Evaluations of Graph Polynomials

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Abstract. A graph polynomial $p(G, \bar{X})$ can code numeric information about the underlying graph G in various ways: as its degree, as one of its specific coefficients or as evaluations at specific points $\bar{X} = \bar{x}_0$. In this paper we study the question how to prove that a given graph parameter, say $\omega(G)$, the size of the maximal clique of G, cannot be a fixed coefficient or the evaluation at any point of the Tutte polynomial, the interlace polynomial, or any graph polynomial of some infinite family of graph polynomials.

Our result is very general. We give a sufficient condition in terms of the connection matrix of graph parameter $f(G)$ which implies that it cannot be the evaluation of any graph polynomial which is invariantly definable in $CMSOL$, the Monadic Second Order Logic augmented with modular counting quantifiers. This criterion covers most of the graph polynomials known from the literature.

1 Introduction

1.1 Graph Parameters and Graph Polynomials

Graph parameters (also called numeric graph invariants) f are functions from the class of all finite graphs $\mathcal{G}$ to some numeric domain which is a commutative ring with 0 and 1, usually the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$ or the reals $\mathbb{R}$. Graph properties are a special case where the value is 0 or 1.

Graph polynomials are functions p from $\mathcal{G}$ into a polynomial ring, usually $\mathbb{Z}[\bar{X}]$, where $\bar{X}$ is a fixed finite set of indeterminates. Graph polynomials are a way to encode infinitely many graph parameters. Every evaluation of the polynomial $p(G; \bar{X})$ at some point $\bar{X} = \bar{x_0}$ is a graph parameter. So are the coefficients of $p(G; \bar{X})$, the total degree or the degree of monomials where the coefficient satisfies certain properties, and the zeros of $p(G; \bar{X})$.

A particular graph polynomial is considered interesting if it encodes many useful graph parameters. Let $G = (V(G), E(G))$ be a graph. The characteristic polynomial $P(G, X)$ of a graph is defined as the characteristic polynomial (in

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the sense of linear algebra) of the adjacency matrix A_G of G. The coefficients of $P(G, X)$ are defined by

$$det(X \cdot \mathbf{1} - A_G) = \sum_{i=0}^{n} c_i(G) \cdot X^i.$$

It is well known that $n = |V(G)|$, $-c_2(G) = |E(G)|$, and $-c_3(G)$ equals twice the number of triangles of G. The second largest zero $\lambda_2(G)$ of $P(G; X)$ gives a lower bound to the conductivity of G, cf. [GR01]. All these are graph parameters.

The Tutte polynomial of G is defined as

$$T(G; X, Y) = \sum_{F \subseteq E(G)} (X-1)^{r\langle E \rangle - r\langle F \rangle}(Y-1)^{n\langle F \rangle} \tag{1}$$

where $k\langle F \rangle$ is the number of connected components of the spanning subgraph defined by F, $r\langle F \rangle = |V| - k\langle F \rangle$ is its rank and $n\langle F \rangle = |F| - |V| + k\langle F \rangle$ is its nullity.

The Tutte polynomial $T(G; X, Y)$ has remarkable evaluations which count certain configurations of the graph G, cf. [Wel93].

(i) $T(G; 1, 1)$ is the number of spanning trees of G,
(ii) $T(G; 1, 2)$ is the number of connected spanning subgraphs of G,
(iii) $T(G; 2, 1)$ is the number of spanning forest of G,
(iv) $T(G; 2, 2) = 2^{|E|}$ is the number of spanning subgraphs of G,
(v) $T(G; 1-k, 0)$ is the number of proper k-vertex colorings of G,
(vi) $T(G; 2, 0)$ is the number of acyclic orientations of G,
(vii) $T(G; 0, -2)$ is the number of Eulerian orientations of G.

All these are also graph parameters which take values in $\mathbb{N}$. More sophisticated evaluations of the Tutte polynomial can be found in [Goo06, Goo08].

We shall show, 4, that the maximal size of a clique in G, $\omega(G)$, is not an evaluation of the Tutte polynomial. The same can be shown for the number of cliques of maximal size.

