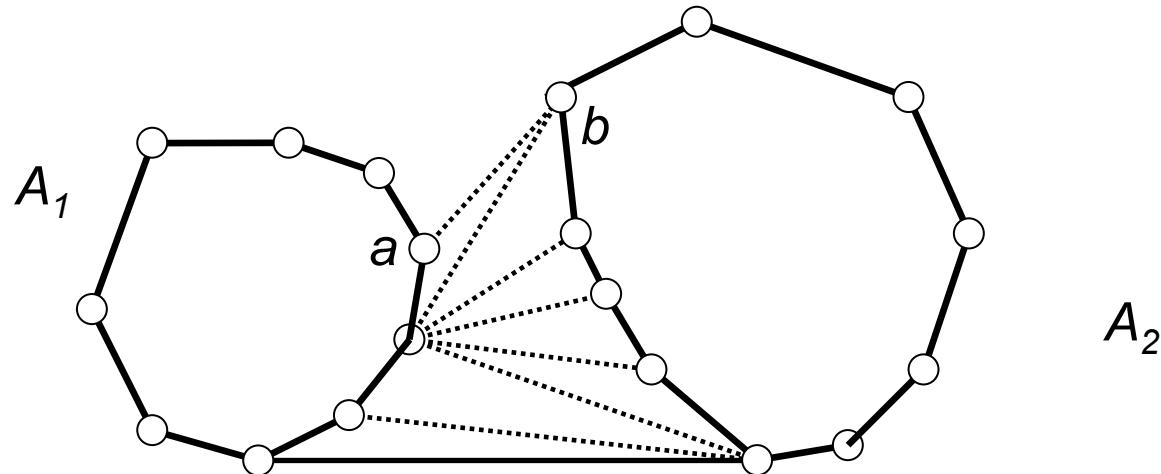
# Finding the Lower Bitangent (1)

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- $a$   $A_1$
- $b$   $A_2$
- $ab$ 
  - $ab$   $A_1$
  - $a \quad pred(a) \quad a$
  - $ab$   $A_2$
  - $b \quad succ(b) \quad b$
- $ab$

# Finding the Lower Bitangent (2)

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# **D&C in 2D: Complexity (1)**

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- 
- 
- 
- 

A

## **D&C in 2D: Complexity (2)**

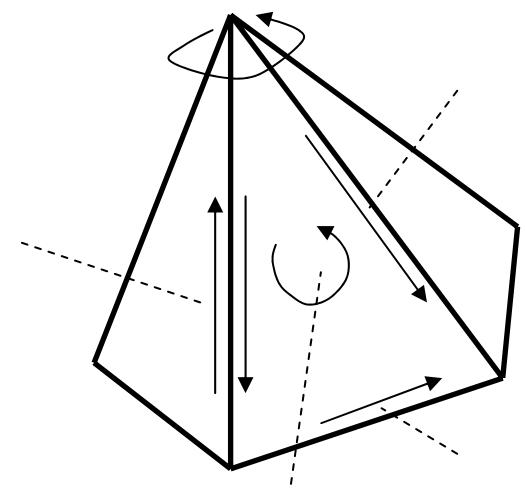
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# **Representation of a 3-polytope**

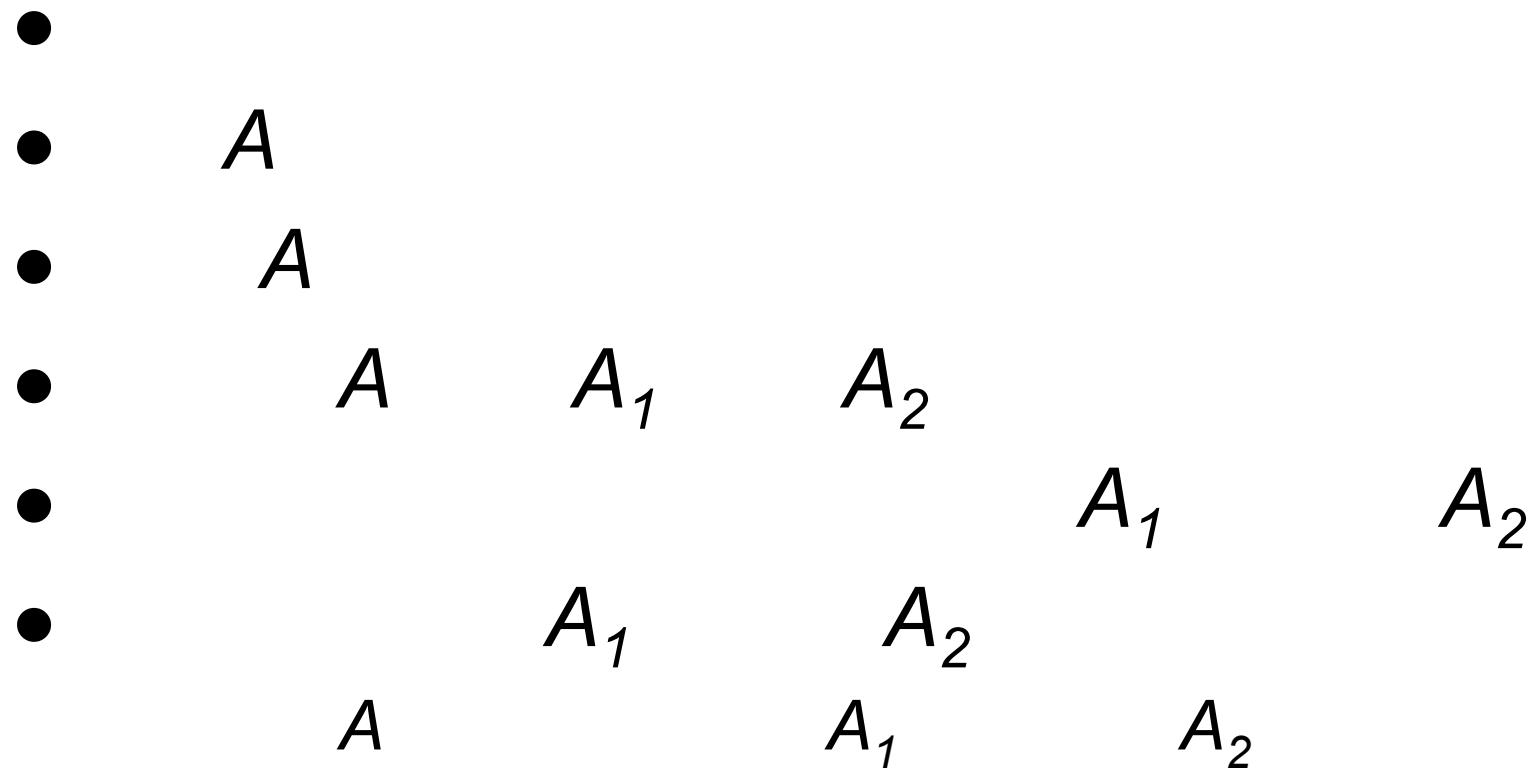
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# Divide-and-Conquer Convex Hulls in Dimension 3: the Algorithm

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# **D&C in 3D Algorithm (cont )**

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- 
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- 



