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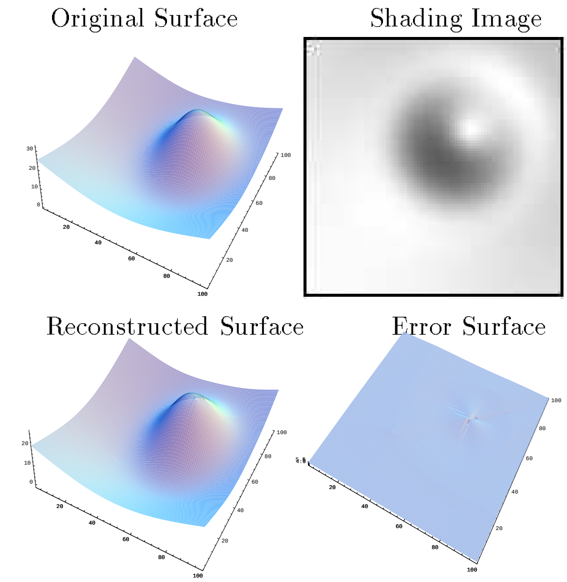

Figure 2: The reconstruction of the surface at the upper left, from its shading image at the upper right is shown at the bottom left. Bottom right is the difference between the original surface and its reconstruction.

5 Experimental Results

We tested the algorithm on a synthetic shading image of the simplest surface with the three basic types of local extremum points: a maximum, a minimum, and a saddle. The oblique light source is given by $\vec{l} = (0.2, 0, 0.96)$. Observe that we do not deal here with self casting shadows (see [14]), nor with solving the global topological structure (see [11, 6, 2]).

The local extremum points cause singularities at the right hand side of the equation since the intensity at their corresponding image locations is equal to zero. This fact should not cause any problem to our numerical algorithm, since one could set the intensity values that are smaller than $O(\Delta x)$ to some $O(\Delta x)$ without reducing the global order of accuracy. Where Δx is the grid spacing (the distance between two grid points). Figure 2 presents the surface, its shading image, the reconstructed surface, and the error, for the oblique light source case. The surface is the solution to Eq. (4) and (5) with a fixed value at the minimum point (one of the singular points).

6 Conclusion

We presented an $O(N \log N)$ algorithm for surface reconstruction from its shading image. The computational complexity bound is data independent (unlike other iterative methods [1, 6]). It is the most efficient sequential algorithm for Horn's original formulation of the shape from shading problem and a natural extension and application of the Fast Marching Method.

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the (x, y) image plane, so that $\vec{l} \cdot (0, 1, 0) = 0$. The shading image is then given by

$$I(x, y) = (l_1, 0, l_3) \cdot \vec{n}, \qquad (3)$$

where $l_1^2 + l_3^2 = 1$. Eq. (3) involves the term z_x. It requires some additional thought to construct a monotonic approximation to this term and an appropriate update rule.

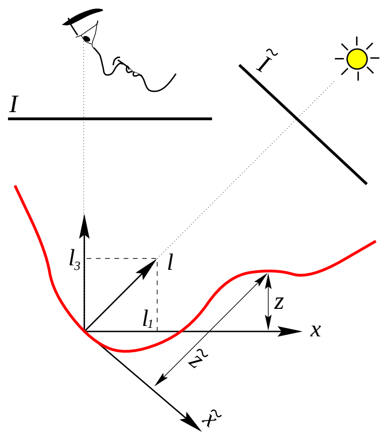

Figure 1: In the oblique light source case, the natural coordinate system is determined by the light source [13].

If we would have had the brightness image in the light source coordinates $\tilde{I}(\tilde{x})$, then the problem would have become the vertical light source case, which is given by the Eikonal equation

$$\tilde{z}_{\tilde{x}}^2 + \tilde{z}_y^2 = \frac{1}{\tilde{I}(\tilde{x}, y)^2} - 1, \qquad (4)$$

see Figure 1

Lee and Rosenfeld [13] suggested the light source coordinates 'to improve' early shape from shading algorithms. In fact adopting this suggestion, it is simple to view the reflectance map 'almost' as an Eikonal equation for which we can design a very efficient numerical method. In the light source coordinate system, the equation to solve looks like the Eikonal equation, yet the right hand side depends on the surface itself via

$$\tilde{I}(\tilde{x}, y) = I(l_3 \tilde{x} + l_1 \tilde{z}, y). \qquad (5)$$

That is, we need to evaluate the value of the surface at a point in order to find the 'brightness' and only then plug it to Eq. (5) and use the Fast Marching Method to solve Eq. (4).