1.2 Coefficients and Evaluation of Graph Polynomials

In this paper we study the question whether a given graph parameter can occur as specific coefficient or the evaluation of a graph polynomial. We do not deal with zeros of graph polynomials. We are mostly interested in negative results: how does one prove that a given graph parameter cannot be a specific coefficient or the evaluation of a family of graph polynomials?

As a simple example, we look at the graph parameter f which is *additive* with respect to the disjoint union, in other words $f(G_1 \sqcup G_2) = f(G_1) + f(G_2)$. This is true for $|V(G)|, |E(G)|$ and also for $k(G)$, $b(G)$, number of connected components and number of blocks (doubly connected components), respectively. They are evaluations of the graph polynomials

$$V(G;X) = \sum_{v \in V(G)} X \quad \text{and} \quad E(G;X) = \sum_{e \in E(G)} X$$

$$C(G;X) = \sum_{U \subseteq V(G): U \in Co} X \quad \text{and} \quad B(G;X) = \sum_{U \subseteq V(G): U \in Bl} X$$

respectively, for $X = 1$. Here Co is the set of connected components, and Bl is the set of blocks respectively.

On the other hand, many graph polynomials $p(G, \bar{X})$, like the Tutte polynomial, are *multiplicative*, i.e., $p(G_1 \sqcup G_2, \bar{X}) = p(G_1, \bar{X}) \cdot p(G_2, \bar{X})$. Clearly, an additive graph parameter cannot be the evaluation of a multiplicative graph polynomial. If we consider instead the graph parameters $2^{|V(G)|}, 2^{|E(G)|}$, we can see that they are evaluations of the, admittedly uninteresting, multiplicative graph polynomial $X^{|V(G)|} \cdot Y^{|E(G)|}$.

More interesting[1] integer-valued graph parameters have the following property: There exist infinite sequences of graphs $(G_i)_{i \in \mathbb{N}}, (H_i)_{i \in \mathbb{N}}$ such that

(i) for all $(i,j) \in \mathbb{N}^2$ $f(G_i \sqcup H_j) = \max f(G_i), f(H_j)$ or $f(G_i \sqcup H_j) = \min f(G_i), f(H_j)$, and
(ii) for all $i \in \mathbb{N}$ the sequence $f_j = f(G_i \sqcup H_j)$ is strictly monotone increasing.

An integer-valued graph parameter f which has this property is called *maximizing*, respectively *minimizing*. In the above property we can replace the disjoint union by the join of two graphs $G_1 \bowtie G_2$ and consider the corresponding property. In this case we say that f is *join-maximizing*, respectively *join-minimizing*. Similarly we speak of *join-additive* and *join-multiplicative* graph parameters.

Example 1. (i) Among the maximizing graph parameters we have: the chromatic number $\chi(G)$, the edge chromatic number $\chi_e(G)$, and the total coloring number $\chi_t(G)$, the size of a maximal clique $\omega(G)$, the size of the maximal degree $\Delta(G)$, the tree-width $tw(G)$, and the clique-width $cw(G)$.
(ii) Among the minimizing graph parameters we have: the minimum degree $\delta(G)$, and the girth $g(G)$, which is the minimum length of a cycle in G.
(iii) $\omega(G)$ is join-additive and maximizing.
(iv) $\alpha(G)$, the size of the maximal independent set, is additive and join-maximizing.

How can we decide whether any of these do or do not occur as the evaluation of a graph polynomial? What about the average degree

$$d(G) = \frac{1}{|V(G)|} \cdot \sum_{v \in V(G)} d_G(v),$$

where $d_G(v)$ denotes the degree of a vertex v of G, which behaves differently than the example listed above?

[1] Almost all graph parameters discussed are taken from [Die96]. One exception is the clique-width, which was introduced in [CO00], and, in connection to graph polynomials, in [Mak04].

