

# Hamiltonian(t) - An Ant-Inspired Heuristic for Recognizing Hamiltonian Graphs

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**Abstract - Given a graph  $G(V, E)$ , we consider the problem of deciding whether  $G$  is Hamiltonian, that is - whether or not there is a simple cycle in  $E$  spanning all vertices in  $V$ . This problem is known to be NP-complete, hence cannot be solved in time polynomial in  $|V|$  unless  $P = NP$ . The problem is a special case of the Travelling Salesperson Problem (TSP), that was extensively studied in the literature, and has recently been attacked by various ant-colony methods. We address the Hamiltonian cycle problem using a new ant-inspired approach, based on repeated covering of the graph. Our method is based on a process in which an ant traverses the graph by moving from vertex to vertex along the edges while leaving traces in the vertices, and deciding on the next step according to the level of traces in the surrounding neighborhood. We show that Hamiltonian cycles are limit cycles of the process, and investigate the average time needed by our ant process to recognize a Hamiltonian graph, on the basis of simulations made over large samples of random graphs with varying density of edges.**

## 1 Introduction

We consider a decision problem in which a graph  $G(V, E)$  over  $n$  vertices is given, and our goal is to decide whether  $G$  is a Hamiltonian graph. That is, we look for an  $n$ -long sequence of vertices  $v_0, v_1, \dots, v_{n-1}$  that visits all vertices in  $V$  such that for all  $0 \leq i < n$ ,  $(v_i, v_{(i+1) \bmod n}) \in E$ . This problem is known to be *NP-complete* [Garey & Johnson 1979], which means that it cannot be solved in time polynomial in  $n$  unless  $P = NP$ . The problem may be considered a special case of the *Travelling Salesperson (TSP) Problem*, which

is to find the shortest path visiting all nodes in a graph, given the distances between all pairs of vertices. Setting the  $(x, y)$  distance to either 1 (if  $(x, y) \in E$ ) or  $\infty$  (otherwise), a Hamiltonian cycle exists if and only if there is an  $n$ -long tour for the salesman that covers all vertices and returns to the starting point. The TSP problem was extensively studied in the literature, and it has recently been attacked by ant-colony methods (e.g. [Dorigo, Maniezzo & Colomi 1996]), achieving a competitive heuristic by combining *local* pheromone-oriented observations with the *global* accumulated experience gained by the whole group of ants. The usage of global information was shown to speed up the process; however it has a cost in communication and complexity of the individual agent.

In the current work we address the Hamiltonian cycle problem using a different ant-inspired approach, based on a repeated covering of the graph, using only *local* information for deciding on the next step. Our method is based on a process in which an ant traverses the graph by moving from vertex to vertex along the edges while leaving traces in the vertices, and deciding on the next step according to the level of traces in the surrounding vertices. This process is shown to have Hamiltonian cycles as limit cycles, that is - once a Hamiltonian cycle has been reached, the ant repeats it forever. We also investigate the average time needed by our ant process to recognize a Hamiltonian graph, by experimenting over a large sample of random Hamiltonian graphs with varying density of edges. A conclusion from these experiments is that for almost all values of edge densities, a Hamiltonian graph is recognized within  $n^3$  steps of the **VAW** process. There is only a rather narrow range of densities (less than 0.03 for  $n = 300$ ) in which the **VAW** does

not recognize a Hamiltonian cycle within the given time. A related graph-theoretical result is a theorem by Posa (see e.g. [Bollobas 1979, pp. 139-142]) saying that an edge density  $\frac{1}{2}n \log n$  is a threshold for Hamiltonicity - namely that almost all graphs with more edges are Hamiltonian, and almost all graphs with less edges are not. Other graph properties (e.g. connectivity) are also known to have similar threshold phenomena. Phase transition across the threshold was reported for various combinatorial problems, and, in particular, for the Hamiltonian cycle problem [Cheesman, Kanefsky & Taylor 1991] [Frank, Gent & Walsh 1998] and for the Euclidean TSP [Gent & Walsh]. However the graphs in those works were relatively small (up to 30 vertices) and their results were concerned in the *existence* of a solution, rather than in an *efficient* way to achieve it, while the algorithm suggested here seems to enable efficiently finding Hamiltonian cycles in large graphs (300 vertices and more) for most values of edge-density. Also, previous works mainly discussed the phase transition that occurs between the middle range, where the problem is hard, to the denser range where the problem becomes easy, while we attempt to also consider the transition between the lower range of edge densities to the middle range. An efficient algorithm for the upper range was presented and proved in [Gurevich & Shelah 1987], which achieves expected time  $O(n/p)$  where  $p$  is the probability of a pair of vertices to be connected by an edge. A different approach was taken in [Broder, Frieze & Shamir 1991], where an algorithm is shown that takes  $O(n^3 \log n)$  time and converges almost surely if  $p > c/n$  where  $c$  is some constant.

Several methods for graph exploration using traces were reported in the literature (e.g. [Even 1979], [Korf 1990], [Koenig & Smirnov 1996], [Tarjan 1972] [Tarry 1895], [Thrun 1992], [Thrun 1992a]), but our method is the first, to our knowledge, that uses a trace-oriented exploration for finding a Hamiltonian cycle.

