Complex Barycentric Coordinates with Applications to Planar Shape Deformation

Ofir Weber, Mirela Ben-Chen and Craig Gotsman

Technion – Israel Institute of Technology

This document contains the proofs and other derivations that were omitted from the paper.

Complex barycentric coordinates – similarity reproduction

Theorem 1: Complex barycentric coordinates $k_j(z)$ reproduce similarity transformations, i.e:

$$\sum_{j=1}^{n} k_j(z) T(z_j) = T(z)$$

where T is a 2D similarity transformation.

Proof:

Similarity transformations can be represented using a linear polynomial over the complex plane in the following way. If the similarity transformation T consists of rotation by positive angle θ, uniform scale s and a translation $t = t_x + it_y$, then according to the rules of complex numbers:

$$T(x+iy) = T(z) = se^{i\theta} z + t = \alpha z + \beta$$

Where α and β are complex numbers. Since complex barycentric coordinates reproduce linear and constant functions by definition, we have:

$$\sum_{j=1}^{n} k_j(z) T(z_j) = \sum_{j=1}^{n} k_j(z)(\alpha z_j + \beta) = \alpha \sum_{j=1}^{n} k_j(z) z_j + \beta \sum_{j=1}^{n} k_j(z) = \alpha z + \beta = T(z)$$

Complex barycentric coordinates – affine reproduction

Theorem 2: Complex barycentric coordinates $k_j(z)$ reproduce affine transformations of z_j which contain non-uniform scale if and only if the complex conjugates of the coordinates $k_j(z)$ also have linear precision, meaning:

$$\sum_{j=1}^{n} \overline{k}_j(z) z_j = z \qquad (1)$$

Proof:

Let T be a 2D transformation which scales the x and y axes non-uniformly: $T = \begin{pmatrix} 2\alpha & 0 \\ 0 & 2\beta \end{pmatrix}$, $\alpha \neq \beta$, $\alpha,\beta \in R$

When applied to complex numbers, T can be described as:

$$T(z) = T(x+iy) = 2\alpha x + i2\beta y = \alpha(z+\overline{z}) + \beta(z-\overline{z})$$

To show that the transform reproduces affine transformations we need to show:

$$\sum_{j=1}^{n} k_j(z)\left(T(z_j)\right) = T(z)$$

$$\sum_{j=1}^{n} k_j(z)\left(\alpha(z_j+\overline{z}_j) + \beta(z_j-\overline{z}_j)\right) = T(z)$$

$$(\alpha+\beta)\sum_{j=1}^{n} k_j(z) z_j + (\alpha-\beta)\sum_{j=1}^{n} k_j(z)\overline{z}_j = \alpha(z+\overline{z}) + \beta(z-\overline{z})$$

$$(\alpha+\beta) z + (\alpha-\beta)\sum_{j=1}^{n} k_j(z)\overline{z}_j = (\alpha+\beta) z + (\alpha-\beta)\overline{z}$$

Since $\alpha \neq \beta$ we have:

$$\sum_{j=1}^{n} k_j(z)\overline{z}_j = \overline{z}$$

and conjugating both sides gives:

$$\sum_{j=1}^{n} \overline{k_j}(z) z_j = z$$

Hence, complex barycentric coordinates reproduce non-uniform scale if and only if (1) holds. Any complex barycentric coordinates reproduce similarity transformations by Theorem 1. Those two facts together imply that complex barycentric coordinates reproduce affine transformations if and only if (1) is satisfied.

Complex three point coordinates

Theorem 5: Any set of complex functions $k_j(z)$ which satisfy

$$\sum_{j=1}^{n} k_j(z)(z_j - z) = 0$$

can be represented in the form:

$$\sum_{j=1}^{n} \left(m_j(z) \frac{B_{j+1}(z)}{A_{j+1}} - m_{j-1}(z) \frac{B_{j-1}(z)}{A_j} \right) (z_j - z) = 0$$

where $m_j(z)$ are arbitrary complex functions over Ω.

