6 introduces an analytic model that describes the propagating $3D$ curve. Tracking such a $3D$ curve is quite a complicated task. However, it is also possible to follow its projection on the plane. Calculating the $3D$ distance maps by tracking the projected evolution from both source and destination points on the given surface, enables us to select the shortest path which is given by the minimal level set in the sum of the two distance maps, see Figure 1.5. The propagation time in this case, indicates distance on the surface, *i.e.,* the geodesic distance.

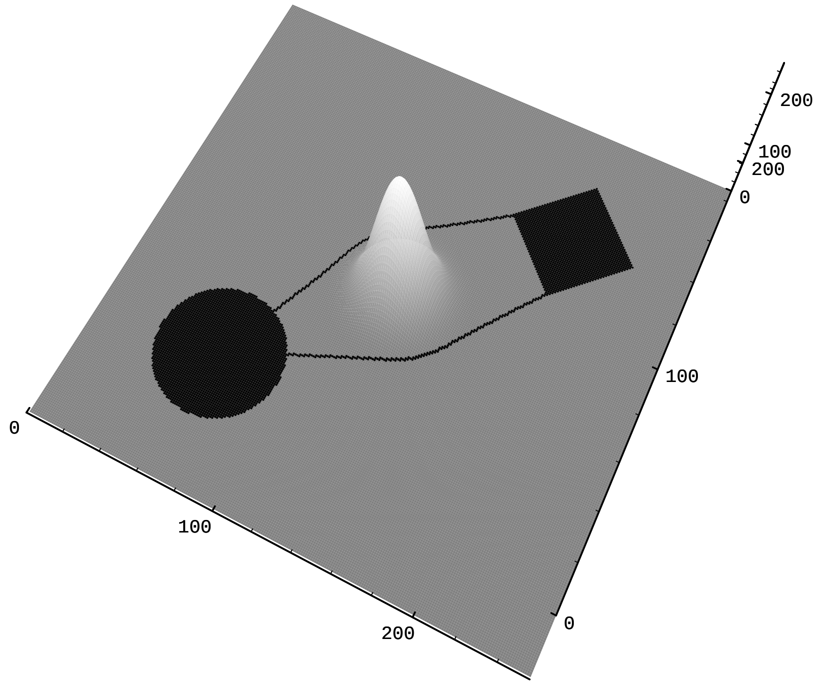

Figure 1.5: Finding the paths of minimal length between the square and circle areas on a Gaussian mountain surface.

1.5.5 Shortening three dimensional curves via two dimensional flows

Given a path connecting two points on a given surface, it is sometimes required to shorten its length locally and to find the closest geodesic to the given curve. In Chapter 7 it is shown that this operation too may be done by propagating a curve along the geodesic curvature [77, 78]. This $3D$ curve propagation may also be performed by tracking its planar projection, and may be used to refine minimal geodesics obtained by other methods, *e.g.* the minimal path estimation obtained by the Kiryati-Szekely algorithm [83].

1.5.6 Distance maps and weighted distance transforms

As stated in Chapter 3 some of the above results may in fact be grouped under the same title of 'generalized distance maps' [69, 76]. While searching for offset curves, one constructs the distance transform. Reconstructing the shape from shading may be shown to be equivalent to calculating a weighted distance transform. Continuous scale morphology, may be shown to result in the distance transform under a given metric, where the the structuring element of the morphological operations defines the unit sphere of the given metric.

1.5.7 Using *Multi–Valued* distance maps in path planning on surfaces with moving obstacles

In Chapter 8 the *multi valued* distance map concept is introduced. A multi valued distance map is defined and used as a tool for computing optimal path for a robot with limited velocity navigating on a surface and avoiding moving obstacles [75]. The distance map on the given surface incorporates the constraints imposed by the moving obstacles and is produced by curve propagation techniques. The basic idea of our method is the use of Hüygens principle leading to a wave front propagating in time and describing the farthest parts the robot could arrive to by moving in all possible ways away from the source region. Clearly, the minimal path to the destination will be determined when this wave front first meets the destination. In some sense, the method proposed searches over all possible spatio-temporal robot movements to determine the time–optimal navigation path and schedule to the required destination. Although it may seem obvious that this process will discover the best navigation course, it is far from trivial to realize how one could actually carry out this program in a computationally efficient way. The limit on the robot velocity is used to reduce the complexity of the problem from a search over a $3D$ configuration space to a search over a $2D$ multi valued array. The analytic analysis as well as efficient numerical algorithms for calculating the multi valued distance map and tracking an optimal path are introduced.