The rest of the paper is organized as follows. In the next section we describe the **VAW** process, show an example of it and prove that Hamiltonian cycles are limit cycles. In the third section we present our experimental work and then conclude with a few open questions.

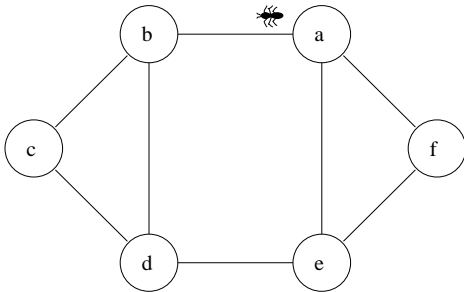
## 2 A Vertex-Ant-Walk Process

Vertex-Ant-Walk (**VAW** for short) is a set of ant-inspired methods for traversing the vertices of a connected graph  $G$  using “smell” traces left on the vertices. In this paper we assume our ant to behave according to the following rule of motion:

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Rule Vertex-Ant-Walk(u vertex;)
A)  $v := u$ 's neighbor with minimal
   value of  $(\mu(\cdot), \tau(\cdot))$ ;
B)  $\mu(u) := \mu(v) + 1$ ;
D)  $\tau(u) := t$ ;
E)  $t := t + 1$ ;
F) go to  $v$ .
end Vertex-Ant-Walk.

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(a,1,1)	(b,1,2)	(d,1,3)	(c,1,4)	(b,2,5)	(a,2,6)
(e,1,7)	(f,1,8)	(e,2,9)	(d,2,10)	(c,2,11)	(b,3,12)
(a,3,13)	(f,2,14)	(e,3,15)	(d,3,16)	(c,3,17)	(b,4,18)
(a,4,19)	(f,3,20)	(e,4,21)	(d,4,22)	(c,4,23)	(b,5,24)
⋮					⋮

Figure 1: A **VAW** ant covers a graph, starting from vertex  $a$ , and arrives at a Hamiltonian limit cycle at time 18. The sequence of vertices is  $abcdcbaefedeb\{afedeb\}^*$ .

The  $(\mu, \tau)$  pair attached to each vertex represents its level of trace:  $\mu$  is the number of visits to the vertex so far and  $\tau$  - the last time the vertex was visited<sup>1</sup>. See Figure 1 for an example of the **VAW** evolution in the course of finding a Hamiltonian cycle.

We shall now show that the **VAW** process has Hamiltonian cycles as limit cycles, that is - once a Hamiltonian cycle has been reached, the ant repeats it forever. For that purpose, let us denote by  $x_t$  the vertex visited at time  $t$  by a **VAW** ant, and by  $P(t) = (x_t, x_{t+1}, x_{t+2}, \dots, x_{t+n})$  the subsequence of  $n + 1$  consecutive vertices in the tour, starting at  $x_t$ .

**Theorem 1** *If, for some  $t_0 > 0$ ,  $P(t_0)$  is a Hamiltonian cycle, then for all  $t > t_0$   $P(t)$  is also a Hamiltonian cycle.*

**Proof:** Assume that  $P(t_0)$  is indeed a Hamiltonian cycle, ending at  $u = x_{t_0+n}$ . The next step, to be taken at time  $t_0 + n + 1$ , depends only on the *relative* values of  $(\mu, \tau)$

<sup>1</sup>Note that this process is slightly different from the **VAW** method presented in [Wagner et. al. 1997], where the trace is increased only if its value is a local minimum. The reason for this change is an attempt to avoid the process from getting stuck in a non-Hamiltonian cycle.

of the vertices. But, since  $P(t_0)$  is a Hamiltonian cycle, all of the  $(\mu, \tau)$  values have changed in the same manner since the last visit to  $u$  ( $n$  steps ago). That is, for all vertices  $v \in V$ :

$$\begin{aligned}\mu_{t_0+n}(v) &= \mu_{t_0}(v) + 1 \\ \tau_{t_0+n}(v) &= \tau_{t_0}(v) + n,\end{aligned}$$

hence, if for vertices  $u, w \in V$  it holds that

$$(\mu_{t_0}(u), \tau_{t_0}(u)) > (\mu_{t_0}(w), \tau_{t_0}(w)),$$

then it should also hold that

$$(\mu_{t_0+n}(u), \tau_{t_0+n}(u)) > (\mu_{t_0+n}(w), \tau_{t_0+n}(w)),$$

so the decision at time  $t_0 + n$  will use the same relative  $(\mu, \tau)$  values and thus will be just the same as the one taken at time  $t_0$ , and the ant follows the same cycle indefinitely.  $\square$

An open question is whether a non-Hamiltonian limit cycle can occur in a Hamiltonian graph.

