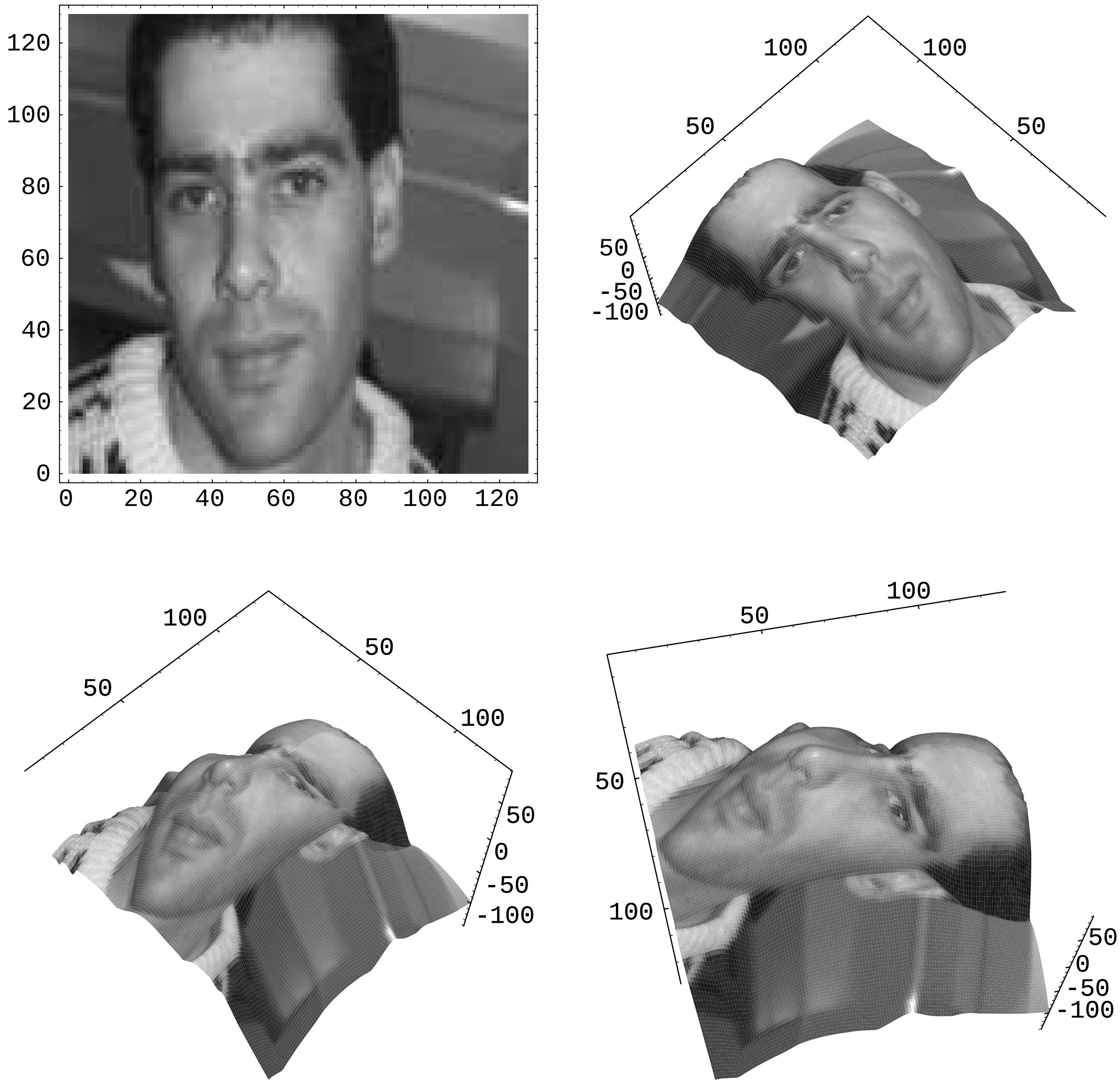

Figure 4.7: "Real" face and the reconstructed surface from three different view points. The process starts at the tip of the nose.

general light source direction problem. A numerically stable implementation in which topological problems in the propagated height contours are often inherently avoided was presented. In this method shocks, cusps and other singularities formed in the contours are also readily dealt with in an efficient numerical scheme originally developed to solve wavefront propagation.

