# CURVE EVOLUTION ON SURFACES

by
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# CURVE EVOLUTION ON SURFACES

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**by**
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*To my wife Michal and my son Noam*

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# Abstract

Many classical problems in Computer Vision, Robotic Navigation, and Image Processing may be formulated as partial differential equations describing the evolution of a curve on a smooth $3D$ manifold. Solving the problems involved in propagating such curves amounts to constructing a solution for these problems. This thesis deals with classical problems like the problem of reconstructing $3D$ objects from their shading images, and the problem of tracking optimal paths for a robot navigating between stationary and moving obstacles on a $3D$ surface. Numerical techniques of propagating curves on $3D$ surfaces are formulated and applied to these problems.

Minimal geodesics which are the paths of minimal length, play an important role in Robotic Navigation as well as many other engineering problems. We use curve evolution procedures to calculate geodesic distance maps on a given surface and to shorten given approximations into local geodesics. Geodesic distances from two points provide the locations of the minimal geodesics connecting these points. Finding the geodesic distance as well as several other problems are solved by curve evolution via level sets techniques. The idea of level set intersection that yields the global solution to the minimal geodesic problem, led us to some other solutions to classical problems like the global skeletonization algorithm, the solution to the Steiner problem on a surface, and the generalized shape from shading algorithm.

In robotic navigation we consider the problem of locating a time optimal path for a robot with limited velocity. A new concept, the 'multi-valued' distance map, is introduced. The multi valued distance map helps in reducing the complexity involved in solving the problem of finding the optimal path of a robot navigating between moving obstacles on a surface.

The problem of reconstructing a $3D$ object from its shading image is a classical problem in computer vision. Here, an extension to the shape from shading via level set evolution for the case of oblique light source is introduced. A generalized shape from shading algorithm for the problem of several singular (brightest) points in the image domain is constructed by introducing the Morse smoothing assumption. Variations on the theme of using curve evolution techniques in solving classical problems like the 'geodesic deformable contours' for shape modeling, edge integration, and continuous scale morphology are shortly described.

Curve evolution refers to the study of curves deforming in time. The deformations are governed by local rules described via partial (usually nonlinear) differential equations, called *evolution equations*. In general, three steps are involved in the research of curve evolution:

The first one is a search for the evolution equation which is best suited to the given problem. The second step is the analysis of the evolution equation. This analysis combines diverse mathematical areas such as differential geometry, numerical analysis, partial differential equations (e.g. Hamilton-Jacobi equations), and hyperbolic conservation laws. The last step involves a numerical implementation of the evolution equation. A good numerical implementation should be consistent with the continuous case, and be as simple, accurate and stable as possible. These three steps will be described in each of the cases we shall mention.

# List of Symbols and Abbreviations

| | |
|---|---|
| $\mathcal{C}(p,t)$ | A family of parametrized planar curves in time: $\mathcal{C}(p,t) : [a,b] \times [0,T) \to \mathbb{R}^2$ with traversal parameter $p$ and time parameter $t$. |
| $s$ | Euclidean arc length parameter of a given curve, $\lvert\mathcal{C}_s\rvert = 1$. |
| $\mathcal{N}$ | The unit normal vector of the curve $\mathcal{C}$. |
| $\mathcal{T}$ | The unit tangent vector of a given curve, $\mathcal{T} = \mathcal{C}_s$. |
| $\kappa\mathcal{N}$ | The curvature vector $\mathcal{C}_{ss}$. |
| $\mathcal{S}$ | A regular surface in $\mathbb{R}^3$, *e.g.* $z(x,y) : \mathbb{R}^2 \to \mathbb{R}$. |
| $p, q$ | The $x$ and $y$ partial derivatives of the function $z(x,y) : \mathbb{R}^2 \to \mathbb{R}$, *i.e.* $p \equiv \partial z/\partial x$ and $q \equiv \partial z/\partial y$. |
| $N$ | The unit normal of a given surface. *e.g.* for $z(x,y) : \mathbb{R}^2 \to \mathbb{R}$, the normal is given by $N = (-p,-q,1)/\sqrt{1+p^2+q^2}$. |
| $\alpha(p,t)$ | A family of $3D$ curves embedded in a given surface: $\alpha(p,t) : [a,b] \times [0,T) \to \mathbb{R}^3$ with a parameter $p$ at time $t$, so that $\alpha \in \mathcal{S}$. |
| $\kappa_g$ | The geodesic curvature, *i.e.* the magnitude of the projection of the curvature vector $\alpha_{ss}$ onto the tangent plane. |
| $\phi(x,y,t)$ | A family of bivariate functions in time: $\phi(x,y,t) : \mathbb{R}^2 \times [0,T) \to \mathbb{R}$, for which the zero level set is the desired planar curve, *i.e.* $\mathcal{C}(t) = \{(x,y) : \phi(x,y,t) = 0\}$, or equivalently $\mathcal{C} = \phi^{-1}(0)$. |
| $\Phi_{ij}^n$ | A discrete approximation of the continuous function $\phi$: $\Phi_{ij}^n \approx \phi(i\Delta x, j\Delta y, n\Delta t)$ |

