Shape representation by metric interpolation

Yonathan Aflalo
Faculty of Electrical Engineering
Technion Institute of Technology

Ron Kimmel
Faculty of Computer Science
Technion Institute of Technology

***Abstract*—Coordinates of vertices in a triangulated surface can be efficiently represented as a set of coefficients that multiply a given basis of functions. One such natural orthonormal basis is provided by the eigenfunctions of the Laplace-Beltrami operator of a given shape. The coefficients in this case are nothing but the result of the scalar inner product of the coordinates treated as a smooth function on the surface of the shape and the eigenfunctions that form the orthonormal basis. Keeping only the significant coefficients allows for efficient representation of a given shape under practical transformations. Selecting the regular metric for the construction of the Laplace-Beltrami operator we notice that while the general shape is preserved, important fine details are often washed out. At the other end, using a scale invariant metric for defining the operator and the corresponding basis, preserves the fine details at the potential expense of loosing the general structure of the shape. Here, we adopt the best of both worlds. By finding the right mix between scale invariant and a regular one we select the metric that serves as the best representation-basis generator for a given shape. We use the mean square error (MSE) to select the optimal space for shape representation, and compare the results to classical spectral shape representation techniques.**

I. INTRODUCTION

One of the most popular computational representations of a geometric structure is a triangulated mesh. The number of triangles describing such an object determines the details captured by such a form. Usually, efficient representations of triangulated surfaces involve in compression of the connectivity of the triangles composing the mesh [1], [2], [8], [17].

The mesh connectivity can be efficiently captured by a small number of coefficients, as proposed for example in [4], [10]. These methods treat the mesh as a graph and exploit the spectral decomposition of the graph Laplacian as a convenient representation basis. It could be effective for uniformly and regularly sampled meshes. Yet, such approaches use only the connectivity and ignore the actual geometry of the set of shapes one is trying to describe. A more appropriate operator is the Laplace-Beltrami that also incorporates information about the mesh geometry [11], [15], [16].

Motivated by this school of thoughts, we first formulate the problem of finding efficient mesh representation via the Laplacian operator. Then, we show that the spectral decomposition of the Laplace-Beltrami is indeed well suited for describing shapes. Next, we study the influence of various metrics on the approximation error, in a mean-squared error l_2 sense, using spectral decomposition of the Laplacian associated with each metric. Finally, we marry the spaces generated by each operator, first by interpolating between the metrics is a manner that minimizes the mean squared error (MSE), and then, by concatenating fragments from each basis, where each fragment is associated to a different part of the shape.

II. EFFICIENT SHAPE REPRESENTATION

In classical signal processing the so called *Fourier Transform* is often used to express a given function as a set of frequency coefficients also known as the function's spectrum. The fundamental assumption in linear signal processing is that any finite dimensional signal can be expressed as a sum of weights or coefficients that multiply the corresponding Fourier basis functions.

Fourier basis functions can be obtained as a solution of the partial differential equation (PDE) $\Delta f = \lambda f$. Where, for example, in one dimension, f is a scalar function $f(t) : \mathbb{R} \to \mathbb{R}$, and $\Delta f = \frac{\partial^2 f}{\partial t^2}$. The Fourier basis is composed of the harmonic functions $\phi_i(t) = \exp(j2\pi it)$. Thereby, a given signal can be expressed by its Fourier coefficients $c_i = \langle f, \phi_i \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the L^2 scalar product.

The Laplace-Beltrami operator (LBO) is an extension of the Laplacian to non-flat manifolds. We can define a metric tensor (g_{ij}) on any given manifold M, where it can be induced, for example, by either global or local parametrization. The LBO general expression, using Einstein summation convention, is given by

$$\Delta_g f = \frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j f,$$

where $g = \det(g_{ij})$ and $g^{ij} = (g_{ij})^{-1}_{ij}$.

The LBO is symmetric and positive definite, and thus admits a spectral decomposition property. That is, there exist an eigenbasis of orthonormal functions $\{\phi_i\}$ and a set of positive real scalars $\{\lambda_i\}$, such that $\Delta_G \phi_i = \lambda_i \phi_i$. Any function defined on the manifold can be expressed as a weighted sum of the basis eigenfunctions. More precisely,

$$f = \sum_i \langle f, \phi_i \rangle \phi_i,$$

where

$$\langle f, h \rangle = \int_M f h d\mu,$$

and $d\mu$ is the infinitesimal volume element of M. The eigenvalue λ_i represents the coefficient of the frequency associated to the i-th eigenvector.