Appendix
Planar Normal Velocity Calculation

Given the simple propagation rule on the plane defined by the light source direction P^l

$$\mathcal{C}^l_t = \hat{\mathbf{t}} \times \hat{l},$$

we want to project this evolving curve to the (x, y)-plane P, and take only the normal component of the velocity vector of the curve in P, see Figure 4.8.

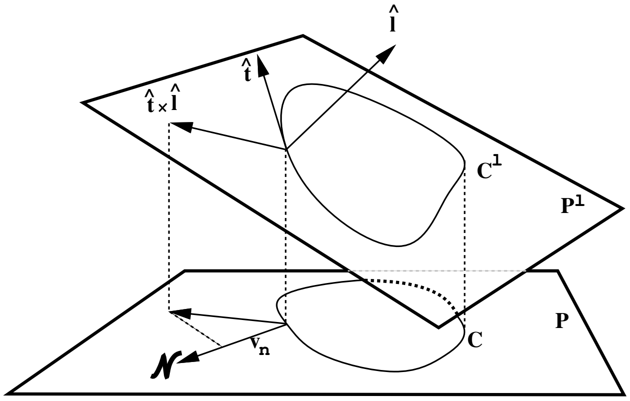

Figure 4.8: $\mathcal{C}^l$ is evolving with a constant unit velocity along its normals directions, and $\mathcal{C}$ is the projection of $\mathcal{C}^l$ to P. The propagation rule that determines the evolution of $\mathcal{C}$ is specified by a velocity v_n along its normals. v_n is achieved by first projecting the $\mathcal{C}^l$ normals to P and then taking only the planar components of that projection.

The above equation may be written as

$$(x_t, y_t, z_t) = \frac{(x_u, y_u, z_u)}{\sqrt{x_u^2 + y_u^2 + z_u^2}} \times \frac{(-p_l, -q_l, 1)}{\sqrt{1 + p_l^2 + q_l^2}} = (w_1, w_2, w_3),$$

where u is the parameter of the curve $\mathcal{C}^l(u, t) = (x(u, t), y(u, t), z(u, t))$.

The projection of this curve evolution on the (x, y)–plane may be written using $\mathcal{C}_t = v_n \, \mathcal{N}$, or

$$(x_t, y_t) = v_n \, \frac{(-y_u, x_u)}{\sqrt{x_u^2 + y_u^2}}.$$

We are looking for the velocity v_n, which is the projection of (w_1, w_2) on the planar normal $\mathcal{N}$. This may be written as

$$v_n = \langle (w_1, w_2), \mathcal{N} \rangle = \frac{-w_1 y_u + w_2 x_u}{\sqrt{x_u^2 + y_u^2}}.$$

By calculating the vector product and eliminating (due to the projection) the $\hat{z}$ component, we get

$$(w_1, w_2) = \frac{(y_u + q_l z_u, -x_u - p_l z_u)}{\sqrt{x_u^2 + y_u^2 + z_u^2}\sqrt{1 + p_l^2 + q_l^2}}.$$

Now, we are ready to calculate v_n,

$$v_n = \frac{(y_u + q_l z_u, -x_u - p_l z_u)}{\sqrt{1 + p_l^2 + q_l^2}\sqrt{x_u^2 + y_u^2 + z_u^2}} \cdot \frac{(-y_u, x_u)}{\sqrt{x_u^2 + y_u^2}}.$$