1.3 Connection Matrices

In [FLS07] the *connection matrix* $M(f,0)$ of a graph parameter f was introduced. Let $(G_i)_{i\in\mathbb{N}}$ be an enumeration of all finite graphs (up to isomorphism). $M(f,0)$ is an infinite matrix where the columns and rows are labeled by finite graphs G_i. Then the entry $M(f,0)_{i,j}$ is defined by $M(f,0)_{i,j} = f(G_i \sqcup G_j)$. We study the matrix $M(f,0)$ as a matrix over the reals $\mathbb{R}$. Similarly, we denote by $M(f,\bowtie)$ the matrix with entries $M(f,\bowtie)_{i,j}$ defined by $M(f,\bowtie)_{i,j} = f(G_i \bowtie G_j)$. Let us denote by $r(f,0)$ and $r(f,\bowtie)$ the rank of $M(f,0)$, respectively of $M(f,\bowtie)$.

Proposition 1. *(i) If f is multiplicative, $r(f,0)=1$.*
(ii) If f is additive, $r(f,0)=2$, unless $M(f,0)$ is the zero matrix.
(iii) If f is maximizing or minimizing, $r(f,0)$ is infinite.
(iv) Similarly for join-multiplicative, join-additive, join-maximizing and join-minimizing where the matrix is replaced by $M(f,\bowtie)$.
(v) For the average degree $d(G)$ of a graph, $r(d,0)$ is infinite.

Proof. The first three statements are easy. For $f = d(G)$ we have

$$M(d,0)_{i,j} = 2\frac{|E_i| + |E_j|}{|V_i| + |V_j|}.$$

This contains, for graphs with a fixed number e of edges, the Cauchy matrix $(\frac{2e}{i+j})_{i,j}$, hence $r(d,0)$ is infinite. □

In [FLS07] it is stated that for the Tutte polynomial $T(G;X,Y)$ and integers m,n the rank $r(T(G,m,n),0)$ is finite. Therefore, no integer valued graph parameter f with $r(f,0)$ infinite is an integer-valued evaluation of the $T(G;X,Y)$. What about rational values, like in the case of $d(G)$? What about other graph polynomials from the vast literature? Can we extend this argument to infinite families of graph polynomials $\mathcal{P}$ and to all their real evaluations?

In [FLS07], for every $k \in \mathbb{N}$ a more general connection matrix $M(f,k)$ and its rank $r(f,k)$ is introduced, and the finiteness of this rank is established for many graph parameters. We shall discuss this in Section 2. Our goal is to find a general criterion which allows us to conclude that for a graph parameter f the rank $r(f,k)$ is finite.

1.4 Main Result

In [Mak04, Mak07], the class of graph polynomials invariantly definable in Monadic Second Order Logic with Modular Counting, *CMSOL*, was introduced. A precise definition is given in Section 3 . For now it suffices to know that the Tutte polynomial, the matching polynomial, the characteristic polynomial, the interlace polynomial, and virtually all graph polynomials in the literature, are invariantly *CMSOL*-definable.

Theorem 2. *Let $p(G,\bar{X})$ be an invariantly CMSOL-definable graph polynomial with values in $\mathbb{R}[\bar{X}]$ with m indeterminates. There are numbers $\gamma(k) \in \mathbb{N}$ depending on the polynomial p and k only, and $\beta \in \mathbb{N}$, such that for all $\bar{x}_0 \in \mathbb{R}^m$, we have $r(p(G,\bar{x}_0),k) \leq \gamma(k)$. and $r(p(G,\bar{x}_0),\bowtie) \leq \beta$.*

Theorem 3. *Let $p(G, \bar{X})$ be an invariantly CMSOL-definable graph polynomial with values in $\mathbb{R}[\bar{X}]$ with m indeterminates $X_1, \ldots, X_m$, and let*

$$X_1^{\alpha_1} \cdot X_2^{\alpha_2} \cdot \ldots \cdot X_m^{\alpha_m}$$

be a specific monomial of $p(G, \bar{X})$ with coefficient $c_\alpha(G)$, where $\alpha = (\alpha_1, \ldots, \alpha_m)$.

Then there is an invariantly CMSOL-definable graph polynomial $p_\alpha(G, \bar{X})$ such that $c_\alpha(G)$ is an evaluation of $p_\alpha(G, \bar{X})$.