# Merging in 3D: Coloring Faces

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- purple  
blue

$C_1$     $C_2$

$C_1$     $C_2$

$C_1$     $C_2$

*red*

$C_1$     $C_2$

$C_1$     $C_2$

*blue*

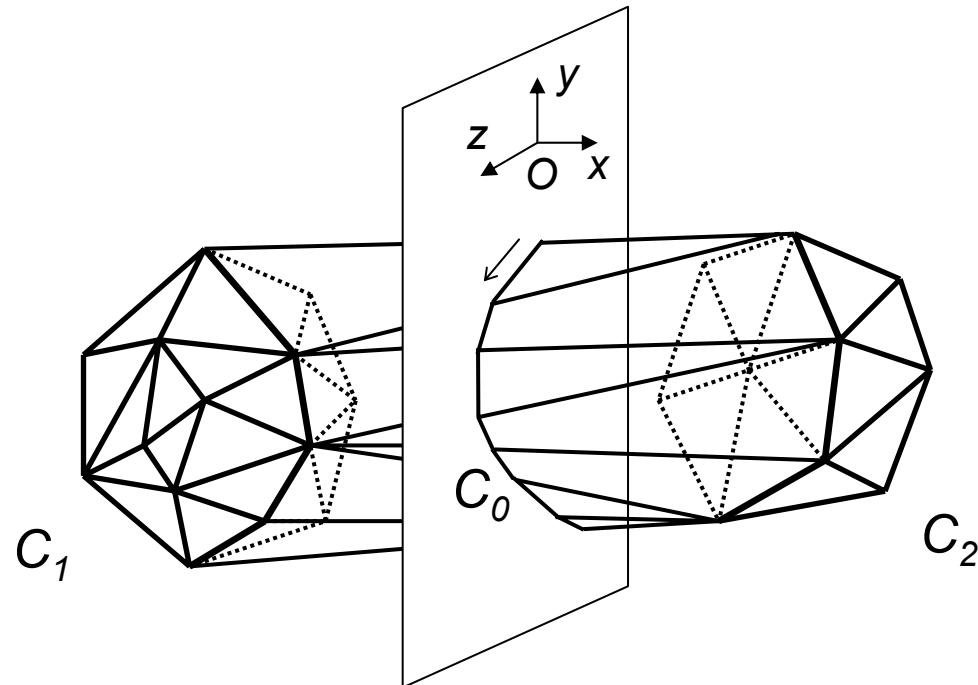
$C$

$C$

*blue*

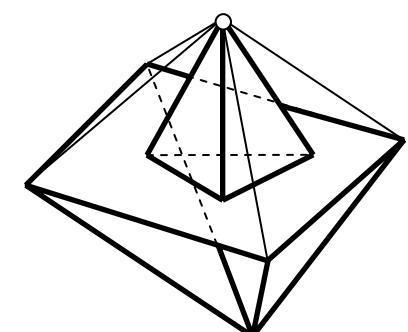
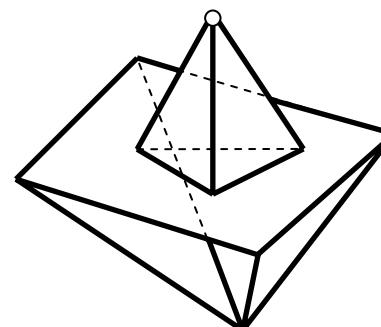
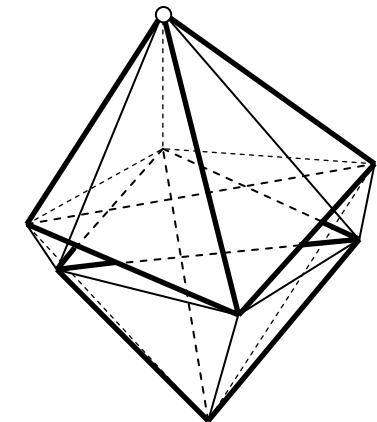
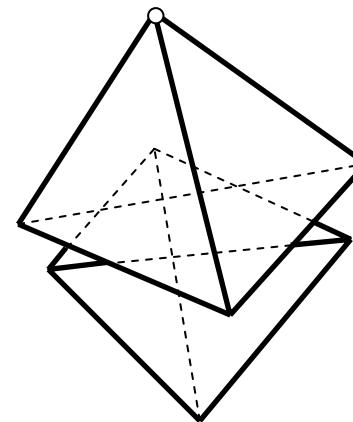
# Merging in 3D Example

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# Purple Faces Examples

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# The New Faces

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- C  $C_1$
- $C_2$
- $H_0$
- 
- 
-

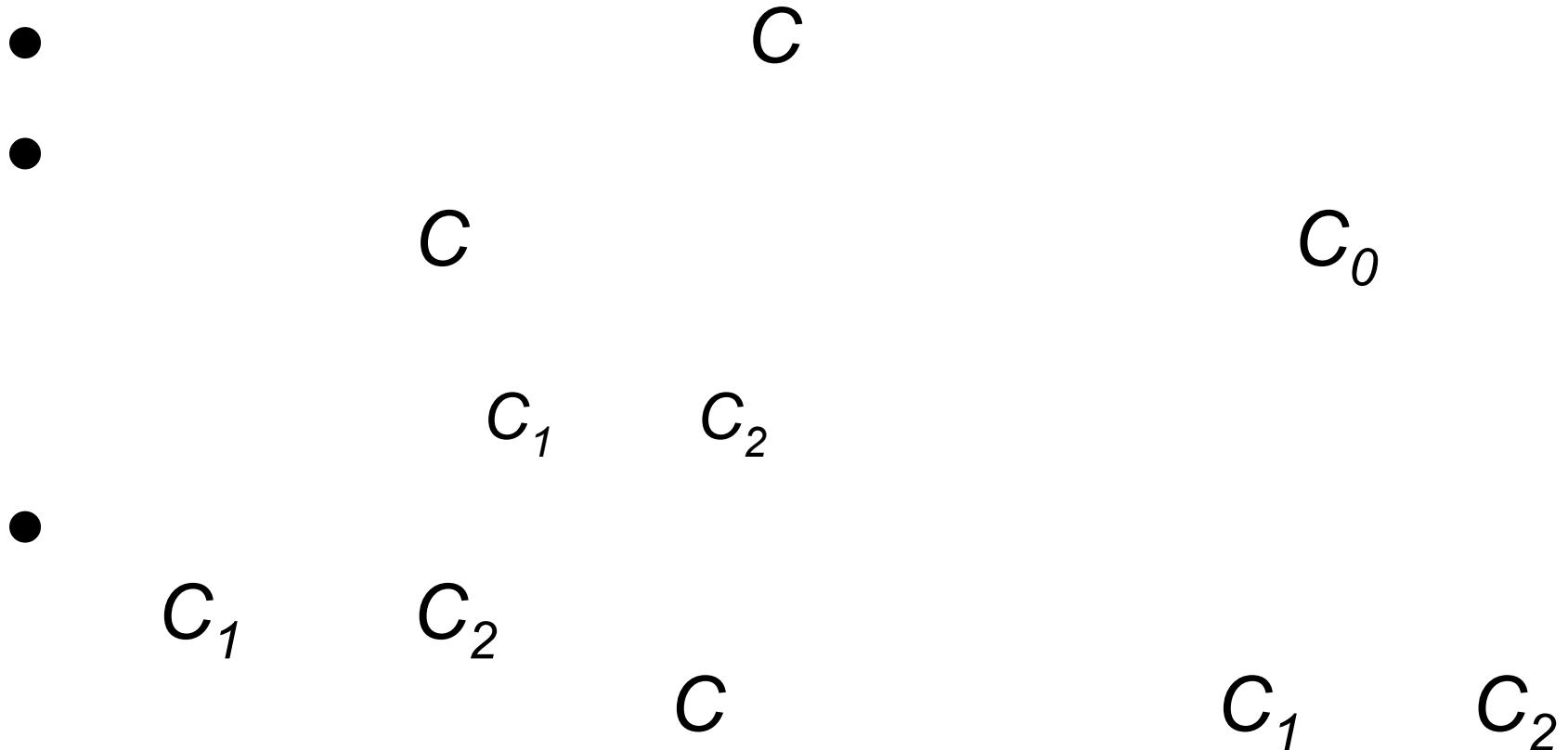
## The New Faces (cont d)