Proof:

Let m_1 be an arbitrary complex function: m_1: Ω → C, and define m_j , $j = 2..n$, recursively:

$$m_j(z) = \frac{A_{j+1} A_j k_j(z) + m_{j-1} B_{j-1}(z) A_{j+1}}{B_{j+1}(z) A_j}$$

Define:

$$\hat{k}_j(z) = m_j(z) \frac{B_{j+1}(z)}{A_{j+1}} - m_{j-1}(z) \frac{B_{j-1}(z)}{A_j}$$

We claim that $\hat{k}_j(z) = k_j(z)$. This holds by construction for $j = 2..n$. Let us show that it also holds for $j = 1$.

From Theorem 4 (in the paper) we have:

$$\sum_{j=1}^{n} \hat{k}_j(z)(z_j - z) = 0$$

hence:

$$\hat{w}_1(z)(z_1 - z) = \sum_{j=2}^{n} \hat{w}_j(z)(z_j - z) = \sum_{j=2}^{n} w_j(z)(z_j - z) = w_1(z)(z_1 - z)$$

Since $z_1 - z \neq 0$, we have that $\hat{w}_1(z) = w_1(z)$, which concludes the proof.

Limits of discrete Cauchy transform on the boundary

Theorem A1: The limits of the discrete Cauchy transform $C_j(w)$ have constant and linear precision, meaning: for all $w \in S$, the following holds:

(a) $\sum_{j=1}^{n} C_j(w) = 1$

(b) $\sum_{j=1}^{n} C_j(w) z_j = w$

Proof:

(a) By definition we have:

$$\sum_{j=1}^{n} C_j(w) \triangleq \sum_{j=1}^{n} \left(\lim_{w^{in} \to w} C_j(w^{in}) \right)$$

Using the rules of limits, we get:

$$\sum_{j=1}^{n} \left(\lim_{w^{in} \to w} C_j(w^{in}) \right) = \lim_{w^{in} \to w} \sum_{j=1}^{n} C_j(w^{in}) = \lim_{w^{in} \to w} 1 = 1$$

which concludes the proof of (a).

(b) Again, by definition:

$$\sum_{j=1}^{n} C_j(w) z_j \triangleq \sum_{j=1}^{n} \left(\lim_{w^{in} \to w} C_j(w^{in}) \right) z_j$$

Using the rules of limits, we get:

$$\sum_{j=1}^{n} \left(\lim_{w^{in} \to w} C_j(w^{in}) \right) z_j = \lim_{w^{in} \to w} \sum_{j=1}^{n} C_j(w^{in}) z_j = \lim_{w^{in} \to w} w^{in} = w$$

which concludes the proof of (b).

Constant and linear precision of discrete Szego coordinates

Theorem A2: The discrete Szego coordinates $G_j(w)$ have constant and linear precision, meaning: for all $w \in \Omega$ the following holds:

(a) $\sum_{j=1}^{n} G_j(w) = 1$

(b) $\sum_{j=1}^{n} G_j(w) z_j = w$

Proof:

Using the definition of the Szego coordinates:

$$\sum_{j=1}^{n} G_j(w) = \sum_{j=1}^{n}\sum_{k=1}^{n} C_k(w) M_{k,j} = \sum_{k=1}^{n} C_k(w)\left(\sum_{j=1}^{n} M_{k,j}\right) = \sum_{k=1}^{n} C_k(w)\left(MI_{nx1}\right)_k \quad (2)$$

where I is a column vector of ones.

