Broadcast Control for Swarm of Agents with Bearing Only and Limited Visibility Sensors

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Abstract

We suggest a control mechanism for leading a team of mobile, oblivious, identical and indistinguishable agents in desired directions. The agents are assumed to have a compass, i.e. a common North direction, and bearing only sensing within a limited visibility range, and may receive a direction-control broadcast with some given probability. We prove that, under the suggested control mechanism, the swarm of agents gathers to a small disk in the plane and moves in the desired direction with an expected velocity dependent on the probability of receiving the control signal.

1 Introduction

Directing a swarm of simple and low-cost agents, toward a given location in the environment using a global broadcast signal that may be "heard" by only some of the agents, is an interesting challenge. Prior studies on swarm motion control were either based on using global potential fileds or external forces [2, 5, 7, 6], or on using leader following processes and either linear interaction dynamics or some non-linear setting [8, 9].

We here address the problem of controlling a swarm of oblivious, anonymous (identical and indistinguishable) agents without explicit inter-agent communication that are capable of sensing only their neighbours' bearing within a limited visibility range. We assume that all agents have a common compass direction (North), hence the exogenous control signal is an azimuth angle that specifies the desired direction of motion in the plane.

We propose and analyse a semi-synchronised discrete time model, where each agent of the swarm has a probability $\gamma > 0$ to "hear" the control signal at each time step. As stated before, the control signal specifies a unit vector $\hat{C}$ in the desired direction of motion.

We start by presenting the agents' local motion law, an extension of the interaction rules presented in [4]. Under this model, each agent jumps inside an allowable region, originally designed for maintaining visibility with its neighbours. Without the presence of the control, this motion law implements a gathering algorithm for swarms of agents with the given capabilities. The proposed law is first shown to gather swarms to a small region even in the presence of the control signal. Then, we prove that, once the swarm's constellation has a complete visibility graph, it moves with a constant expected velocity in the desired direction broadcast by a controlling entity. We end this paper by presenting simulation results, and discussing possible future work.

2 The motion law

We consider a system of n identical, anonymous, and oblivious agents in the $\mathbb{R}^2$-plane specified by their time varying locations $\{p_i(k)\}_{i=1,2,\ldots,n}$. We assume that the agents sense only the direction to their neighbours (i.e. bearing only sensing), hence their information about neighbours is partial. Their steps in time are determined by the set of unit vectors pointing to their current neighbours and the control signal, if received. The neighbours of each agent i, at any time step k are defined as the set of agents located within a given visibility range V form the position $p_i(k)$, and are denoted by the set $N_i(k)$. The neighbourhood relation between all the agents is conveniently described by a time dependent visibility-graph.

Our agents move according to the motion law presented in [4]. At time step k, an agent may be active with a strictly positive probability δ. Each active agent not "surrounded" by neighbours, jumps to a random point choosen according to a uniform distribution in an allowable region defined by the geometry of the vectors pointing to its neighbours and the control. The allowable region of an agent is defined as the area the agent can move into, without losing visibility with any of its neighbours. Gorgon and Bruckstein proved that in the absence of control such a motion law gathers the swarm to a constellation having a complete visibility graph within a finite expected number of time steps [3], and recently Manor and Bruckstein proved that a small modification of the motion law can further be shown to gather the swarm to a disk of a radius equal to σ (assumed to be $\sigma \leq V/2$), where σ is the agents' maximal allowed step size. This process too happens (without control!) within a finite expected number of time steps. See e.g. [4].

In order to adjust this gathering motion law to a control law, we assume that each agent currently "hearing" a control signal acts as if it has another neighbouring agent located in the direction provided by the control signal. This approach to a controlled scenario greatly helps in the analysis of the dynamics of the controlled swarm.

The formal law of motion:
Let $\psi_i(k)$ be the angle of the minimal sector anchored at agent i's position at time step k that contains all its neighbours, and let $\hat{\psi}_i(k)$ be a unit vector in the direction of the bisector of the angle $\psi_i(k)$. If an agent i at time step k has $\psi_i(k) \geq \pi$, it is considered "surrounded" by neighbours.

In the sequel let us denote a disk of radius r centered at a point c by $D_r(c)$.

