Convex Hulls in Two and Three Dimensions

The Advanced CG Course

Outline

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-

CH with Divide-and-Conquer

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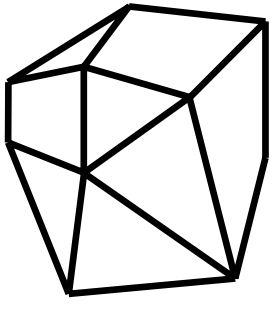
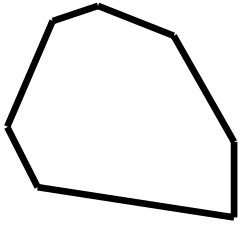
-
-

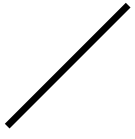
Polytope a reminder

- *finite*

polytope *d-polytope*







Supporting Hyperplane

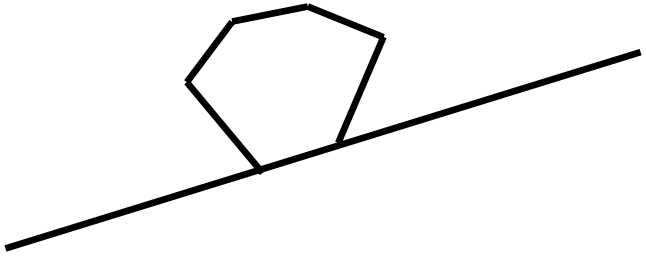
- H

  H^+ H^-

- H supports d P

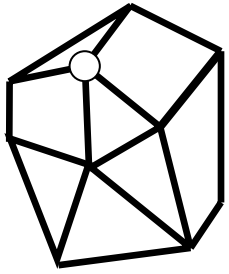
  $H \cap P$

  H^+ H^-

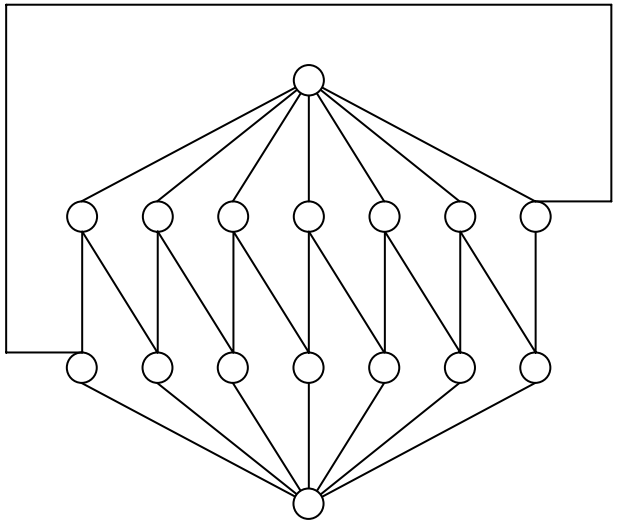


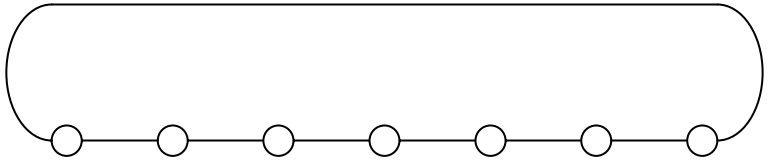
Faces of a Polytope a reminder

- d P
  H *face*
  P
- k k
  vertex
  edge
  facet



Representation of a 2-polytope





Divide-and-Conquer Convex Hulls in Dimension 2: the Algorithm

- A
- A
- A A_1 A_2
- A_1 A_2
- A_1 A_2
  A A_1 A_2

D&C in 2D Algorithm (cont)

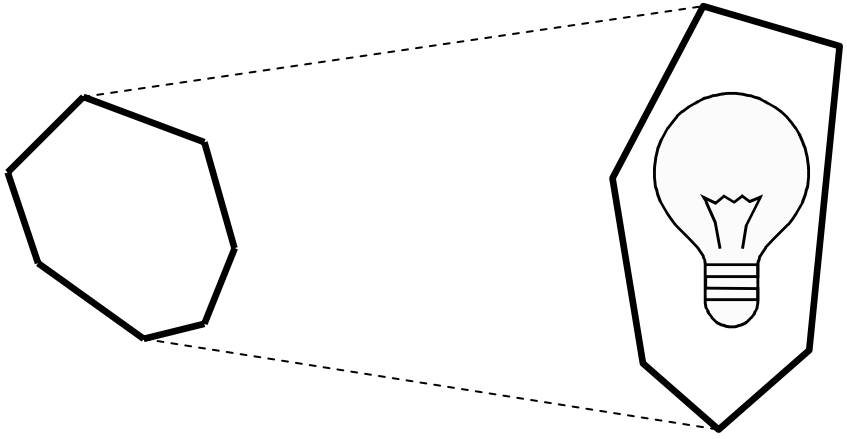
-
-
-

Merging in 2D: Coloring Edges

-
- A_1 *red* A_2 A
- A_1 *blue* A_2 A
- A_2 A_1

Coloring Edges (cont)

