Image Processing via the Beltrami Operator ⋆

R. Kimmel,[1] *R. Malladi,*[1] *and N. Sochen*[2]
[1] Lawrence Berkeley National Laboratory University of California, Berkeley, CA 94720.
[2] Electrical Engineering Dept. Technion, Haifa 32000, Israel.

Abstract. We present a framework for enhancing images while preserving either the edge or the orientation-dependent texture information present in them. We do this by treating images as manifolds in a feature-space. This geometrical interpretation leads to a natural way for grey level, color, movies, volumetric medical data, and color-texture image enhancement. Following this, we invoke the Polyakov action from high-energy physics, and develop a minimization procedure through a geometric flow. This flow, based on manifold volume minimization yields a natural enhancement procedure. We apply this framework to edge-preserving denoising of grey value and color images, for volumetric medical data, and orientation-preserving flows for grey level and color texture images.

1 Introduction

In this paper, we present a general framework for processing images of various types like grey scale, color, and those that have orientation-dependent information such as textures. We do this by treating images as embedded maps that flow towards minimal surfaces. In other words, our view on images is that they are $2D$ or $3D$ manifolds embedded in higher dimensional space; for example a grey-scale image is a surface in (x, y, I) space and a color image is a surface embedded in a $5D$ space, i.e. the (x, y, I^r, I^g, I^b). We then use the Polyakov action, that is a general way of measuring area for a manifold embedded in a given space. The edge-preserving enhancement procedure is a result of minimizing this "action" and is expressed via a geometric flow. Our framework has the following properties: (1) It is the most general way of writing the geometrical scale-space and enhancement algorithms for grey-scale, color, volumetric, time-varying, and texture images, (2) it unifies many existing partial differential equation based schemes for image processing, and (3) the schemes are edge-preserving and hence suitable for segmentation tasks.

Texture plays an important role in the understanding process of many images, specially those that involve natural scenes. Preserving the orientation information while diffusing a given texture image is important in certain cases, say in denoising a fingerprint image. We imagine a procedure that preserves domains

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of constant/homogeneous texture, enhances the texture in each domain, and thereby enhances the boundaries between neighboring domains with different textures. Weickert in [23, 24] presents a coherence enhancing flow based on a structure tensor idea. In Section 4 we first link the coherence enhancing flow to the Beltrami geometric framework. Then we extend the method by inverting the diffusion direction across the edge for better enhancement and sharpening results.

2 Images as Embedded Maps that flow toward Harmonic Maps

Our geometric framework finds a seamless link between the L_1 ([19, 3] TV and its variants) and the L_2 norms (used in [13] and its variants) based on the geometry of the image and its interpretation as a surface[2]. The aspect ratio between the gray level and the xy plane, used as a parameter, enables us to switch between the two commonly used norms; see [22] for details.

In this work, we also propose a flow in a rich feature space which is different from the image space. Other flows in similar feature spaces were recently proposed in [20, 18, 5, 21, 25]; see also [23, 24] for orientation preserving flows. All these approaches begin with a flat metric [7] that does not yield a meaningful minimization process when going to more than one channel[3]. The main difference between these schemes and the one we propose is the geometric interpretation of the information as a manifold flowing so as to minimize its volume. Our geometric perspective of a color image as a surface embedded in a higher dimensional space enabled us to define a simple and natural coupling in the multi-channel color space. Other schemes have also considered image as a surface [2, 8, 26, 12], some even used the image information to build a Riemannian metric for segmentation [4]. However, these methods were not generalized to feature space or any co-dimension higher than one. We now describe the details of our framework.