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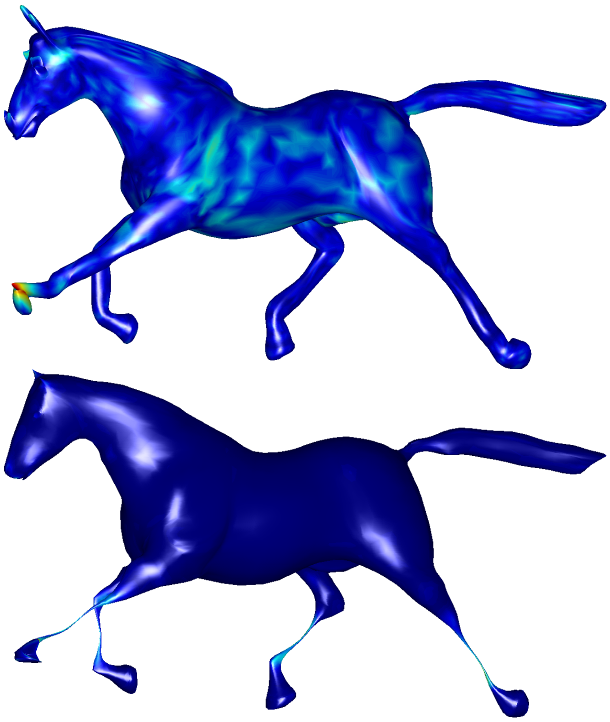

Fig. 1. Reconstruction of the shape of an Horse (8431 vertices) using 300 vectors of the graph-laplacian (top) and the Laplace-Beltrami (bottom) spectral decomposition. Color represents the displacement error at each vertex; where the darker the color the smaller is the error.

Roughly speaking, the scalar product $\langle f, \phi_i \rangle$ gets smaller as the function f becomes smoother, and when ϕ_i are ordered properly, goes to zero as i goes to infinity. The Laplace-Beltrami eigen-decomposition was found to be useful in shape and data processing and analysis [5], [7], [9], [11], [15], [16].

The surface of a three dimensional shape can be considered as a non-flat two-dimensional manifold. There exist several methods to approximate the LBO Δ_g, see for example [18] for an axiomatic analysis of desired properties and possible realizations, and [13] for the celebrated cotangent weights approximation. Lévy [11] formalized the decomposition of the Laplace Beltrami operator in cotangent-weight form as a generalized eigendecomposition problem.

Here, we propose to follow the philosophy according to which the coordinates of a given surface S can be represented as a linear combination of its Laplace-Beltrami eigenvectors. The idea of spectral coordinates projection was exploited in [4], [10], where the authors focus on the graph laplacian rather than the Beltrami operator. In order to demonstrate the potential gain in using the Laplace-Beltrami rather than the graph laplacian, Figure 1 displays two representations of a surface using the first 300 vectors extracted from the graph laplacian and the LBO.

We define the MSE between a surface S and its spectral representation $\hat{S}$ by

$$\text{MSE}(S, \hat{S}) = \int_S \left((x-\hat{x})^2 + (y-\hat{y})^2 + (z-\hat{z})^2\right) da$$

where x, y, z and $(\hat{x}, \hat{y}, \hat{z})$ represent the coordinates of S (resp. $\hat{s}$) and da is an area element on S. The MSE for the graph laplacian model is 0.8017 while the Laplace-Beltrami model yields a much lower error of 0.607. We notice that, in terms of MSE, the Laplace-Beltrami as a basis generator is better suited for this problem. Our next goal is to find an even better metric for the representation using, as an example, two other metrics; the equi-affine metric defined in [14] or a scale invariant metric introduced in [3].

III. Scale invariant metric construction

Here, following [3] we briefly review the construction of a scaling invariant metric that we later use in order to build a feature sensitive representation basis. Let $S(u, v)$ be a parametrized surface $S : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$. The length of a parametrized curve $C(p)$ in S is given by

$$\begin{aligned} l(C) &= \int_C |C_p| dp = \int_C |S_u u_p + S_v v_p| dp \\ &= \int_C \sqrt{|S_u|^2 du^2 + 2\langle S_u, S_v \rangle du dv + |S_v|^2 dv^2}. \end{aligned}$$