- *red* A_1 A_2 *blue*



Merging in 2D: Coloring Vertices

- A_1

  A_2

  Red A

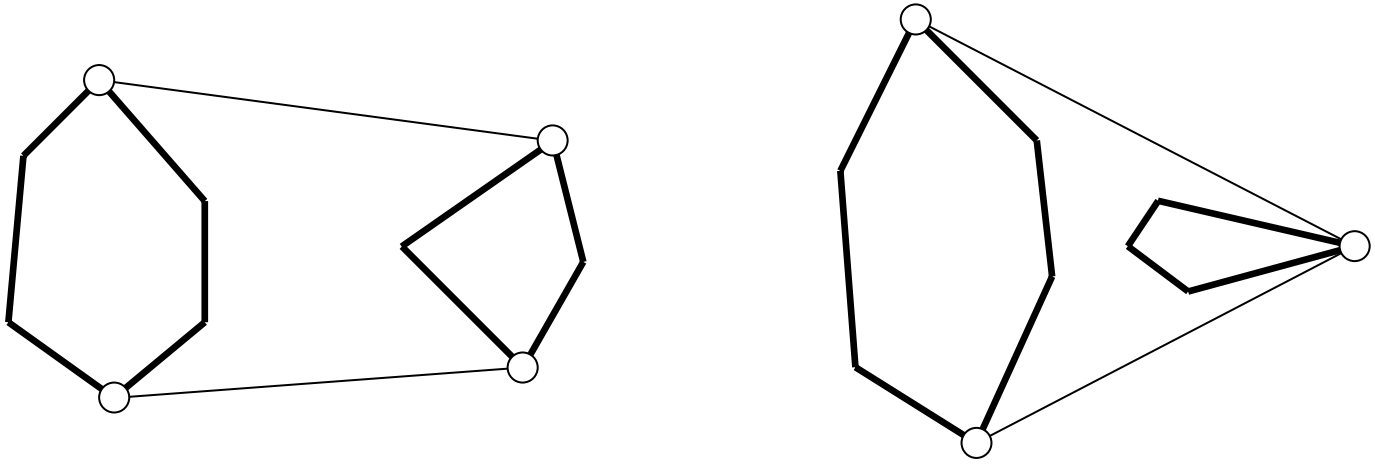
  Blue A

  Purple A

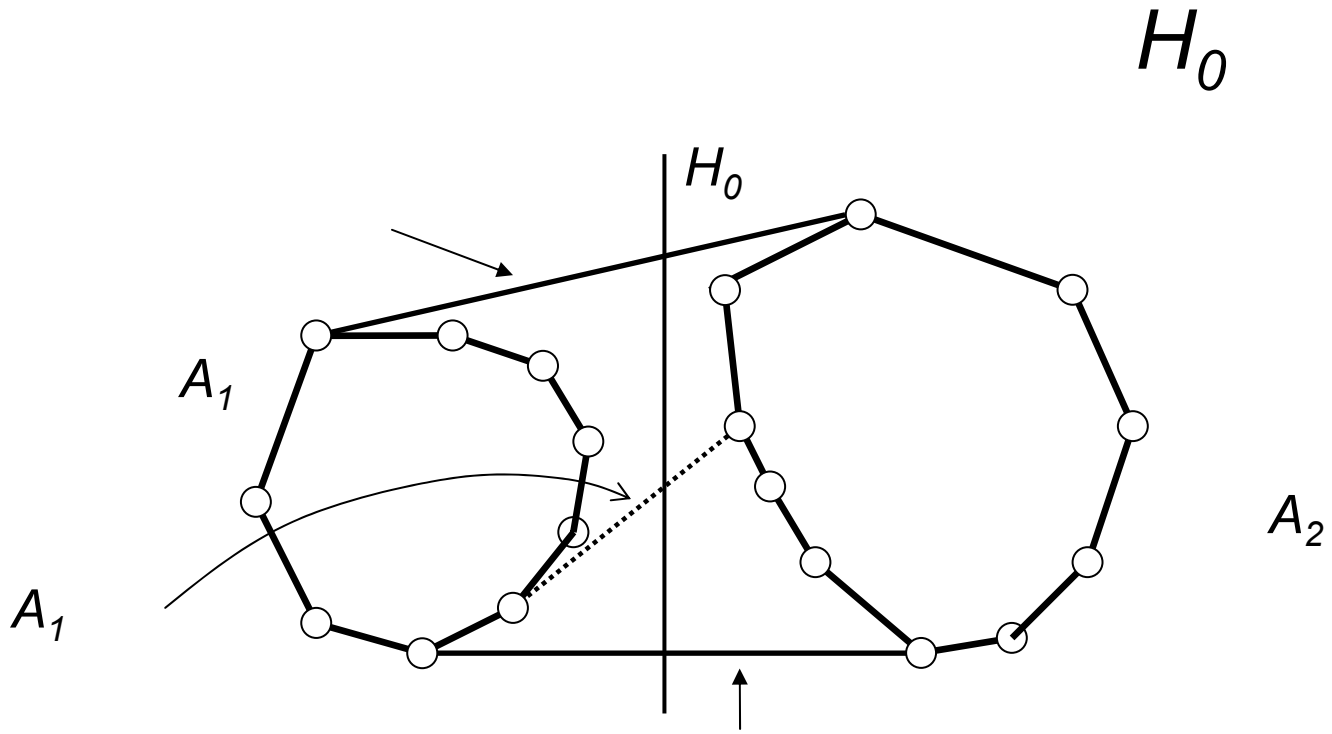
- A_2

  A_1

Coloring Vertices (cont)

- 

Merging Convex Hulls

- 

Finding the Lower Bitangent (1)

- a A_1
- b A_2
- ab

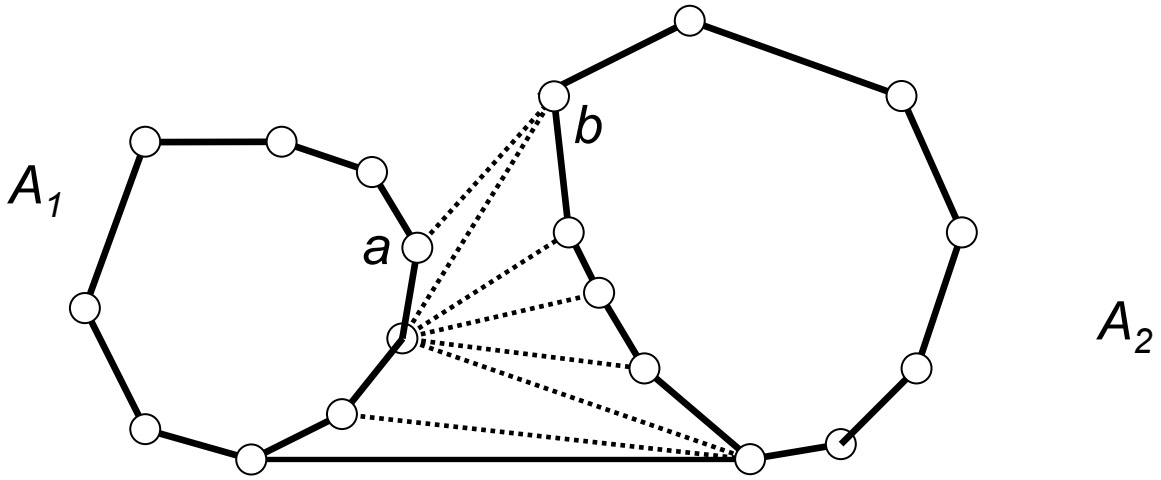
  ab A_1

  a $pred(a)$ a

  ab A_2

  b $succ(b)$ b
- ab

Finding the Lower Bitangent (2)



-
-

D&C in 2D: Complexity (1)

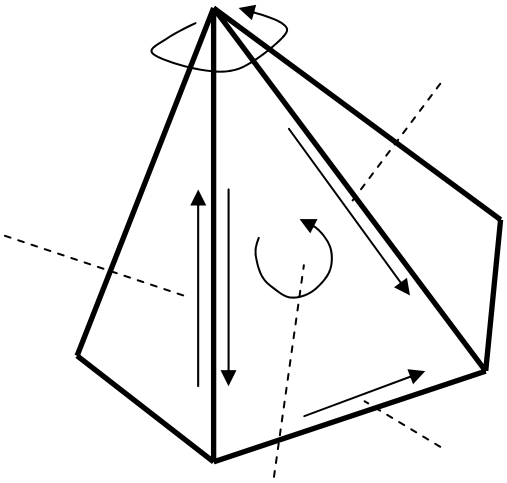
-
-
- A
-

D&C in 2D: Complexity (2)