An agent i is **active** at each time step k with a strictly positive probability δ, and receives the control signal with probability γ. If an **active** agent i does not currently "hear" the control signal, and is not surrounded by its neighbours, i.e $\psi_i(k) < \pi$, it jumps to a uniformly selected random point inside its current "allowable region", defined as follows:

$$ar_i(k) = \bigcap_{j \in N_i(k)} D_{\frac{\sigma}{2}} \left(p_i(k) + \frac{\sigma}{2} U_{ij}(k) \right) =$$

$$D_{\frac{\sigma}{2}} \left(p_i(k) + \frac{\sigma}{2} U_i^-(k) \right) \cap D_{\frac{\sigma}{2}} \left(p_i(k) + \frac{\sigma}{2} U_i^+(k) \right) \quad (1)$$

where recall that $\sigma < V/2$ and $U_{ij}(k)$ is a unit vector pointing from $p_i(k)$ to $p_j(k)$. In (1), $U_i^-(k)$ and $U_i^+(k)$ are unit vectors pointing form agent i to its extremal right and left neighbours, i.e. those defining the sector $\psi_i(k)$ (see Figure 1).

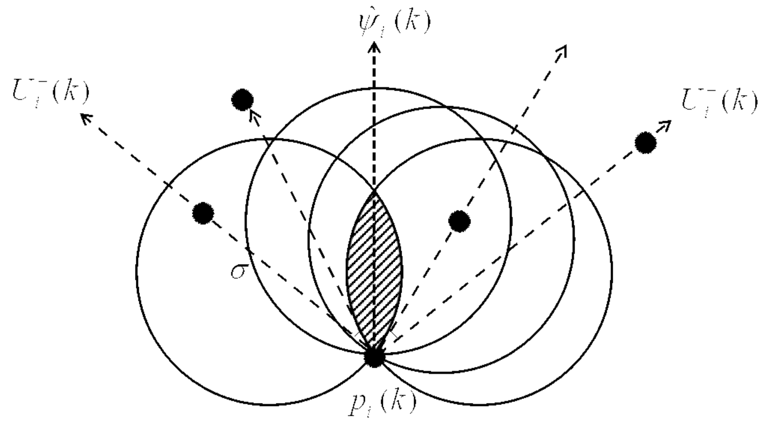

Figure 1: The allowable region of an agent (dashed area) is the intersection of all disks $D_{\frac{\sigma}{2}} \left(p_i(k) + \frac{\sigma}{2} U_i^j(k) \right)$ where $j \in N_i(k)$. Notice that it is equal to the intersection of the two disks associated with the "extremal" neighbours.

If agent i is **active** and receives the control signal, it adds another virtual agent, v, to its neighbour set. The virtual is presumably in the control direction $\hat{C}$ relative to i's position. Then, it proceeds to determine its next position considering the extended neighbour set $v \cup N_i(k)$. We denote the allowable region resulting from the extended set of neighbours by $ar_i^v(k)$. Formally, the local control law for the motion of agent i is therefore the following:

$$p_i(k+1) = \begin{cases} p_i(k) & \text{if } \chi_i(k) = 0 \text{ or } \psi_i(k) \geq \pi \\ \text{a random point in } ar_i(k) & \chi_i(k) = 1 \text{ and } \xi_i(k) = 0 \\ \text{a random point in } ar_i^v(k) & \chi_i(k) = 1 \text{ and } \xi_i(k) = 1 \end{cases} \tag{2}$$

where $\chi_i(k)$ is a binary variable equal to 1 or 0 with probability δ or $(1-\delta)$ determining whether the agent i is active at time step k, and $\xi_i(k)$ is a binary variable equal to 1 or 0 with probability γ or $(1-\gamma)$ modelling the probabilistic "reception" of the control signal by agent i at time step k.

Under the above proposed rule of motion, we have the following straightforward initial result:

***Lemma* 1.** *If all agents move inside their allowable regions, none of them will lose visibility with their neighbours.*

Proof. Consider a pair of neighbour agents i and j. Each one of these agents may jump into its own allowable region, which by (1) is contained in the $\sigma/2$-disk which centered in the direction of its pair, i.e.

$$ar_i(k) \subset D_{\frac{\sigma}{2}}\left(p_i(k) + \frac{\sigma}{2}U_{ij}(k)\right) \text{ and } ar_j(k) \subset D_{\frac{\sigma}{2}}\left(p_j(k) + \frac{\sigma}{2}U_{ji}(k)\right)$$

Since $\sigma \leq V/2$, we have that both disks $D_{\sigma/2}\left(p_i(k) + \frac{\sigma}{2}U_{ij}(k)\right)$ and $D_{\sigma/2}\left(p_j(k) + \frac{\sigma}{2}U_{ji}(k)\right)$ are contained in a disk of radius $V/2$ centered at the agents' average location, $(p_i(t) + p_j(t))/2$. Hence, the next possible location for agents i and j, inside their allowable regions, will again result in a less than V distance between them.