Remark 1. Theorem 3 is also valid for monomials of the form

$$X_1^{n_1(G)-\alpha_1} \cdot X_2^{n_2(G)-\alpha_2} \cdot \ldots \cdot X_m^{n_m(G)-\alpha_m},$$

where $n_i(G) = |V(G)|$ or $n_i(G) = |E(G)|$. This can be used to get, for example, the coefficient of $X^{|V(G)|-3}$ of the characteristic polynomial.

We want to apply Theorems 2 and 3 to graph parameters which are discussed in, say, [Die96]. To do so we use Proposition 1.

Corollary 4. *The following graph parameters are not a specific coefficient nor an evaluation of some CMSOL-definable graph polynomial.*

(i) $d(G)$, the average rank of a graph G.
(ii) Any graph parameter f which is maximizing, minimizing, join-maximizing or join-minimizing.

Remark 2. The degree of a graph polynomial is a graph parameter which behaves differently than evaluations or coefficients. The degree of the characteristic polynomial $P(G; X)$ is $|V(G)|$.

(i) On the other hand, let $\omega_i(G)$ be the number of induced subgraphs of size i which are cliques. Consider the polynomial $\Omega(G; X) = \sum_i \omega_i(G) \cdot X^i$. Then the degree of $\Omega(G; X)$ is $\omega(G)$. However, $\omega(G)$ is maximizing, so by Theorems 2 and 3, $\omega(G)$ cannot be a fixed coefficient or evaluation of any invariantly *CMSOL*-definable polynomial.
(ii) Similarly, the size of a maximal independent set $\alpha(G)$, is the degree of the polynomial $Ind(G; X) = \sum_i ind_i(G) \cdot X^i$ where $ind_i(G)$ is the number of independent sets of size i. However, $\alpha(G)$ is join-maximizing, so by Theorems 2 and 3, $\alpha(G)$ cannot be a fixed coefficient or evaluation of any invariantly *CMSOL*-definable polynomial.

Both $\Omega(G; X)$ and $Ind(G; X)$ are both *CMSOL*-definable polynomials, as we shall see in Section 3.

The remainder of the paper is organized as follows: In the next section we discuss further generalizations of connection matrices and their use, and in Section 3 we present the necessary material on *CMSOL*-definable polynomials. In Section 4 we discuss further research.

2 Connection Matrices $M(f,k)$

In [FLS07] more general connection matrices $M(f,k)$ are introduced for every natural number k. Instead of looking at all graphs and their disjoint unions, they look at graphs with k vertices labeled with distinct labels from the set $[k] = \{0, \ldots, k-1\}$. Let $\mathcal{G}_k$ denote the class of k-labeled graphs. On $\mathcal{G}_k$ they define the operation $G_1 \sqcup_k G_2$, which is like the disjoint union, but vertices with corresponding labels are identified. This may create multiple edges. Let $(G_i^k)_{i \in \mathbb{N}}$ be an enumeration of all graphs from $\mathcal{G}_k$. The matrix $M(f,k)$ now has the entries $M(f,k)_{i,j} = f(G_i^k \sqcup_k G_j^k)$, and $r(f,k)$ denotes its rank.

Example 2. (i) For $\lambda \in \mathbb{N}^+$, let $\chi(G;\lambda)$ denote the number of proper λ-colorings of G. In [FLS07] it is shown that $r(\chi(G;\lambda), k) = B_{k,\lambda}$, where $B_{k,\lambda}$, denotes the number of partitions of a k-element set into at most λ parts.

(ii) Let $\omega_{conn}(G)$ be defined by

$$\omega_{conn}(G) = \begin{cases} \omega(G) & \text{if } G \text{ is connected} \\ 0 & \text{else} \end{cases}$$

where $\omega(G)$ is the size of the maximal clique of G. It is easy to see that $r(\omega_{conn}, 0) = 2$, but

$$\omega_{conn}(G_0 \sqcup_1 G_1) = \max\{\omega_{conn}(G_0), \omega_{conn}(G_1)\},$$

holds if both G_0 and G_1 are connected, hence, using Proposition 1, $r(\omega_{conn}, 1)$ is infinite.