-
-
-
-

Representation of a 3-polytope

-



Divide-and-Conquer Convex Hulls in Dimension 3: the Algorithm

-
- A
- A
- A A_1 A_2
- A_1 A_2
- A_1 A_2
- A A_1 A_2

D&C in 3D Algorithm (cont)

-
-
-

Merging in 3D: Coloring Faces

- C_1 C_2

  C_1 C_2

  C *red*

  C_1 C_2 C *blue*

  C_1 C_2 C

  purple

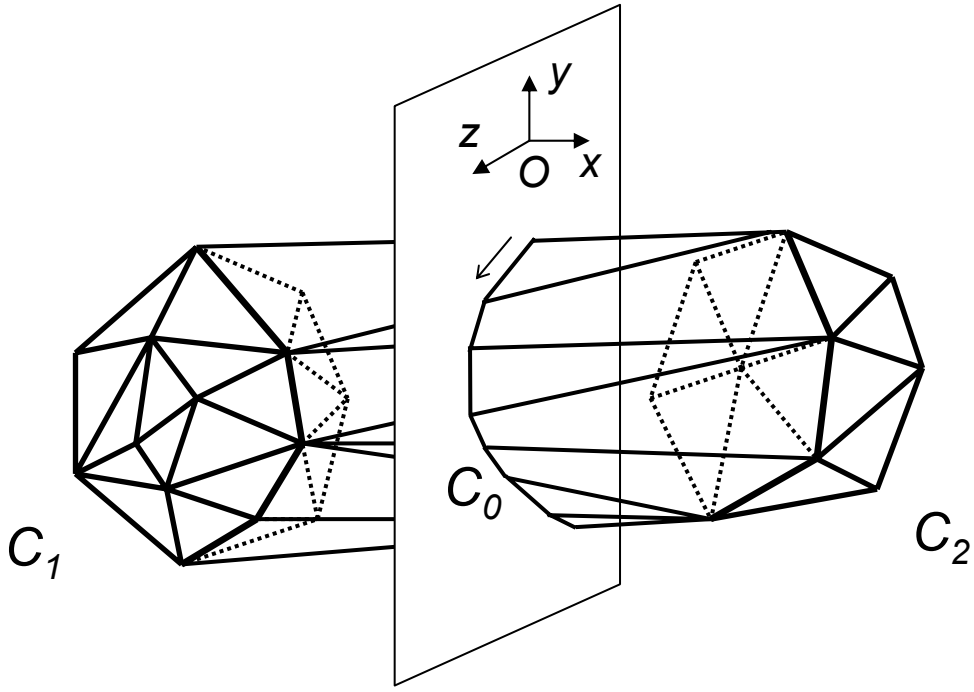
  blue

  C_1 C_2 C

  purple

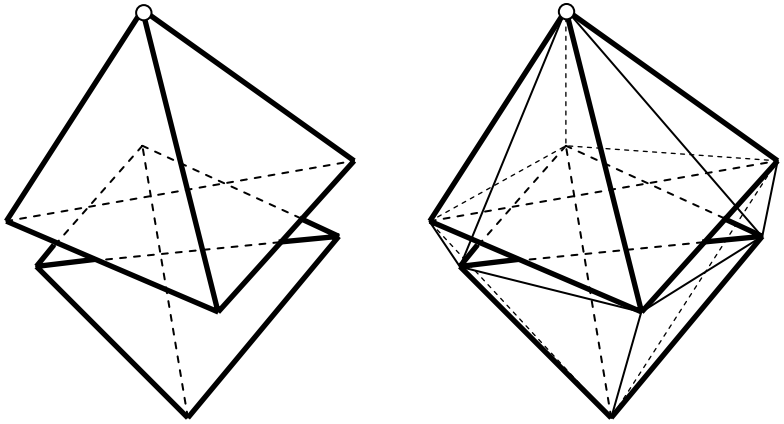
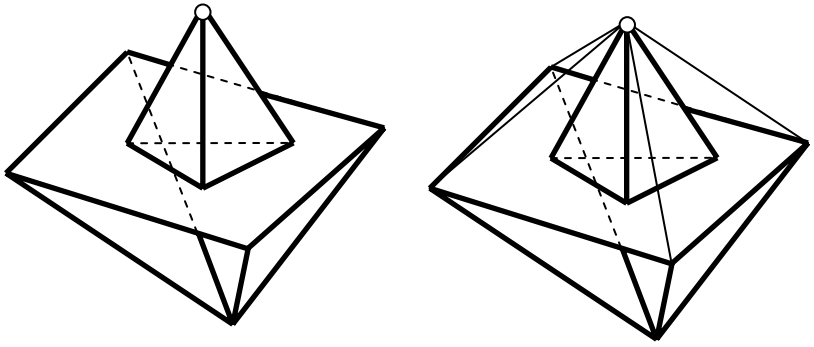
  blue

Merging in 3D Example



- _____

Purple Faces Examples

- 
- 

The New Faces

- C C_1 C_2
- H_0
-
-

The New Faces (cont d)

- H_0

  C_0

- C_0

  C

- C

  C_0

The Merging Algorithm

- C
- C C_0

  C_1 C_2
- C_1 C_2

  C C_1 C_2

Finding the First New Edge

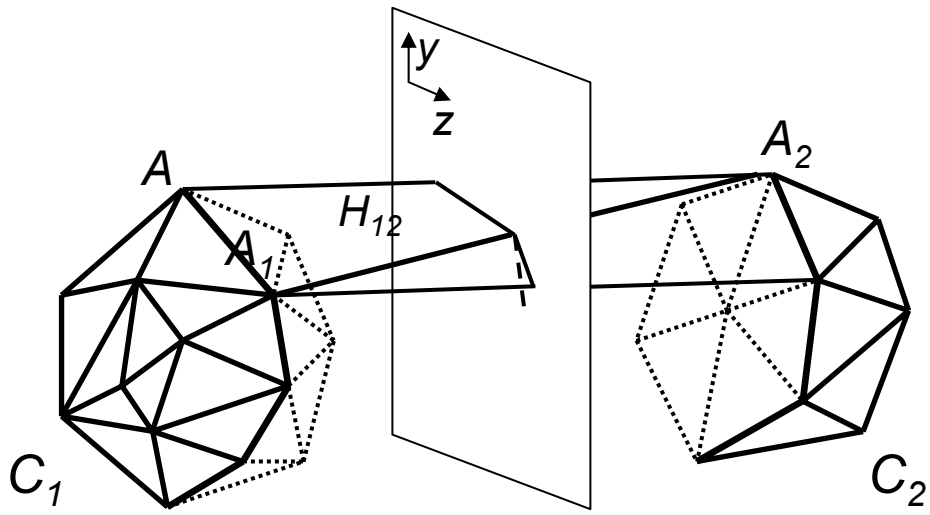
- C'_1 C'_2 C_1 C_2 $z=0$
- C'_1 C'_2 $U'_1U'_2$ U'_1 U'_2 U_1 U_2
- U_1 U_2 U_1U_2 C_1 C_2

