A General Framework for Low Level Vision

Nir Sochen Ron Kimmel Ravi Malladi

Abstract— **We introduce a new geometrical framework based on which natural flows for image scale space and enhancement are presented. We consider intensity images as surfaces in the (x, I) space. The image is thereby a 2D surface in 3D space for gray level images, and 2D surfaces in 5D for color images. The new formulation unifies many classical schemes and algorithms via a simple scaling of the intensity contrast, and results in new and efficient schemes. Extensions to multi dimensional signals become natural and lead to powerful denoising and scale space algorithms.**

Keywords—**Scale-space, Non-linear image diffusion, Image smoothing, Image enhancement, Color image processing**

I. Introduction

THE importance of dynamics of image geometry in the perception and understanding of images is by now well established in computer vision. Geometry, symmetry and dynamics are also the main issues in physics. Borrowing ideas from high-energy physics, we propose in this paper a geometrical framework for low-level vision. The two main ingredients of this framework are 1) defining images as embedding maps between two Riemannian manifolds. 2) An action functional that provides a measure on the space of these maps. This action is the natural generalization of the L2 Euclidean norm to non-Euclidean manifolds and is known as the Polyakov action in physics. The justification for the use of this functional in computer vision is twofold: It unifies many seemingly unrelated scale space methods on one hand and provides new and improved ways to smooth and denoise images on the other. It will lead us in this paper to the construction of image enhancement procedures for gray and color images. The framework also integrates many existing denoising and scale space procedures by a change of a single parameter that switches between the Euclidean L1 and L2 norms.

Motivated by [2], [31], we consider low level vision as an input to output process. For example, the most common input is a gray level image; namely a map from a two dimensional surface to a three dimensional space ($\mathbb{R}^3$). We have at each point of the xy coordinate plane an intensity $I(x, y)$. The $\mathbb{R}^3$ *space-feature* has Cartesian coordinates (x, y, I) where x and y are the *spatial* coordinates and I is the feature coordinate [1]. The output of the low level process in most models consists of 1) A smoothed image from which reliable features can be extracted by local, and therefore differential operators, and 2) A segmentation, that is, either a decomposition of the image domain into homogeneous regions with boundaries, or a set of boundary points – an "edge map".

The process assumes the existence of layers serving as operators such that the information is processed locally in the layers and forwarded to the next layer with no interaction between distant layers. This means that the output has the form $\mathbf{X}(\Sigma, t)$ which is the solution of $\partial_t \mathbf{X} = O\mathbf{X}$, where O is a local differential operator, and the input image is given as initial condition. This process yields a one-parameter family of images on the basis of an input image. Normally such a family is called a scale-space (see [35] and references therein).

The importance of edges that are obtained from the intensity gradient is acknowledged, and gradient based edge detectors are a basic operation in many computer vision applications. Edge detectors appear by now in almost all image processing tools. The importance of edges in scale space construction is also obvious. Boundaries between objects should survive as long as possible along the scale space, while homogeneous regions should be simplified and flattened in a more rapid way. We propose here a new non-linear diffusion algorithm which does exactly that.

Another important question, for which there is only partial answers, is how to treat multi valued images. A color image is a good example since we actually talk about 3 images (Red, Green, Blue) that are composed into one. Should one treat such images as multi valued functions as proposed in [14]?

We attempt to answer the above question by viewing images as embedding maps, that flow towards minimal surfaces. We go two dimensions higher than most of the classical schemes, and instead of dealing with isophotes as planar curves we deal with the whole image as a surface. For example, a gray level image is no longer considered as a function but as a two dimensional surface in three dimensional space. This idea is quite old [20], [46] for gray level images, yet, to the best of our knowledge, it was never carried on to higher dimensions. As another example, we will consider a color image as 2D surfaces now in 5D. We thank the editors for communicating to us a related effort that is published in this issue, see [47].

We have chosen to present our ideas in the following order: Section II introduces the basic concepts of a metric and the induced metric and presents a measure on maps between Riemannian manifolds that we borrowed from high energy physics. This measure provides a general framework for non-linear diffusion in computer vision, as shown in the following sections. In Section III we introduce a new flow that we have chosen to name *Beltrami flow*, present a geometric interpretation in the simplest 3D case, its relation to previous models, and two examples of the Beltrami flow for color images. Then, in Section IV we refer to other models that are the result of the same action through different

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[1] While in this paper the feature coordinate is simply the zeroth jet space $j^0 I$, we use the term feature space to leave room for a more general cases like texture [24], etc.

choices of the image metric and the minimization variables. We also study the geometrical properties of a generalized version of the mean curvature flow that is closely related to the proposed framework. We conclude in Section V with a summarizing discussion.

II. Polyakov Action and Harmonic Maps

A. The geometry of a map

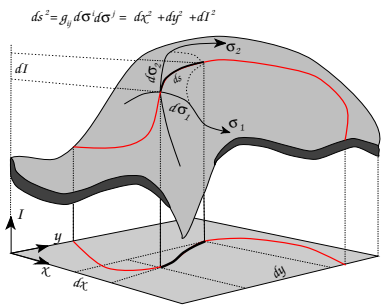

Fig. 1. Length element of a surface curve ds, may be defined either as a function of a local metric defined on the surface (σ_1, σ_2), or as a function of the coordinates of the space in which the surface is embedded (x, y, I).