Using the Euclidean arc-length s, we also have

$$l(C) = \int_C ds.$$

Hence, the usual metric definition for an infinitesimal Euclidean distances on a surface can be deduced from the two previous equations to be

$$\begin{aligned} ds^2 &= |S_u|^2 du^2 + 2\langle S_u, S_v \rangle du dv + |S_v|^2 dv^2 \\ &= g_{ij} d\omega^i d\omega^j, \end{aligned}$$

where we used Einstein summation convention, $\omega^1 = u$, $\omega^2 = v$, and $g_{ij} = \langle S_{\omega^i}, S_{\omega^j} \rangle$. The metric $g_{ij} = \langle S_i, S_j \rangle$ in shorthand notations, is known as the *regular metric*. We can generalize this result to any arc-length, using an appropriate metric, and then measure the length of any curve on the surface with respect to the specific metric.

Next, consider a curve $C(s)$ in the plane for which a scale invariant arc-length θ can be easily defined by the change of the angle between the tangent vector C_s and the x-axis. θ can be easily computed from the curvature definition

$$|\kappa| = |C_{ss}| = \left| \frac{d\theta}{ds} \right|,$$

thereby $\theta(s) = \int_C |\kappa| ds$. Extending the scale invariance definition to metrics of surfaces is more tricky, as there are two principal curvatures, κ_1 and κ_2, for each surface point. Still, if similarity invariance is desired (scale and isometry) it was shown in [3] that an invariant metric can be defined by

$$\tilde{g}_{ij} = |K| g_{ij},$$

where $K = \kappa_1 \kappa_2$ is the Gaussian curvature. The invariance of the $\tilde{g}$ is simple to prove. We use the regular metric $g_{ij}(S) = \langle S_i, S_j \rangle$, thus a uniformly scale surface βS would

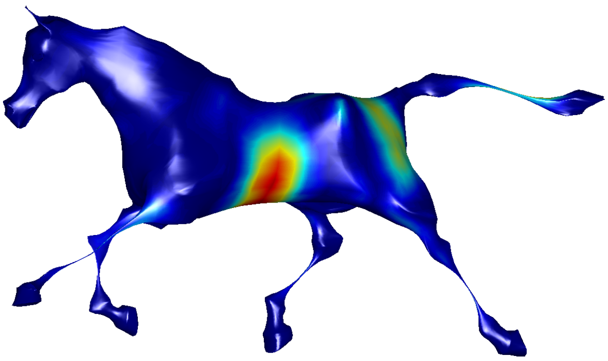

Fig. 2. Reconstruction of the shape of an horse (8431 vertices) using 300 eigenvectors of the scale invariant Laplace-Beltrami operator. Color represents the error.

have $g_{ij}(\beta S) = \beta^2 g_{ij}(S)$. The Gaussian curvature can be defined by the metric derivatives according to which

$$K(S) = -\frac{1}{2\sqrt{g_{11}g_{22}}}\left(\frac{\partial}{\partial_u}\frac{(g_{22})_u}{\sqrt{g_{11}g_{22}}} + \frac{\partial}{\partial_v}\frac{(g_{11})_v}{\sqrt{g_{11}g_{22}}}\right).$$

It is then simple to see that $K(\beta S) = \beta^{-2}K(S)$ and thus $\tilde{g}_{ij}(\beta S) = K(\beta S)g_{ij}(\beta S) = \beta^{-2}K(S)\beta^2 g_{ij}(S) = \tilde{g}_{ij}(S)$ which proves the invariance. Moreover, as K is invariant to isometries, the property is preserved by the metric $\tilde{g}$ which is referred to as similarity (isometry and scale) invariant.

Scale invariant Laplace Beltrami operator would produce scale invariant eigenfunctions that would weigh equally small and large surface features. In the next section we would try to marry between the similarity invariant and the regular metric such that one captures the features while the other takes care for the global structure.

IV. Experimental results: Searching for the best metric

We use the scale invariant metric $\tilde{g}$ to define $\Delta_{\tilde{g}}$. The representation using the first 300 eigenfunctions and corresponding coefficients is shown on Figure 2. Note, that the reconstruction of the shape from the 300 first eigenvectors is indeed inaccurate. Nevertheless, even-though the error is high when the Gaussian curvature vanishes, the detailed parts with effective curvature are captured quite accurately. Actually, the scale invariance of the metric allows the features to become dominant which in a sense complements the regular metric. The global MSE is 0.8677 which is even worse than the graph laplacian option.