Finding the Other New Faces (1)

- *gift-wrapping*

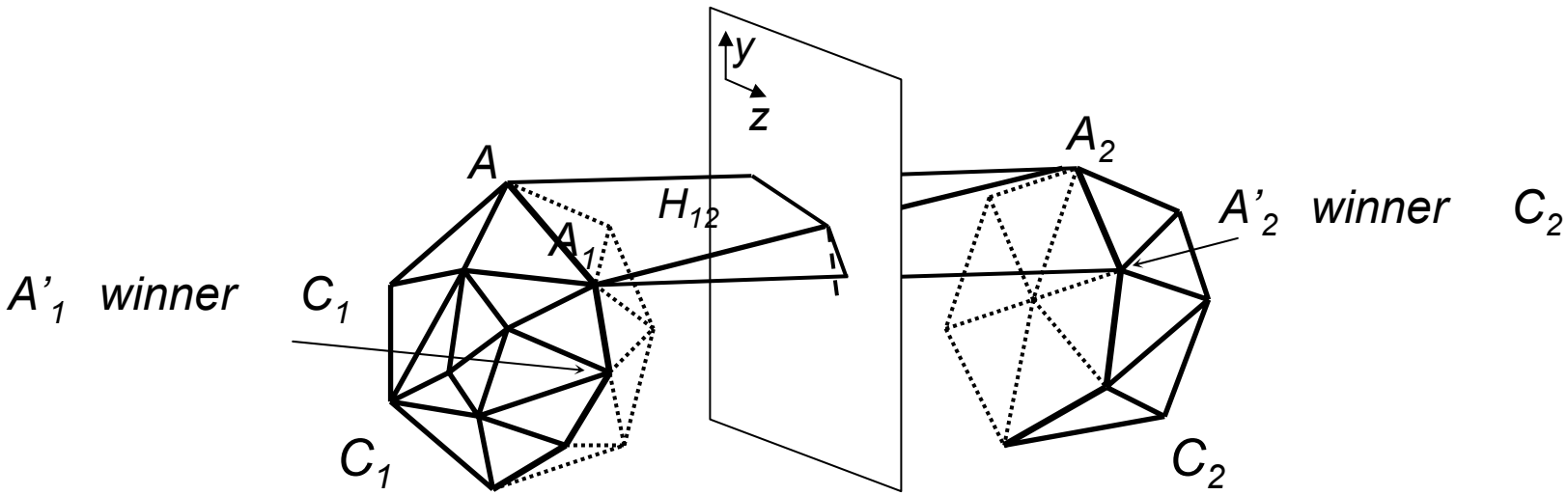
H_{12}

A_1A_2A



Finding the Other New Faces (2)

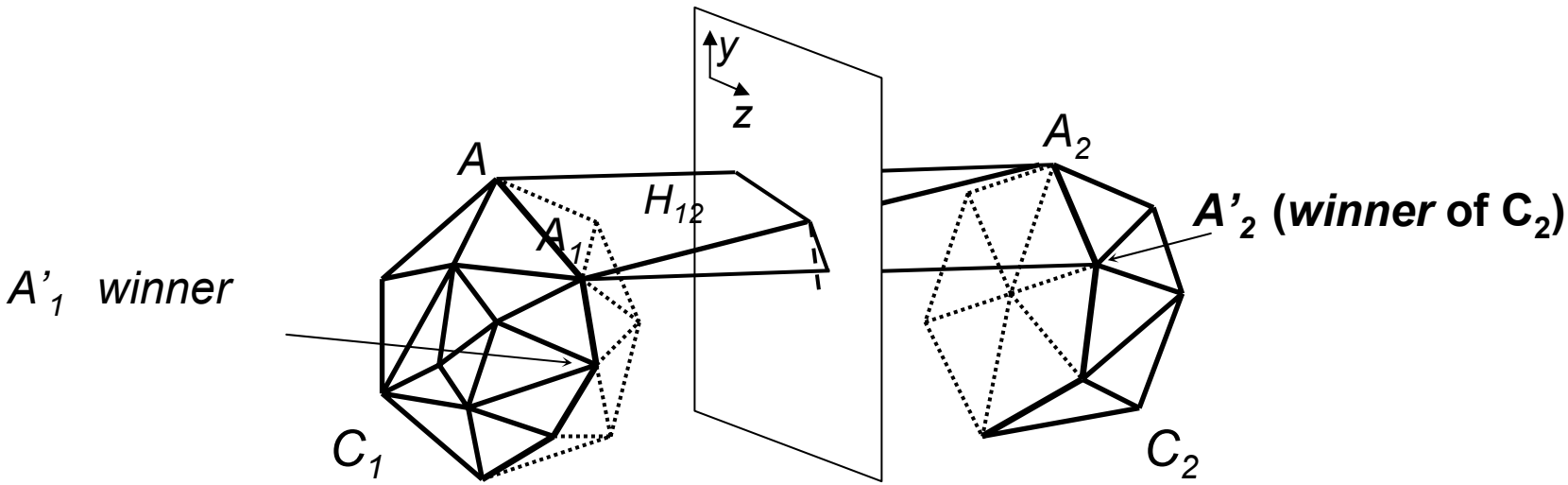
- A_1A_2 A_1A_2A' A' C_1 C_2 H A_1A_2 A'_1 A'_2 winner C_1 C_2 A_1A_2



Choosing between A_1 and A_2

- H_i $A_1 A_2 A'_i$
- A'_i

H_{12} H_i



Finding the Winner (for C_1)