1.5.8 Skeletons via level sets

Skeletons are thin, exact descriptors of shapes. Defining the distance of a point from a curve as the infimum of distances between the point to the set of curve points. The skeleton of a shape is the set of internal points whose distance to the boundary is realized in more than one boundary point. Each point of the skeleton is associated with a width descriptor corresponding to its distance from the boundary.

Being a stick figure, or naive description of the shape, skeletons are perceptually appealing. From a pattern recognition point of view, skeletons provide a unique combination of boundary and area information. Although mathematically well defined (in the continuous plane), it has always been a problem to implement skeletons on computers. This situation has brought numerous suggestions of solutions referred to as skeletonization or thinning algorithms.

Having a stable scheme describing distances in the digital plane, solves many of the inherent problems of skeletonization. As shown in Chapter 9, skeletons are located on zero crossing curves of differences of distance transforms from boundary segments [79]. Applying simple differential geometry results to skeletons, it is possible to find a necessary and sufficient partition of the boundary to segments whose distance transforms participate in the specification of the skeleton location. See Figure 1.6.

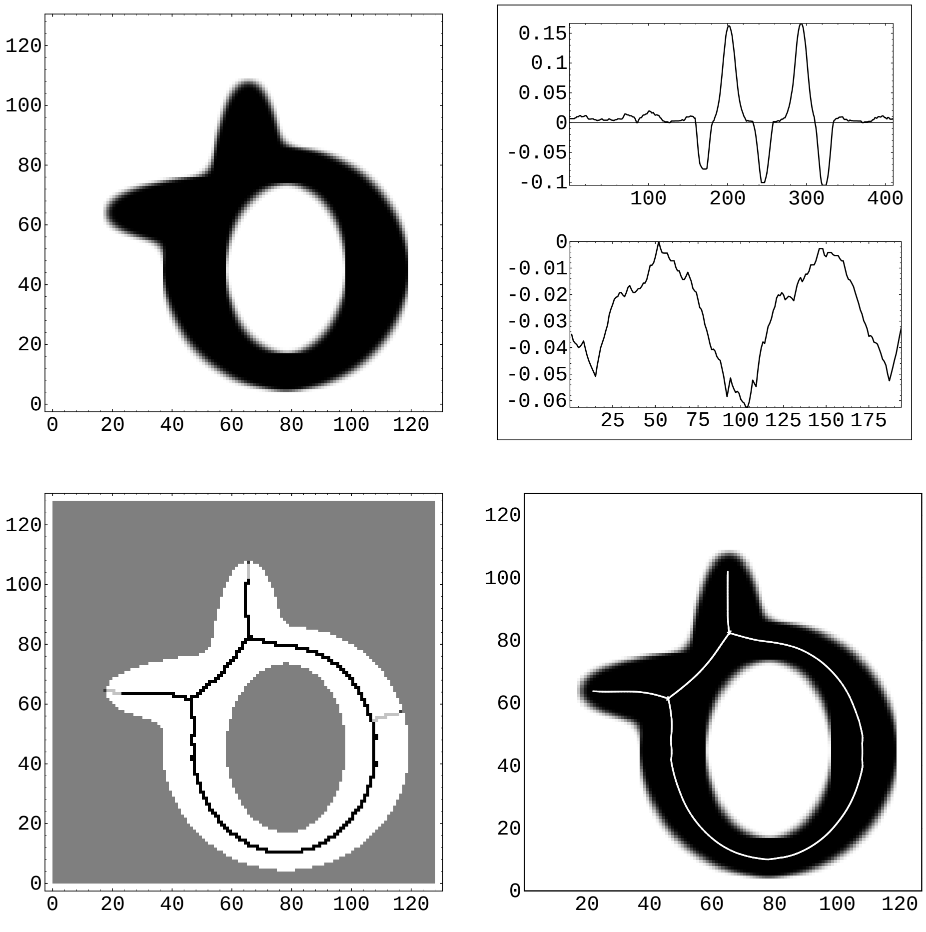

Figure 1.6: Finding the skeleton of the shape in the upper left frame is done by first locating the curvature positive maxima along the boundaries on the upper right. Then, by calculating the distance from each boundary segment, the skeleton may be determined with sub-pixel accuracy as shown in the lower right frame.