The basic concept of Riemannian differential geometry is distance. The natural question in this context is how do we measure distances? We will first take the important example $\mathbf{X} : \Sigma \rightarrow \mathbb{R}^3$. Denote the local coordinates on the two dimensional manifold Σ by (σ^1, σ^2), these are analogous to arc length for the one dimensional manifold, i.e. a curve, see Fig. 1. The map $\mathbf{X}$ is explicitly given by $(X^1(\sigma^1, \sigma^2), X^2(\sigma^1, \sigma^2), X^3(\sigma^1, \sigma^2))$. Since the local coordinates σ^i are curvilinear, the squared distance is given by a positive definite symmetric bilinear form called the metric whose components we denote by $g_{\mu\nu}(\sigma^1, \sigma^2)$:

$$ds^2 = g_{\mu\nu} d\sigma^\mu d\sigma^\nu \equiv g_{11}(d\sigma^1)^2 + 2g_{12} d\sigma^1 d\sigma^2 + g_{22}(d\sigma^2)^2,$$

where we used Einstein summation convention in the second equality; identical indices that appear one up and one down are summed over. We will denote the inverse of the metric by $g^{\mu\nu}$, so that $g^{\mu\nu} g_{\nu\gamma} = \delta^\mu_\gamma$, where δ^μ_γ is the Kronecker delta.

Let $\mathbf{X} : \Sigma \rightarrow M$ be an embedding of (Σ, g) in (M, h), where Σ and M are Riemannian manifolds and g and h are their metrics respectively. We can use the knowledge of the metric on M and the map $\mathbf{X}$ to construct the metric on Σ. This procedure, which is denoted formally as $(g_{\mu\nu})_\Sigma = \mathbf{X}^*(h_{ij})_M$, is called the *pullback* for obvious reasons and is given explicitly as follow:

$$g_{\mu\nu}(\sigma^1, \sigma^2) = h_{ij}(\mathbf{X}) \partial_\mu X^i \partial_\nu X^j, \qquad (1)$$

where $i, j = 1, ..., \dim M$ are being summed over, and $\partial_\mu X^i \equiv \partial X^i(\sigma^1, \sigma^2)/\partial\sigma^\mu$.

Take for example a grey level image which is, from our point of view, the embedding of a surface described as a graph in $\mathbb{R}^3$:

$$\mathbf{X} : (\sigma^1, \sigma^2) \rightarrow (x = \sigma^1, y = \sigma^2, z = I(\sigma^1, \sigma^2)), \qquad (2)$$

where (x, y, z) are Cartesian coordinates. Using Eqn. (1) we get

$$(g_{\mu\nu}) = \begin{pmatrix} 1 + I_x^2 & I_x I_y \\ I_x I_y & 1 + I_y^2 \end{pmatrix}, \qquad (3)$$

where we used the identification $x \equiv \sigma^1$ and $y \equiv \sigma^2$ in the map $\mathbf{X}$.

Actually, we can understand this result in an intuitive way: Eq. (1) means that the distance measured on the surface by the local coordinates is equal to the distance measured in the embedding coordinates, see Fig. 1. Under the above identification, we can write $ds^2 = dx^2 + dy^2 + dI^2 = dx^2 + dy^2 + (I_x dx + I_y dy)^2 = (1 + I_x^2) dx^2 + 2 I_x I_y dx dy + (1 + I_y^2) dy^2$.

Next we provide a measure on the space of these maps.

B. The measure on maps

In this subsection, we present a general framework for non-linear diffusion in computer vision. We will show in the sequel that many known methods fall naturally into this framework and how to derive new ones. The equations will be derived by a minimization problem from an action functional. The functional in question depends on *both* the image manifold and the embedding space. Denote by (Σ, g) the image manifold and its metric and by (M, h) the space-feature manifold and its metric, then the map $\mathbf{X} : \Sigma \rightarrow M$ has the following weight [34]

$$S[X^i, g_{\mu\nu}, h_{ij}] = \int d^m\sigma \sqrt{g} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j h_{ij}(\mathbf{X}), \qquad (4)$$

where m is the dimension of Σ, g is the determinant of the image metric, $g^{\mu\nu}$ is the inverse of the image metric, the range of indices is $\mu, \nu = 1, \ldots, \dim\Sigma$, and $i, j = 1, \ldots, \dim M$. The metric of the embedding space is h_{ij}.

To gain some intuition about this functional, let us take the example of a surface embedded in $\mathbb{R}^3$ and treat both the metric $(g_{\mu\nu})$ and the spatial coordinates of the embedding space as free parameters, and let us fix them to

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad x = \sigma^1 \quad , \quad y = \sigma^2. \qquad (5)$$

We also adopt in $\mathbb{R}^3$ the Cartesian coordinates (i.e. $h_{ij} = \delta_{ij}$). Then we get the Euclidean L2 norm:

$$S[I, g_{\mu\nu} = \delta_{\mu\nu}, h_{ij} = \delta_{ij}] = \int d^2\sigma (|\nabla x|^2 + |\nabla y|^2 + |\nabla I|^2). \qquad (6)$$

If we now minimize with respect to I, we will get the usual heat operator acting on I. We see that the Polyakov action is the generalization of the L2 norm to curved spaces. Here, $d^m\sigma\sqrt{g}$ is the volume element (area element for d=2) of Σ – the image manifold , and $g^{\mu\nu}\partial_\mu X^i \partial_\nu X^j h_{ij}(\mathbf{X})$ is the generalization of $|\nabla I|^2$ to maps between non-Euclidean manifolds. Note that the volume element as well as the rest of the expression is reparameterization invariant. This means that they are invariant under a smooth transformation $\sigma^\mu \rightarrow \tilde{\sigma}^\mu(\sigma^1, \sigma^2)$. The Polyakov action really depends

on the geometrical objects and not on the way we describe them via our parameterization of the coordinates.