Since H is a sampling matrix over the polygon, each of its rows contains t and 1-t for some t. Hence, the rows of H sum to unity, and we have:

$$H_{kxn} I_{nx1} = I_{kx1} \quad (3)$$

Since C_b have constant precision according to Theorem A1:

$$CI_{nx1} = I_{kx1}$$

Multiplying by the pseudo-inverse on both sides:

$$I_{nx1} = \left(C^* C\right)^{-1} C^* I_{kx1} \quad (4)$$

Plugging in the expression for M, and (3) and (4):

$$MI_{nx1} = (C^* C)^{-1} C^* H I_{nx1} = (C^* C)^{-1} C^* I_{kx1} = I_{nx1}$$

And back to (2):

$$\sum_{j=1}^{n} G_j(w) = \sum_{k=1}^{n} C_k(w)\left(MI_{nx1}\right) = \sum_{k=1}^{n} C_k(w) = 1$$

This completes the proof of (a).

Moving to (b), we have:

$$\sum_{j=1}^{n} G_j(w) z_j = \sum_{j=1}^{n}\sum_{k=1}^{n} C_k(w) M_{k,j} z_j = \sum_{k=1}^{n} C_k(w) \sum_{j=1}^{n} M_{k,j} z_j = \sum_{k=1}^{n} C_k(w)\left(Mz\right)_k \quad (5)$$

where z is the complex vector $z = (z_1, z_2, \ldots, z_n)$.

Since C is reproducing according to Theorem A1, i.e.:

$$\sum_{j=1}^{n} C_j(w) z_j = w$$

for $w \in S$, we have that:

$$Cz = Hz$$

Multiplying both sides by the pseudo-inverse of C we get:

$$z = \left(C^* C\right)^{-1} C^* H z$$

Plugging in the definition of M we have: $z = Mz$. Note that this means that z is an eigenvector of M.

Plugging this back into (5) gives:

$$\sum_{j=1}^{n} G_j(w) z_j = \sum_{k=1}^{n} C_k(w)\left(Mz\right)_k = \sum_{k=1}^{n} C_k(w) z_k = w$$

which completes the proof of (b).

<u>Second derivatives of the Cauchy transform on the boundary</u>

The second derivatives of the discrete Cauchy transform are:

$$g''(z) = \sum_{j=1}^{n} w_j(z) z_j$$

$$w_j(z) = \frac{1}{2\pi i}\left(\frac{1}{B_{j-1}(z) B_j(z)} - \frac{1}{B_j(z) B_{j+1}(z)}\right)$$

Since the logarithm function has been eliminated, the derivatives are no longer multi-valued, and except at the vertices, the second derivative is well-defined, hence, for all $z \in S$, $z \neq \{z_1, z_2, \ldots, z_n\}$, the following holds:

$$w_j(z) = \lim_{z^{in} \to z} w_j(z^{in}) = w_j(z)$$

Theorem A3: The second derivatives of the Cauchy transform satisfy: for all $z \in S$, such that $m \notin \{z_1, z_2, \ldots, z_n\}$, the following holds:

(a) $\sum_{j=1}^{n} w_j(z) = 0$

(b) $\sum_{j=1}^{n} w_j(z) z_j = 0$

Proof:

(a) From the definition of w_j we have:

$$\sum_{j=1}^{n} w_j(z) = \frac{1}{2\pi i}\sum_{j=1}^{n}\left(\frac{1}{B_{j-1}(z)B_j(z)} - \frac{1}{B_j(z)B_{j+1}(z)}\right)$$

Splitting into two sums, and changing the summation index gives:

$$\sum_{j=1}^{n} w_j(z) = \frac{1}{2\pi i}\left(\sum_{j=1}^{n}\frac{1}{B_{j-1}(z)B_j(z)} - \sum_{j=1}^{n}\frac{1}{B_j(z)B_{j+1}(z)}\right) =$$

$$= \frac{1}{2\pi i}\left(\sum_{j=1}^{n}\frac{1}{B_{j-1}(z)B_j(z)} - \sum_{j=1}^{n}\frac{1}{B_{j-1}(z)B_j(z)}\right) = 0$$