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- $H_0$
- $C_0$
- $C$
- $C_0$

# The Merging Algorithm

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# Finding the First New Edge

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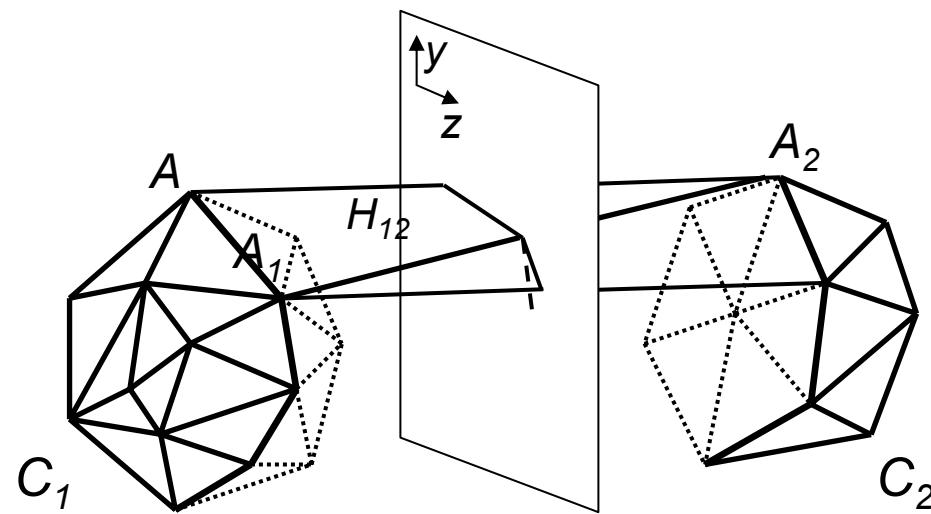
- $C'_1$        $C'_2$        $C_1$   
 $C_2$        $z=0$
- $C'_1$        $C'_2$
- $U'_2$        $U'_1 U'_2 U'_1$   
 $U_1$        $U_2$
- $U_1$        $U_2$   
 $C_1$        $C_2$   
 $U_1 U_2$