(iii) If we replace connected by m-connected, we get a graph parameter ω_{m-conn} with $r(\omega_{m-conn}, i)$ finite for $i \leq m$, and infinite otherwise.
This shows that the $r(f,k)$ can switch from finite to infinite at any stage.

Problem 1. What can we say in general about the behavior of $r(f,k)$ as a function of k?

Definition 1. *Let f be a graph parameter. f is* k-additive *if for all k-labeled graphs $G_1, G_2 \in \mathcal{G}_k$ we have $f(G_1 \sqcup_k G_2) = f(G_1) + f(G_2)$.*
Similarly, f is k-multiplicative, k-maximizing *or* k-minimizing *if the corresponding properties hold with the disjoint union replace by $\sqcup_k$.*

Similarly to Proposition 1 we have

Proposition 5. *If f is k-multiplicative, $r(f,k) = 1$.*
If f is k-additive and $M(f,k)$ is not the zero matrix, $r(f,k) = 2$.
If f is k-maximizing or k-minimizing, $r(f,k)$ is infinite.

Let $pm(G)$ denote the number of perfect matchings of G. In [FLS07], it is shown that $r(pm, k) = 2^k$. For the proof they define auxiliary graph parameters $pm_A(G)$ for k-graphs as follows: Let $A \subseteq [k]$ be a set of labels. Denote by

$pm_A(G)$ the number of matchings in G that match all the unlabeled vertices and the vertices with label in A, but not any of the other labeled vertices. Then we have

$$pm(G_1 \sqcup_k G_2) = \sum_{A_1 \cap A_2 = \emptyset, A_1 \cup A_2 = [k]} pm_{A_1}(G_1) \cdot pm_{A_2}(G_2)$$

We generalize this as follows:

Definition 2. *A graph parameter f is* weakly (k,γ)-multiplicative*, if there exists a finite set of graph parameters $f_i : i \leq \gamma$ with $i, \gamma \in \mathbb{N}$ with $f = f_0$, and a matrix $M^k \in \mathbb{R}^{\gamma \times \gamma}$, such that $f_0(G_1 \sqcup_k G_2) = \sum_{i,j} f_i(G_1) M^k_{ij} f_j(G_2)$.*
In other words, $f(G_1 \sqcup_k G_2)$ is given by a quadratic form defined by $M^k_{i,j}$ of rank at most γ. The number $\gamma = \gamma(k)$ may depend on k. Similarly we define weakly $(\bowtie, \gamma)$-multiplicative*, where sqcup$_k$ is replaced by the join and the quadratic form is given by $M^{\bowtie}$.*

Proposition 6. *Let f be a graph parameter which is weakly (k,γ)-multiplicative. Then $r(f,k) \leq \gamma$ where $\gamma = \gamma(k)$ depends on k.*

The following theorem is proven in [Mak04, Theorem 6.4]:

Theorem 7. *Let f be a graph parameter which is the evaluation $f(G) = p(G, \bar{x}_0)$ of an invariantly CMSOL definable graph polynomial $p(G, \bar{X})$. Then f is weakly $(k, \gamma(k))$-multiplicative and weakly $(\bowtie, \beta)$-multiplicative for $\gamma : \mathbb{N} \to \mathbb{N}$ and $\beta \in \mathbb{N}$, which depend on the polynomial p, but not on $\bar{x}_0$.*

Remark 3. The function $\gamma(k)$ may grow super-exponentially. The best general upper bounds known contains iterated exponentials, cf. [Mak04].

Theorem 2 now follows immediately from Theorem 7 and Proposition 6.

3 *CMSOL*-Definable Graph Polynomials

In this section we give the definition of invariantly *CMSOL*-definable polynomials *in normal form.* A full treatment is given in [Mak04] . There we give a more general definition of invariantly *CMSOL*-definable polynomials, and show that every such polynomial can be written in normal form. The more general definition makes it easier to verify that a given graph polynomial is indeed invariantly *CMSOL*-definable. All we need for the proof of our two main theorems is the definition of this normal form and Theorem 7.