Using the chain rule $z_u = z_x x_u + z_y y_u = p_l x_u + q_l y_u$, we get

$$v_n = \sqrt{\frac{x_u^2(1 + p_l^2) + y_u^2(1 + q_l^2) + 2p_l q_l x_u y_u}{(1 + p_l^2 + q_l^2)(x_u^2 + y_u^2)}}.$$

Writing the planar normal as its components $\mathcal{N} = (n_1, n_2) = \frac{(-y_u, x_u)}{\sqrt{x_u^2 + y_u^2}}$, we conclude with

$$v_n = \frac{1}{\sqrt{1 + p_l^2 + q_l^2}} \sqrt{n_1^2(1 + q_l^2) + n_2^2(1 + p_l^2) - n_1 n_2 2 p_l q_l}. \tag{4.11}$$

It was noted in Chapter 2 that the planar unit normal to any level set of a given continuous function ϕ can be written as $\mathcal{N} = \frac{\nabla\phi}{||\nabla\phi||}$. Using this notation in Equation (4.11) we obtain

$$v_n = \frac{1}{\sqrt{\phi_x^2 + \phi_y^2}\sqrt{1 + p_l^2 + q_l^2}} \sqrt{\phi_x^2(1 + q_l^2) + \phi_y^2(1 + p_l^2) - \phi_x \phi_y 2 p_l q_l}.$$

Chapter 5

Generalized Shape form Shading

This chapter presents a solution to the problem of reconstructing a smooth three dimensional object from its two dimensional gray level image in the case of several singular points [71]. An algorithm based on topological properties of simple smooth surfaces is provided to solve the problem of global reconstruction. Classifying singular points in the shading image as maxima minima and two kinds of saddle points, serves as the key to the solution of the problem. The global reconstruction procedure, being deterministic and using topological properties of the surface performs better than other approaches proposed so far that are based on classification of singular points according to the behavior of characteristics in their neighborhood. The proposed algorithm is simple, easy to implement and lends itself to a parallel implementation.

5.1 Introducing the Generalized Problem

As pointed out in the previous chapter, the problem of reconstructing the three dimensional shape of an object from its shading (gray level) image was and still is of great interest to the computer vision community. Horn [55], formulated the problem as a first order non linear partial differential equation, and attempted to solve it by using the classical characteristic–strip expansion method. This direct approach suffers from numerical difficulties of inaccuracy and instability due to its local nature. The search for better approaches led to the development of a series of algorithms that solve regularized versions of the problem (see [57, 58], which represent only few of the many published works concerning such approaches.) Direct iterative procedures were recently presented in [117, 105], followed by [6, 41], the problem being reformulated to yield iterative numerical schemes that could be proven to converge to the 'correct' solution. However, these algorithms were aimed at solving the local problem, and some external information, like the nature of singular points, was necessary in order to patch the local solutions together.

The shaded image induces a relation between surface normals and the light source direction. According to the so called *Lambertian* shading rule, the image, *i.e.* the two dimensional array of gray levels at each pixel in the image, maps the cosine of the angle between the light source direction and the surface normal. The shape from shading problem is the inverse problem of reconstructing the $3D$ surface from this data. In order to resolve ambiguities, smoothness assumptions as well as other 'clues' in the image were explored, such as apparent contours, grazing light edges, self shadow edges [92], and (last but most important for our discussion) the nature of singular points. Singular points are those points in the image where there is no ambiguity in the normal direction. The normal of the surface at a singular point is known to be in the direction of the light source. These brightest points are the local minima, local maxima and saddle points (with respect to the light source direction) of the surface. Horn [55] suggested to use small circles around singular points as initial conditions for the characteristic strip expansion method. Some attempts to use those points as a key to the global reconstruction were made in [42, 106], where the behavior of characteristics around a singular point is used to determine the nature of that point. The local solution to the shape from shading problem is extended from maxima to minima and so forth, based on that classification procedure. This method requires choosing a singular point that is known not to be a saddle. Extending the solution involves the inspection and classification of singular points on the boundary of the 'area of attraction' of a singular point. These operations make no use of global topological properties of the surface, and may therefore lead to mistakes.