Given the above functional, we have to choose the minimization. We may choose for example to minimize with respect to the embedding alone. In this case the metric $g_{\mu\nu}$ is treated as a parameter of the theory and may be fixed by hand. Another choice is to vary only with respect to the feature coordinates of the embedding space, or we may choose to vary with respect to the image metric as well. We will see that these different choices yield different flows. Some flows are recognized as existing methods like the heat flow, a generalized Perona-Malik flow, or the mean-curvature flow. Other choices are new and will be described below in detail.

Another important point is the choice of the embedding space and its geometry. In general, we need information about the task at hand in order to fix the right geometry. Take for example the grey level images. It is clear that the intensity I is not on equal footing as x and y. In fact the relative scale of I with respect to the spatial coordinates (x, y) is to be specified. This can be interpreted as taking the metric of the embedding space as follows:

$$(h_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta^2 \end{pmatrix}. \tag{7}$$

We will see below that different limits of this ratio β interpolate between the flows that originate from the Euclidean L1 and L2 norms.

Using standard methods in variational calculus, the Euler-Lagrange equations with respect to the embedding are (see [42] for derivation):

$$-\frac{1}{2\sqrt{g}} h^{il} \frac{\delta S}{\delta X^l} = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu X^i) + \Gamma^i_{jk} \partial_\mu X^j \partial_\nu X^k g^{\mu\nu}, \tag{8}$$

where Γ^i_{jk} are the Levi-Civita connection coefficients with respect to the metric h_{ij} that describes the geometry of the embedding space (see [44], [42] for a definition of the Levi-Civita connection).

Our proposal is to view scale-space as the gradient descent:

$$X^i_t \equiv \frac{\partial X^i}{\partial t} = -\frac{1}{2\sqrt{g}} h^{il} \frac{\delta S}{\delta X^l}. \tag{9}$$

Few remarks are in order. First, notice that we used our freedom to multiply the Euler-Lagrange equations by a strictly positive function and a positive definite matrix. This factor is the simplest one that does not change the minimization solution while giving a reparameterization invariant expression. This choice guarantees that the flow is geometric and does not depend on the parameterization. We will see below that the Perona-Malik flow, for example, corresponds to another choice of the pre-factor, namely 1. The operator that is acting on X^i in the first term of Eqn. (8) is the natural generalization of the Laplacian from flat spaces to manifolds and is called *the second order differential parameter of Beltrami* [27], or in short *Beltrami operator*, and we will denote it by Δ_g. When the embedding is in a Euclidean space with Cartesian coordinate system the connection elements are zero. If the embedding space is not Euclidean or if the Coordinate system we use is not Cartesian we have to include the Levi-Civita connection term since it is no longer equal to zero.

In general for any manifolds Σ and M, the map $\mathbf{X} : \Sigma \to M$ that minimizes the action S with respect to the embedding is called a *harmonic map*. The harmonic map is the natural generalization of the geodesic curve and the minimal surface to higher dimensional manifolds and for different embedding spaces. We have here a framework that can treat curves, surfaces, and higher dimensional image data embedded in gray, color and higher dimensional and geometrically non-trivial embedding spaces.

III. The Beltrami flow

In this section, we present a new and natural flow. The image is regarded as an embedding map $\mathbf{X} : \Sigma \to \mathbb{R}^n$, where Σ is a two dimensional manifold. We treat grey-level and color images as examples and then compare to related works. Explicitly, the maps for grey-level and color images are

$$\mathbf{X} = \left(x(\sigma^1, \sigma^2), y(\sigma^1, \sigma^2), I(\sigma^1, \sigma^2)\right)$$

and

$$\mathbf{X} = \left(x(\sigma^1, \sigma^2), y(\sigma^1, \sigma^2), \{I^i(\sigma^1, \sigma^2)\}_{i=1}^3\right), \tag{10}$$

respectively. In the above map we have denoted (r, g, b) by $(1, 2, 3)$ for convenience, or in general notation by i. We minimize our action in Eqn. (4) with respect to the metric and with respect to (I^r, I^g, I^b). The coordinates x and y are parameters from this view point and are identified as usual with σ^1 and σ^2 respectively. We note that there are obviously better selections to color space definition rather than the RGB flat space. Nevertheless, we get good results even from this oversimplified assumption.

Minimizing the metric gives, as we have seen, the induced metric which is given for grey level image in Eqn. (3) and for color images by

$$(g_{\mu\nu}) = \begin{pmatrix} 1 + \sum_{i=1}^3 (I^i_x)^2 & \sum_{i=1}^3 I^i_x I^i_y \\ \sum_{i=1}^3 I^i_x I^i_y & 1 + \sum_{i=1}^3 (I^i_y)^2 \end{pmatrix}, \tag{11}$$

and $g = \det(g_{ij}) = g_{11} g_{22} - g_{12}^2$. Note that this metric differs from the Di Zenzo matrix [14] (which is not a metric since it is not positive definite) by the addition of 1 to g_{11} and g_{22}. The source of the difference lies in the map used to describe the image; Di Zenzo used $\mathbf{X} : \Sigma \to \mathbb{R}^3$ while we use $\mathbf{X} : \Sigma \to \mathbb{R}^5$.

The action functional under our choice of the metric is the Nambu functional

$$\begin{aligned} S &= \int d^2\sigma \sqrt{\det(\partial_\mu X^i \partial_\nu X_i)} \\ &= \int d^2\sigma \sqrt{1 + \sum_i |\nabla I^i|^2 + \frac{1}{2} \sum_{ij} (\nabla I^i, \nabla I^j)^2}, \end{aligned} \tag{12}$$

where $(\nabla I^i, \nabla I^j)$ stands for the magnitude of the vector product of the vectors ∇I^i and ∇I^j. For grey level

images the last term vanishes and we are left with $S = \int d^2\sigma\sqrt{1+|\nabla I|^2}$. The action in Eqn. (12) is simply the area of the image surface.