# Finding the Other New Faces (1)

- *gift-wrapping*

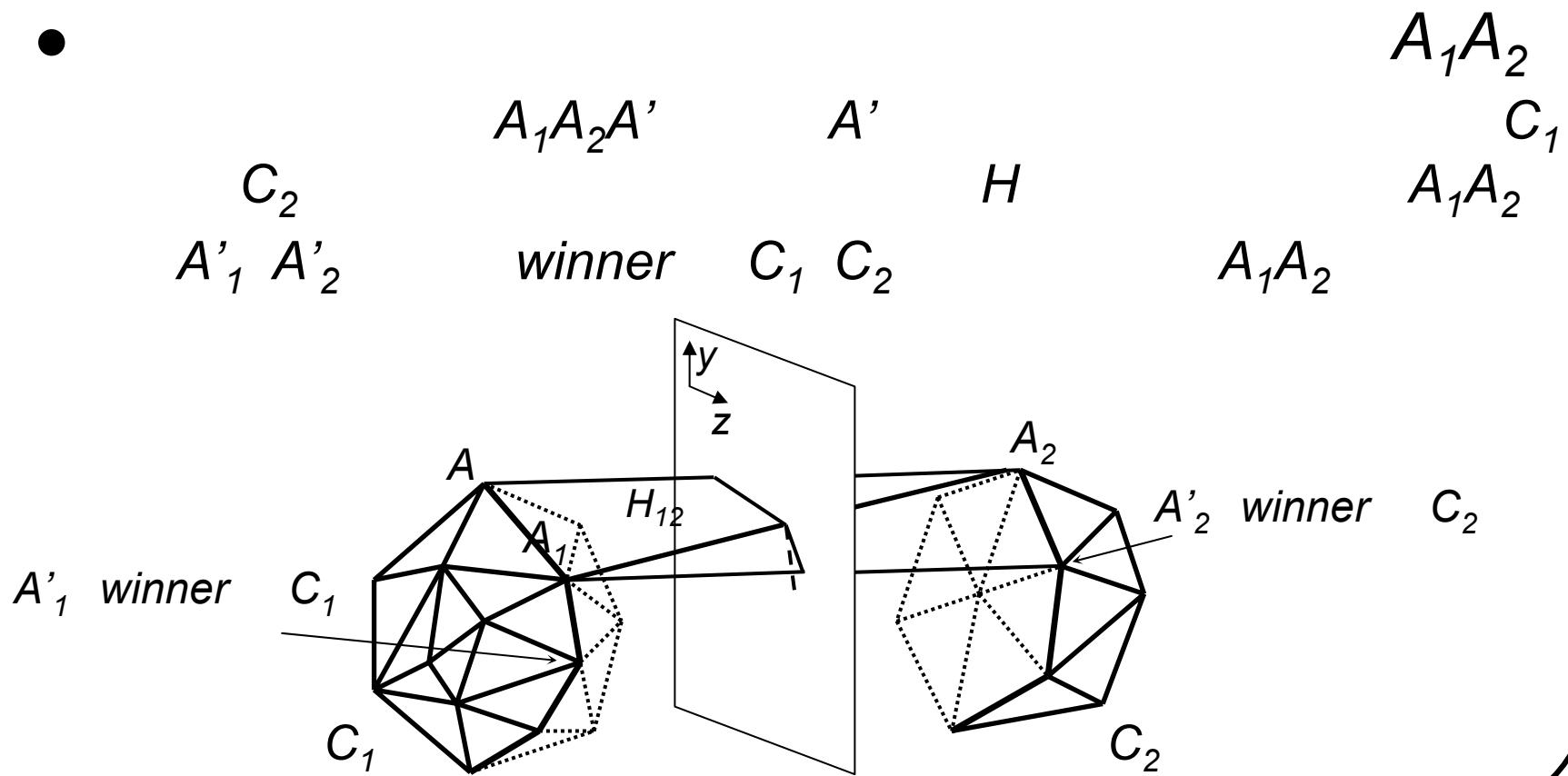
$H_{12}$

$A_1 A_2 A$



# Finding the Other New Faces (2)

- 



# Choosing between $A_1$ and $A_2$

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- $H_i$   $A_1 A_2 A'$
  - $A'_i$   $H_{12}$   $H_i$
- 
- The diagram shows two sets of points,  $C_1$  and  $C_2$ , represented by black-outlined polyhedra. A point  $A$  is marked on the top face of  $C_1$ . A point  $A'_1$  is labeled as the "winner" of  $C_1$ . A point  $A_2$  is marked on the top face of  $C_2$ . A point  $A'_2$  is labeled as the "winner" of  $C_2$ . A vertical plane passes through the centers of  $C_1$  and  $C_2$ , containing a horizontal line segment  $H_{12}$ . A coordinate system is shown with axes  $y$  and  $z$ . Dotted lines connect  $A$  and  $A'_1$  to the plane  $H_{12}$ , and similarly for  $A_2$  and  $A'_2$ .

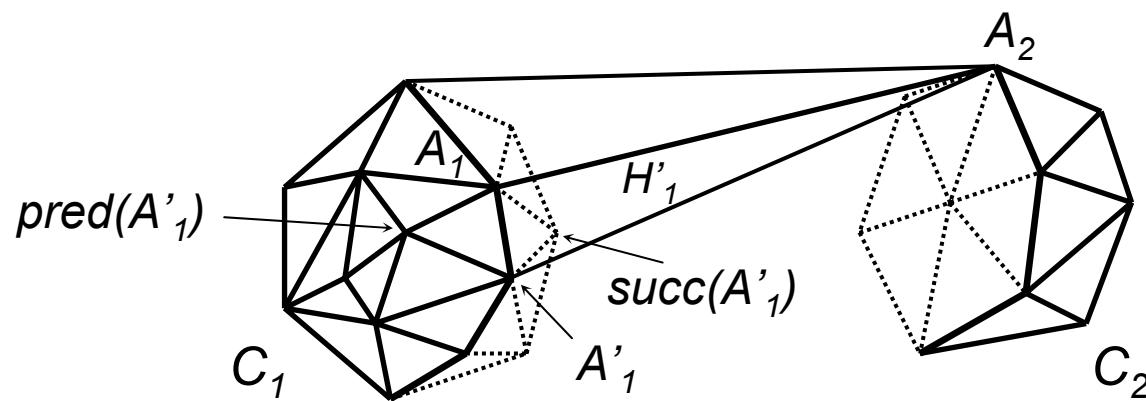
# Finding the Winner (for $C_1$ )

- $A'_1 \quad A_1$   
 $\text{pred } A'_1 \quad \text{succ } A'_1$   
 $A'_1 \quad A'_1$   
 $H'_1 \quad A_1 \quad A_1 A'_1 A_2 H'_1 +$   
 $A_2$   

The diagram illustrates the geometric representation of sets  $C_1$  and  $C_2$ . Two polytopes,  $C_1$  and  $C_2$ , are shown as wireframe meshes. They are interconnected by several edges:  $A_1$  connects a vertex in  $C_1$  to a vertex in  $C_2$ ;  $A_2$  connects another vertex in  $C_1$  to a vertex in  $C_2$ ; and  $H'_1$  is a horizontal edge connecting corresponding vertices in  $C_1$  and  $C_2$ . Dotted lines represent the sets  $\text{pred}(A'_1)$  and  $\text{succ}(A'_1)$ , which are the sets of vertices in  $C_1$  and  $C_2$  respectively that are adjacent to the edge  $A'_1$ .

# The Winner Lemma

- $\frac{\text{winner } C_1}{A_1 A_2}$   
 $A_1$   
 $\text{succ } A'_1$        $A'_1$        $C_1$   
 $\text{pred } A'_1$        $H'_1 +$



# The Winner Proof

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- $\xrightarrow{\hspace{1cm}}$
- $A'_1$   $A_1 A_2 \rightarrow$
- $A'_1 A_1 A_2$   $C_1 A_2 \rightarrow$
- $H'_1$   $C_1 A_2 \rightarrow$
- $H'_1$   $C_1 \rightarrow$
- $A_1 A'_1$   $C_1$   $pred\ A'_1$
- $succ\ A'_1$   $H'_1^+$

# The Winner Proof (cont d)

---

- ←
- $A'_1$   
 $\text{pred } A'_1$
- $C_1$   
 $\text{succ } A'_1$
- $A_1$   
 $H'_1 +$
- $C_1$   
 $A_1 A'_1 \text{succ } A'_1$
- $A_1 A'_1 \text{pred } A'_1 \rightarrow$
- $C_1$   
 $H'_1 +$
- $H'_1$
- $C_1 \rightarrow$
- $A'_1 A_1 A_2$   
 $A'_1$
- $C_1$   
 $A_2 \rightarrow$
- $A_1 A_2 \square$