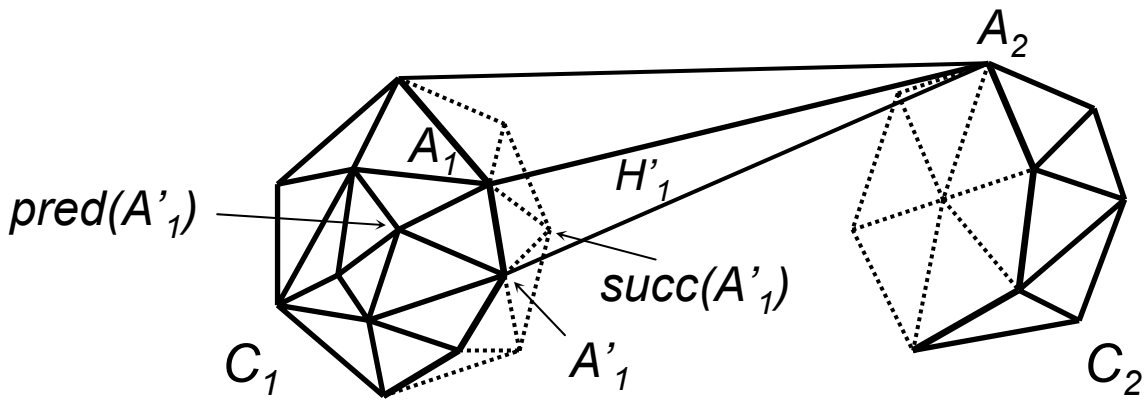
- A'_1 A_1

  pred A'_1 succ A'_1

  A'_1 A'_1

  A_1

  H'_1 $A_1A'_1A_2$ H'^{+}_1



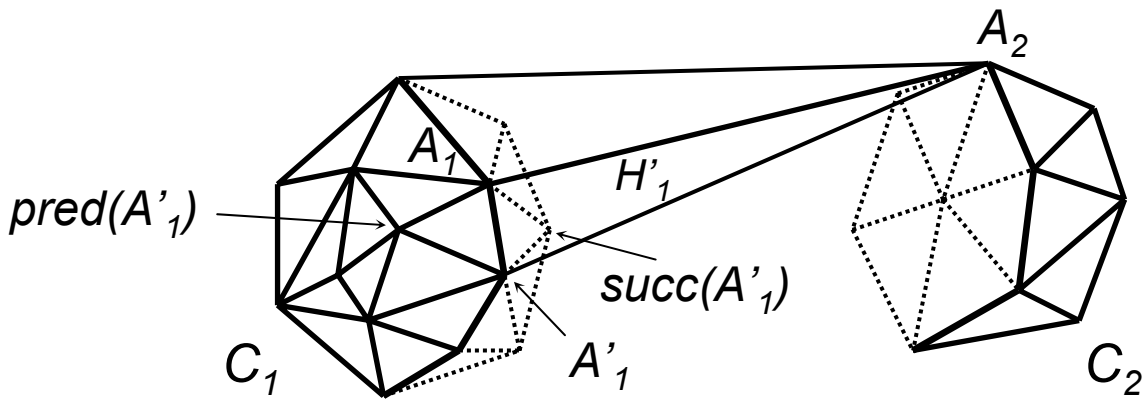
The Winner Lemma

- _____ *winner* C_1

  A_1A_2 A'_1 C_1

  A_1 *pred* A'_1

  succ A'_1 H'^{+}_1



The Winner Proof

- $\underline{\qquad\qquad \rightarrow}$

A'_1 $\qquad$ $A_1 A_2 \rightarrow$

$A'_1 A_1 A_2$ $\qquad$ $C_1 \quad A_2 \quad \rightarrow$

H'_1 $\qquad$ $C_1 \quad A_2 \quad \rightarrow$

H'_1 $\qquad$ $C_1 \rightarrow$

$A_1 A'_1$ $\qquad$ C_1 $\qquad$ $pred\ A'_1$

$succ\ A'_1$ $\qquad$ H'^{+}_1

The Winner Proof (cont d)

- ←

A'_1 C_1 A_1

$pred\ A'_1$ $succ\ A'_1$ H'^{+}_1

C_1

$A_1A'_1 succ\ A'_1$

$A_1A'_1 pred\ A'_1$ ➔

C_1 H'^{+}_1 H'_1 C_1 ➔

$A'_1A_1A_2$ C_1 A_2 ➔

A'_1 A_1A_2 □

Finding the Winner (cont d)