Since the allowable region of an agent i is the intersection between all the $\sigma/2$-disks associated with its neighbours

$$ar_i(k) = \bigcap_{j \in N_i(k)} D_{\frac{\sigma}{2}}(c_j i(k)) \supseteq ar_i^v(k) = \bigcap_{j \in N_i^v(k)} D_{\frac{\sigma}{2}}(c_j i(k))$$

even in a controlled situation, it will remain at time $(k+1)$ at a distance less than V from all its current neighbours. □

3 Gathering to a small region

Before dealing with the movement of the system under the broadcast control, we shall show that, even in a controlled scenario, a swarm of agents acting according to the dynamics given by (2) gathers to within a disk of radius equal to σ in a finite expected number of time steps. To do so, we first recall bring the essence of the gathering proof given in [4]. The proof states that a swarm of agents acting by motion law (2) gathers to a disk of diameter V, and then to a disk of radius σ, without a broadcast control (the case of $\gamma = 0$).

The gathering proof in [4] was done in two steps. Theorem 2 (in [4]) states that any constellation having a connected visibility graph reaches a complete visibility graph within a finite expected number of time-steps. Henceforth, all agents remain confined to a disk of diameter V. Then, Theorem 3 (in [4]) states that a constellation having a complete visibility graph further shrinks to within a disk of radius σ in an additional finite expected number of time-steps.

The outline of the proof of Theorem 2 is that at each time-step there is a strictly positive probability of $\delta(1-\delta)^{n-1}$ for the agent located at the sharpest corner of the constellation's convex-hull to be the only active agent. This agent, when jumping, has a probability bounded away from zero by a constant to reduce its distance from $\bar{p}(k)$, the centroid at time step k of the agents' constellation, by at least a positive constant quantity s^*. As a consequence of this jump, $\mathcal{L}(P(k))$, the sum of all agents' squared distances of from $\bar{p}(k)$, is reduced by at least ${s^*}^2/n$.

We always assume initial constellations with a connected visibility graph. Hence, by Lemma 1 the visibility graphs remain connected, and $\mathcal{L}(P(k))$ is bounded from above. Therefore, $\mathcal{L}(P(k))$ may reach a value of less than $V^2/4$ within a finite number of time steps with strictly positive (though very small!) probability. This happens when a long sequence of decreasing steps occurs. However, once $\mathcal{L}(P(k))$ reaches $V^2/4$, the visibility graph of the constellation is necessarily complete, and by Lemma 1, it remains complete, i.e. all the agents will henceforth be confined to a disc of diameter V.

The proof of Theorem 3 in [4], analyses the dynamics of the system considering the minimal enclosing circle of the agents positions, its radius and center being denoted by $R(k)$ and $C(k)$. In the proof we show that, any agent located at a distance greater than $\sigma/2$ from $C(k)$ can not increase its distance from $C(k)$, and if an agent is located at a distance smaller than or equal to $\sigma/2$ from $C(k)$, it can not jump to a distance greater than σ from $C(k)$. Therefore, if $R(k) > \sigma$, it cannot increase. Furthermore we show that, there are at least two agents located on the circumference of the minimal enclosing circle, or within a close proximity to it, and located at corners of the constellation's convex-hull with angles bounded below π by a constant as well. These agents have strictly positive probabilities, δ, to be **active**, and therefore will jump to locations closer to $C(k)$ with strictly positive probability. Hence, if $R(k) > \sigma$, the radius of the smallest enclosing circle drops significantly with a strictly positive probability within a batch of $\lceil n/2 \rceil$ time-steps. Once $R(k)$ reaches σ it cannot exceed it.

Notice that these proofs are based on agents being in special situations, occurring at some time steps. These agents have strictly positive probabilities to be **active** (δ), and in fact to be the only **active** agent at a time step ($\delta(1-\delta)^{n-1}$). If we add the broadcast control to the model ($\gamma > 0$), we still may use the same proofs assuming further that these agents, in the special situations, do not "hear" the broadcast. Therefore, the above mentioned probabilities should be updated to $\delta(1-\gamma)$ and $\delta(1-\delta)^{n-1}(1-\gamma)$ respectively, leaving them strictly positive, by the assumptions of the model we discuss in this paper, and, as a consequence, we have that our system still gathers within a finite expected number of time steps, even in the presence of broadcast control.

4 Random dynamics analysis

In this section we analyse the dynamics of a constellation of agents with complete visibility graph. We start with the dynamics of the swarm without the presence of a control broadcast.