$I(x,y)$ A gray level image: $I(x,y) : \mathbb{R}^2 \to \mathbb{R}$. In some cases referred to as shading image.

$\hat{l}$ A unit vector indicating the light source direction in $\mathbb{R}^3$.

$\pi\circ$ The planar projection operator: $\pi \circ (x,y,z) = (x,y)$.

$\langle a,b\rangle$ The scalar product operator: $\langle a,b\rangle = a_1b_1 + a_2b_2 + ...$

$\times$ The vector product operator.

$\nabla$ The gradient operator $\nabla = (\partial/\partial x, \partial/\partial y)$.

$|a|$ The magnitude operator $|a| = \sqrt{\langle a,a\rangle}$.

$a_t$ The subscripts $t,s,x,y,p,u,v$ indicate partial derivatives.
*e.g.*, $\mathcal{C}_s = \dfrac{\partial \mathcal{C}}{\partial s}$, and $\phi_{xy} = \dfrac{\partial^2\phi}{\partial x\partial y}$.

# Chapter 1

# Introduction

## 1.1 Outline of This Thesis

In this chapter we briefly review the problems we dealt with and point out the main contributions of this thesis. One short section is dedicated to each of the various problems and their solutions.

Numerical analysis of conservation laws play an important role in the implementation of curve evolution equations. The second chapter introduces the basic concepts in numerical analysis and presents the relation between curve evolution, Hamilton-Jacoby partial differential equations, and differential conservation laws. This close relation enables us to introduce finite difference approximations based on the theory of conservation laws into curve evolution.

Detailed background on distance maps and morphology, the shape from shading problem, skeletons, geodesic deformable contours, and robotic navigation may be found in the rest of the chapters.

A rough sketch of the relation between the problems and the techniques involved in their solutions is presented in Figure 1.1. Following the diagram, curve evolution via level sets is first used to produce the different kinds of distance maps. The level sets of linear combinations of the calculated distance transforms are then used to track the minimal geodesics, and to solve other interesting problems. The minimal geodesics may then be refined by the geodesic curvature flow.

## 1.2 Contribution and Main Results

Let us first summarize the main the contribution of this thesis

- A framework for the computation of weighted distance transforms, geodesic distance transforms on $3D$ surfaces, and distance transforms for any given metric via curve evolution is introduced.