Next, we attempt to use the best of both scale-invariant and regular metric, by interpolating between the two metrics. We define the interpolated metric as

$$\hat{g}_{ij} = |K|^{\alpha} g_{ij},$$

where $\hat{g}$ represents the regular metric, K is the Gaussian curvature, and $\alpha \in [0, 1]$ is a scalar we use to minimize the representation error. The MSE as a function of α is shown in

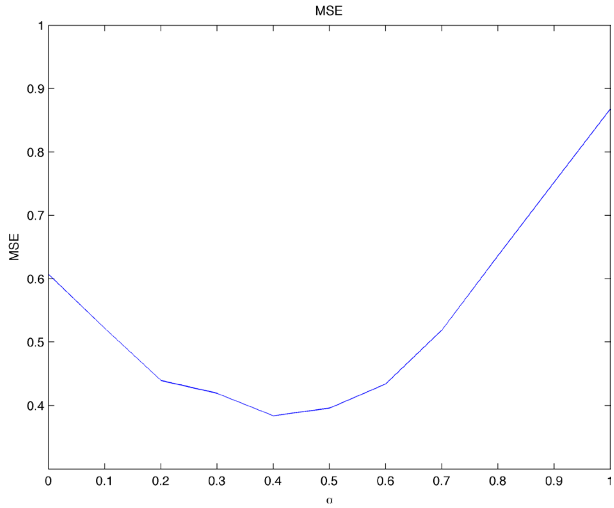

Fig. 3. The reconstructed MSE of a horse with the metric $\hat{g} = K^{\alpha}g$. The x-axis represents α interpolating between the regular metric (left) and the scale invariant one (right).

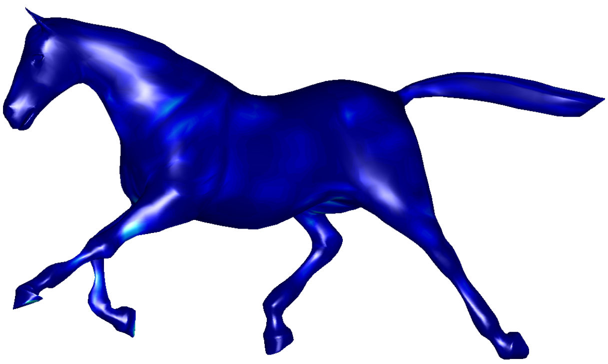

Fig. 4. Optimal reconstruction of a horse (8431 vertices) using 300 eigenvectors of the LBO with optimally interpolated metric ($\alpha = 0.4$). The darker the color the higher is the error.

Figure 3 with the best reconstruction shown in Figure 4 with an MSE of 0.3654.

We repeated the experiment with the Equi-Affine invariant metric and obtain the result shown in the Figure 5, with MSE of 0.6494.

Next, we repeated the experiment with other shapes like the Armadillo in Figure 6, and the Centaur in Figure 7. The best results were always obtained for the interpolated scale-regular metric.

The proposed technique is useful for efficient representation of families of almost isometric shapes for which a single representation basis could be considered. Assume we have a family of isometric shapes with given correspondences. Those can be obtained by various methods [5], [6], [9], [12]. We can then order the coordinates x, y, z of each shape such that all appear in a consistent order. We then compute the projection of the coordinates on the selected eigenvectors of the Laplace-Beltrami operator and use only these coefficients. Good results where obtained for various postures of the horse compared to

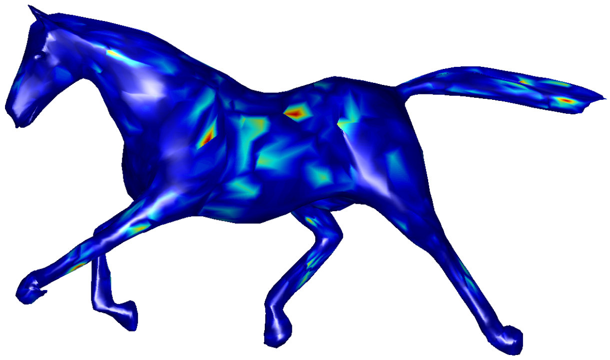

Fig. 5. Reconstruction of the shape of a horse (8431 vertices) using 300 eigenvectors of the LBO defined by the interpolated optimized equi-affine invariant metric ($\alpha = 0.3$).