In order to characterize the dynamics of the system, we rely on a Theorem from [1] on the convergence in probability of random variables that are sums of uniformly bounded random increments. Given a random variable X_k so that

$$X_k = \sum_{i=1}^{k} Y_i, \text{ i.e. } Y_k = X_k - X_{k-1}$$

If Y_k satisfies

$$\mathbb{E}\left[Y_k | \mathcal{F}_{k-1}\right] = 0$$

where $\mathcal{F}_{k-1}$ is the sigma filed generated all prior realization of the process and

$$\sum_{i=1}^{k} Var\{Y_i\} \overset{a.s.}{\to} \infty$$

(i.e. the variances sum to infinity with probability 1) then

$$\frac{X_{\tilde{k}}}{\sqrt{\nu}} \overset{p}{\to} N(0,1)$$

where $\tilde{k}$ is defined as

$$\tilde{k} = \min\{k : \sum_{k'=1}^{k} \mathbb{E}\{\|Y_{k'}\|^2\} > \nu\}$$

i.e. $\tilde{k}$ is a stopping time defined by ν.

We use the linearity of expectation to prove that once the gathering process shrank the constellation of agents to within a disk of radius $V/2$ the expected centroid will either remain in place when no broadcast is heard or it will drift in the direction of the control vector with a velocity equal to $\delta\gamma\sigma/(2n)$.

The above Theorem will further prove that the distribution of the swarm centroid location about the expected trajectory converges to a Gaussian with variance increasing linearly with k.

4.1 No broadcast control (or equivalently $\gamma = 0$)

We first prove that the expected centroid of the agents' constellation does not move in time (see also [4]).

Let us analyse the long term behaviour of the random variable vectors $\bar{p}(k) = 1/n \sum_{i=1}^{n} p_i(k)$ in time. Let $\Delta p_i(k) = p_i(k+1) - p_i(k)$ be the step of agent i at time step k. Then, the vector $\bar{p}(k+1)$ obeys

$$\bar{p}(k+1) = \frac{1}{n}\sum_{i=1}^{n} p_i(k+1) = \frac{1}{n}\sum_{i=1}^{n}(p_i(k) + \Delta p_i(k)) = \bar{p}(k) + \frac{1}{n}\sum_{i=1}^{n} \Delta p_i(k)$$

Therefore, we have to consider the sum of the jumps the agents make at each time-step.

Let $\bar{ar}_i(k)$ be the "mean" location of the current allowable region of agent i. If agent i is not located at a corner of the system's convex-hull, i.e. we have that $\psi_i(k) \geq \pi$, it cannot jump, hence $p_i(k+1) = p_i(k)$. Otherwise, $\psi_i(k)$ is equal to $\varphi_i(k) < \pi$, the inner-angle of the convex-hull corner defined by agent i. By (1), $\bar{ar}_i(k)$ is located at the center of $ar_i(k)$.

$$\bar{ar}_i(k) = \iint\limits_{v \in ar_i(k)} v \mathrm{d}v = p_i(k) + \frac{\sigma}{2} \cos(\frac{\varphi_i(k)}{2}) \hat{\psi}_i(k)$$

An agent i located at a corner of the convex-hull stays put with probability $1-\delta$ and jumps with probability δ to a uniformly distributed random point in $ar_i(k)$, therefore its expected position at the next time-step is

$$\begin{aligned}
&\mathbb{E}(p_i(k+1)) = p_i(k)(1-\delta) + \bar{ar}_i(k)\delta = \\
&p_i(k)(1-\delta) + \left(p_i(k) + \tfrac{\sigma}{2}\cos(\tfrac{\varphi_i(k)}{2})\hat{\psi}_i(k)\right)\delta = \\
&p_i(k) + \delta\tfrac{\sigma}{4}\left(U_i^-(k) + U_i^+(k)\right)
\end{aligned} \tag{3}$$

Assuming the indices of the agents on $\partial CH(P(k))$, the set of agents defining the convex-hull of the constellation $P(k)$, are ordered by the sequence of corners in the convex-hull. We have that the extremal left neighbour of agent $i \in \partial CH(P(k))$ is $i+1 \in \partial CH(P(k))$, and the extremal right neighbour of $i+1$ is i, i.e.

$$U_i^+(k) = -U_{i+1}^-(k)$$

Therefore, the expected position of the agent's centroid at the next time step coincides with its current position, since

$$\begin{aligned}
&\mathbb{E}(\bar{p}(k+1)) = \sum_i \mathbb{E}(p_i(k+1)) = \\
&\sum_i p_i(k) + \delta\tfrac{\sigma}{4} \sum_{i \in \partial CH(k)} \left(U_i^-(k) + U_i^+(k)\right) = \bar{p}(k) + 0
\end{aligned} \tag{4}$$

Hence by the linearity, the expected position of the constellation's centroid is stationary once the visibility graph is complete.