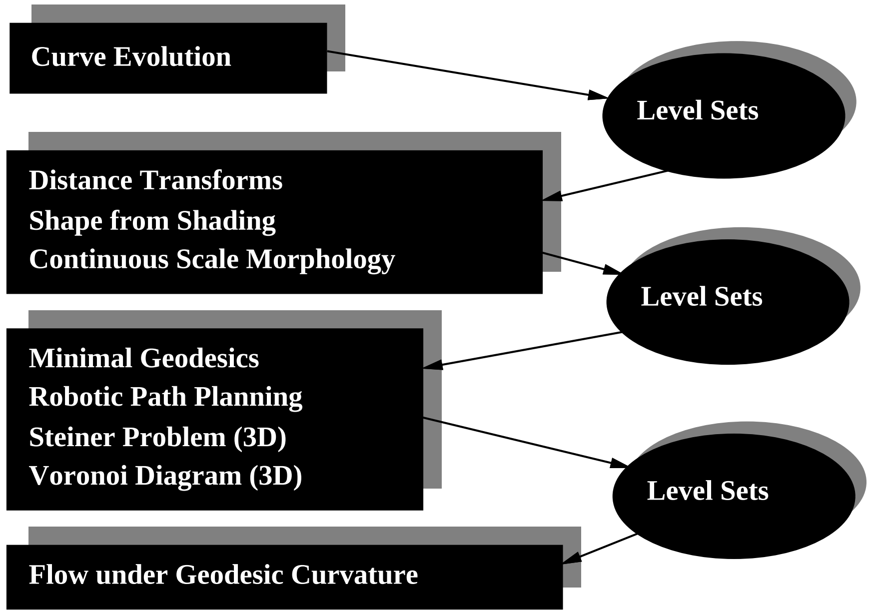


Figure 1.1: The relation between the problems and the techniques involved in their solutions

- The equivalence of the metric dependent distance transform and the continuous scale morphology for convex structuring elements is presented.
- The shape from shading curve evolution algorithm [17] is generalized for the case of oblique light source.
- A global shape from shading algorithm for the problem of several singular points in the image domain is constructed by introducing the Morse smoothness assumption. This result answers one of the basic open question in the field of Computer Vision.
- A global approach for tracking minimal geodesics on $3D$ surfaces [64, 65], and an alternative approach for refining initial rough guesses into accurate final solutions, are introduced. A coarse to fine procedure is produced by integrating the Kiryati-Szekely shortest path graph-search based algorithm and the geodesic curvature flow scheme into one framework.
- A novel global skeletonization algorithm is introduced. This classical problem in Computer Vision is approached from a different direction achieving better stability and accuracy.
- A geodesic framework for deformable models in shape modeling is constructed. The 'geodesic active contours' shown to behave better than previous deformable models for object segmentation and shape modeling. A new application for the weighted distance transform for edge integration is presented.

- The 'Multi-valued' distance map concept is introduced for robotic navigation in the presence of moving obstacles.

- The Eulerian formulation and a proper numerical scheme for each of the curve evolution solutions is developed.

## 1.3 Recent Developments in Curve Evolution

Recently researchers in the field of image processing and computer vision started to pay attention to new ways of analyzing and representing two-dimensional, stationary or moving images, via planar curve evolution [18]. In fact, any image can be viewed as a set of level curves "evolving" with the height parameter. Even such a simple description is quite useful in a variety of situations.

Several image analysis algorithms nowadays are based on propagating planar curves in the image plane according to local variations in the grey-level of the image. Those planar contours might be, for example, the level sets on the surface of an object whose shaded image we are trying to interpret so as to recover its three-dimensional structure. The Shape-from-Shading field is indeed a good example illustrating the way curve propagation algorithms found a very interesting application [17]. Their usefulness in this and other applications was further enhanced by the recent development, in the field of numerical analysis, of a "miraculous" algorithm for the stable propagation of planar curves according to a variety of rules [111]. This algorithm together with some recent results in the theory of curve evolution resulted in the 'affine curvature flow' that was found to be a powerful tool for image smoothing and deblurring [120].

Other fields in which there were immediate consequences of having a stable and efficient way to propagate curves, are Computer Aided Design, Robotics, Shape Analysis and Computer Graphics.

In CAD there is a need to find offset curves and surfaces, implying fixed-speed curve propagation. Geodesic deformable models where introduced for shape modeling and analysis. In Computer Graphics, Pnueli and Bruckstein found an interesting application in the design of a clever half-toning method they named **Digi**$_{D}$*ürer*, that aims to emulate the work of classical engravers [114, 115, 123], see Figure 1.2.

In Robotics, where one often needs to find a path for robots that need to move from a source to a certain destination, one could determine shortest routes by propagating a wave of possibilities, and finding out the way its wavefront reaches the destination point. This is like Feynman's particles sniffing all possible paths before deciding on the trajectory of minimal action [46]. This, by the way, can be done even in the presence of moving obstacles. Last, but not least, we shall mention the field of Mathematical Morphology, where there is a need to precisely compute various types of distance functions, to enable erosion or dilation of shapes.