3.1 The Logic *CMSOL*

We consider now *ordered k-graphs* of the form $G = (V, E, R, \leq, a_0, \ldots, a_{k-1})$, where V and E are finite sets of vertices and edges, respectively, $\leq$ is a linear order on V, and $R \subseteq V \times E \times V$ is the graph incidence relation. This allows us also to have directed graphs with multiple edges. $a_i : i \in [k]$ are the labeled elements of V.

We define the logic *CMSOL* for these graphs inductively. We have first order variables $x_i : i \in \mathbb{N}$ which range over elements of $V \cup E$, and (monadic) second order variables $U_i : i \in \mathbb{N}$, which range over subsets of $V \cup E$. Terms $t, t', \ldots$ are either first order variables or one of the constants $a_i : i \in [k]$. Atomic formulas are of the form $t = t'$, $t \leq t'$, $R(t, t', t'')$, $U_i(t)$ and have the natural interpretation. Formulas are built inductively using the connectives $\vee, \wedge, \rightarrow, \leftrightarrow, \neg$, and the quantifiers $\forall x_i$, $\exists x_i$, $\forall U_i$, $\exists U_i$ with their natural interpretation. Additionally we have quantifiers $C_{a,b} x_i$ for each $a, b \in \mathbb{N}$. The formula $C_{a,b} x_i \phi(x_i)$ is interpreted by the statement "the number of elements satisfying $\phi(x_i)$ equals a modulo b".

The following can be written as *CMSOL*-formulas:

Example 3. (i) The formula $\Phi_{match}(U)$ which says that U is a matching.
(ii) The formula $\Phi_{pm}(U)$ which says that U is a perfect matching.
(iii) The formula $\Phi_{fconn}(U, x)$ which says that x is the first element of some connected component of the spanning subgraph $[U]_G$ with edge set $U \subseteq E$.
(iv) The formula Φ_{euler} which says that every vertex has even degree and the graph is connected.
(v) The formula $\Phi_{ham}(U)$ which says that U is a Hamiltonian path.

3.2 Simple Invariantly *CMSOL*-Definable Polynomials

A *simple CMSOL*-definable polynomial in one indeterminate X is of the form

$$\sum_{U:\Phi(U)} \prod_{x:U(x)} X$$

where $\Phi(U)$ is an *CMSOL*-formula. Φ is called an iteration formula. It is *invariantly CMSOL*-definable, if its value does not depend on the ordering $\leq$ of G.

Remark 4. A formula is *order invariant* if its truth does not depend on the specific order. It is easy to see that a formula with quantifiers $C_{a,b}$ is equivalent to some order-invariant formula without such quantifiers. For a polynomial to be invariantly *CMSOL*-definable, it is not required that the iteration formulas are all order invariant.

Example 4. (i) The polynomials $Ind(G; X)$ and $\Omega(G; X)$, as defined in Remark 2, are *CMSOL*-definable.
(ii) For $\Phi = \Phi_{pm}(U)$ we get the matching polynomial. For $|V|$ even, its coefficient of $X^{\frac{|V|}{2}}$ is the number of perfect matchings $pm(G)$. Now $pm(G)$ can be written as

$$\sum_{U:\Phi_{pm}(U)} \prod_{x:U(x)} 1$$

which is an evaluation of a simple *CMSOL*-definable polynomial.

(iii) The polynomial $X^{k(G)}$, where $k(G)$ is the number of connected components of G, can be written as

$$\sum_{U:U=E} \prod_{x:\Phi_{fconn}(U,x)} X$$

It is an invariantly *CMSOL*-definable polynomial.