Now, we compare our norm to that proposed by Shah in [41]: $\int\sqrt{\sum_{i=1}|\nabla I^i|^2}$. We notice that the proposed area norm in Eqn. (12), includes an extra term that does not appear in Shah's norm and other previous norms in the literature. The term $\sum_{ij}(\nabla I^i, \nabla I^j)^2$ measures the directional difference of the gradient between different channels. The minimization of a norm that includes this term, directs different channels to align together as they become smoother and simpler in scale. One should recognize this cross correlation of orientation between the channels as a very important feature; overcoming the color fluctuations along edges as a result of a lossy JPEG standard compression is a good example.

Minimizing Eqn. (12) with respect to I^i gives the Beltrami flow

$$I^i_t = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu I^i) \equiv \mathbf{H}^i. \tag{13}$$

It means that the velocity in the I^i direction is proportional to the component of the mean curvature vector in the I^i direction. Note the difference between Eqn. (13) and the mean curvature flow in Eqn. (20). Here we only move the feature coordinates while keeping x and y fixed, where as in the mean curvature flow we move all coordinates. The projection of the mean curvature vector on the feature coordinates is an edge preserving procedure. Intuitively it is obvious. Each point on the image surface moves with a velocity that depends on the mean curvature and the I^i components of the normal to the surface at that point. Since along the edges the normal to the surface lies almost entirely in the x-y plane, I^i hardly changes along the edges while the flow drives other regions of the image towards a minimal surface at a more rapid rate.

For a simple implementation of the Beltrami flow in color we first compute the following matrices: I^i_x, I^i_y, p^i, and q^i given by

$$\begin{aligned} p^i &= g_{22}I^i_x - g_{12}I^i_y, \\ q^i &= -g_{12}I^i_x + g_{11}I^i_y. \end{aligned}$$

Then the evolution is given by

$$I^i_t = \frac{1}{g}\left(p^i_x + q^i_y\right) - \frac{1}{2g^2}\left(g_x p^i + g_y q^i\right) \tag{14}$$

where $g_x = \partial_x g$ ($g_y = \partial_y g$).

For grey-scale case, we get the following expression after plugging the explicit form of $(g_{\mu\nu})$:

$$I_t = \frac{(1+I_y^2)I_{xx} - 2I_xI_yI_{xy} + (1+I_x^2)I_{yy}}{(1+I_x^2+I_y^2)^2}. \tag{15}$$

Let us further explore the geometry of the flow and relate it to other known methods.

A. Geometric Flows Towards Minimal Surfaces

A minimal surface is the surface with least area that satisfies given boundary conditions. It has nice geometrical properties, and is often used as a natural model of various physical phenomena, e.g. soap bubbles 'Plateau's problem,' in computer aided design, in structural design, and recently even for medical imaging [6]. It was realized by J. L. Lagrange in 1762 [28], that the mean curvature equal to zero is the Euler Lagrange equation for area minimization. Hence, the mean curvature flow is the most efficient flow towards a minimal surface. Numerical schemes for the mean curvature flow, and the construction of minimal surfaces under constraints, where studied since the beginning of the modern age of numerical analysis [13], and is still the subject of ongoing numerical research [11], [12], [8].

For constructing the mean curvature flow of the image as a surface, we follow three steps: 1) Let the surface $\mathcal{S}$ evolve according to the geometric flow $\frac{\partial \mathcal{S}}{\partial t} = \vec{F}$, where $\vec{F}$ is an arbitrary smooth flow field. The geometric deformation of $\mathcal{S}$ may be equivalently written as $\frac{\partial \mathcal{S}}{\partial t} = \langle\vec{F}, \vec{\mathcal{N}}\rangle\vec{\mathcal{N}}$, where $\vec{\mathcal{N}}$ is the unit normal of the surface at each point, and $\langle\vec{F}, \vec{\mathcal{N}}\rangle$ is the inner product (the projection of $\vec{F}$ on $\vec{\mathcal{N}}$). 2) The mean curvature flow is given by: $\frac{\partial \mathcal{S}}{\partial t} = H\vec{\mathcal{N}}$, where H is the mean curvature of $\mathcal{S}$ at every point. 3) Considering the image function $I(x,y)$, as a parameterized surface $\mathcal{S} = (x, y, I(x,y))$, and using the relation in step 1, we may write the mean curvature flow as: $\frac{\partial \mathcal{S}}{\partial t} = \frac{H}{\langle\vec{\mathcal{N}}, \vec{z}\rangle}\vec{z}$, for any smooth vector field $\vec{z}$ defined on the surface. Especially, we may choose $\vec{z}$ as the $\hat{I}$ direction, i.e. $\vec{z} = (0,0,1)$. In this case

$$\frac{1}{\langle\vec{\mathcal{N}}, \vec{z}\rangle}\cdot\vec{z} = \sqrt{1+I_x^2+I_y^2}\cdot(0,0,1) = \sqrt{g}(0,0,1). \tag{16}$$

Fixing the (x,y) parameterization along the flow, we have $\mathcal{S}_t = \frac{\partial}{\partial t}(x,y,I(x,y)) = (0,0,I_t(x,y))$. Thus, for tracking the evolving surface, it is enough to evolve I via $\frac{\partial I}{\partial t} = H\sqrt{1+I_x^2+I_y^2}$, where the mean curvature H is given as a function of the image I, see Fig. 2, and Eqn. (22). See [10], [11] for the derivation of H (as D.L. Chopp summarizes the original derivation by J.L. Lagrange from 1762).