2.1 The Metric

The basic concept of Riemannian differential geometry is distance. Let us start with the map $\mathbf{X} : \Sigma \rightarrow \mathbb{R}^3$, where Σ is a $2D$ manifold. We denote the local coordinates on the two dimensional manifold Σ by (σ^1, σ^2). The map X is explicitly given by $(X^1(\sigma^1, \sigma^2), X^2(\sigma^1, \sigma^2), X^3(\sigma^1, \sigma^2))$. Since the local coordinates σ^i are curvilinear, and not orthogonal in general, the distance square between two close points on Σ, $p = (\sigma^1, \sigma^2)$ and $p + (d\sigma^1, d\sigma^2)$ is not $ds^2 = d\sigma_1^2 + d\sigma_2^2$. In fact, the squared distance is given by a positive definite symmetric bilinear form called the metric, whose components we denote by $g_{\mu\nu}(\sigma^1, \sigma^2)$, i.e.

$$ds^2 = g_{\mu\nu} d\sigma^\mu d\sigma^\nu = g_{11}(d\sigma^1)^2 + 2g_{12} d\sigma^1 d\sigma^2 + g_{22}(d\sigma^2)^2, \qquad (1)$$

[2] TV (Total Variation) schemes are based on minimizing the L_1 norm, namely $\int |\nabla I|$, the L_2 norm minimizes $\int |\nabla I|^2$, while the area of the gray level image surface is given by $\int \sqrt{1 + |\nabla I|^2}$.

[3] This flat metric is called 'structure tensor' in [23, 24].

where we used Einstein summation convention in the second equality.

2.2 Polyakov Action

Let us briefly review our framework for non-linear diffusion in computer vision. The equations are derived by a minimization problem from an action functional. The functional in question depends on *both* the image manifold and the embedding space. Denote by (Σ, g) the image manifold and its metric and by (M, h) the space-feature manifold and its metric, then the map $\mathbf{X} : \Sigma \to M$ has the following weight

$$S[X^i, g_{\mu\nu}, h_{ij}] = \int d^m\sigma \sqrt{g} g^{\mu\nu} \partial\mu X^i \partial\nu X^j h_{ij}(\mathbf{X}), \tag{2}$$

where m is the dimension of Σ, g is the determinant of the image metric, $(g^{\mu\nu})$ is the inverse of the image metric, the range of indices is $\mu, \nu = 1, \ldots, \dim \Sigma$, and $i, j = 1, \ldots, \dim M$, and (h_{ij}) is the metric of the embedding space. This functional, for $m = 2$, was first proposed by Polyakov [16] in the context of high energy physics.

Given the above functional, we have to choose the minimization. We may choose for example to minimize with respect to the embedding alone. In this case the metric $(g_{\mu\nu})$ is treated as a parameter and may be fixed by hand. Another choice is to vary only with respect to the feature coordinates of the embedding space, or we may choose to vary the image metric as well. In [22] we show how different choices yield different flows. Some flows are recognized as existing methods, other choices are new and will be described below.

Using standard methods in variational calculus (see [22]), the Euler-Lagrange equations with respect to the embedding are:

$$X^i_t = -\frac{1}{2\sqrt{g}} h^{il} \frac{\delta S}{\delta X^l} = \frac{1}{\sqrt{g}} \partial\mu (\sqrt{g} g^{\mu\nu} \partial\nu X^i). \tag{3}$$

The operator that is acting on X^i is the natural generalization of the Laplacian from flat spaces to manifolds and is called *the second order differential parameter of Beltrami* [10], or for short *Beltrami operator*, and is denoted by Δ_g. For the grey scale image case,the flow $I_t = \Delta_g I$, is *edge-preserving*. The generalization to any manifold embedded with arbitrary co-dimension is given by using Eq. 3 for all the embedding coordinates and the induced metric; see [22] for more details. In what follows we apply this operator to construct an orientation-preserving flow on texture images. But first let us look at the color image case more closely.

3 Color

We apply the Beltrami flow to the 5 dimensional space-feature needed in color images. The embedding space-feature space is taken to be Euclidean with Cartesian coordinate system. The image, thus, is the map $f : \Sigma \to \mathbb{R}^5$ where Σ is a two dimensional manifold. Explicitly the map is

$$f = \left(X^1(\sigma^1,\sigma^2), X^2(\sigma^1,\sigma^2), I^r(\sigma^1,\sigma^2), I^g(\sigma^1,\sigma^2), I^b(\sigma^1,\sigma^2)\right).$$

Note that there are obvious better selections to color space definition rather than the RGB flat space.