3.3 Invariantly *CMSOL*-Definable Polynomials in Normal Form

A *CMSOL*-definable polynomial in indeterminates $X_1, \ldots, X_\ell$ in *normal form* has the form

$$\sum_{U_1:\Phi_1(U_1)} \sum_{U_2:\Phi_2(U_2)} \cdots \sum_{U_{\ell_1}:\Phi_{\ell_1}(U_{\ell_1})} \left(\prod_{\bar{x}_1:\phi_1(\bar{x}_1)} X_1 \prod_{\bar{x}_2:\phi_2(\bar{x}_2)} X_2 \ldots \prod_{\bar{x}_\ell:\phi_\ell(\bar{x}_\ell)} X_\ell \right)$$

where all the formulas Φ_i and ϕ_i are *CMSOL*-formulas with the iteration variables indicated. There may be additional parameters in the formulas. However, Φ_i may not contain the variables U_j for $j > i$, and ϕ_i may not contain $\bar{x}_j$ for $j > i$. Both Φ_i and ϕ_i are referred to as iteration formulas. It is invariantly *CMSOL*-definable if its values do not depend on the order of G.

Example 5. (i) The Tutte polynomial, as defined by (1), is not given in a good way to show that it is invariantly *CMSOL*-definable. We look instead at the Tutte polynomial in its form as a partition function

$$Z(G, X, Y) = \sum_{U:U\subseteq E} X^{k[U]_G} Y^{|U|} = \sum_{U:U\subseteq E} \left(\prod_{x_1:\Phi_{fconn}(U,x_1)} X \prod_{x_2:U(x_2)} Y \right)$$

which shows that it is invariantly *CMSOL*-definable. It is related to the Tutte polynomial by the equation

$$T(G; X, Y) = (X-1)^{-k(G)} \cdot (Y-1)^{-|V(G)|} \cdot Z(G; (X-1)(Y-1), (Y-1)).$$

Another way to prove that $T(G; X, Y)$ is invariantly *CMSOL*-definable, is to use its definition via its spanning tree expansion, cf. [Bol99, Chapter 10] and [Mak05].

(ii) To see that the characteristic polynomial $P(G; X)$ is of the required form, we need a few definitions. An *elementary subgraph* of a graph G is a subgraph (not necessarily induced) which consists only of isolated vertices and cycles. If H is an elementary subgraph of G, we denote by $k(H)$ its number of connected components, and $c(H)$ the number of its cycles. With this notation we have, [Big93, Proposition 7.4], that the coefficients of $P(G; X) = \sum_i c_i X^{n-i}$ can be expressed as

$$c_i = (-1)^i \cdot \sum_{H:|V(H)|=i} (-1)^{k(H)} \cdot 2^{c(H)}$$

where the summation is over all elementary subgraphs $H = (V(H), E(H))$ of G of size i. Therefore we have

$$P(G;X) = \sum_{V(H)} \sum_{E(H)} (-1)^{|V(H)|} \cdot (-1)^{k(H)} \cdot 2^{c(H)} \cdot \prod_{v \notin V(H)} X$$

which is an evaluation of

$$\hat{P}(G;W,X,Y,Z) = \sum_{V(H),E(H)} \left(W^{|V(H)|} \cdot Y^{k(H)} \cdot Z^{c(H)} \cdot \prod_{v \notin V(H)} X \right)$$

at $W = -1, Y = -1, Z = 2$, and which is *CMSOL*-definable.

(iii) The interlace polynomial defined in [ABS04] is also a *CMSOL*-definable polynomial, but to see this one has to use the quantifier $C_{0,2}$ which says that the cardinality of a certain set is even, cf. [Cou].

3.4 Proof of Theorem 3

We are left to prove Theorem 3. Let a graph polynomial $p(G;X,Y)$ be given in normal form with two indeterminates X, Y, and two iteration formulas for summation. The general case is similar.

$$p(G;X,Y) = \sum_{U_1:\Phi_1(U_1)} \sum_{U_2:\Phi_2(U_2)} \left(\prod_{\bar{x}_1:\phi_1(U_1,U_2,\bar{x}_1)} X \prod_{\bar{x}_2:\phi_2(U_1,U_2,\bar{x}_2)} Y \right)$$

Let $X^a \cdot Y^b$ be a monomial with coefficient $c_{a,b}$. Then we have

$$c_{a,b} = \sum_{U_1:\Phi_1(U_1)} \sum_{U_2:\Phi_2(U_2)} \left(\sum_{A:(A=\emptyset)\wedge\psi_{1,a}(U_1,U_2)\wedge\psi_{2,b}(U_1,U_2)} 1 \right),$$

where $\psi_{i,c}(U_1, U_2)$ says "There are exactly c many tuples $\bar{x}$ satisfying $\phi_i(U_1, U_2, \bar{x})$". The summation over $A = \emptyset$ ensures that the last sum contains at most one summand. Clearly, this is an evaluation of an invariantly *CMSOL*−definable polynomial. □

3.5 Not Invariantly *CMSOL*-Definable Graph Polynomials

Here we give an example of a graph polynomial which is not invariantly *CMSOL*-definable.