In order to overcome this dependence, we use the directional propagation and 'adopt' the smallest $\tilde{z}$ value from all the neighbors of the updated grid point. The update step then reads

- Let $\tilde{z}_1 = \min\{\tilde{z}_{i-1,j}, \tilde{z}_{i+1,j}\}$ and $\tilde{z}_2 = \min\{\tilde{z}_{i,j-1}, \tilde{z}_{i,j+1}\}$;
- Let $k = l_3 i + l_1 \min\{\tilde{z}_1, \tilde{z}_2\}$;
- IF $|\tilde{z}_1 - \tilde{z}_2| < f_{kj}$ THEN $\tilde{z}_{ij} = \frac{\tilde{z}_1 + \tilde{z}_2 + \sqrt{2f_{kj}^2 - (\tilde{z}_1 - \tilde{z}_2)^2}}{2}$;

  ELSE $\tilde{z}_{ij} = \min\{\tilde{z}_1, \tilde{z}_2\} + f_{kj}$;

Where $\tilde{z}_{ij} = z(i\Delta\tilde{x}, j\Delta y)$, and $f_{kj} = f(k\Delta x, j\Delta y)$. Again, without loss of generality we assume $\Delta\tilde{x} = \Delta y = 1$, and $f(x, y) = I(x, y)^{-2} - 1$. The numerical algorithm in this case is still consistent, one pass since the smallest $\tilde{z}$ neighbor will never change its value, and is thus within the fast marching framework. The map between the light source coordinates $(\tilde{x}, y, \tilde{z})$ and the image coordinates (x, y, z) is a simple rotation given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l_3 & 0 & l_1 \\ 0 & 1 & 0 \\ -l_1 & 0 & l_3 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ y \\ \tilde{z} \end{pmatrix}.$$

We have thereby extended the Fast Marching Method to the case of $|\nabla z| = F(z)$ relevant to the oblique light source shape from shading problem. A consistent solution can be computed with $O(N \log N)$, where N is the total number of pixels (grid points).

information always flow form small to large values of the solution z. Therefore, the surface z may be reconstructed by first setting all z values to ∞, and the correct hight value at the local minimum points. In case the hight values at these locations are unknown, then a global topology solver can be applied [11, 2].

Assume, for simplicity, that we deal with a single known minimum point. An alternate scanning directions of the numerical grid, while solving the quadratic equation (1) for z_{ij} at each visited grid point, would eventually converge. Actually, the rate of convergence depends on the complexity of the surface z_{ij}. If we reconstruct a connected spirals like surface, then there is a need for $O(N)$ scans in the worst case, that yields a total computational complexity bound of $O(N^2)$. Note again that this is a worst case analysis. The main point is that for alternating scanning directions based methods the complexity is data dependent and ranges between $O(N)$ and $O(N^2)$. For simple surfaces, convergence can be achieved in few iterations.

Assume without loss of generality that $\Delta x = \Delta y = 1$, and set initially all $z_{ij} = \infty$ besides the minimum point that is set to zero. Then the update step for z_{ij} can be written as the following simple procedure

- Let $z_1 = \min\{z_{i-1,j}, z_{i+1,j}\}$ and $z_2 = \min\{z_{i,j-1}, z_{i,j+1}\}$;
- IF $|z_1 - z_2| < f_{ij}$ THEN $z_{ij} = \frac{z_1+z_2+\sqrt{2f_{ij}^2-(z_1-z_2)^2}}{2}$;

  ELSE $z_{ij} = \min\{z_1, z_2\} + f_{ij}$;

The fast marching method introduces order to the update steps. Points are updated and accepted by their values from small to large. The selection of the smallest point among the set of candidate points and the update of its neighboring grid points involves an $O(\log N)$ worst case complexity, that yields a total of $O(N \log N)$ worst case computational complexity. The order of updates is similar to that of Dijkstra's graph search algorithm [5, 16], and is based on a heap structure of the points at the front. The main difference from Dijkstra's graph search method is the numerical update step. Actually, one may use the finite numerical accuracy to avoid the ordering and reduce the total complexity to $O(N)$.

Our shading image is usually given on a rectangular pixels grid. Therefore, the Fast Marching Method can be directly applied to solve the shape from shading problem with a vertical light source. However, for the general oblique light source, the model to be solved reads $|\nabla z(x, y)| = f(x, y, z(x, y))$ as shown in the following section. Observe that for this more general case, the right hand side depends on $z(x, y)$. We will show how to include this partial differential equation, which is not an Eikonal equation anymore, within the Fast Marching framework. Full details on the Fast Marching Method are given in [18].

4 Shape from Shading: Oblique Light Source

Let us focus on the oblique light source case in which the light source direction is different than that of the viewer. Recall, that the shading image for this Lambertian case is given by

$$I(x, y) = \vec{l} \cdot \vec{n},$$

where $\vec{l} = (l_1, l_2, l_3)$ is the light source direction, and $\vec{n}$ the unit normal to the surface $z(x, y)$ that we want to reconstruct is given by

$$\vec{n} = \frac{(-z_x, -z_y, 1)}{\sqrt{1 + z_x^2 + z_y^2}}. \tag{2}$$

We use our freedom to choose the coordinate system so that $l_2 = 0$, this is done by rotating

$O(N \log N)$ computational steps sequential algorithm for solving the Eikonal equation on a rectangular grid, where N is the total number of grid points. This algorithm, known as the 'Fast Marching Method,' relies on a systematic causality relationship based on upwinding, coupled with a heap structure for efficiently ordering the updated points.