Next, we rely on the Theorem from [1] described above to prove that the constellation's centroid distribution converges in probability to a distribution with projections on the x and y axes (and in fact on any direction)of normal distributions. We refer the reader to [4] for a detailed proof.

4.2 Behavior of the swarm under broadcast control ($\gamma > 0$)

We next show that, in the presence of broadcast control, the centroid of a swarm of agents acting by motion law (2) moves in the desired direction with a constant expected velocity (expected displacement per time step).

Furthermore, we prove that the distribution of the system centroid at time step k, when projected on any arbitrary direction, convergences in probability to a distribution similar to that of a biased 1D random-walk variable.

Let $\hat{C}$ be the desired direction broadcast to the swarm. We next calculate the centroid of the allowable regions of the agents of the set $\partial CH(k)$ when the broadcast is received. We denote the centroid of the allowable region $ar_i^v(k)$ by $\bar{a}r_i^v(k)$.

Let $\hat{C}^{\perp}$ be a unit vector orthogonal to $\hat{C}$. Let S^u/S^d be the set of agents located at positions with the maximal/minimal projection on $\hat{C}^{\perp}$, and let u/d be an agent of the set S^u/S^d located at a position with the minimal projection on $\hat{C}$, i.e.

$$u = arg \min_{i \in S^u} \left\{ p_i^{\intercal}(k)\hat{C} \right\}, \text{ where } S^u = arg \max_i \left\{ p_i^{\intercal}(k)\hat{C}^{\perp} \right\}$$

$$d = arg \min_{i \in S^d} \left\{ p_i^{\intercal}(k)\hat{C} \right\}, \text{ where } S^d = arg \min_i \left\{ p_i^{\intercal}(k)\hat{C}^{\perp} \right\}$$

Furthermore, let the agents from the right and left sides of the segment $[p_d(k), p_u(k)]$ be the sets R and L, see Figure 2.

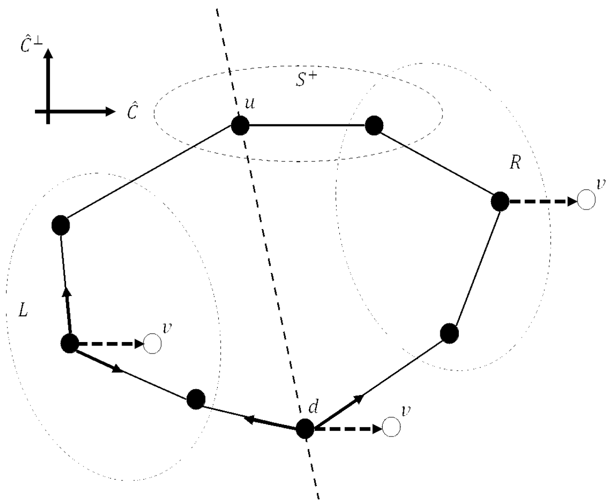

Figure 2: The agents of the convex-hull divided into subsets based on the influence they may have when receiving the control signal. The allowable regions of agents u and d may be influenced by adding to their neighbour sets a virtual agents in the direction $\hat{C}$. The allowable regions of the agents in the set L is not affected, and while those of the agents in the set R vanish.

The allowable region of an agent i hearing the broadcast is the intersection of all the disks $D_{\sigma/2}(p_i + \frac{\sigma}{2}U_i^j(k))$, where $j \in N_i^v(k)$ (including the virtual agent v), which is equal to the intersection between the two disks associated with the extremal agents (in case $\psi_i(k) < \pi$). Hence, the allowable regions of the agents of the set L are not affected by the virtual agent v, and the allowable regions of the agents of the set R vanish because of it. Furthermore, the virtual agent is the extremal right agent of u, and extremal left agent of d.