In this chapter we address the problem of global shape from shading, where no boundary nor initial conditions are assumed besides the shading image itself, based on the topological properties of simple smooth Morse functions. We rely on the theory of differential topology to classify the singular points. The 'correct solution' is selected from all possible solutions as the one that satisfies the global topological properties and agrees with the local shading equation. There is no need to classify any of the singular points, and all those points take part in the process of merging areas until all possible solutions are found.

The structure of this chapter is as follows: In Section 5.2, a local shape from shading via level set algorithm based on [17, 66, 69, 76] is shortly referred to, see Chapter 4 for more details. It was shown in Chapter 3 [72, 76, 117] that the local shape from shading problem is similar to the weighted distance transform. Starting from all the singular points we calculate the weighted distance transform for each of those points. Section 5.3 briefly reviews the topological properties of simple smooth surfaces. Those properties are used in Section 5.4 to formulate the global shape from shading algorithm. The algorithm implements a 'logical combination' of the weighted distance transforms of all the singular points. Some examples that clarify the algorithm are presented in Sections 5.5, followed by some remarks on the complexity of the algorithm in Section 5.6.

5.2 Local Shape from Shading

We have chosen to use the simple case where the light source and the viewer are located at the same place. Consider a smooth surface $z(x, y)$. According to the *Lambertian* shading rule, the brightness map (shading image) $I(x, y)$ is equal to the inner product of the light source direction $\hat{l} = (0, 0, 1)$ and the normal $N(x, y)$ to the surface. This relation is known as the *irradiance equation* and may be written as

$$I(x, y) \; = \; R(p, q) \; = \; \langle \hat{l}, N \rangle \; = \; \frac{1}{\sqrt{1 + p^2 + q^2}},$$

where $p = \partial z / \partial x$ and $q = \partial z / \partial y$ are the partial derivatives of the surface. The extension to the 'oblique' light source becomes simple considering light source coordinates as was done in in the previous chapter and in [72], see also [42, 85].

Starting from a small circle around a singular point one may extend the solution by using the equal height contour evolution as suggested by Bruckstein in [17]. Bruckstein observed that given an equal height contour $\mathcal{C}(p) : S^1 \to \mathbb{R}^2$, it may serve as the initial condition, $\mathcal{C}(p, 0) = \mathcal{C}(p)$, for the curve evolution equation

$$\frac{\partial \mathcal{C}(p, t)}{\partial t} = \frac{I}{\sqrt{1 - I^2}} \mathcal{N}, \tag{5.1}$$

where $\mathcal{N}$ is the planar normal of the evolving curve. The evolving curve tracks the equal height contours of the surface, and thereby reconstructs the surface. The 'time', t, in the evolution equation represents the height. This direct shape from shading method was recently reformulated via level sets propagation and numerically implemented by the Osher–Sethian algorithm [111, 127], in [66, 72]. In [66] the equal height contour propagation via level sets was introduced and compared to other shape from shading algorithms, see also [81]. The algorithm was applied to real images and its performance under different types of noise was demonstrated. The numerical implementation introduced in [66] is consistent with the continuous case and therefore does not suffer from metrication errors. This is an important property in the construction of numerical schemes that is overlooked in some shape from shading numerical algorithms.

Recently it was also shown that the local shape from shading problem is equivalent to the problem of computing the weighted distance transform [69, 76], see also [92, 105, 117]. Equation (5.1), describes the propagation of an equal cost contour where the cost function, defined over the image domain $\Omega \in \mathbb{R}^2$, is given as a function of the brightness by $f(x, y) = \sqrt{1/I^2(x, y) - 1}$.