(b) Again, from the definition of w_j:

$$\sum_{j=1}^{n} w_j(z)z_j = \frac{1}{2\pi i}\sum_{j=1}^{n}\left(\frac{1}{B_{j-1}(z)B_j(z)} - \frac{1}{B_j(z)B_{j+1}(z)}\right)z_j =$$

$$= \frac{1}{2\pi i}\sum_{j=1}^{n}\left(\frac{z_j}{B_{j-1}(z)B_j(z)} - \frac{z_{j-1}}{B_{j-1}(z)B_j(z)}\right) =$$

By definition: $B_j = z_j - z$, hence:

$$z_j - z_{j-1} = z_j - z - \left(z_{j-1} - z\right) = B_j - B_{j-1}$$

Plugging this back in the previous expression:

$$\sum_{j=1}^{n} w_j(z)z_j = \frac{1}{2\pi i}\sum_{j=1}^{n}\left(\frac{B_j(z) - B_{j-1}(z)}{B_{j-1}(z)B_j(z)}\right) = \frac{1}{2\pi i}\sum_{j=1}^{n}\left(\frac{1}{B_{j-1}(z)} - \frac{1}{B_j(z)}\right) =$$

$$= \frac{1}{2\pi i}\left(\sum_{j=1}^{n}\frac{1}{B_{j-1}(z)} - \sum_{j=1}^{n}\frac{1}{B_j(z)}\right) = 0$$

which completes the proof.

Constant and linear precision of Point-2-Point Cauchy coordinates

Theorem A4: The point-to-point Cauchy coordinates, with positional constraints $f(w_k) = f_k$ have constant and linear precision. Meaning: for all $m \in \Omega$ the following holds:

(a) $\sum_{j=1}^{p} D_j(m) = 1$

(b) $\sum_{j=1}^{p} D_j(m)w_j = m$

Proof:

Going back to the definition of the point-to-point Cauchy coordinates:

$$\sum_{j=1}^{p} D_j(m) = \sum_{j=1}^{p}\sum_{k=1}^{n} C_k(m)N_{k,j} = \sum_{k=1}^{n} C_k(m)\sum_{j=1}^{p} N_{k,j} = \sum_{k=1}^{n} C_k(m)\left(NI_{px1}\right)_k$$

$$\sum_{j=1}^{p} D_j(m)w_j = \sum_{j=1}^{p}\left(\sum_{k=1}^{n} C_k(m)N_{k,j}\right)w_j = \sum_{k=1}^{n} C_k(m)\sum_{j=1}^{p} N_{k,j}w_j = \sum_{k=1}^{n} C_k(m)\left(Nw\right)_k$$

As was the case for the Szego coordinates, to prove constant and linear precision, it is enough to show that:

$$NI_{px1} = I_{nx1}$$

$$Nw = z$$

Let C be the matrix, whose j-th column is $C_j(w_k)$, where $w_1, w_2, \ldots, w_p \in \Omega$ are the point constraints, and let W be the matrix whose j-th column is $W_j(m)$, where $m=Hz$ are the samples of the polygon. Since the discrete Cauchy coordinates are constant reproducing, we have:

$$CI_{nx1} = I_{px1}$$

In addition, as we showed in Theorem A3:

$$WI_{nx1} = 0_{kx1}$$

Where 0_{kx1} is a column vector of k zeroes, where k is the number of samples on the boundary.

Hence, we have:

$$AI_{nx1} = \begin{pmatrix} C \\ \lambda W \end{pmatrix} I_{nx1} = \begin{pmatrix} I_{px1} \\ 0_{kx1} \end{pmatrix}$$

Multiplying both sides by the pseudo-inverse of A, we have:

$$I_{nx1} = A^{+}\begin{pmatrix} I_{px1} \\ 0_{kx1} \end{pmatrix}$$

Since N is defined to be the first p columns of A^{+}, we get: $I_{nx1} = NI_{px1}$, as required.