### 3 Experiments in using VAW to find a Hamiltonian Cycle

A question of interest is “how long does it take for a VAW process to find a Hamiltonian cycle if one exists?”. As a partial answer, several experiments were performed on random graphs. Our experiments were done over random Hamiltonian graphs  $H_n(p)$ , where  $n$  is the number of vertices and  $p$ -the probability of additional edges beyond the basic cycle  $v_1, v_2, \dots, v_n, v_1$ . Thus we start with an  $n$ -cycle and draw additional edges at random with probability  $p$  for each possible edge. Hence the expected size of  $E$  is  $n + p \frac{n(n-3)}{2}$ . We then run a VAW process on the graph until a Hamiltonian cycle is found or the time approaches an upper bound of  $n^3$ . This experiment is run 50 times and we then record  $s(n, p)$  - the percentage of successes which is 100 times the number of times a Hamiltonian cycle was found within  $n^3$  steps of the process divided by the number of runs, as depicted in Figure 2. The average times of recognitions are plotted against  $p$  in Figure 3. It can be seen that most of the graphs are “easy” to recognize, with the exception of a middle region. One way to explain this phenomenon is by observing that if  $p$  is very small, the cycle is nearly the only content of the graph, while if  $p$  is fairly large, the graph is so dense that almost all permutations in  $S_n$  are Hamiltonian cycles.

In a different set of simulations we ran the process for  $n$  varying between 50 and 500 for various values of  $p$  (the

edge density), and found that for the given cases, the cover time is close to quadratic in  $n$ , see Figure 4. This result is to be compared with the average time of  $O(n^{3.5})$ , reported for a neural network heuristic method for the same problem [Mehta & Fulop 1993], and to the  $O(n^3 \log n)$  algorithm presented in [Broder, Frieze & Shamir 1991].

## 4 Some Open Questions

The Vertex Ant Walk seems to be quite a simple process; however, several facets of its behavior are still challenging, as is shown by the following examples.

- We have shown that the Vertex Ant Walk has the Hamiltonian cycle as its limit cycle; however we do not know if those are the *only* limit cycles, that is whether or not there exist limit cycles of the process which are longer than  $n$  vertices.
- Simulations have shown that the VAW is an interesting heuristic, its performance being comparable and even better than the neural network method. From simulation results one may guess that for most values of  $p$ , VAW on a graph from  $G(n, p)$  converges within time  $n^3$ ; but can an upper bound be proved on the convergence time ?
- A probabilistic version of VAW rule does not determine the next neighbor specifically, but assigns each neighbor a probability according to its current  $(\mu, \tau)$  mark, e.g. the probability of jumping from  $u$  to  $v$  may be

$$\text{Prob}(u \rightarrow v) = \frac{1/(1 + \mu(v))}{\sum_{w \in N(u)} 1/(1 + \mu(w))},$$

where, clearly,  $\sum_{w \in N(u)} \text{Prob}(u \rightarrow w) = 1$ . ( $N(u)$  stands for the set of vertices  $v \in V$  such that  $(u, v) \in E$ ). Is such a semi-random process faster (in recognizing a Hamiltonian graph), on the average, or slower than the deterministic one ?

## 5 Acknowledgement

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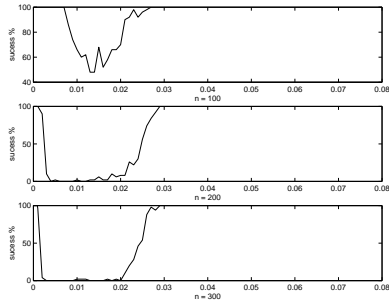


Figure 2: Percentage of success in recognizing Hamiltonian cycles (that is finding one within  $n^3$  steps) by the VAW process, plotted against the edge density for vertex number of 100,200 and 300. The hard cases seem to lie between the densities 0.2 and 0.4.

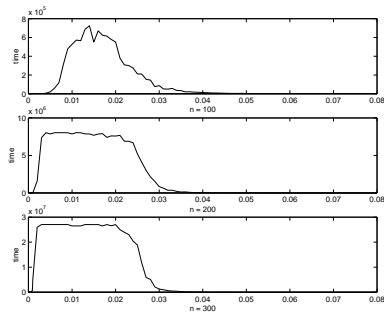


Figure 3: Times for finding a Hamiltonian cycle by the VAW process, plotted against the edge density for vertex number of 100,200 and 300. The hard cases seem to lie between the densities 0.2 and 0.4, in which the plot shows  $\max(t, n^3)$ .

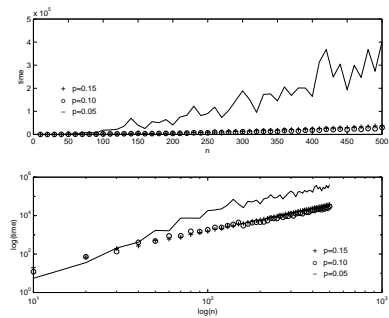


Figure 4: Times for finding a Hamiltonian cycle by the VAW process, plotted against the number of vertices, for edge densities 0.05, 0.10 and 0.15. Times are averaged on 10 runs of the process, and from the log-log plot it can be seen that, for the given cases, the cover time is less than quadratic in  $n$ .