Having only one singular point that is known to be a minimum $\mathbf{m} \in \mathbb{R}^2$, the surface may be reconstructed by calculating the weighted distance map from that singular point. In this case, the problem may be solved by the level sets propagation algorithm. The reconstructed surface is the weighted distance transform of the minimum point. The height ('weighted distance') at each point $\mathbf{x} \in \mathbb{R}^2$ is determined by [117]

$$z(\mathbf{x}) = \inf_{\mathcal{L}(s) \in L} \left\{ \int_0^{|\mathcal{L}|} f(\mathcal{L}(s)) ds \; : \; |\frac{\partial \mathcal{L}}{\partial s}| = 1, \; \mathcal{L}(0) = \mathbf{x}, \; \mathcal{L}(|\mathcal{L}|) = \mathbf{m} \right\},$$

where L is the set of all curves connecting point $\mathbf{x}$ to the given minimum point $\mathbf{m}$.

Other non direct (iterative) methods that solve this local shape from shading problem were recently presented, see *e.g.* [6, 117, 42], and may replace our core level–set based procedure that solves the local problem. The basis for our global algorithm is the knowledge of 'how to combine' the local solutions together.

5.3 Topological Properties of Smooth Surfaces

Our basic assumption is that the object we try to reconstruct has a simple and smooth surface described by a Morse function formally defined in [15]. Such a surface is assumed to have no pathological singularities like the *monkey-saddle*, see Figure 5.1. All singular points

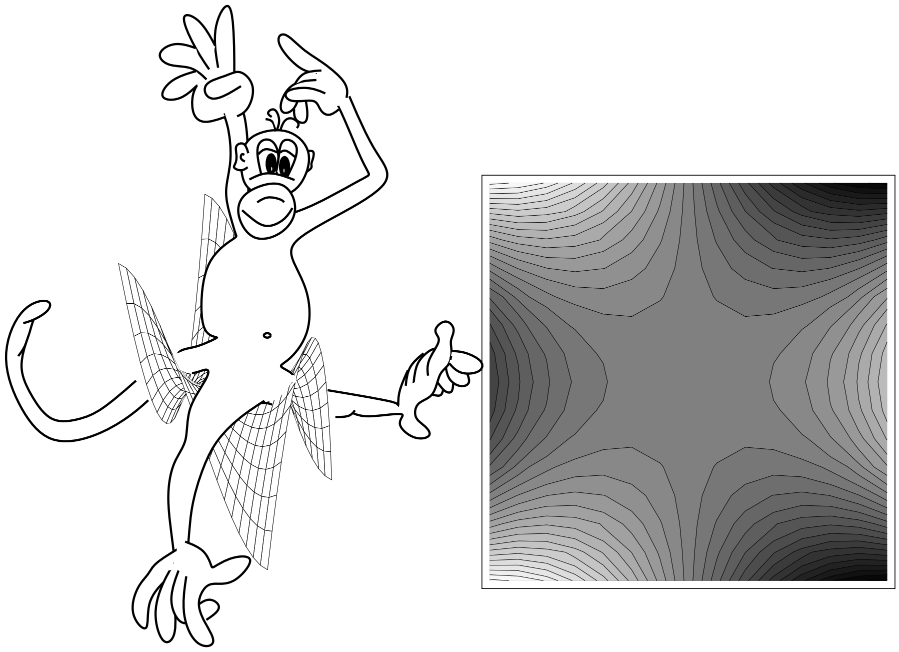

Figure 5.1: A pathological case of a singular point known as the *monkey saddle*. The surface on the left, and its equal height contours painted on a gray-level map which represents the heights on the right.

of a Morse function may be locally described, up to a mere change of coordinates in the plane [15], by

$$d \pm x^2 \pm y^2,$$

where d is some constant. This local behavior in the neighborhood of singular points yields three local types of surfaces as presented in Figure 5.2. Being able to classify the singular points greatly helps solving the global shape from shading problem. In fact, given the height of all minimum points $\{z(\mathbf{m}) \mid \mathbf{m} \text{ is a minimum}\}$, the global solution is given by [117].