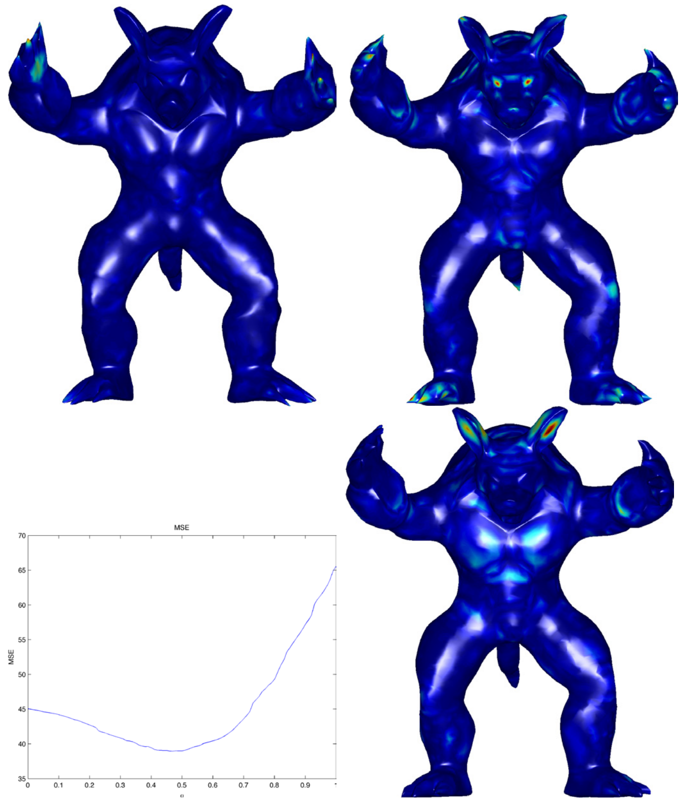

Fig. 6. Left to right, top to bottom: Reconstruction using graph laplacian (MSE $= 115$), the LBO (MSE $= 51$), the MSE of the reconstructed Armadillo w.r.t. the metric $\hat{g} = |K|^{\alpha} g$, and the optimal reconstruction of the Armadillo (10k vertices) using 500 eigenvectors of the LBO w.r.t. the interpolated optimal scale invariant metric (obtained for $\alpha = 0.47$, MSE $= 38$). The darker the color the lower is the error at a point.

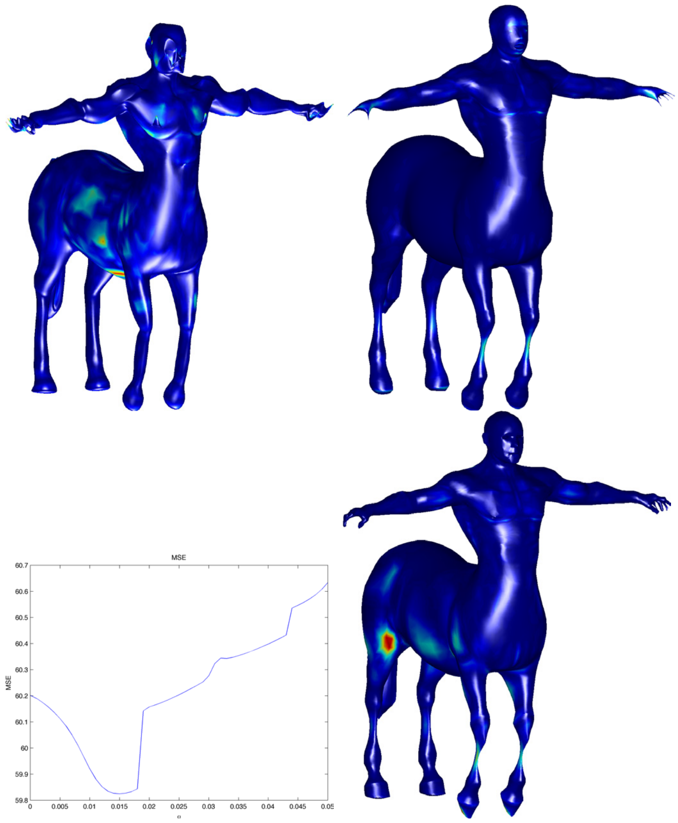

Fig. 7. Left to right, top to bottom: Reconstruction using graph laplacian (MSE $= 197.4$), the LBO (MSE $= 91.66$), the MSE of the reconstructed Centaur w.r.t. the metric $\hat{g} = |K|^{\alpha} g$, and the optimal reconstruction of the Centaur (15k vertices) using 500 eigenvectors of the LBO w.r.t. the interpolated optimal scale invariant metric (obtained for $\alpha = 0.015$, MSE $= 59.8$).