Substituting the explicit equation for mean curvature, we end up with the equation

$$I_t = \frac{(1+I_y^2)I_{xx} - 2I_xI_yI_{xy} + (1+I_x^2)I_{yy}}{1+I_x^2+I_y^2}, \tag{17}$$

with the initial condition $I(x,y,t=0) = I_0(x,y)$. Using the notation of Beltrami second order operator Δ_g and the metric g, Equation (17) may be read as $I_t = g\Delta_g I$. This equation was studied in depth in [16], [32] where the existence and uniqueness of weak solutions was proved under some mild conditions on the behavior of the curvature on the boundary and the smoothness of the initial condition. The Beltrami flow itself (selective mean curvature flow) $I_t = \Delta_g I$ is given explicitly for the simple 2D case in Eqn. (15). The difference between the two flows is the

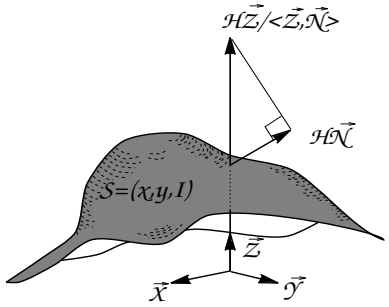

Fig. 2. Consider the surface mean curvature flow $\mathcal{S}_t = H\vec{\mathcal{N}}$, mean curvature H in the surface normal direction $\vec{\mathcal{N}}$. A geometrically equivalent flow is the flow $\partial(x,y,I)/\partial t = H(1+|\nabla I|^2)^{1/2}\cdot(0,0,1)$ which yields the mean curvature flow when projected onto the normal.

factor g. This factor has an important significance in keeping the flow geometrical, that is, it depends on geometrical objects and not on coordinates used in describing them. It also serves as an edge detector by behaving like an edge-preserving flow; see Fig. 3.

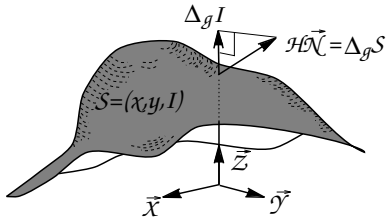

Fig. 3. Consider the mean curvature H in the surface normal direction $\vec{\mathcal{N}}$. It can also be expressed as $H\vec{\mathcal{N}} = \Delta_g\mathcal{S}$. Beltrami operator that operates on I: $\Delta_g I$, is the third component of this vector: Projection onto the I ($\vec{z}$) direction.

B. Related works

In [17], the authors propose a similar flow for grey level images: $I_t = H = \sqrt{g}\Delta_g$. Geometrically, they rotate the curvature normal vector so that it coincides with the $\vec{z}$ axis. This equation was studied extensively by mathematicians [29], [19], [15] where the existence and uniqueness of weak solutions was discussed. It is located somewhere between the mean curvature flow for the image as a surface $I_t = g\Delta_g I = H\sqrt{g}$ that was used in [30] to denoise images, and our Beltrami flow, which in 2D case simplifies to $I_t = \Delta_g I = H/\sqrt{g}$. All of the above flows lead towards a minimal surface, yet our proposed framework better preserves the edges, naturally extends to any number of dimensions, and is reparameterization invariant.

Let us show next the direct relation to TV methods [36] and especially for the regularization introduced by Vogel and Oman [45], and efficiently implemented for changing the regularization ratio (from large to small) in [8]. We will show that by modifying the aspect ratio between the intensity and the xy coordinates we are able to switch between norms. It is possible to obtain the TV norm, travel through minimal surfaces, and end up with potential surfaces at the other limit.

The regularized TV is defined by: $\min\int\sqrt{\beta^2+|\nabla I|^2}$, where β is a real number, subject to constraints that are used to monitor the drifting of the evolving image away from the initial one. Contrast scaling of $I \to \beta I$, we have $\nabla I \to \beta\nabla I$ and the TV norm becomes $\int\sqrt{1+|\nabla I|^2}$. This is exactly an area minimization towards a minimal surface that could be realized through mean curvature flow with constraints imposed by the noise variance and scale. In other words, the regularized TV is in fact a flow towards a minimal surface with respect to the scaled surface $(x,y,\beta I)$. The ratio between the image size (resolution) and the gray level is taken in an arbitrary way for creating an artificial Euclidean metric, therefore, setting this ratio to β brings us to the minimal surface computation. It is important to note that the β ratio should be determined for every image processing algorithm. The β ratio may be introduced via Polyakov action by defining the embedding metric h_{ij} to be as in Eqn. (7). The only way to avoid the β ratio dependence is to construct planar curve evolution for the gray level sets, such that embedding is preserved [1], [39], [22]. This was called 'contrast invariance' in [2]. Yet, these schemes are pure smoothing schemes that do not preserve edges.

We note that it is possible to impose constraints on the functional that modify the flow like the variance constraints of the Rudin-Osher-Fatemi total variation (TV) method [36]. We have just shown that large β ratio leads to potential surfaces, while at small ratio we have the TV norm. We have thereby linked together many classical schemes via a selection of one parameter, that is, the image gray level scale with respect to its xy coordinates. This scale is determined arbitrarily anyhow in most of the current schemes.

Note that for color images we have a different situation. First we can have three different regularization ratios one for each channel. Second, even when we take a common ratio for all channels, in the limit $\beta \to \infty$ we get an action that does not agree with the color TV [38], [37], [3], [41]. This can be seen easily by observing Eqn. 12 where only the third term survives in this limit. This is the term that contributes the most for coupling between the color channels. The other limit $\beta \to 0$ gives a channel by channel linear diffusion.

Because of space limitations we refer the reader to our papers [42], [25], [23], [43] for comparison with other methods suggested recently for non-linear color image processing like [7], [38], [37], [3], [41].