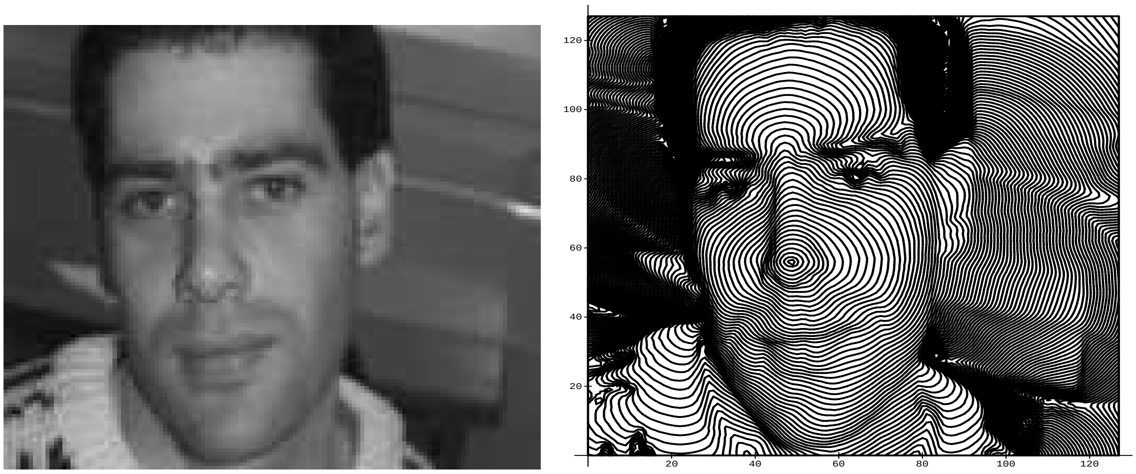


Figure 1.2: Propagating a planar curve with a velocity proportional to the image gray levels results in an artistic approach for halftoning.

## 1.4 Curve Evolution

The solution for some of the problems that we describe is based on the ability to find a new curve-evolution-based formulation to the problem. This new formulation is of the form of a differential equation that describes the propagation of a planar curve in time, under the constraints imposed by the problem. While propagating a planar curve one must often overcome various problems such as topological changes, *e.g.* a single curve that splits into two separate curves, and numerical problems that may be caused by the type of curve representation used, *e.g.* the problem of determining the offset curve to a polynomial parametric curve.

As we show in Chapter 2 the implicit representation of the propagating curve solves numerical and topological problems of the propagation. Tracking the *zero level set* of the bivariate function $\phi(x, y)$ propagating in time, overcomes these problems in an elegant way, and leads to the desired numerical scheme. This new formulation for the implementation of propagating curves is due to Osher and Sethian [111], who called it the *Eulerian* formulation.

In the sequel, some classical problems are presented, and the curve evolution solutions to these problems are shortly described. In this chapter we shall try to give the motivation while the following chapters and cited papers may shed more (analytic) light on these problems and their solutions.

## 1.5 Variations on a Theme

### 1.5.1 Shape from shading

A classical problem in the area of computer vision is how to reconstruct a $3D$ surface $z(x, y)$ from a given gray–level picture $I(x, y)$. In Chapter 4 it is shown that under reasonable assumptions about the light source and the object reflection properties, it is possible to solve this problem by using the image data to control the evolution of a planar curve so as to track the equal height contours of the object [17, 62, 66, 81]. Those equal height contours refer to equal heights with respect to the light source direction [72]. Some analytic manipulations on the relations between the contours and the data leads to an evolution rule for a planar curve. This evolution rule, in which the propagation time indicates the height with respect to the light source direction, is determined by the gray-level image and the local nature of the curve, see Chapter 4 for more details.

In Chapter 5 [67, 70, 71] we shown how to use "weighted distance transforms" implied by the shape from shading curve evolution for each of the singular points in the shading image to solve the global shape from shading problem for smooth surfaces (Morse functions). See Figure 1.3.

### 1.5.2 Continuous scale morphology

In the field of shape theory, it is often required to analyze a shape by activating some "morphological" operations that make use of a "structuring element" with some given shape, see Figure 1.4. In Chapter 3 we explore the problem of morphological operators in which the element may be of any convex shape with variable sizes [13, 119]. This problem too may be reformulated as the problem of activating a propagation rule for the shape boundary. The evolution rule for the shape's boundary is determined by the structuring element's shape, and the time of evolution in this case represents the size of the element.

### 1.5.3 Shape offsets or prairie fire propagation

In *CAD* (computer aided design) one often encounters the need to find the offset of a given curve. A simple algorithm that solves this problem may be constructed by considering a curve that propagates with a constant velocity along its normal direction at each point [68, 8]. The propagation time represents the "offset distance" from the given curve.