- A_1 A_1A_2

  A'_1 C_1

  A_1

  $\leftarrow$

  pred *succ* H'^{+}_1

- A_1 ____

Incidence of a Vertex to Several Pivots - Lemma

- _____ A_1 C_1

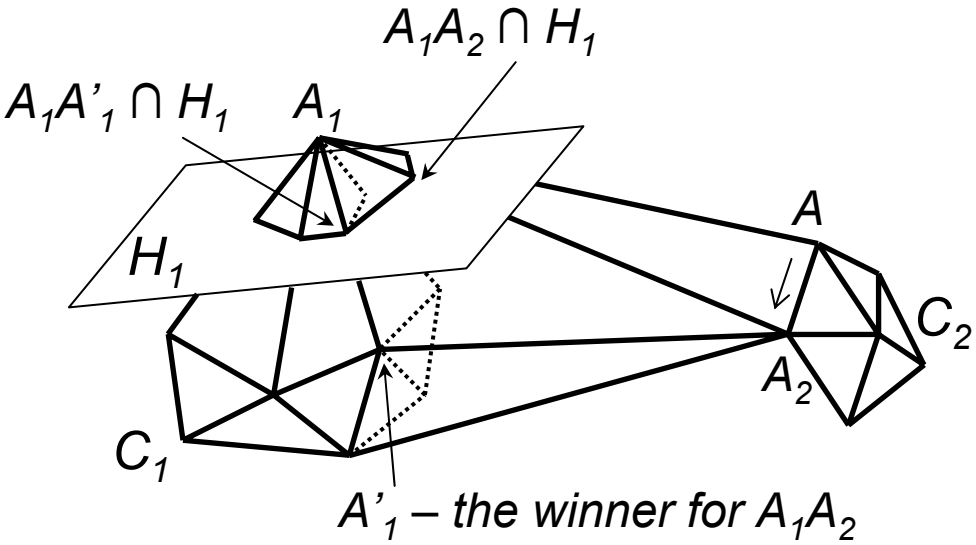
- A_1 C_1
  A_2 C_2

Incidence of a Vertex to Several Pivots Proof

- H_1

A_1

C_1 C_2 A_1

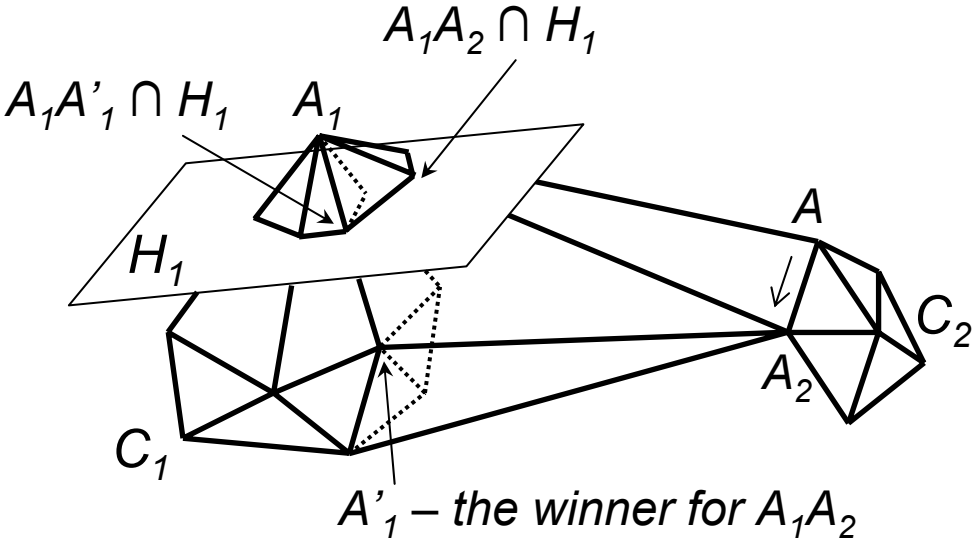


Incidence of a Vertex to Several Pivots Proof (cont d)

- A_1A_2

A'_1

H_1 $C_1 \cap H_1$ $A_1A'_1 \cap H_1$

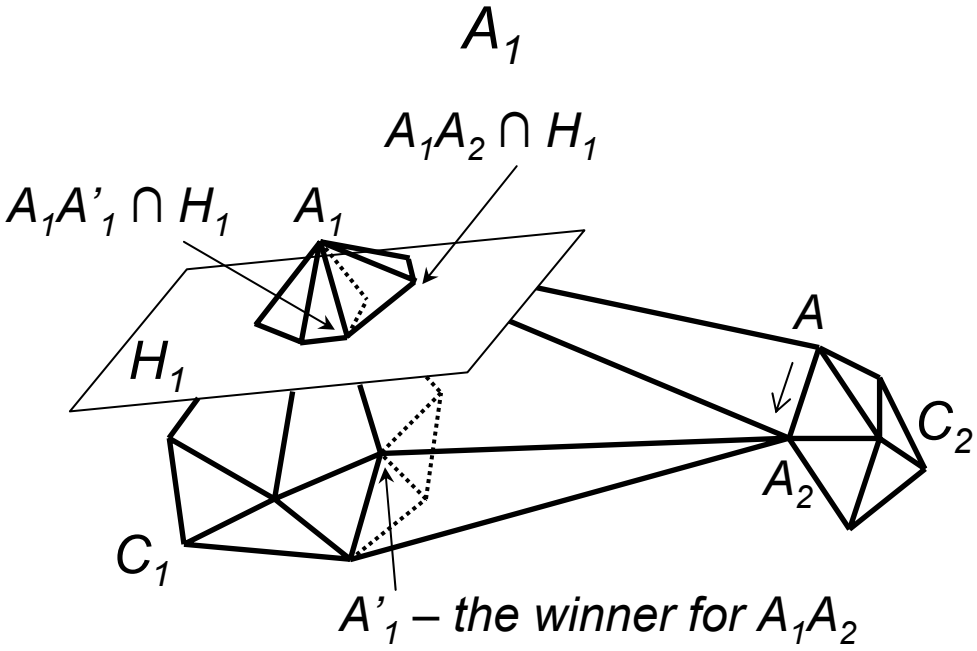


Incidence of a Vertex to Several Pivots Proof (cont d)

- H_1

$C_1 \cap H_1$

$C_1 \cap H_1$



The Process of Finding Winners

- C_1

  A_1

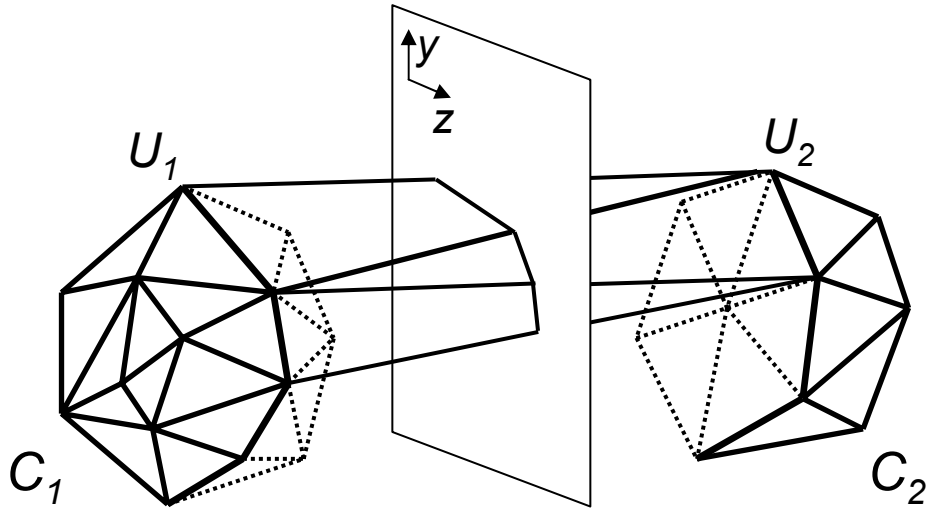
  C_1 A_1

- A_1 A_1

- A_1

Finding Winners (cont d)

- $U_1 U_2$



Reconstruction of conv(C_1 U C_2)

- C_1 C_2

- C

C_1 C_2

Complexity of the Algorithm (1)

- C_1 C_2

  n_1 n_2

- n_1+n_2

  C_1 C_2

  n_1+n_2

Complexity of the Algorithm (2)

-

C_1 C_2

n_1+n_2

Complexity of the Algorithm (3)