C. Beltrami Flow in Color Space: Results

We now present some results of denoising color images using our model. Spatial derivatives are approximated using central differences and an explicit Euler step is employed to reach the solution. We represent the image in the RGB space; however, other representations and different numerical schemes (as in [8]) are possible.

In the first example, we corrupt a given image with Gaussian noise and denoise it using our method. The left image in Fig. 4 shows an image corrupted with noise and the im-

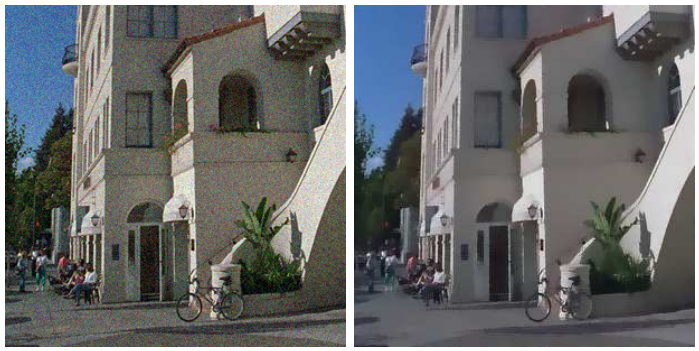

Fig. 4. Reconstruction of a color images corrupted with Gaussian noise (this is a color image).

Fig. 5. Reconstruction of an image that has been corrupted by JPEG compression algorithm (this is a color image).

age on the right depicts its reconstruction. In the second example, we consider noise artifacts introduced by lossy compression algorithms such as JPEG. In Fig. 5, the left image shows a JPEG compressed image and the right image is its "corrected" version using our Beltrami flow.

IV. Choices that Lead to Known Methods

We will survey in this section different choices for the dynamic and parametric degrees of freedom in the action functional.

A. Linear scale-space

Recently, Florac *et al.* [18] invoked reparameterization invariance in vision. The basic motivation in their work is to give a formulation of the linear scale-space, which is based on the linear heat flow, that lends itself to treatment in different coordinate systems. They also noted on the possibility to use a non-flat metric, and raised the idea of using an image induced metric.

In order to have reparameterization invariance one has to write an invariant differential operator. The simplest second order invariant differential operator is the Beltrami operator. The major difference then between our approach and the one given in [18] is the class of metrics allowed. Since a change in parameterization can not change the geometry of the problem, and since they are interested in a linear scale-space, they only allow metrics for which the Riemann tensor vanishes, that is metrics of a flat space.

Our point of view is that an image is a surface embedded in $\mathbb{R}^n$ (or a more general Riemannian manifold). From this perspective the natural metric to choose is the induced metric of the surface. This metric is never flat for a significant image.

B. Generalized Perona-Malik flows

We fix, as in the linear case, the xy coordinates and vary the action with respect to I while the metric is arbitrary for the time being. Using the Euler-Lagrange equation without any pre-factor, we get the following flow

$$I_t = \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu I.$$

We assume now that the image is a d dimensional manifold embedded in $\mathbb{R}^{d+1}$. The task at hand is to find the right choice of the metric to reproduce the Perona-Malik flow. We select $(g_{\mu\nu}) = \tilde{f}\mathcal{I}_d$, where $\mathcal{I}_d$ is the identity matrix. The determinant is $g = (\tilde{f})^d$, and consequently the flow becomes

$$I_t = \sum_{\mu=1}^{n} \partial_\mu \tilde{f}^{\frac{d}{2}-1} \partial_\mu I.$$

For any dimension different from two we can choose $\tilde{f}^{\frac{d}{2}-1} = C(I)$ to get

$$I_t = \mathrm{div}\left(C(I)\nabla I\right),$$

which is the basic idea of Perona and Malik [33]. If we further specify $\tilde{f}^{\frac{d}{2}-1} = C(I) = \frac{f(I_0)}{|\nabla I|}$, where I_0 is the original image, we arrive at

$$I_t = \mathrm{div}\left(f(I_0)\frac{\nabla I}{|\nabla I|}\right),$$

which is the core (up to the $|\nabla I|$ normalization factor) of what is known in the literature as the geodesic active contours [4], [5], [6], [21], [40]. Note that this works only for dimension different from two. Examples of higher dimensional manifolds in vision and image processing are 3D images and movies as 3 dimensional manifolds [26], 3D movie as 4D manifold and texture as a 4D manifold embedded in 6D space [24].

A simple way to get the 2-D Perona-Malik flow is to go one dimension higher: Imagine a map which is the embedding of a 3-D hyperplane as follows $(x, y, z, I(x, y))$, note that I depends only on x and y. Now choose a metric which is zero except the diagonal elements $(f^{-1}(x, y), f^{-1}(x, y), f^2(x, y))$, so that the determinant is 1 and the diagonal of the inverse metric matrix reads (f, f, f^{-2}). Since both the metric and the intensity do not depend on z then the derivative with respect to z vanishes and we get the 2-D Perona-Malik flow: $I_t = \partial_x(fI_x) + \partial_y(fI_y)$. In fact, f can depend on z since $\partial_z(f(x, y, z)I_z) = 0$ if $I_z = 0$, so that we can identify z with the parameter in the Perona Malik diffusion function, e.g. $f = exp(-z(I_x + I_y)^2)$. Our approach gives the z and f a special form which has a well defined geometrical meaning and it is derived from a minimization of an action functional.

C. The mean curvature flow

In this subsection, we choose to minimize with respect to all the embedding variables in the action. We also choose the induced metric as the image metric.