## Finding the Winner (cont d)

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- 

$A_1$                        $A_1 A_2$

$A'_1$

$C_1$

$A_1$

←

*pred*

*succ*

$H'_1 +$

- 

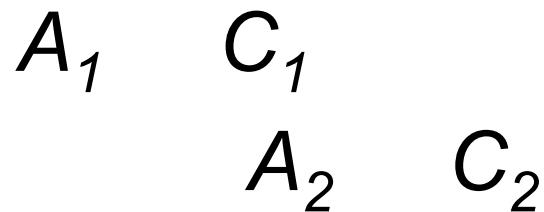
$A_1$

—

# **Incidence of a Vertex to Several Pivots - Lemma**

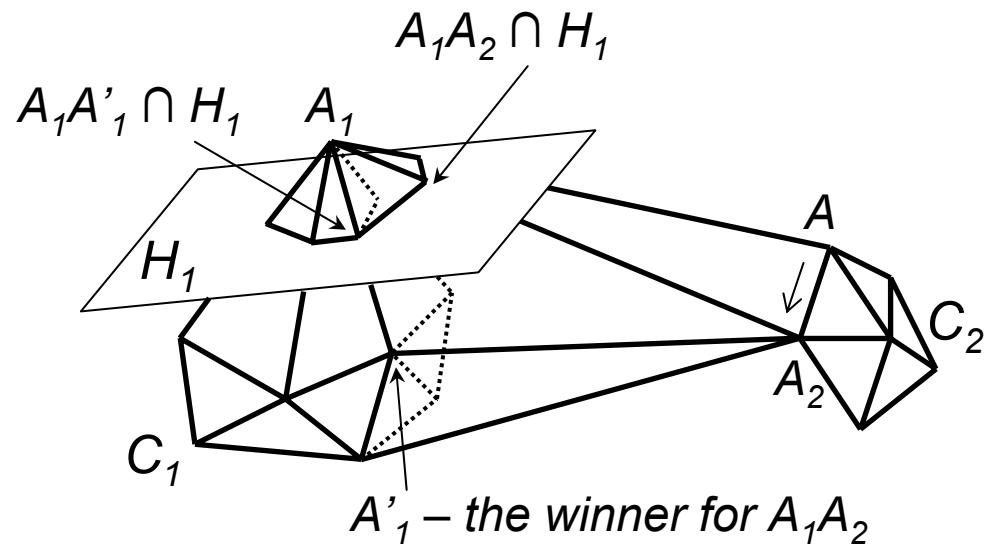
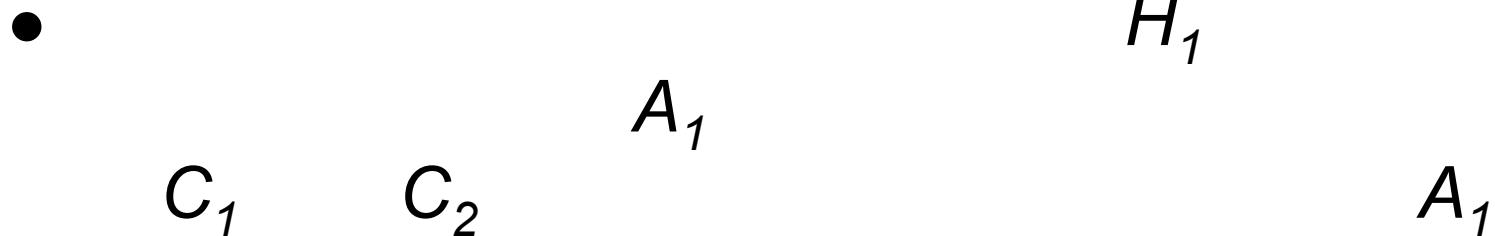
---

-   $A_1$   $C_1$

-   $A_1$   $C_1$   
 $A_2$   $C_2$

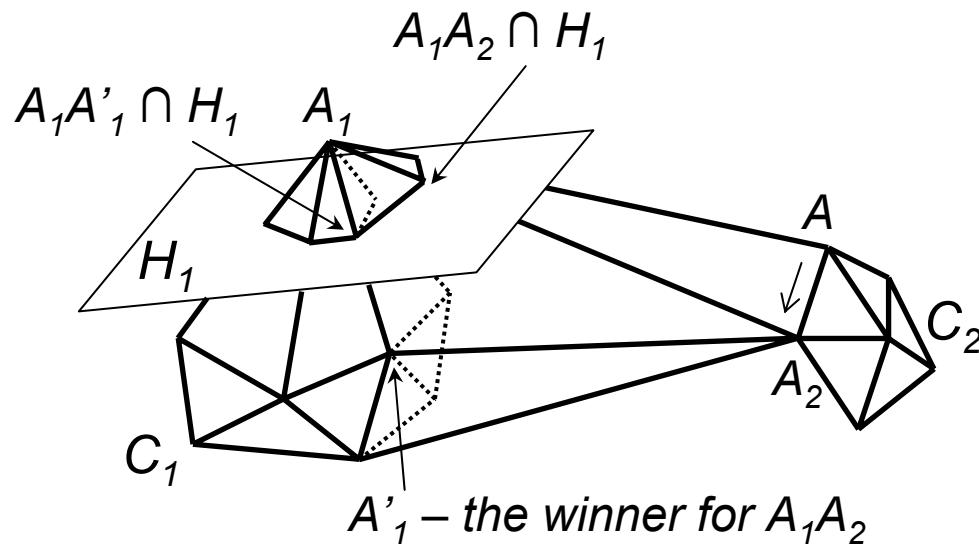
# Incidence of a Vertex to Several Pivots Proof

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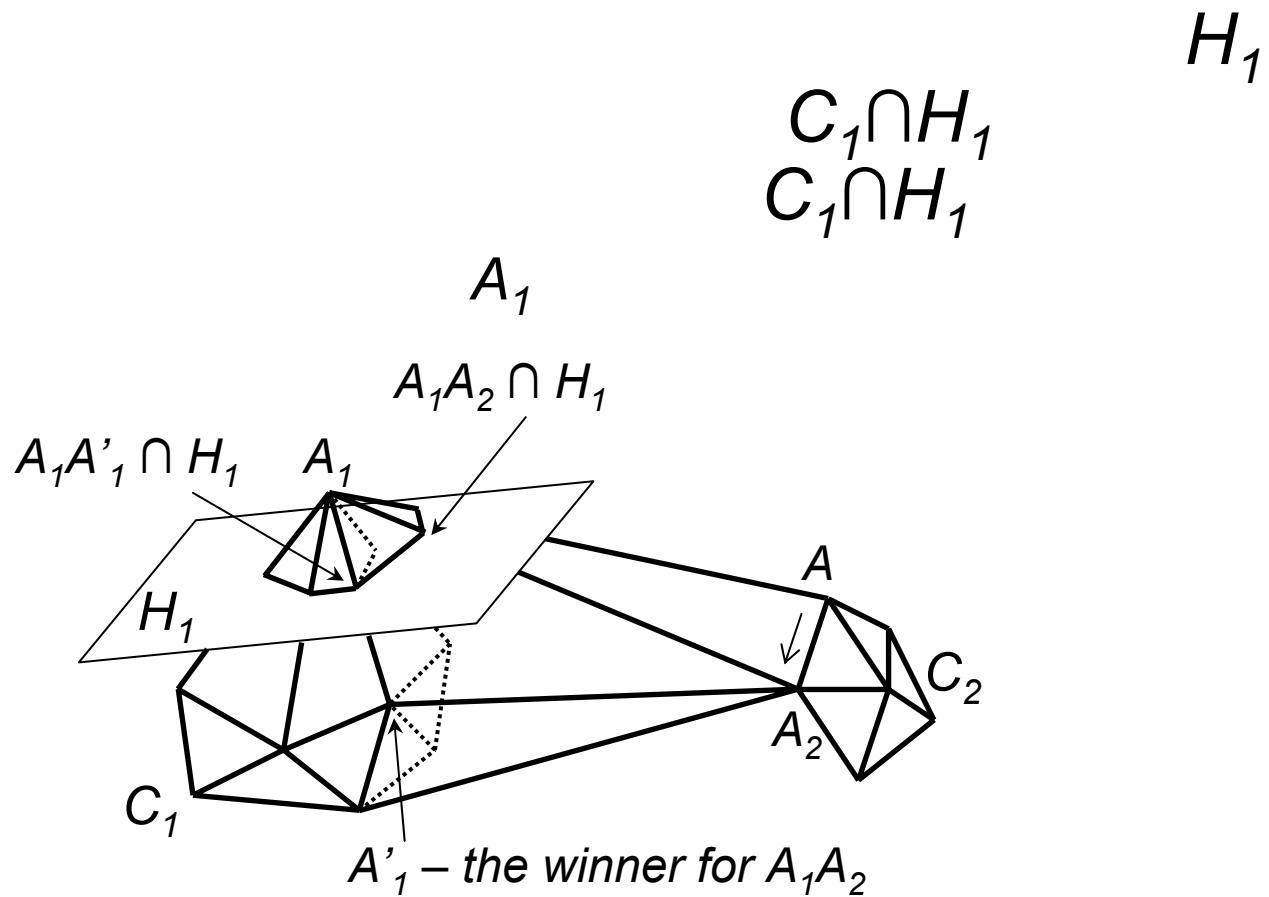
# Incidence of a Vertex to Several Pivots Proof (cont d)