Definition 3. *(i) A proper vertex coloring is* harmonious*, if each pair of colors appears at most once along an edge. We denote by $\chi_{harm}(G)$ the least k such that G has a harmonious proper k-coloring.*

(ii) A proper vertex coloring is complete*, if each pair of colors appears at least once along an edge. We denote by $\chi_{comp}(G)$ the largest k such that G has a complete proper k-coloring.*

(iii) Let $\chi_{harm}(G;k)$ and $\chi_{comp}(G;k)$ denote the number of harmonious, respectively complete proper k-colorings of G.

Proposition 8. *(i) $\chi_{harm}(G;k)$ is a polynomial in k.*
(ii) $\chi_{comp}(G;k)$ is not a polynomial in k.

Proof. (i) follows from [MZ06], but it is not difficult to prove it directly.
(ii) $\chi_{comp}(G;k) = 0$ for large enough k. □

Theorem 9. *$\chi_{harm}(G)$ and $\chi_{comp}(G)$ are graph parameters which are not evaluations of invariantly CMSOL-definable graph polynomials.*

Proof. $\chi_{comp}(G)$ is maximizing, so we can apply Proposition 4.

For $\chi_{harm}(G)$ we observe that, for stars S_n, a set of n edges which meet all in one single vertex, we have

$$\chi_{harm}(S_n \sqcup S_m) = \max\{\chi_{harm}(S_m), \chi_{harm}(S_n)\} + 1.$$

Now the argument proceeds like in the case a maximizing graph parameter.

Theorem 10. *$\chi_{harm}(G;k)$ is not an invariantly CMSOL-definable graph polynomial.*

Proof. Let L_i denote the graph which consists of i vertex disjoint edges. We look at $M(\chi_{harm}(G,k),0)$ restricted to the graphs $L_i, i \in \mathbb{N}$, which we denote by $M_L(k)$ and its rank by $r_L(k)$. We note that $\chi_{harm}(L_i \sqcup L_j) = 0$ iff $i+j > \binom{k}{2}$. Therefore, $r_L(k) = \binom{k}{2}$ which is not bounded, contradicting Theorem 2.

Remark 1. *It is shown in [EM95], that computing $\chi_{harm}(G)$ is* **NP***-complete already for trees. This, together with the fact, proven in [Mak05], that evaluations of invariantly CMSOL-definable graph polynomials are polynomial time for graphs of tree-width at most k, shows that $\chi_{harm}(G;X)$ is not invariantly CMSOL-definable, unless* $\mathbf{P} = \mathbf{NP}$. *Our proof above eliminates the complexity theoretic hypothesis* $\mathbf{P} = \mathbf{NP}$.

4 Conclusions and Open Problems

We have shown that many standard graph parameters studied in the literature cannot appear as evaluations of *CMSOL*-definable graph polynomials, which covers most of the graph polynomials studied in the literature, cf. [Mak07]. We have also exhibited for the first a natural graph polynomial which is not *CMSOL*-definable.

In [FLS07] the graph parameters f which can occur as evaluations of partition functions arising from counting weighted graph homomorphisms are characterized as exactly those parameters which are multiplicative, and such that for all $k \in \mathbb{N}$ we have that $r(f,k) \leq |V|^k$ and the matrices $M(f,k)$ are positive semi-definite. It is easy to see that the partition functions are evaluations of invariantly *CMSOL*-definable graph polynomials of a very special kind.

It remains a challenging problem to characterize those graph parameters which do occur as evaluations of invariantly *CMSOL*-definable polynomials.

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