$$z(\mathbf{x}) = \inf\{z(\mathbf{m}) + \int_0^{|\mathcal{L}|} f(\mathcal{L}(s))ds \quad : \quad \mathcal{L}(s) \in L_{\mathbf{m}},$$
$$|\frac{\partial \mathcal{L}}{\partial s}| = 1,\ \mathcal{L}(0) = \mathbf{x},\ \mathcal{L}(|\mathcal{L}|) = \mathbf{m},\ \mathbf{m} \text{ is a minimum}\},$$

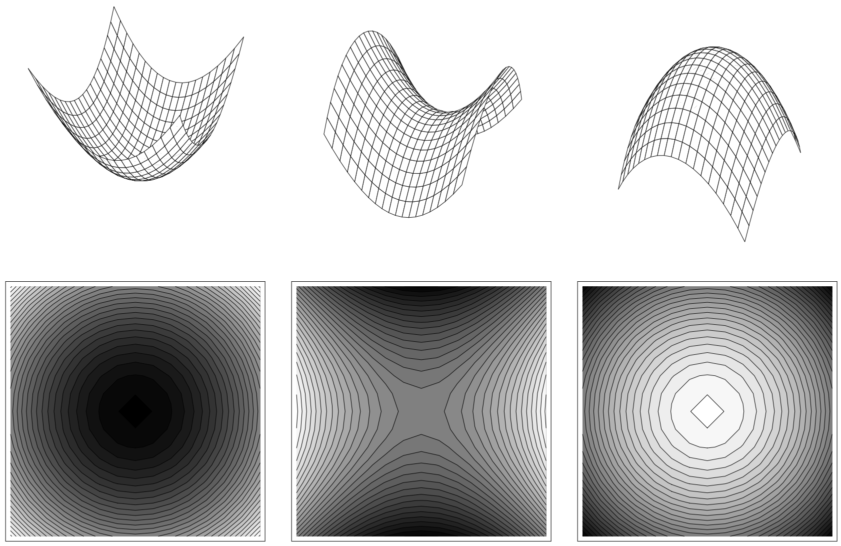

Figure 5.2: The three types of singular points that characterize the Morse function are the minimum, the saddle and the maximum points. The lower frames display the equal height contours painted on a gray-level image that represents the heights.

where $L_{\mathbf{m}}$ is the set of all planar curves starting at $\mathbf{x}$ and ending at $\mathbf{m}$, see also [105].

Propagating from an assumed local minimum singular point, the evolving contour meets several kinds of singular points. There are only three types of such singularities that are of interest to us. According to the mountaineers theorem [53, 96], the number of extrema located within a closed equal height contour of a smooth surface exceeds by one the number of saddle points within that contour. Therefore, when tracking an equal height contour that starts as a small circle around a minimum, the first singular point that the propagating contour meets must be a saddle point. There are two possible ways for the contour to reach a saddle point. The first one is when the saddle is reached from two opposite sides, see Figure 5.3 upper row.

In this case the propagation may proceed with no difficulties. After encountering such a saddle point the contour splits into two separate curves and the internal curve may encounter a maximum point while propagating inwards and vanish at that point without violating any of the topology conditions. We shall refer to such a saddle as the 'saddle of first type'.

In the second case, see Figure 5.3 lower row, the saddle point is touched from one side only. In this case a topology–based procedure is needed to correct the contour propagation to cause the interior to obey the mountaineers theorem. Propagating the contour beyond that point will cause that point to become a shoulder (inflection), a pathological phenomenon not permitted in Morse functions. However, note that propagating beyond that point causes false results only in a well defined area which is a subset of the region swept by the propagating equal height contour. The rest of the contour is not affected and can still be used in the reconstruction process. This fact helps us in producing the global algorithm while keeping