1.5.9 Geodesic active contours

One of the main problems in image analysis is the segmentation problem. Given several objects in an image it is necessary to integrate their boundaries in order to achieve good model of the objects under inspection. This problem was addressed in many ways over the years, starting with simple thresholding, region growing, and deformable contours based on energy minimization along a given contour called ‘snakes’.

In Chapter 10 a novel geometric model that starts form a user defined contour and segments objects in various type of images [23, 22] is introduced. The idea is to minimize a total ‘non-edge’ penalty function integrated along the curve. The relation to the classical snakes and to recent geometric models is explored, showing better behavior of the proposed method over its ‘ancestors’: The classical snakes and the recent geometric models.

The tumor in the Figure 1.7 is an *acousticus neurinoma*, and includes the triangular shaped portion at the top left part. The detection process is presented on the zoom out part of the tumor on the right. For comparison, the same image was also applied to the model developed in [21, 95]. Due to the large variation of the gradient along the object boundaries and the high noise in the image, the curve did not stop at the correct position, it shrinks to a point and the tumor was not detected.

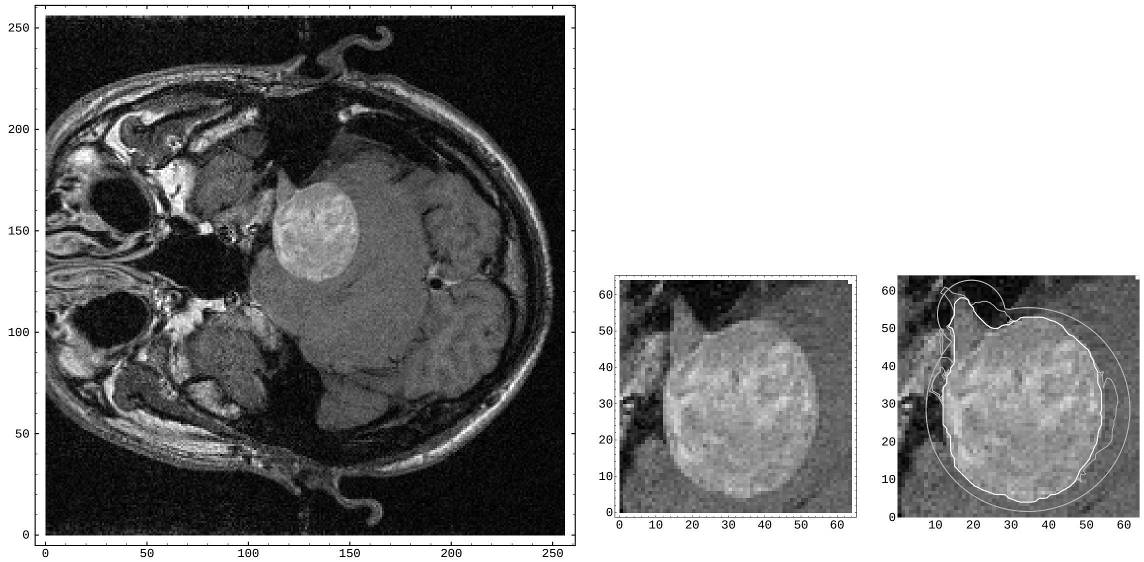

Figure 1.7: An example of tumor detection in MRI via geodesic active contours. The tumor in the image on the left is an *acousticus neurinoma*, and includes the triangular shaped portion at the top left part. For this image, an inward deforming contour was used. The tumor portion on the right is shown after zoom out for better presentation. The gray contours are the positions of the evolving curve in time, while the white contour is the final result of segmenting the tumor.

This way of finding local geodesics in a potential function defined by an edge detection operator requires an initial contour. In some other cases, it is desired to locate the minimal geodesic connecting two points along the boundary of an object. In Chapter 11 an approach of integrating edges by locating the minimal geodesic is explored[33]. See Figure 1.8.