- $A_1A_2$   
 $A'_1$   
 $H_1$        $C_1 \cap H_1$        $A_1A'_1 \cap H_1$



# **Incidence of a Vertex to Several Pivots Proof (cont d)**

- 1



# The Process of Finding Winners

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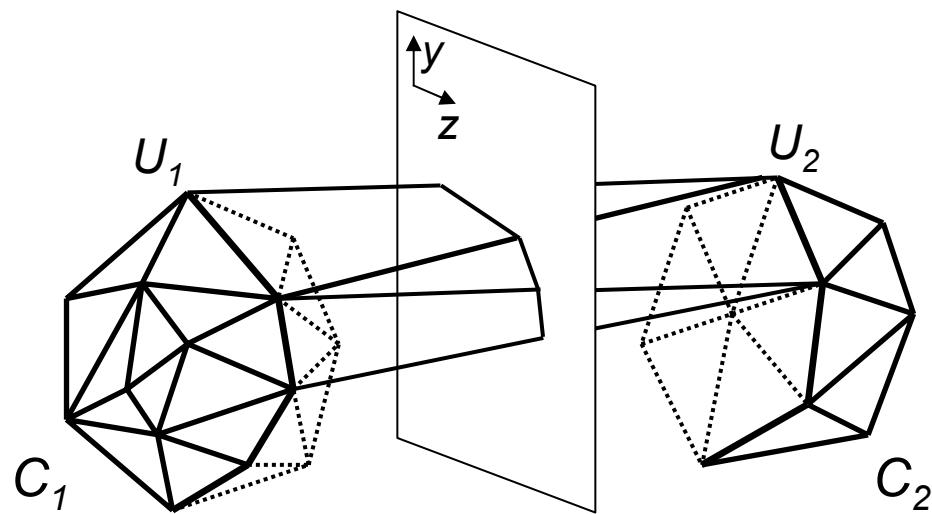
- $C_1$
- $A_1$        $C_1$        $A_1$
- $A_1$        $A_1$

# Finding Winners (cont d)

---

- 

$U_1 U_2$



# **Reconstruction of $\text{conv}(\mathbf{C}_1 \cup \mathbf{C}_2)$**

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- $C_1$   $C_2$

—

- $C_1$   $C_2$   $C$

# **Complexity of the Algorithm (1)**

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- 

$$\begin{array}{cc} C_1 & C_2 \\ n_1 & n_2 \end{array}$$

- 

$$\begin{array}{cc} & n_1+n_2 \\ C_1 & C_2 \end{array}$$

$$n_1+n_2$$

## **Complexity of the Algorithm (2)**

---

- 

$C_1$

$C_2$

$n_1 + n_2$

# **Complexity of the Algorithm (3)**

---

- $C$   
 $C_1 \quad C_2$
- $n_1 + n_2$   
 $n_1 + n_2$
- $n_1 + n_2$   
 $O(n \log n)$

# **Convex hull of a Polygonal Line**

---

•

$x$

•

*any*

# Polygonal Line - Definitions

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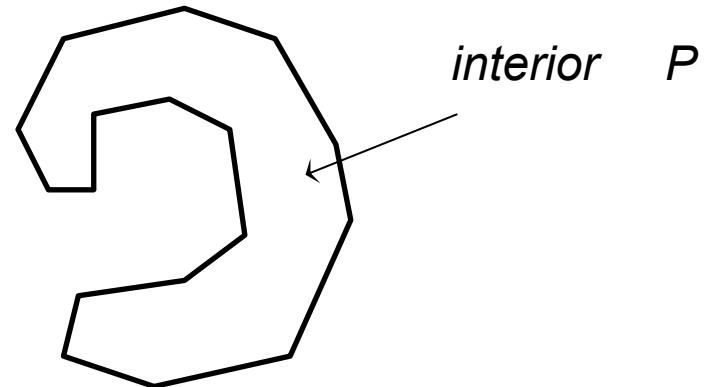
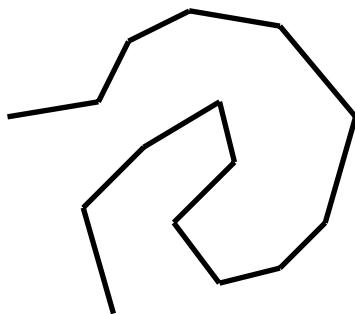
- *vertices*
- *edges*
- *simple*
- *closed*

# Polygonal Lines and Polygons

- The diagram shows a polygon labeled  $P$ . The interior of the polygon is shaded gray, while the exterior is white. The word "interior" is written above the polygon, and the word "exterior" is written below it.

# Examples of Polygonal Lines

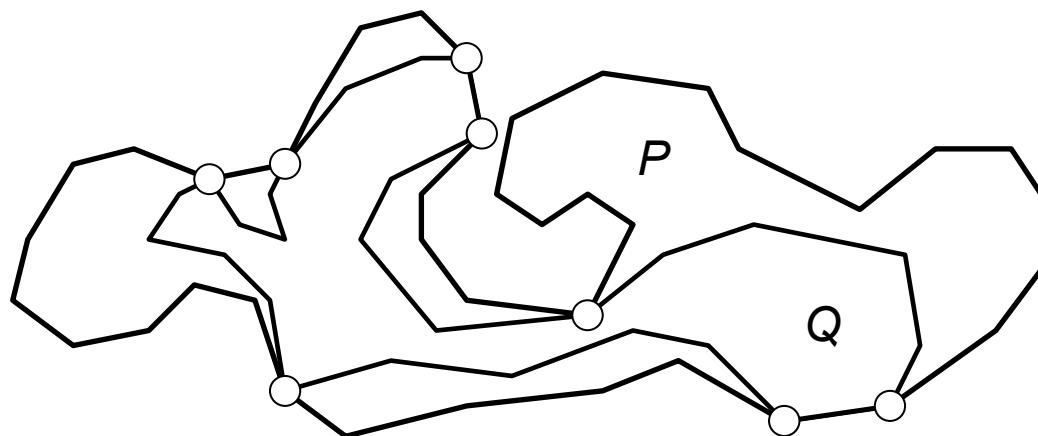
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# Theorem 9.4.1

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•  $P$   $Q$   
 $Q$   
 $P$   $P$   $Q$