We minimize our action (2) with respect to (I^r, I^g, I^b). For convenience we denote below (r, g, b) by $(1, 2, 3)$, or in general notation i. The induced metric is given in this case as follows:

$$\begin{aligned} g_{11} &= 1 + (I^1_x)^2 + (I^2_x)^2 + (I^3_x)^2, \\ g_{12} &= I^1_x I^1_y + I^2_x I^2_y + I^3_x I^3_y, \\ g_{22} &= 1 + (I^1_y)^2 + (I^2_y)^2 + (I^3_y)^2. \end{aligned} \tag{4}$$

The action functional under this choice of the metric is the Euler functional $S = \int d^2\sigma\sqrt{g}$. It is simply the area of the image surface. Minimization with respect to I^i gives the Beltrami flow

$$I^i_t = \frac{1}{\sqrt{g}}\partial\mu(\sqrt{g}g^{\mu\nu}\partial\nu I^i), \tag{5}$$

which is a flow towards a minimal surface that preserves edges. As an example, we show the result of color denoising in Fig. 1.

4 The Metric as a Structure Tensor

In [9, 11], Gabor considered an image enhancement procedure based on a single small time step along a directional flow. It is based on the anisotropic flow via the <u>inverse</u> second directional derivative in the 'edge' direction (∇I direction) and the geometric heat equation (second derivative in the direction parallel to the edge). The same idea of steering the diffusion direction motivated many recent works[4]. Cottet and Germain [6] used a smoothed version of the image to direct the diffusion, while Weickert [23] smoothed also the structure tensor $\nabla I \nabla I^T$ and then manipulated its eigenvalues to steer the smoothing direction. Eliminating one eigenvalue from a structure tensor $\sum_i \nabla I^i \nabla I^{i^T}$, was used in in [21], in which the tensors are not necessarily positive definite. However, in [24], the eigenvalues are manipulated to result in a positive definite tensor.

Motivated by all of these results we will first link the anisotropic orientation diffusion (coherence enhancement) to the geometric framework, and then invert the diffusion direction across the edge. Let us first show that the diffusion directions can be deduced from the smoothed metric coefficients $g_{\mu\nu}$ and may thus be included within the Beltrami framework under the right choice of directional diffusion coefficients.

The induced metric $(g_{\mu\nu})$ is a symmetric uniformly positive definite matrix that captures the geometry of the image surface. Let λ_1 and λ_2 be the largest and

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[4] See [17] for many interesting extensions and applications of the locally isotropic flow.