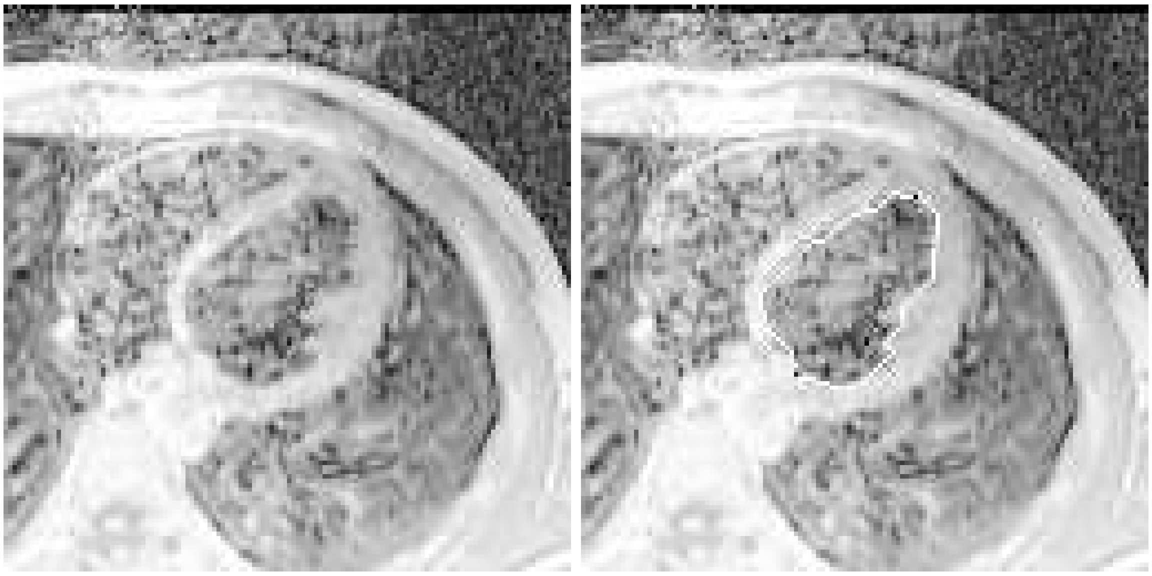

Figure 1.8: An MR heart image on the left. The white contour between the two black end points on the right is the segmentation result of the desired ventricle

Chapter 2

Numerical Schemes and the Eulerian Formulation

The procedures required for in solving some of the classical problems we deal with are in fact procedures for solving partial differential equations (PDEs). In this chapter we give a brief introduction to numerical analysis issues that were found relevant for approximation, when solving such PDE's. We present the basics of how to select the proper numerical scheme for approximating a given evolution equation. Planar curve evolutions are reformulated as Hamilton-Jacobi (HJ) equations. Then, using the close relation between hyperbolic conservation laws and HJ equations, numerical schemes that make use of this relation are discussed. The purpose of this chapter is to stress the practical usage of the numerical methods, therefore basic concepts and recipes are presented in a simple and often simplistic way that we hope will still be of help for potential future implementers of such methods.

2.1 Helpful Literature

The numerical analysis literature that best fits our needs, deals with numerical approximations of Hamilton Jacobi equations. As shown in [111] these equations are closely related in nature to hyperbolic conservations laws. Therefore, numerical techniques that were developed for approximating the evolution of differential conservation laws may readily be adapted to the type of equations we shall encounter.

The book of LeVeque [87] is a great help as an introduction to numerical methods for conservation laws. The interested reader could find more detailed information of the basic concepts and definitions in this book. Formal definitions and limitations of numerical methodologies as applied to conservation laws in fluid dynamics may be found in [133]. More information about conservation laws and the theory of shock waves, is given by one of the founders of this theory, Peter D. Lax, in his lecture notes [84]. The theory of shock waves and its applications in the analysis of gas dynamics is given by Smoller in the third part of his book [132].

The relation between conservation laws and the evolution of curves was introduced by Osher and Sethian in their classic paper [111]. In this paper, Osher and Sethian present a new formulation for curve evolution by considering the evolution of a higher dimensional function in which the curve is embedded as a level set. The relation of this evolution process to conservation laws is explored, stable and efficient numerical schemes being proposed. Other numerical schemes approximating the same type of PDE's may be found in [108].