We have already seen in (3) that the centroid of a non-surrounded agent's allowable region is located at the current position of the agent plus the sum of the two unit vectors pointing to its extremal neighbours multiplied by $\sigma/4$. Hence, assuming that an agent $i \in \partial CH(k)$ "hears" the signal, we have that the centroid of its allowable region is as follows:

$$\bar{ar}_i^v(k) = p_i(k) + \frac{\sigma}{4} \begin{cases} U_i^-(k) + U_i^+(k) & i \in L \\ 0 & i \in R \\ U_i^-(k) + \hat{C} & i = d \\ \hat{C} + U_i^+(k) & i = u \end{cases}$$

Thereby, its next expected position is

$$\mathbb{E}\{p_i(k+1)\} =$$

$$p_i(k) + \tfrac{\delta\sigma}{4}(1-\gamma)\left(U_i^-(k) + U_i^+(k)\right) + \tfrac{\delta\sigma}{4}\gamma \begin{cases} U_i^-(k) + U_i^+(k) & i \in L \\ 0 & i \in R \\ U_i^-(k) + \hat{C} & i = d \\ \hat{C} + U_i^+(k) & i = u \end{cases} \tag{5}$$

Using (5), we can calculate the next expected centroid of the agents' constellation for any time step k, as follows:

$$\mathbb{E}\{\bar{p}(k+1)\} = \frac{1}{n}\sum_i \mathbb{E}\{p_i(k+1)\} =$$

$$\bar{p}(k) + \frac{\delta\sigma}{4n}\gamma \left(\sum_{i\in L} \left(U_i^-(k) + U_i^+(k)\right) + U_u^-(k) + U_d^+(k) + 2\hat{C} \right) =$$

$$\bar{p}(k) + \frac{\delta\sigma}{4n}\gamma \left(\sum_{u\le i<d} U_i^-(k) + \sum_{u<i\le d} U_i^+(k) + 2\hat{C} \right)$$

Notice that we assume that the agents located at the convex-hull corners are marked by successive indices in a counter-clockwise increasing order, and therefore $u < d$. Using the fact that a pair of agents i and $i+1$ located at consequent corners of the convex-hull are the extremal right and left neighbours of each other, we here too have,

$$U_i^+(k) = -U_{i+1}^-(k)$$

Hence,

$$\mathbb{E}\{\bar{p}(k+1)\} = \bar{p}(k) + \frac{\delta\sigma}{4n}\gamma \left(\sum_{u\le i<d} U_i^-(k) - \sum_{u\le i<d} U_i^-(k) + 2\hat{C} \right) = \bar{p}(k) + \frac{\delta\gamma\sigma}{2n}\hat{C} \tag{6}$$

This prove that, a constellation of agents with a complete visibility graph moves with an expected velocity of $\frac{\delta\gamma\sigma}{2n}$ in the direction of $\hat{C}$. We verified this result in multiple simulations, the results being displayed in Figure 3.

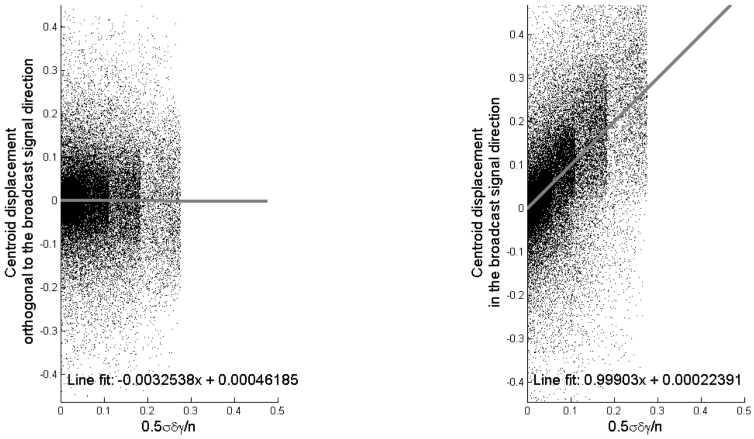

Figure 3: The displacement of a swarm centroid vs $\frac{\delta\gamma\sigma}{2n}$. Results of 60000 samples extracted from multiple simulations with various number of agents and values of *gamma* and δ. The results are compatible with the theoretic expected velocities $\delta\gamma/(2n)$.

Let $\Delta\bar{p}(k)$, be the constellation centroid displacement at time step k, i.e $\Delta\bar{p}(k) = \bar{p}(k+1) - \bar{p}(k)$. Let S_k and X_k be the projections of the distributions of $\bar{p}(k) - k\frac{\delta\gamma\sigma}{2n}\hat{C}$ and $\Delta\bar{p}(k) - \frac{\delta\gamma\sigma}{2n}\hat{C}$ on a unit vector with an arbitrary direction U. Then, by (6), we have that $\mathbb{E}\{\bar{X}_{k+1}|X_k\} = 0$.