This is, of course, also Blum's prairie fire propagation model for finding shape skeleton, *i.e.* the shock fronts of the propagation rule.

### 1.5.4 Minimal geodesics on surfaces

This important problem in the field of robotic navigation may be solved by considering an equal distance contour propagating from a point on a given surface [63, 64, 65]. Chapter

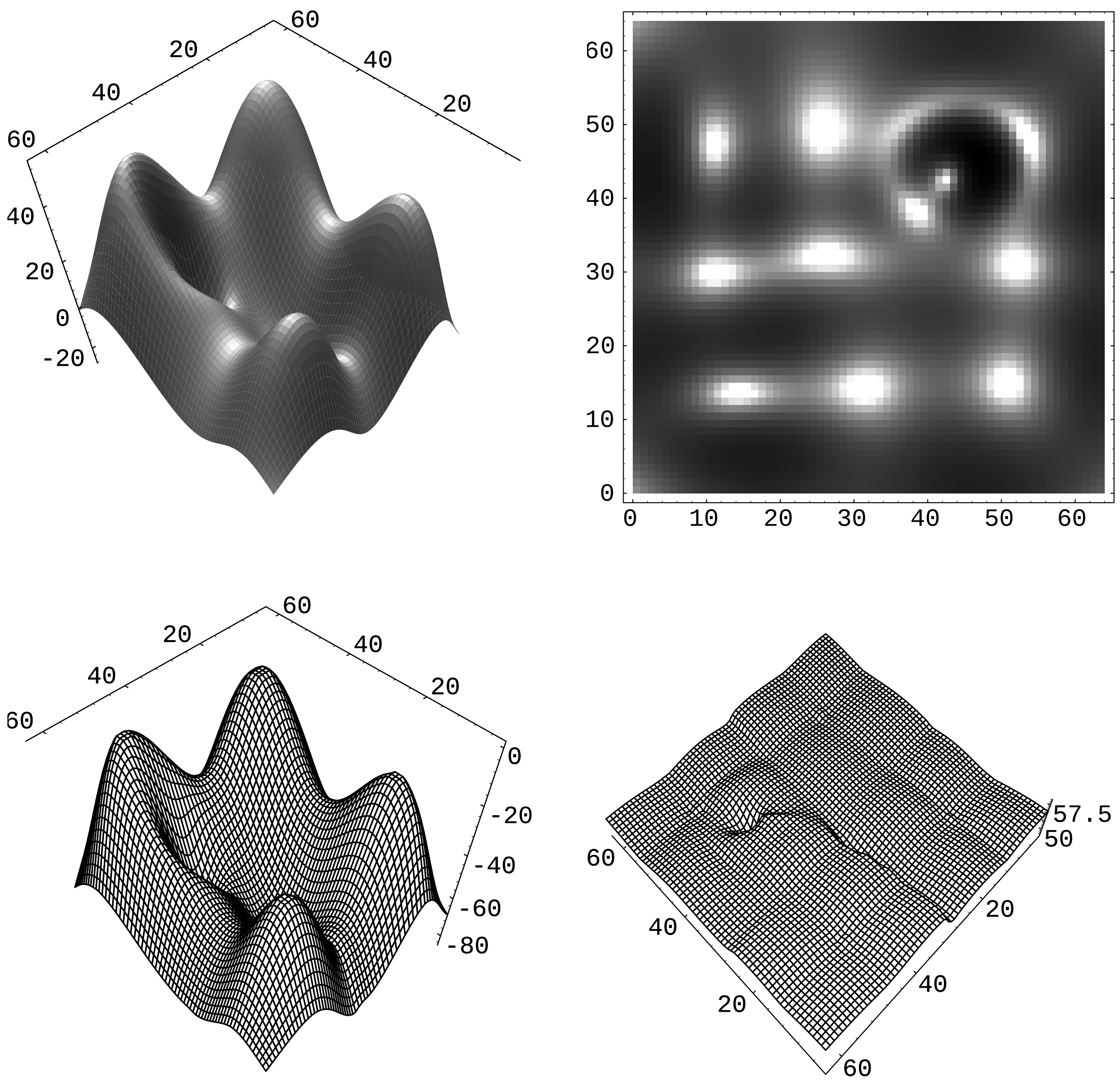


Figure 1.3: A smooth synthetic surface on the upper left produces the shading image on the upper right frame. The reconstruction of the $3D$ shape from the shading image, based on Morse smoothness assumption is displayed on the lower left, and the error of subtracting the reconstruction from the original surface on the lower right.