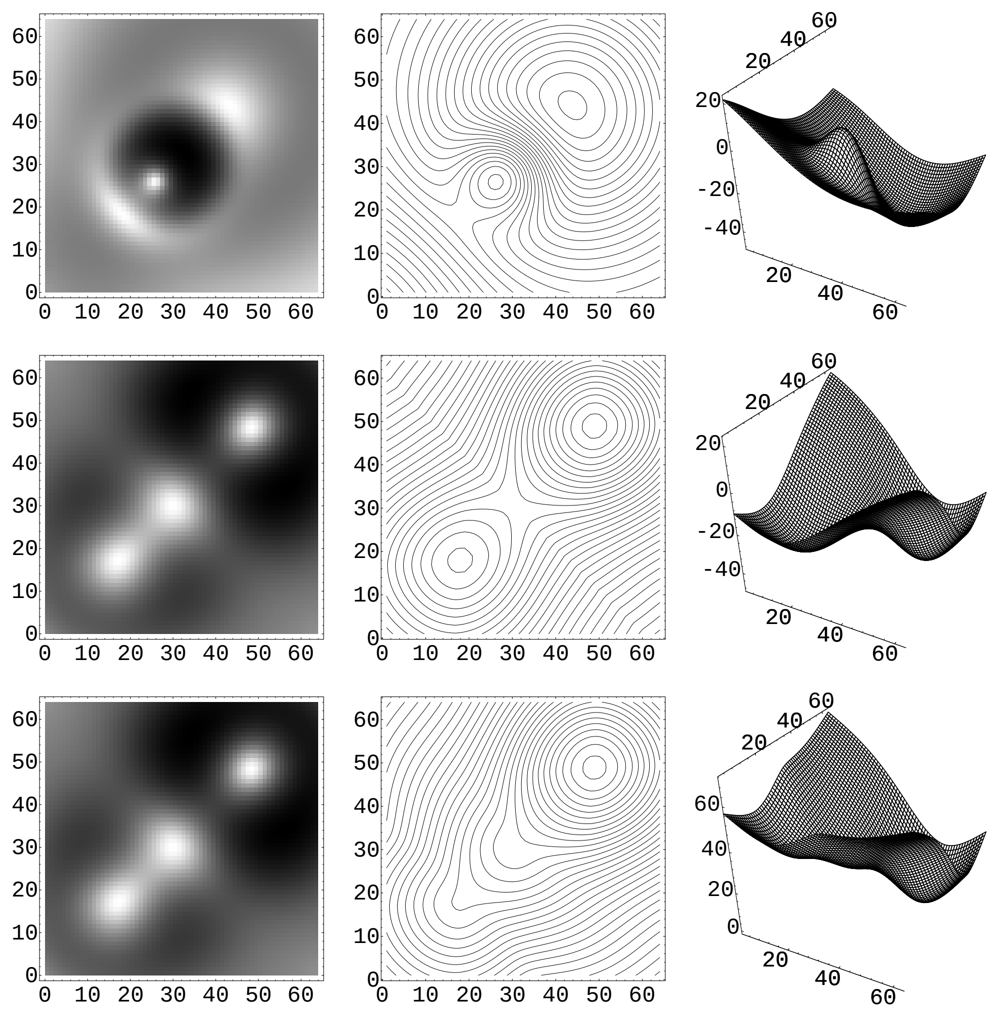

Figure 5.3: Left column: the shading images. Middle column: the equal height contours. Right column: the $3D$ surface. **Upper row:** Starting the propagation of the equal height contour at the upper right singular (brightest) point, the propagating equal height contour reaches the saddle (the lowest left singular point) form two opposite sides, splits into two contours, and the mountaineers theorem remains valid when the propagation continues. **Middle row:** The shading image, the equal height contours of the original surface and the original surface itself are shown. **Lower row (Saddle of inflection type):** Starting the propagation at the upper right singular point, the propagating equal height contour reaches the saddle (that corresponds to the middle singular point) from only one side. Continuing the propagation will cause that saddle to appear as a higher order local extrema that looks like a shoulder, which is forbidden according to our smoothness assumptions.