An important property of the solution that distinguishes it from graph search based methods is its converges to the continuous physical (viscosity) solution as the rectangular numerical grid is refined. Tsitsiklis, in [19], also solved the Eikonal equation on a rectangular grid with the same computational complexity, by iteratively solving a 'cost to go' optimization problem for the dynamically sorted grid points.

In this note we use Sethian's Fast Marching Method and modify it to construct a numerical solution for the oblique light source shape from shading problem.

2 Shape from Shading

Let us first review the shading image formation model for a 3D Lambertian object. Assume, that the object we try to reconstruct is given as a function $z(x, y) : R^2 \to R$, whose surface normal at each point is given by $\vec{n}(x, y) : R^2 \to S^2$. Next, let the light source direction be given by $\vec{l} \in S^2$. Then, the intensity image, $I(x, y) : R^2 \to R$, for an orthographic projection of the object is given by the inner product of the light source direction and the surface normal,

$$I(x, y) = \vec{l} \cdot \vec{n}(x, y).$$

For the simple vertical light source case $\vec{l} = (0, 0, 1)$, in which the light source is located near the viewer, the shading image is given by

$$I(x, y) = \frac{1}{\sqrt{1 + z_x^2 + z_y^2}}.$$

The problem in hand is the reconstruction of $z(x, y)$ from its gradient magnitude at each point that is given by

$$|\nabla z(x, y)| = \sqrt{(I(x, y))^{-2} - 1}.$$

This equation is known as the *Eikonal equation.* See [20] for a 'shading from shape' Eikonal based technique. It was shown in [2] for the three singular points case, and in [11] for the more general case, that with a simple smoothness assumption, the reconstruction problem can be solved for surfaces with complicated topologies as long as the surface normals are known to be pointing outwards along the boundaries of a given domain (e.g. the image boundaries). In the following sections we deal with the problem of how to reconstruct a shape from its shading image in a computationally efficient and numerically consistent way.

3 Sethian's Fast Marching Method

The Fast Marching Method is an $O(N \log N)$ numerical algorithm for solving the Eikonal equation, e.g. $|\nabla z(x, y)| = f(x, y)$. The first version of the algorithm is based on the following numerical approximation of the Eikonal equation

$$\begin{aligned} &\Big(\max(D_{ij}^{-x} z, -D_{ij}^{+x} z, 0)\Big)^2 + \\ &\quad \Big(\max(D_{ij}^{-y} z, -D_{ij}^{+y} z, 0)\Big)^2 \;=\; f_{ij}^2, \quad (1) \end{aligned}$$

where $z_{ij} = z(i\Delta x, j\Delta y)$, and $D_{ij}^{-x} z = (z_{ij} - z_{i-1,j})/\Delta x$ is the standard backwards derivative approximation, $D_{ij}^{+x} z = (z_{i+1,j} - z_{ij})/\Delta x$ is the standard forward derivative approximation in the x direction, and similarly for the y direction. This numerical approximation selects the correct viscosity solution for the shape from shading problem as proven by Rouy and Tourin [15]. One important observation is that

Optimal Algorithm for Shape from Shading

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Abstract

An optimal numerical algorithm for the reconstruction of a surface from its shading image is presented. The algorithm solves the 3D reconstruction from a single shading image problem. The shading image is treated as a penalty function and the hight of the reconstructed surface is a weighted distance. A first order numerical scheme based on Sethian's Fast Marching Method is used to compute the reconstructed surface. The surface is a viscosity solution of an Eikonal equation for the vertical light source case. For the oblique light source case, the surface is the viscosity solution to a different partial differential equation. A modification of the Fast Marching Method yields a numerically consistent, computationally optimal, and practically fast algorithm for the classical shape from shading problem.

1 Introduction

One of the earliest problems in the field of computer vision is the reconstruction of a three dimensional object from its single gray level image. The problem, for the case of a diffusive reflectance model of the surface, also known as Lambertian reflectance, is recognized as the 'shape from shading problem' [7, 8]. Various numerical schemes were proposed over the years, most of these methods were based on variational principles that require an additional smoothness or additional regularization terms that introduce second order derivatives into the minimization process. These terms yield an over-smoothed reconstructions, see for example the methods in [9]. Only two early direct models for the shape from shading did not incorporate extra smoothness terms, the first is the characteristic strips expansion method that Horn used when he first introduced the problem [7], the second is Bruckstein's equal hight contours tracking model [3]. Unfortunately, the first numerical implementations of these algorithms suffered from numerical instabilities.

New numerical algorithms based on recent results in curve evolution theory, control theory, and the viscosity framework [4], were applied to the shape from shading problem in [15, 6, 10, 12]. In these advanced numerical algorithms the smoothness assumption is embedded within the scheme without the need for an extra smoothness as a penalty.

Recently, Sethian [18, 17] introduced an