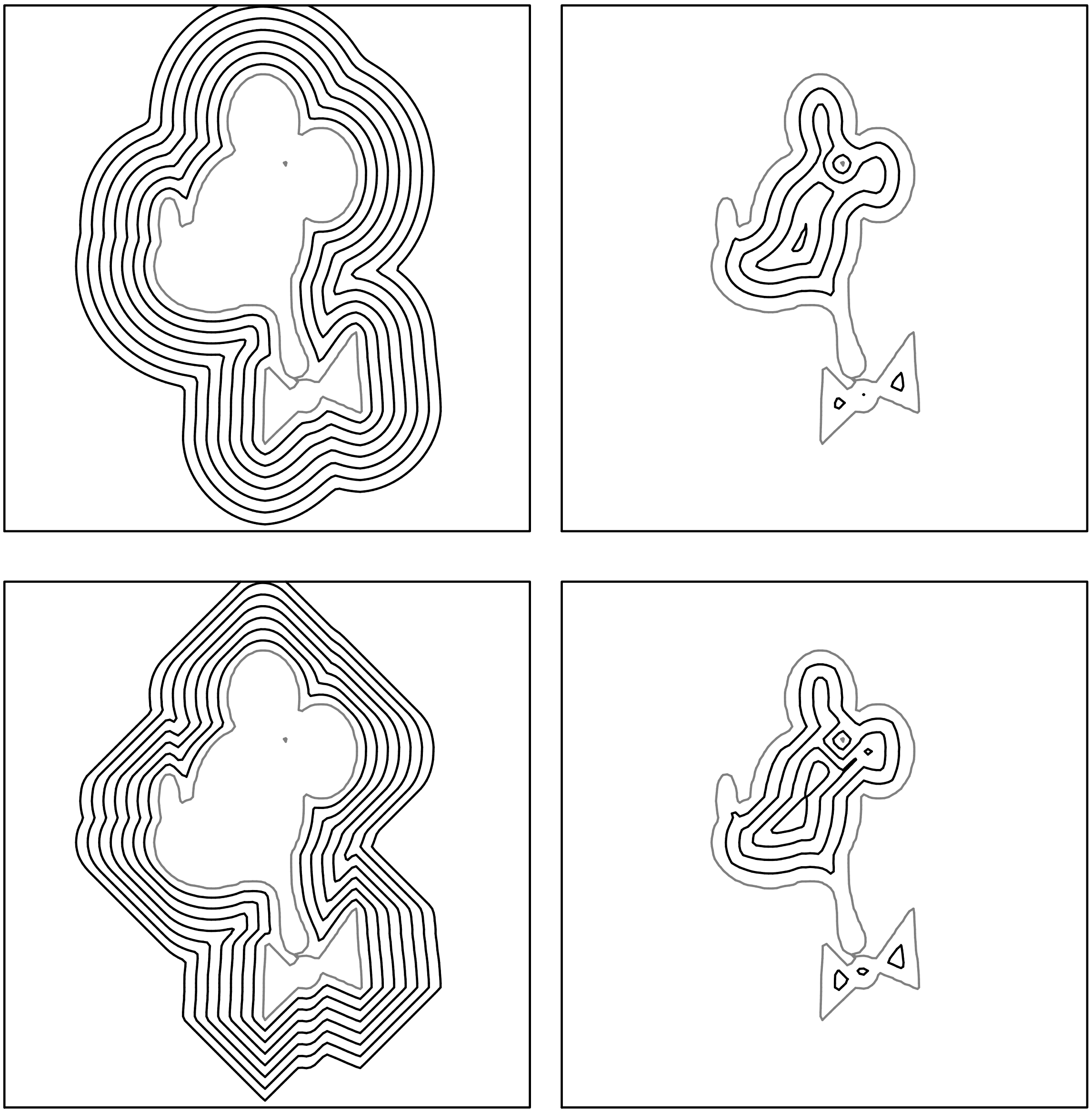

Figure 1.4: A dilation and erosion operations with diamond and circle structuring elements of different scales.