- C

  C_1 C_2

  n_1+n_2

  n_1+n_2
- n_1+n_2
- **O(n log n)**

Convex hull of a Polygonal Line

- x
- *any*

Polygonal Line - Definitions

- *vertices*

edges

- *simple*

closed

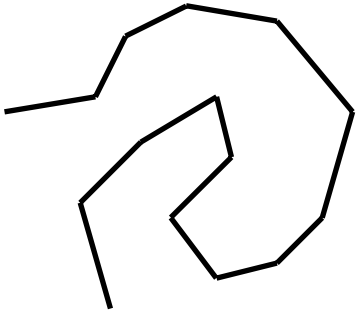
Polygonal Lines and Polygons

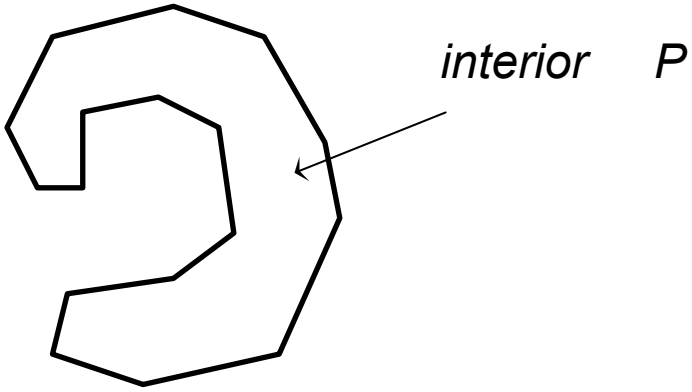
- *polygon*
- *P* *P*

interior *P*

exterior *P*

Examples of Polygonal Lines



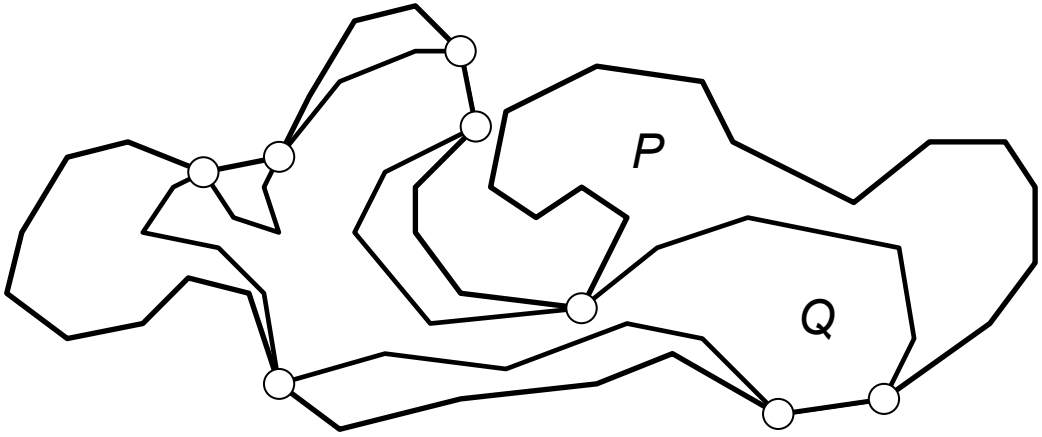


Theorem 9.4.1

- P Q

  Q

  P P Q



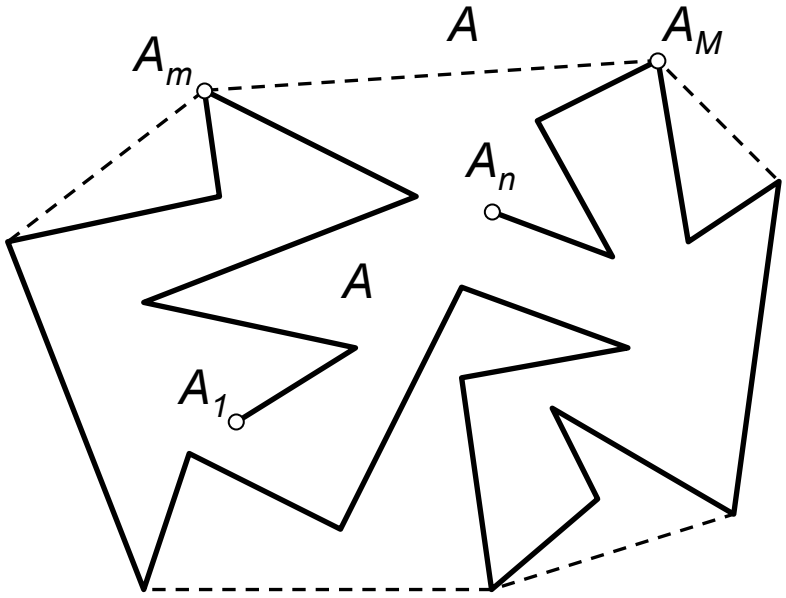
Ranks of Vertices

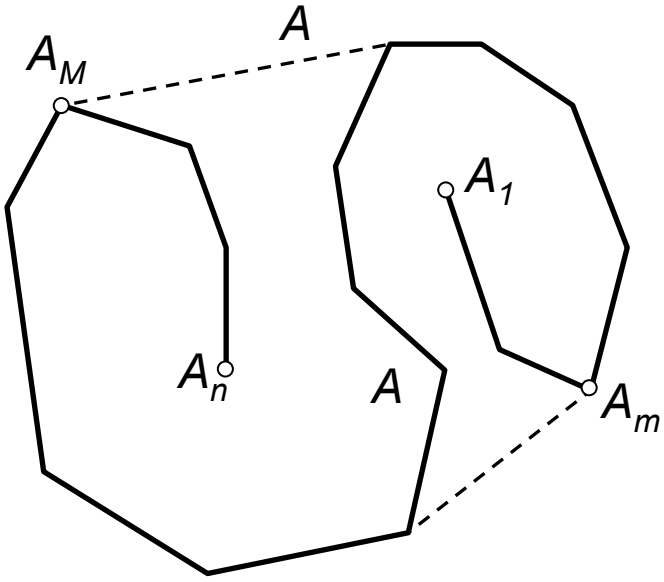
- A n
- A A_1 A_2 A_n
  A
- A A
- A_m A_M A
  A

Ranks of Vertices (example)