the graph laplacian, as shown in Figure 8

V. Conclusions

The coordinates of a surface can be treated as a smooth function on the surface, for which there should be some compact representation. Here, we experimented with such description spaces that were constructed from the Laplace-Beltrami operator with respect to a specific metric. The best metric was found to be somewhere in-between a regular and a similarity invariant metric. Such a basis exhibits isometry invariance and was indeed proven to be effective when animating an articulated object, like a horse in motion. The choice of a metric was designed axiomatically, and the right balance was selected and justified empirically. This is a small step towards more efficient representation domains that treat objects with non-regular numerical support. Incorporating other invariant measures with empirical tuning of parameters could potentially lead to even more efficient descriptions of geometric structures, which is a venue we plan to explore in the future.

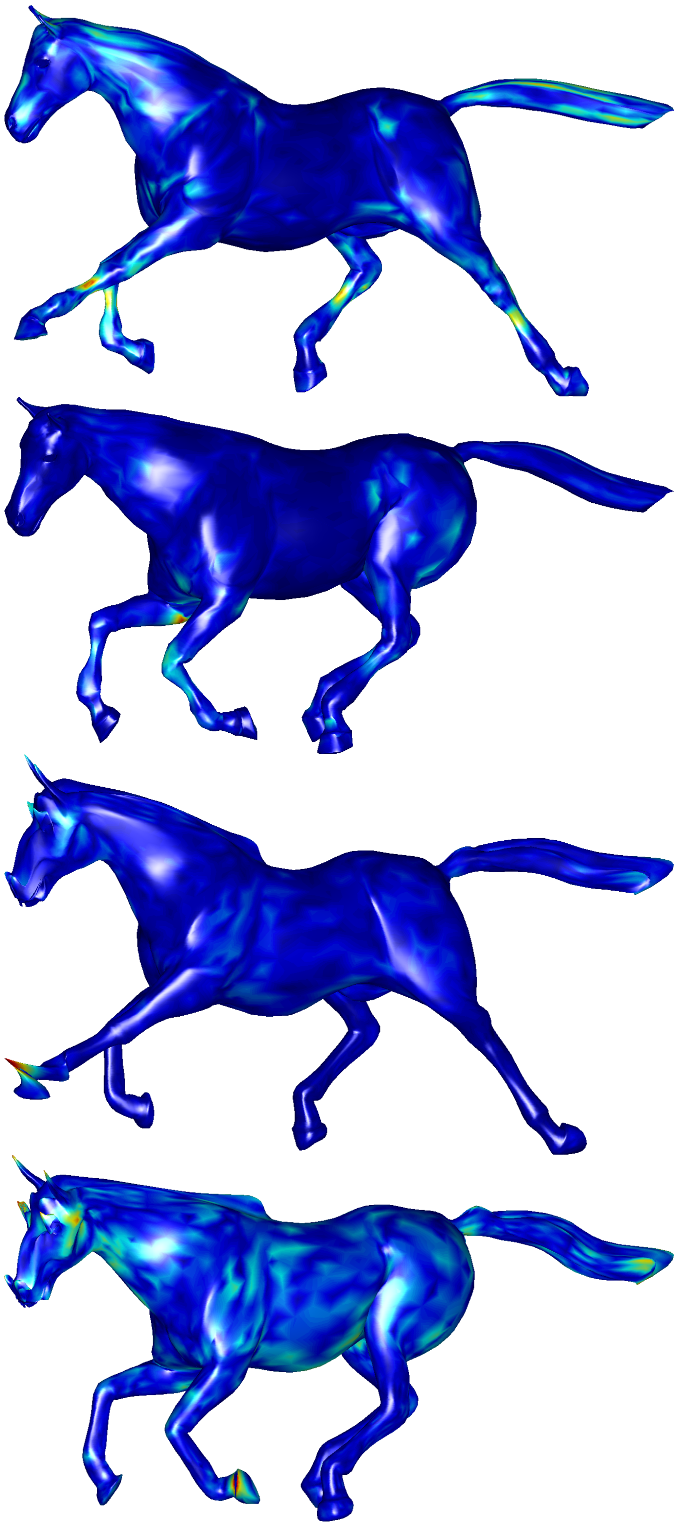

Fig. 8. Efficient reconstruction of a running horse by projecting coordinates of each shape to the first 300 eigenvectors of the LBO w.r.t. the optimal interpolated metric (top, average MSE 0.33) and the graph laplacian (bottom, average MSE 0.83).

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