The search for better numerical schemes in this field is still an ongoing concern of many researchers. Although several new techniques have been introduced since the book of LeVeque was published, we still feel that "the mathematical theory is lagging behind the state-of-the-art computational methods" [87].

2.2 Basic Definitions

The continuous case analysis is of course very important when analyzing PDE's. However (although accurate analysis serves an important role in understanding the behavior of the equation) when implementing a numerical approximation of such an equation on a digital computer one must address several other topics as well.

An example of a very simple, yet very important, question is how to approximate $u_x(x)$, the first derivative of the function $u(x) : \mathbb{R} \to \mathbb{R}$ in the x direction. Let us simplify the problem and assume that $u(x)$ is sampled by taking uniform samples of its values at equal distances of Δx. Denote u_i to be its i-th sample, *i.e.* $u_i \equiv u(i\Delta x)$, and $D^x u_i$ as the *finite difference approximation* of the function u at the point $x = i\Delta x$. In approximating u_x one should consider computation efficiency, accuracy and consistency with the continuous case. Using the samples it is possible to interpolate a smooth function passing through the function values at the sample points. A very simple approximation is the *centered* difference finite approximation, given by

$$D^x u_i \equiv \frac{u_{i+1} - u_{i-1}}{2\Delta x}.$$

It is based on the Taylor series expansion, with a truncation error of $\mathcal{O}(\Delta x^2)$.

The *forward* finite approximation is similarly defined as

$$D_+^x u_i \equiv \frac{u_{i+1} - u_i}{\Delta x},$$

and the *backwards* approximation:

$$D_-^x u_i \equiv \frac{u_i - u_{i-1}}{\Delta x}.$$

In both cases above, the truncation error is of $\mathcal{O}(\Delta x)$.

Taking $\Delta x \to 0$ the approximation obviously converges to the continuous case for the case of smooth functions. Convergence to the continuous case is an important issue that is referred to as *consistency* with the continuous case.

2.3 Conservation Laws and Hamilton-Jacobi Equations

The curve evolution equations are differential rules describing the change of the curve, or its evolution, in 'time'. As we shall see in the next section there is a formulation that puts curve evolution equations into a closely related formulation having the flavor of conservation laws. Following [84]: A conservation law asserts that the rate of change of the *total amount* of substance contained in a fixed domain G is equal to the *flux* of that substance across the boundary of G. Denoting the *density* of that substance by u, and the flux by f, the conservation law is

$$\frac{d}{dt}\int_G u dx = -\int_{\partial G}\langle f, n\rangle dS,$$

where n denotes the outwards normal to G and dS the surface element on ∂G, which is the boundary of G, so that the integral on the right measures the outflow–hence the minus sign. Applying the divergence theorem, taking d/dt under the integral sign, dividing by the volume of G and shrinking G to a point where all partial derivatives of u and f are continuous we obtain the *differential conservation law*:

$$u_t + \nabla f = 0.$$

Consider the simple $1D$ case in which the integral (by x and t) version of a conservation law gets the explicit form of:

$$\int_{x_0}^{x_1}(u(x,t_1) - u(x,t_0))dx + \int_{t_0}^{t_1}(f(x_1,t) - f(x_0,t))dt = 0.$$

A solution u is called a *generalized solution* of the conservation law if it satisfies the above integral form for every interval (x_0, x_1) and every time interval (t_0, t_1). Taking $x_1 \to x_0$, $t_1 \to t_0$, and dividing by the volume $dxdt = (x_1 - x_0)(t_1 - t_0)$, we obtain the $1D$ differential conservation law:

$$u_t + f_x = 0.$$

For $f_x = (H(u))_x$, (*i.e.,* assuming f is a function of u given by $H(u)$) a *weak solution* of the above equation is defined as $u(x,t)$ that satisfies [127]

$$\frac{d}{dt}\int_{x_0}^{x_1} u(x,t)dx = H(u(x_0,t)) - H(u(x_1,t)).$$

Weak solutions are useful in handling non smooth data. Observe further that u need not be differentiable to satisfy the above form, and they are not unique. Thus, we are left with the problem of selecting a special 'physically correct' weak solution.

The Hamilton–Jacobi (HJ) equation in $\mathbb{R}^d$ has the form

$$\phi_t + H(\phi_{x_1}, ..., \phi_{x_d}) = 0, \qquad \phi(x,0) = \phi_0(x).$$