# Ranks of Vertices

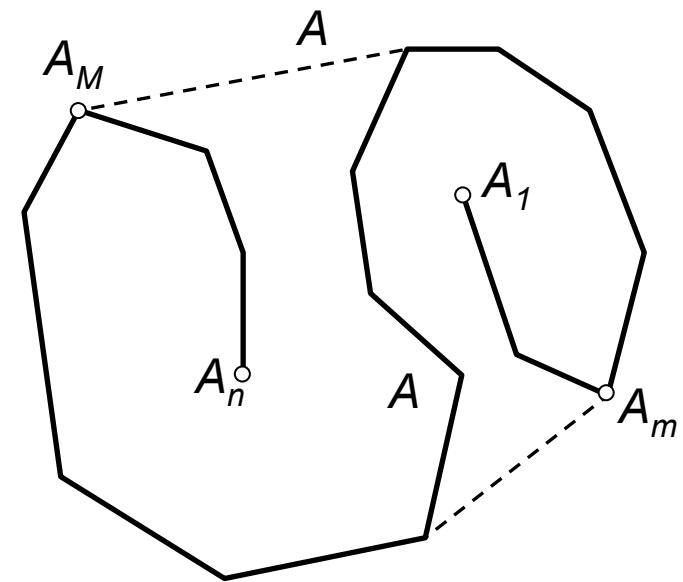
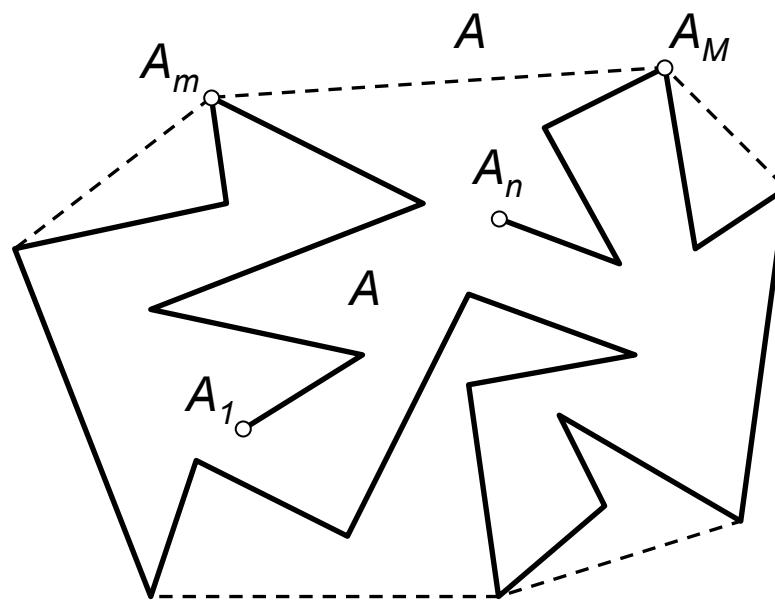
---

- $A$   $n$
- $A$   $A_1$   $A_2$   $\dots$   $A_n$   $A$
- $A$   $A$
- $A_m$   $A_M$   $A$   
 $A$

# Ranks of Vertices (example)

- $A_m \ A_M$

$A$   
 $A$



## **Corollary 9.4.2**

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- 

$A$

$A_m$

$A_M$

$A$

$A_m$

$A$

$A_M$

# The Algorithm that Builds Convex Hull of a Polygonal Line

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$A$

$A$

$A_1$   $A_2$

$A_n$

$A$



$A$

# **The Algorithm Data Structures**

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- $A_i$   $A$
- $A_i$   $A_i$
- $A_i$   $A$

# The Algorithm in general

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- 

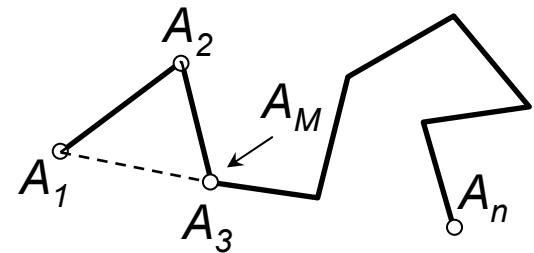
$A_i$

$A_{i-1}$

$A_i$

$A_1 A_2 A_3$

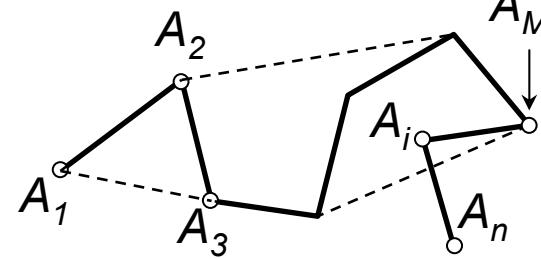
$A_3$



$A_i$

$A_{i-1}$

$A_M$



# The 2 Phases of the Algorithm

---

-   
 $A_i$   
 $A_{i-1}$
-   
 $A_i$   
 $A_{i-1}$

## Lemma 9.4.3

---

- $\begin{array}{ccc} \text{pred}(A_M) & \text{succ}(A_M) \\ & A_M \\ & & A_{i-1} \end{array}$
- $\begin{array}{ccc} H_p^+ & H_s^+ \\ & \\ \text{pred}(A_M)A_M & & A_M \text{succ}(A_M) \\ & A_{i-1} & \\ & & & A_{i-1} \end{array}$
- $\begin{array}{ccccc} \hline & A_i & & A_{i-1} & A_i \\ & & H_p^+ \cap H_s^+. & & \end{array}$