Clearly, the increments X_k are uniformly bounded by $n\sigma$. We next prove that, the sum of their variances tends to infinity with probability 1. Denote the distribution of the step of an agent i at time step k by $\Delta p_i(k) = p_i(k+1) - p_i(k)$, and let $Var(A)$ be the variance of a random variable A. Then, due to the fact that any pair of random variables $\Delta p_i(k)$ and $\Delta p_j(k)$ are conditionally independent for $i \neq j$, we have that

$$Var(X_k|P(k)) = Var(U^\intercal \Delta\bar{p}(k)|P(k)) =$$

$$Var(\frac{1}{n}U^\intercal \sum_i \Delta p_i(k) \geq \frac{1}{n^2} Var(U^\intercal \Delta p_s(k)|P(k))$$

where $P(k) = \{p_1(k), p_2(k), ..., p_1(k)\}$, and s is the agent located at the sharpest corner occupied by an agent from the set $\{L, u, d\}$.

The minimal value $Var(U^\intercal \Delta p_s(k))$ can assume is for a unit vector U orthogonal to $\hat{\psi}_s(k)$, i.e $U^\intercal \hat{\psi}_s(k) = 0$. Then, we have that:

$$Var(U^\intercal \Delta p_s(k)) \geq 2\delta^2 \frac{\int\limits_{\frac{\sigma}{2}\sin\left(\frac{\psi_s(k)}{2}\right)}^{\frac{\sigma}{2}} (x - \frac{\sigma}{2}\sin\left(\frac{\psi_s(k)}{2}\right))^2 \sqrt{\left(\frac{\sigma}{2}\right)^2 - x^2}dx}{\left(\frac{\sigma}{2}\right)^2 \left(\frac{\pi - \psi_s(k)}{2} + \frac{sin(\psi_s(k))}{2}\right)} =$$

$$-2\delta^2 \frac{\int_0^{\frac{\pi-\psi_s(k)}{2}} \left(\frac{\sigma}{2}\right)^4 \cos^3(\theta)\sin(\theta)d\theta}{\left(\frac{\sigma}{2}\right)^2 \left(\frac{\pi-\psi_s(k)}{2} + \frac{sin(\psi_s(k))}{2}\right)} =$$

$$\delta^2 \left(\frac{\sigma}{2}\right)^2 \frac{1-\cos^4\left(\frac{\pi-\psi_s(k)}{2}\right)}{\frac{\pi-\psi_s(k)}{2} - \frac{1}{2}\sin(\pi-\psi_s(k))}$$

Recall that u and d are the agents with the maximal and minimal projections on $\hat{C}^{\perp}$, and L is the set of agents on the convex-hull from the left side of the line crossing through the positions of agents u and d. Denote the virtual agents of u and d by v_u and v_d. Then considering the convex-hull of positions of the set $\{L, u, d, v_u, v_d\}$, we have that the sum of its inner angles is $\pi(m+2-2)$, assuming the cardinality of this set is $m+2$. Furthermore, the sum of inner-angles of the convex hull corners without those associate with v_u and v_d is $\pi(m+2-2)-\pi$, and their average value is $\pi(m-1)/m = \pi(1-1/m)$. Let n be the number of real agents in the system, then we have that $\pi(1-1/m) \leq \pi(1-1/n) = \varphi_*$. Therefore, $\psi_s(k)$, the angle of the sharpest corner of that convex-hull, without referring the corners associated with the virtual agents, is, clearly, upper bounded by φ_*.

Hence, we have that

$$Var(X_k) \geq Var(U^{\intercal}\Delta p_s(k)) \geq \delta^2 \left(\frac{\sigma}{2}\right)^2 \frac{1-\cos^4\left(\frac{\pi-\varphi_*}{2}\right)}{\frac{\pi-\varphi_*}{2} - \frac{1}{2}\sin(\pi-\varphi_*)} \triangleq Var^*$$

and as a consequence

$$\sum_{k=1}^{\infty} Var\{X_k\} \to \infty$$

Then, from Theorem 35.11 in [1], we have that

$$\frac{S_{\tilde{k}}}{\sqrt{\nu}} \xrightarrow{p} N(0,1)$$

where $\tilde{k}$ define as the stopping time as ν goes to infinity

$$\tilde{k} = \min\{t : \sum_{k=1}^{t} Var(X_k) > \nu\}$$

Recall that all the variables X_k are uniformly bounded by $n\sigma$. Hence, their variances are bounded by $(n\sigma)^2$, and we may assume that the mean value of the increments X_ks' variances converges to a finite constant value η^2, i.e

$$\frac{1}{k}\sum_k Var(X_k) \to \eta^2$$

Then, we have that

$$\frac{S_\nu}{\sqrt{\nu}\eta} \xrightarrow{p} N(0,1)$$

i.e the distribution of S_n converges in probability to the distribution of a random-walk with steps of the size η, so that the projection of the random vector $\bar{p}(k) - \frac{\delta\gamma\sigma}{2n}$ on an arbitrary (constant) direction U converges to a normal distribution with variance $k\eta^2$.