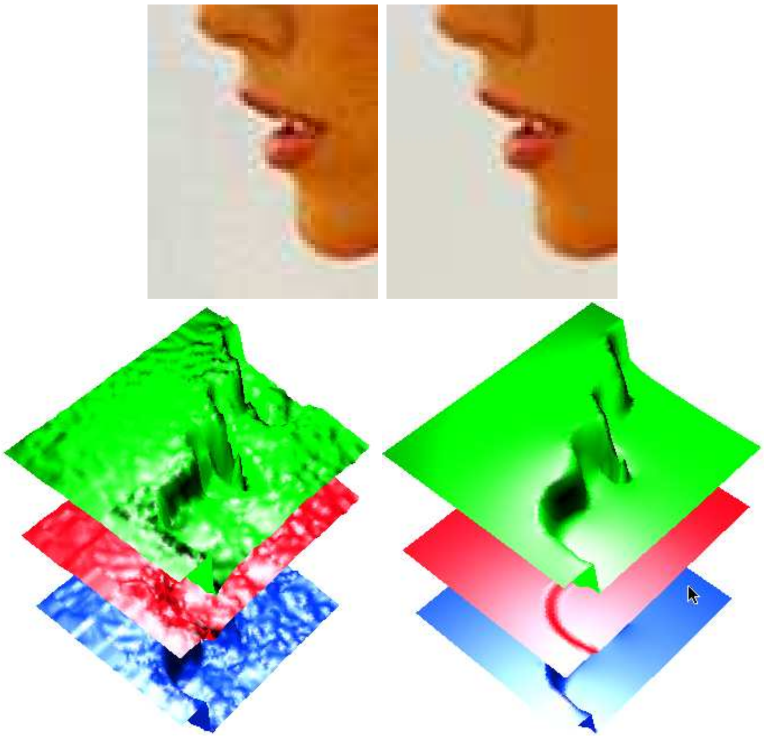

Fig. 1. Color results: The top row shows noisy image on the left and the denoised one on the right. To better depict the edge-preserving property of our method, in the bottom row we render as surfaces the three color channels of both the noisy and the reconstructed image.

the smallest eigenvalues of $(g_{\mu\nu})$, respectively. Since $(g_{\mu\nu})$ is a symmetric positive matrix its corresponding eigenvectors u_1 and u_2 can be chosen orthonormal. Let $U \equiv (u_1|u_2)$, and $\Lambda \equiv \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then we readily have the equality $(g_{\mu\nu}) = U\Lambda U^T$. Note also that

$$(g^{\mu\nu}) \equiv (g_{\mu\nu})^{-1} = U\Lambda^{-1}U^T = U \begin{pmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{pmatrix} U^T, \tag{6}$$

and that $g \equiv \det(g_{\mu\nu}) = \lambda_1\lambda_2$.

We will use the image metric in its natural geometric interpretation, i.e. as a structure tensor. The coherence enhancement Beltrami flow $\mathbf{I}_t = \Delta_{\hat{g}}\mathbf{I}$ for color-texture images is then given as follows:

1. Compute the metric coefficients $g_{\mu\nu}$. For the N channel case (for color $N = 3$) we have (see Eq. (4)) $g_{\mu\nu} = \delta_{\mu\nu} + \sum_{k=1}^{N} I^k_\mu I^k_\nu$.

2. Diffuse the $g_{\mu\nu}$ coefficients by convolving with a Gaussian of variance ρ, thereby $\tilde{g}_{\mu\nu} = G_\rho * g_{\mu\nu}$. For $2D$ images $G_\rho = e^{-(x^2+y^2)/\rho^2}$.
3. Change the eigenvalues, λ_1, λ_2, $\lambda_1 > \lambda_2$, of $(\tilde{g}_{\mu\nu})$ so that $\lambda_1 = \alpha^{-1}$ and $\lambda_2 = \alpha$, for some given positive scalar $\alpha \ll 1$. This yields a new metric $\hat{g}_{\mu\nu}$ that is given by: $(\hat{g}_{\mu\nu}) = \tilde{U} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \tilde{U}^T = \tilde{U} \Lambda_\alpha \tilde{U}^T$.
4. Evolve the k-th channel via Beltrami flow, that by the selection $\hat{g} \equiv \det(\hat{g}_{\mu\nu}) = \lambda_1 \lambda_2 = \alpha^{-1}\alpha = 1$ now reads

$$
\begin{aligned}
I_t^k &= \Delta_{\hat{g}} I^k \equiv \frac{1}{\sqrt{\hat{g}}} \partial_\mu \sqrt{\hat{g}} \hat{g}^{\mu\nu} \partial_\nu I^k = \partial_\mu \hat{g}^{\mu\nu} \partial_\nu I^k \\
&= \operatorname{div}\left(\tilde{U} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \tilde{U}^T \nabla I^k \right) = \operatorname{div}\left(\tilde{U} \Lambda_\alpha \tilde{U}^T \nabla I^k \right). \qquad (7)
\end{aligned}
$$

Note again that both for gray level and color images the above flow is similar to the coherence-enhancing anisotropic diffusion with the important property of a uniformly positive definite diffusion tensor. For color images, $(g_{\mu\nu}) = \mathcal{I} + \sum_i \nabla I^i \nabla I^{i^T}$, where $\mathcal{I}$ is the identity matrix, and I^i are the color channels $((I^r, I^g, I^b) \equiv (I^1, I^2, I^3))$. In this case all that is done is the identity added to the structure tensors $\nabla I \nabla I^T$ for gray and $\sum_i \nabla I^i \nabla I^{i^T}$ for color. This addition does not change the eigenvectors and thus the above flow is equivalent to Weickert schemes [23, 24]. Next, we introduce a new inverse/direct diffusion model.