Going back to the action in Eqn. (4) and minimizing with respect to each one of the embedding coordinates X^i, we get the Euler Lagrange equations (see [42] for derivation):

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu X^i) + \Gamma^i_{jk}\partial_\mu X^j \partial_\nu X^k g^{\mu\nu} = 0. \qquad (18)$$

We take the image metric to be $g_{\mu\nu} = \partial_\mu X^i \partial_\nu X^j h_{ij}$ which is by definition the induced metric. For the case of grey-level image (i.e. $\mathbf{X} : \Sigma \rightarrow R^3$), it is given explicitly in Eqn. (3).

Substituting the induced metric in Eq. (9) we get the generalized mean curvature flow, namely

$$X^i_t = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu X^i) + \Gamma^i_{jk}\partial_\mu X^j \partial_\nu X^k g^{\mu\nu} \equiv \mathbf{H}^i. \qquad (19)$$

where $\mathbf{H}$ is the mean curvature vector by definition [9], [44].

For embedding of a manifold in $\mathbb{R}^n$ with Cartesian coordinate system the affine connection is identically zero and we are left with the Laplace-Beltrami operator:

$$X^i_t = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu X^i) \qquad (20)$$

Plugging the explicit expression of the induced metric (Eqn. (3)) for the case $\mathbf{X} : \Sigma \rightarrow R^3$, in the above equation, we obtain

$$\mathbf{X}_t = \mathbf{H} = H\vec{\mathcal{N}}, \qquad (21)$$

where $\mathbf{H}$ is the mean curvature vector that can be written for surfaces as the mean curvature H times the unit normal to the surface $\vec{\mathcal{N}}$: [2]

$$\begin{aligned} H &= \frac{(1+I_x^2)I_{yy} - 2I_xI_yI_{xy} + (1+I_y^2)I_{xx}}{g^{\frac{3}{2}}} \\ \vec{\mathcal{N}} &= \frac{1}{\sqrt{g}}(-I_x, -I_y, 1)^T \end{aligned} \qquad (22)$$

where $g = 1 + I_x^2 + I_y^2$.

The fact that this choice gives us the mean curvature flow should not be a surprise, since if we check how the choice of metric $g_{\mu\nu}$ effects the action functional, we notice that

$$S = \int d^2\sigma\sqrt{g} = \int d^2\sigma\sqrt{\det(\partial_\mu X^i \partial_\nu X^j h_{ij})}, \qquad (23)$$

which is the Euler functional that describes the area of the surface (also known in high energy physics as the Nambu action). The geometrical meaning of this flow is evident. Each point of the surface moves in the direction of the normal with velocity proportional to the mean curvature.

[2]Note that some definitions of the mean curvature include a factor of 2 that we omit in our definition.

If the embedding space is not Euclidean or if we use a non-Cartesian coordinate system we have to use the more general flow, Eqn. (19). In this way we generalize the mean curvature flow to any dimension, codimension, and geometry.

V. Concluding Remarks

Inventing a perceptually good smoothing process which is compatible with a segmentation process, and formulating a meaningful scale space for images is not an easy task, and is actually what low level vision research is about. Here we tried to address these questions and to come up with a new framework that both introduces new procedures and unifies many previous results. There are still many open questions to be asked, like what is the right aspect ratio between the intensity and the image plane? Or in a more general sense, a deeper question that both the fields of string theory and computer vision try to answer, is what is the 'right' embedding space h_{ij}?

The question of what is the 'right norm' when dealing with images is indeed not trivial, and the right answer probably depends on the application. For example, the answer for the 'right' color metric h_{ij} is the consequence of empirical results, experimental data, and the application. Here we covered some of the gaps between the two classical L_p norms in a geometrical way and proposed a new approach to deal with multi dimensional images. We used recent results from high energy physics that yield promising algorithms for enhancement, segmentation and scale space.

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References

[1] L Alvarez, F Guichard, P L Lions, and J M Morel. Axioms and fundamental equations of image processing. *Arch. Rational Mechanics*, 123, 1993.

[2] L Alvarez and J M Morel. Morphological approach to multiscale analysis: From principles to equations. In B M ter Haar Romeny, editor, *Geometric–Driven Diffusion in Computer Vision*. Kluwer Academic Publishers, The Netherlands, 1994.

[3] P Blomgren and T F Chan. Color TV: Total variation methods for restoration of vector valued images. cam TR, UCLA, 1996.

[4] V Caselles, R Kimmel, and G Sapiro. Geodesic active contours. In *Proceedings ICCV'95*, pages 694–699, Boston, Massachusetts, June 1995.

[5] V Caselles, R Kimmel, and G Sapiro. Geodesic active contours. *IJCV*, 22(1):61–79, 1997.

[6] V Caselles, R Kimmel, G Sapiro, and C Sbert. Minimal surfaces based object segmentation. *EEE Trans. on PAMI*, 19:394–398, 1997.

[7] A Chambolle. Partial differential equations and image processing. In *Proceedings IEEE ICIP*, Austin, Texas, November 1994.

[8] T F Chan, G H Golub, and P Mulet. A nonlinear primal-dual method for total variation-based image restoration. *Presented at AMS/SIAM workshop on Linear and Nonlinear CG methods*, July 1995.

[9] I Chavel. *Riemanian geometry: A modern introduction*. Cambridge University Press, 1993.

[10] D L Chopp. Computing minimal surfaces via level set curvature flow. Ph.D Thesis, Lawrence Berkeley Lab. and Dep. of Math. LBL-30685, Uni. of CA. Berkeley, May 1991.

[11] D L Chopp. Computing minimal surfaces via level set curvature flow. *J. of Computational Physics*, 106(1):77–91, May 1993.

[12] D L Chopp and J A Sethian. Flow under curvature: Singularity formation, minimal surfaces, and geodesics. *Jour. Exper. Math.*, 2(4):235–255, 1993.