In a similar fashion, since C is linear reproducing, combined with Theorem A3, we have:

$$Az = \begin{pmatrix} C \\ \lambda W \end{pmatrix} z = \begin{pmatrix} w_{px1} \\ 0_{kx1} \end{pmatrix}$$

Again, multiplying both sides by the pseudo-inverse of A, and using the definition of N we get:

$$z = A^{+} \begin{pmatrix} w_{px1} \\ 0_{kx1} \end{pmatrix} = Nw$$

which concludes the proof.

The MLS complex barycentric coordinates

When inspecting the MLS expression for the "as-similar-as-possible" deformation, it is relatively straight-forward to see that all the expressions can be replaced by their complex representations. Let $p_i \in \Omega$ be the positions of the constrained points, and $q_i \in \mathbb{C}$ their target position. Let the following expressions be defined as in the MLS [SMW06] paper:

$$w_i(z) = \frac{1}{|p_i - z|^{2\alpha}} \quad , \quad w^*(z) = \sum_i w_i(z) \quad , \quad p^*(z) = \frac{1}{w^*(z)} \sum_i w_i(z) p_i \quad , \quad \hat{p}_i(z) = p_i - p^*(z)$$

$$, \quad q^*(z) = \frac{1}{w^*(z)} \sum_i w_i(z) q_i \quad , \quad \hat{q}_i(z) = q_i - q^*(z)$$

In addition, let:

$$\mu(z) = \sum_i w_i(z) \hat{p}_i(z) \overline{\hat{p}_i(z)} \quad , \quad A_i(z) = w_i(z) \hat{p}_i \overline{(z - p^*(z))}$$

where $\overline{z}$ is the conjugate of z.

The MLS deformation is defined as:

$$f_{mls}(z) = \sum_i \hat{q}_i(z) \left(\frac{1}{\mu(z)} \overline{A_i(z)} \right) + q^*(z)$$

Plugging in our expressions:

$$f_{mls}(z) = \sum_i \left(q_i - \frac{1}{w^*(z)} \sum_j w_j(z) q_j \right) \left(\frac{1}{\mu(z)} \overline{A_i(z)} \right) + \frac{1}{w^*(z)} \sum w_i(z) q_i$$

Rearranging, to obtain the coefficients of q_i:

$$f_{mls}(z) = \sum_i \left(q_i - \frac{1}{w^*(z)} \sum_j w_j(z) q_j \right) \frac{\overline{A_i(z)}}{\mu(z)} + \frac{1}{w^*(z)} \sum w_i(z) q_i =$$

$$= \sum_i q_i \frac{\overline{A_i(z)}}{\mu(z)} - \sum_i \left(\frac{1}{w^*(z)} \sum_j w_j(z) q_j \right) \frac{\overline{A_i(z)}}{\mu(z)} + \frac{1}{w^*(z)} \sum w_i(z) q_i =$$

$$= \sum_i q_i \left(\frac{\overline{A_i(z)}}{\mu(z)} + \frac{w_i(z)}{w^*(z)} \right) - \frac{1}{w^*(z)} \sum_j w_j(z) q_j \sum_i \frac{\overline{A_i(z)}}{\mu(z)}$$

Changing the summation indices, and rearranging again:

$$f_{mls}(z) = \sum_i q_i \left(\frac{\overline{A_i(z)}}{\mu(z)} + \frac{w_i(z)}{w^*(z)} \right) - \frac{1}{w^*(z)} \sum_i w_i(z) q_i \sum_j \frac{\overline{A_j(z)}}{\mu(z)} =$$

$$= \sum_i q_i \left(\frac{\overline{A_i(z)}}{\mu(z)} + \frac{w_i(z)}{w^*(z)} - \frac{w_i(z)}{w^*(z)} \sum_j \frac{\overline{A_j(z)}}{\mu(z)} \right)$$

Finally:

$$f_{mls}(z) = \sum_i M_i(z) q_i$$

$$M_i(z) = \frac{w_i(z)}{w^*(z)} \left(1 - \sum_j \frac{\overline{A_j(z)}}{\mu(z)} \right) + \frac{\overline{A_i(z)}}{\mu(z)}$$