4.1 Beyond a Metric: Inverse Diffusion Across the Edge

Let us take one step further, and exit our 'metric' framework by defining $(g_{\mu\nu})$ to be a non-singular symmetric matrix with one positive and one negative eigenvalues. That is, instead of a small diffusion we introduce a controlled inverse diffusion across the edge. Here we extend Gabor's idea [9, 11] of inverting the diffusion along the gradient direction.

Inverting the heat equation is an inherently unstable process. However, if we keep smoothing the metric coefficients, and apply the heat operator in the perpendicular direction we get a coherence-enhancing flow with sharper edges that is stable for a short duration of time. The idea is simply to change the sign of one of the modified eigenvalues in the algorithm described in the previous subsection. In other words, in step 3 of the previous scheme we change the eigenvalues of $(\tilde{g}_{\mu\nu})$ such that the largest eigenvalue λ_1 is now $\lambda_1 = -\alpha^{-1}$ and $\lambda_2 = \alpha$, for some given positive scalar $\alpha < 1$.

For the gray level case with $\rho = 0$ it simplifies to highly unstable inverse heat equation. However, as ρ increases the smoothing along the edges becomes fundamental and the scheme is similar in its spirit to that of [9]; also see [14, 1].

4.2 Color Orientation-Enhancing Results

In [23] the coherence enhancement flow was applied on several color masterpieces by van Gogh, which resulted in a 'coherence enhancement of expressionism'. In the next example we attempt to 'enhance and sharpen impressionism'. We apply first the anisotropic oriented diffusion flow and then the new oriented diffusion along/inverse diffusion across the edge on a color painting by Claude Monet, see Fig. 2.

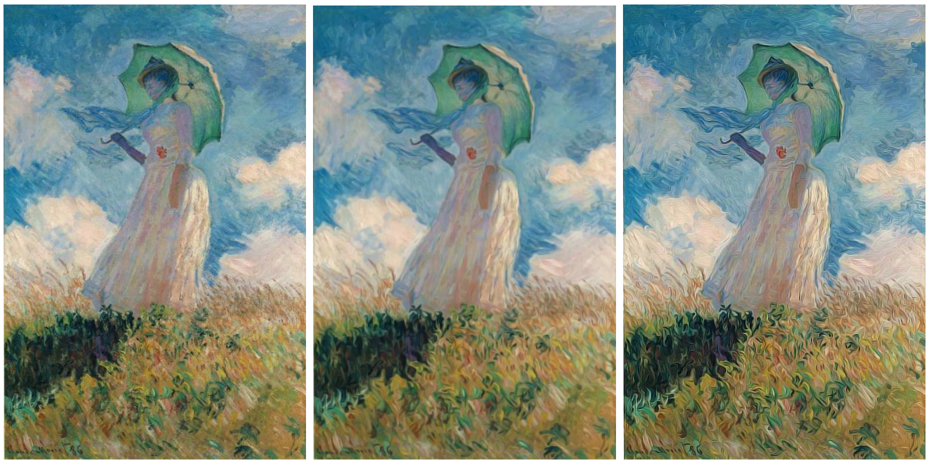

Fig. 2. Top: Original picture "Femme à l'ombrelle tournée vers la gauche," by Claude Monet 1875 ("woman with umbrella turning left") 521×784 (left), the result of orientation-preserving diffusion (middle), and the result of inverse/direct diffusion flow ($\rho = 4$) (right).

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