We ran multiple simulations, and used the results to estimate η. We present Gaussian fits of the swarms' average positions at time step 1000 in Figure 4. Interestingly, the average random step size, η, is best fit to the following model:

$$\eta \propto \frac{1}{n(1+\gamma)}$$

Note that in [4], we have seen that without the control η is inversely dependent on the numbers of agents. So that in the controlled case the centroid of the swarm behave is as if each agent adds its own virtual agent to the constellation. This result is analysed in Figure 5.

5 Discussion

We showed that a flock/swarm of identical, anonymous and oblivious agents having limited visibility and bearing only sensing with an initial constellation having connected visibility graph, can be directed by exogenous control to move in desired directions. This may be achieved via a simple broadcast control mechanism. Furthermore, we showed that the random constellation's centroid move in the desired direction with a speed determined by the probability that agents "hear" the broadcast control. In fact, centroid of the flock performs a motion that is similar in probability to a biased random walk.

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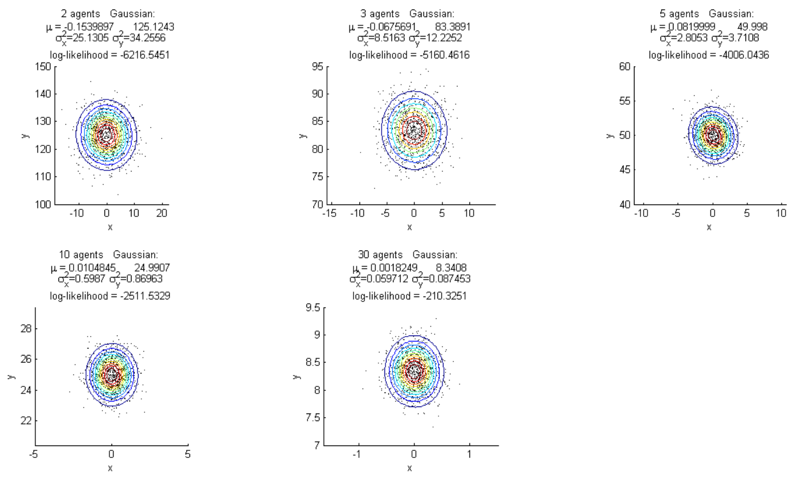

Figure 4: dynamics under broadcast control. Gaussian fit for the average position of the systems' constellations at time-step 1000. This results are for 1000 simulations with random initial constellations of 2,3,5,10,30 agents.

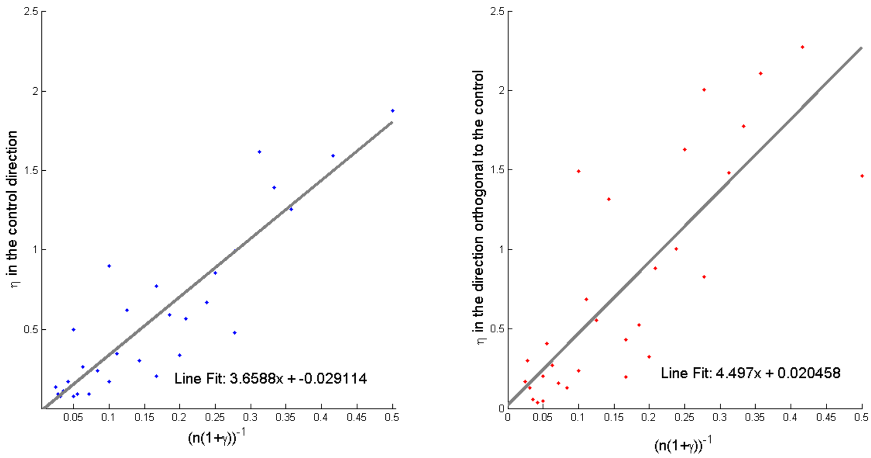

Figure 5: Linear fit of η, the quantity considered as the step size of the drifted random-walk under broadcast control, vs $\frac{1}{n(1+\gamma)}$. Each dot on the graph is the η resulted from distributions of 1000 simulations with random initial constellations. The distributions are for all the following numbers of agents $n = (2, 3, 5, 10, 20)$, with the following broadcast receiving probabilities $\gamma = (0.2, 0.4, 0.6, 0.8, 1)$.