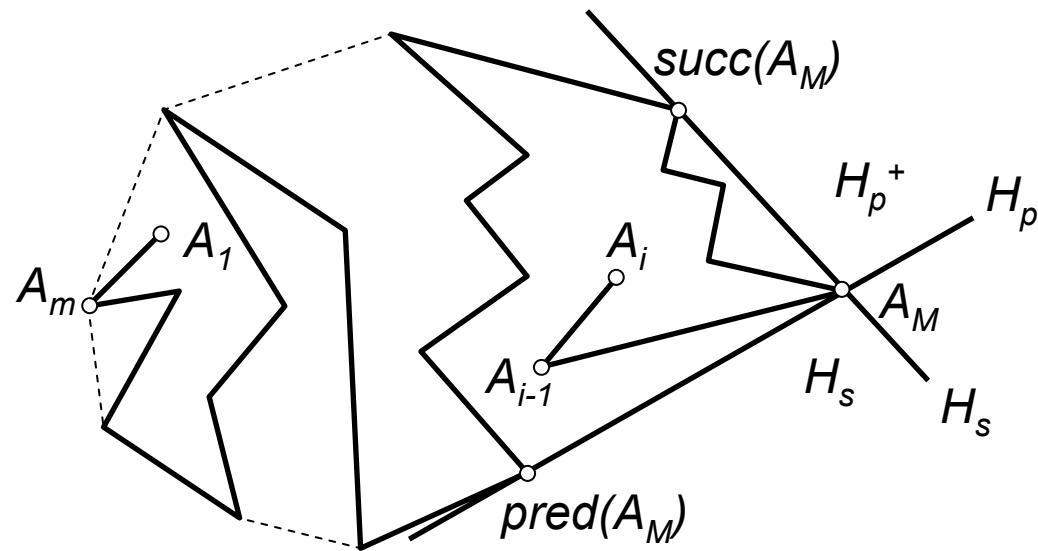
## Lemma 9.4.3    Example

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- $A_i$

$$H_p^+ \cap H_s^+$$

$$A_i$$



## Lemma 9.4.3 Proof

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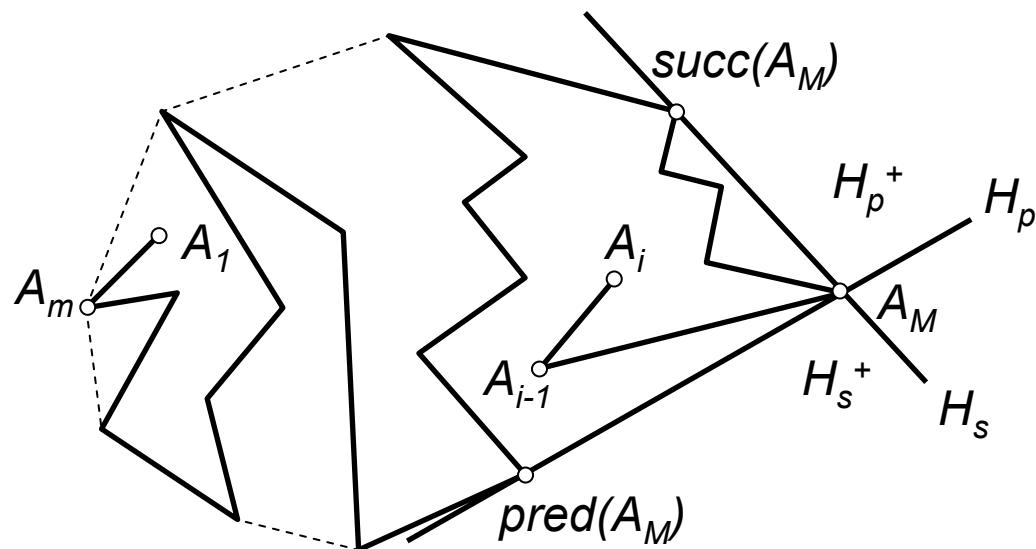
•  $\xrightarrow{\quad}$

$A_i$

$H_p^+ \cap H_s^+$

$A_{i-1}$

$A_{i-1}$   
 $H_p^+ \cap H_s^+$ .



## Lemma 9.4.3 Proof (cont d)

•    ←

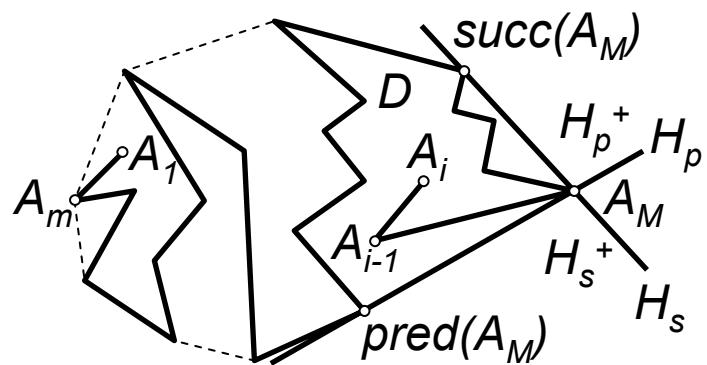
$A$   
 $\text{succ}(A_M)$   
 $A_M \text{succ}(A_M)$

$\text{pred}(A_M)$   
 $\text{pred}(A_M) A_M$   
 $A_{i-1}$

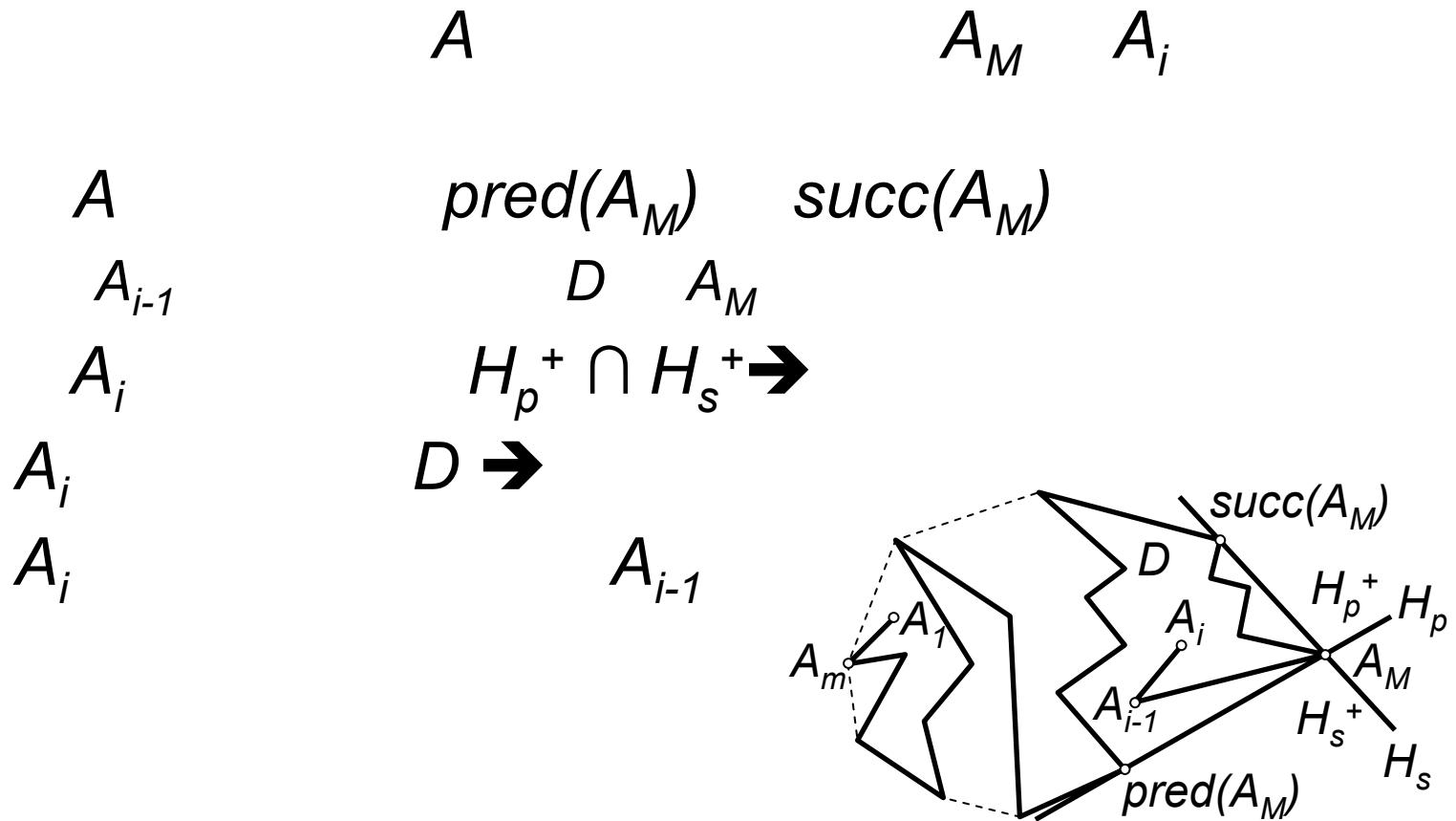
$D$

$D$

$A_{i-1}$



## Lemma 9.4.3 Proof (cont d)



# **Complexity of the Algorithm**

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- 

$$pred(A_M)A_MA_i \quad A_Msucc(A_M)A_i$$

- 

$$A_{i-1} \quad \text{red} \quad A_i$$

-