- A_m A_M A

A





Corollary 9.4.2

- A

A_m A_M

A

A_M

A_m

A

The Algorithm that Builds Convex Hull of a Polygonal Line

- A

  A A_1 A_2 A_n

  A

- A

The Algorithm Data Structures

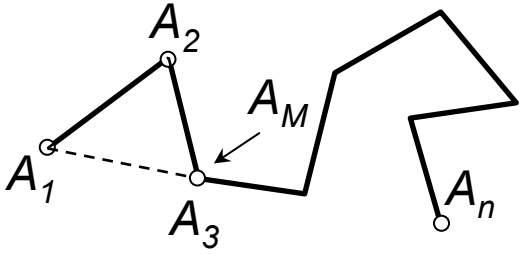
- A_i A
- A_i A_i
- A_i

A

The Algorithm in general

- $A_1 A_2 A_3$

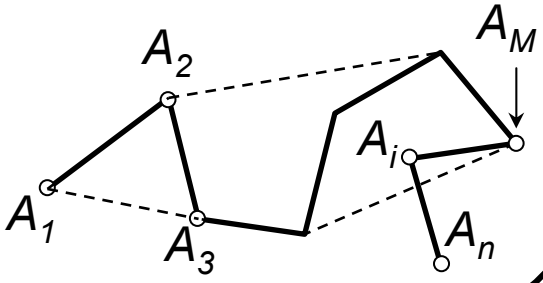
A_3



- A_i

A_{i-1}

A_i



A_i

A_{i-1}

The 2 Phases of the Algorithm

-
- _____ A_i A_{i-1}
- _____ A_i A_{i-1}

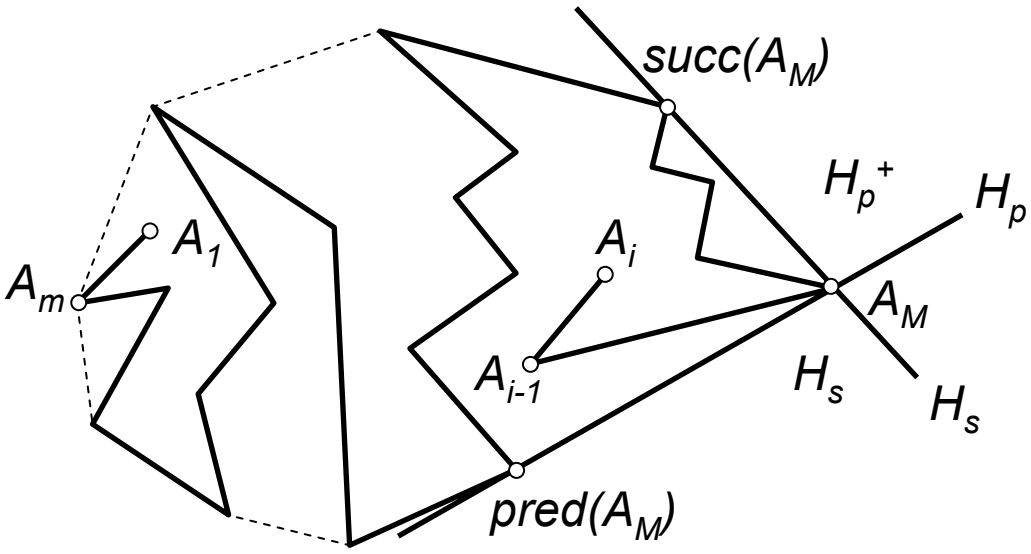
Lemma 9.4.3

- $pred(A_M)$ $succ(A_M)$
  A_M
  A_{i-1}
- H_p^+ H_s^+
  A_{i-1}
  $pred(A_M)A_M$ $A_M succ(A_M)$
  A_{i-1}
- _____ A_i A_{i-1} A_i
  $H_p^+ \cap H_s^+$.

Lemma 9.4.3 Example

- A_i A_{i-1} A_i

$$H_p^+ \cap H_s^+$$

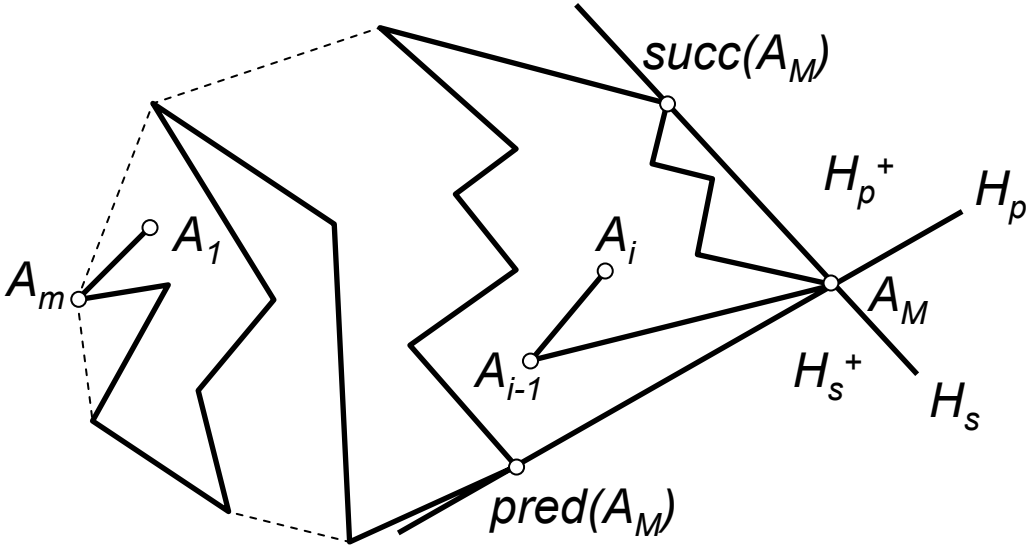


Lemma 9.4.3 Proof

- $\underline{\qquad\qquad \rightarrow}$

A_i $\qquad$ A_{i-1} $\qquad$ A_{i-1}

$H_p^+ \cap H_s^+$ $\qquad$ A_i $\qquad$ $H_p^+ \cap H_s^+$.



Lemma 9.4.3 Proof (cont d)

- ←

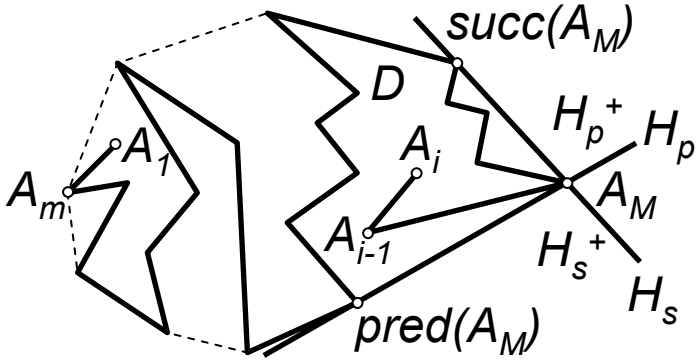
A $pred(A_M)$

$succ(A_M)$ $pred(A_M)A_M$

$A_M succ(A_M)$ A_{i-1}

D

D A_{i-1}



Lemma 9.4.3 Proof (cont d)

A A_M A_i

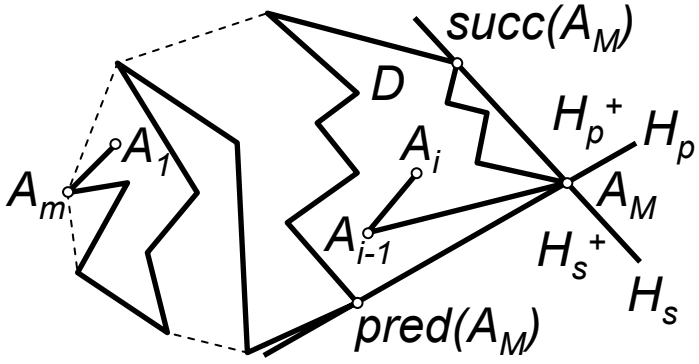
A $pred(A_M)$ $succ(A_M)$

A_{i-1} D A_M

A_i $H_p^+ \cap H_s^+ \rightarrow$

A_i $D \rightarrow$

A_i A_{i-1}



Complexity of the Algorithm

- $pred(A_M)A_M A_i$ $A_M succ(A_M)A_i$
- A_{i-1} red A_i