[13] P Concus. Numerical solution of the minimal surface equation. *Mathematics of Computation*, 21:340–350, 1967.

[14] S Di Zenzo. A note on the gradient of a multi image. *Computer Vision, Graphics, and Image Processing*, 33:116–125, 1986.

[15] K Ecker. Estimates for evolutionary surfaces of prescribed mean curvarure. *Math. Zeitschrift*, 180:179–192, 1982.

[16] K Ecker and G Huisken. Mean curvature motion of entire graphs. *Ann. Math.*, 130:453–471, 1989.

[17] A I El-Fallah, G E Ford, V R Algazi, and R R Estes. The invariance of edges and corners under mean curvature diffusions of images. In *Processing III SPIE*, volume 2421, pages 2–14, 1994.

[18] L M J Florack, A H Salden, , B M ter Haar Romeny, J J Koendrink, and M A Viergever. Nonlinear scale-space. In B M ter Haar Romeny, editor, *Geometric–Driven Diffusion in Computer Vision*. Kluwer Academic Publishers, The Netherlands, 1994.

[19] C Gerhardt. Evolutionary surfaces of prescribed mean curvature. *J. Diff. Geom.*, 36:139–172, 1980.

[20] W E L Grimson. *From images to surfaces*. MIT Press, Cambridge, USA, 1981.

[21] S Kichenassamy, A Kumar, P Olver, A Tannenbaum, and A Yezzi. Gradient flows and geometric active contour models. In *Proceedings ICCV'95*, Boston, Massachusetts, June 1995.

[22] R Kimmel. Intrinsic scale space for images on surfaces: The geodesic curvature flow. In *Lecture Notes In Computer Science: First International Conference on Scale-Space Theory in Computer Vision*, volume 1252, pages 212–223. Springer-Verlag, 1997.

[23] R Kimmel. What is a natural norm for multi channel image processing. LBNL report, Berkeley Labs. UC, CA 94720, March 1997.

[24] R Kimmel, N Sochen, and R Malladi. On the geometry of texture. Report LBNL-39640, UC-405, Berkeley Labs. UC, CA 94720, November 1996.

[25] R Kimmel, N Sochen, and R Malladi. From high energy physics to low level vision. In *Lecture Notes In Computer Science: First International Conference on Scale-Space Theory in Computer Vision*, volume 1252, pages 236–247. Springer-Verlag, 1997.

[26] R Kimmel, N Sochen, and R Malladi. Images as embedding maps and minimal surfaces: Movies, color, and volumetric medical images. In *Proc. of IEEE CVPR'97*, pages 350–355, Puerto Rico, June 1997.

[27] E Kreyszing. *Differential Geometry*. Dover Publications, Inc., New York, 1991.

[28] J L Lagrange. *Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies*, volume I. Gauthier-Villars, Paris, 1867. translated by D. J. Struick.

[29] A Lichnewski and R Temam. Pseudo solutions of the time dependent minimal surface problem. *Journal of Differential Equations*, 30:340–364, 1978.

[30] R Malladi and J A Sethian. Image processing: Flows under min/max curvature and mean curvature. *Graphical Models and Image Processing*, 58(2):127–141, March 1996.

[31] D Marr. *Vision*. Freeman, San Francisco, 1982.

[32] V I Oliker and N N Urlatseva. Evolution of nonparametric surfaces with speed depending on curvarure ii. the mean curvature case. *Comm. Pure Applied Math.*, 46:97–135, 1993.

[33] P Perona and J Malik. Scale-space and edge detection using anisotropic diffusion. *IEEE-PAMI*, 12:629–639, 1990.

[34] A M Polyakov. Quantum geometry of bosonic strings. *Physics Letters B*, 103B(3):207–210, 1981.

[35] In B M ter Haar Romeny, editor, *Geometric–Driven Diffusion in Computer Vision*. Kluwer Academic Publishers, The Netherlands, 1994.

[36] L Rudin, S Osher, and E Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.

[37] G Sapiro. Vector-valued active contours. In *Proceedings IEEE CVPR'96*, pages 680–685, 1996.

[38] G Sapiro and D L Ringach. Anisotropic diffusion of multivalued images with applications to color filtering. *IEEE Trans. Image Proc.*, 5:1582–1586, 1996.

[39] G Sapiro and A Tannenbaum. Affine invariant scale–space. *International Journal of Computer Vision*, 11(1):25–44, 1993.

[40] J Shah. A common framework for curve evolution, segmentation and anisotropic diffusion. In *Proceedings IEEE CVPR'96*, pages 136–142, 1996.

[41] J Shah. Curve evolution and segmentation functionals: Application to color images. In *Proceedings IEEE ICIP'96*, pages 461–464, 1996.

[42] N Sochen, R Kimmel, and R Malladi. From high energy physics to low level vision. Report LBNL 39243, LBNL, UC Berkeley, CA 94720, August 1996.

[43] N Sochen and Y Y Zeevi. An addaptive local feature scale-space. EE-Technion and TAU HEP report, Technion and Tel-Aviv University, March 1997.

[44] M Spivak. *A Comprehensive Introduction to Differential Geometry*. Publish or Perish, Inc., Berkeley, 1979.

[45] R Vogel and M E Oman. Iterative methods for total variation denoising. *SIAM. J. Sci. Statist. Comput.*, to appear, 1996.

[46] S D Yanowitz and A M Bruckstein. A new method for image segmentation. *Computer Vision, Graphics, and Image Processing*, 46:82–95, 1989.

[47] A. Yezzi. Modified curvature motion for image smoothing and enhancement. *IEEE Trans. IP*, 1997.