MORPHING
PLANAR TRIANGULATIONS

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MORPHING PLANAR TRIANGULATIONS

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THE GENEROUS FINANCIAL HELP OF TECHNION IS GRATEFULLY
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Abstract

Morphing, also known as metamorphosis, is the gradual and continuous transformation of one shape into another. The morphing problem has been investigated in many contexts, e.g., morphing of two-dimensional images, polygons, polylines and freeform curves. Triangulations are ubiquitously used in computer graphics as a representation and a parameterization (e.g., for texture mapping) of surfaces and planar shapes. Two planar triangulations with a predefined correspondence between the vertices are compatible (isomorphic), if they are topologically equivalent. This work describes methods for morphing compatible planar triangulations. Triangulations with identical boundaries are considered as a particular case.

The basic approach is based on the representation of a planar triangulation as a matrix derived from the triangulation topology. The matrix is constructed using barycentric coordinates of the vertices relative to their neighbors. The choice of such a matrix has many degrees of freedom. Morphing the triangulations corresponds to interpolations between matrices. This interpolation may also be performed in different ways. This work extends previous work of Gotsman and Floater on morphing of tilings, where it is shown that the method always yields a valid morph, namely all intermediate graphs are valid planar triangulations. In the context of the morphing problem, this means that the morph is self-intersection free.
Two methods for interpolation between matrices are considered. The first one uses convex combinations of the matrices. The second method performs a more sophisticated ‘interpolation’ between matrices, in which each column of the matrices is treated separately, in a ‘local’ manner.

We investigate properties of the morphs generated using matrices. We study the trajectories traveled by the vertices during the morph and reducibility to a linear trajectory morph. Two schemes, which generate the linear trajectory morph if it is valid, or a morph with trajectories close to linear otherwise, are presented. An efficient method for verification of validity of the linear trajectory morph between two triangulations is proposed. In addition, we show that the approach can be extended to deal with morphs with required characteristics. A morph through a fixed intermediate triangulation can be derived using an extension of this approach. It is also demonstrated how to obtain a morph with a natural evolution of triangle areas.
List of Symbols

\( \mathcal{G} \) – graph, 9
\( \varphi \) – 1–1 mapping of graph vertices onto a set of distinct points in \( \mathbb{R}^2 \), 9
\( \mathcal{T} \) – planar triangulation, 10
\( \partial \mathcal{G} \) – boundary of a planar graph \( \mathcal{G} \), 10
\( \lambda_{i,j} \) – barycentric coordinate of vertex \( i \) relative to vertex \( j \), 17
\( S \) – triangle area, 12
\( A \) – neighborhood matrix, 19
\( x \) – column vector of \( x \)-components of vertex coordinates, 20
\( A(\mathcal{T}) \) – neighborhood matrix corresponding to \( \mathcal{T} \), 22
\( \mathcal{A}(\mathcal{T}) \) – set of all possible \( A(\mathcal{T}) \), 22
\( \mathcal{A}(\mathcal{T}') \) – set of all possible \( A(\mathcal{T}') \) where \( \mathcal{T} \) and \( \mathcal{T}' \) are isomorphic, 22
\( \mathcal{Z} \) – star, a triangulation with a single interior vertex, 45
\( (\rho, \varphi) \) – point in polar coordinates in \( \mathbb{R}^2 \), 47
Chapter 1

Introduction

Morphing, also known as metamorphosis, is the gradual transformation of one shape into another. Morphing has wide practical use in areas such as computer graphics, animation and modeling. Currently, to achieve more spectacular, impressive and accurate results, the morphing process requires a lot of the work to be done manually. A major research challenge is to develop techniques that will automate this process as much as possible.

1.1 Morphing

The morphing problem has been investigated in many contexts, e.g., morphing of two-dimensional images [3, 12], polygons and polylines [19, 18, 20, 13, 4], freeform curves [17] and even voxel-based volumetric representations [6]. The morphing process always consists of solving two main problems. The first one is to find a correspondence between elements (features) of the two shape representations. The second problem is to find trajectories that corresponding elements traverse during the morphing process.
A successful solution to the correspondence problem is important for the resulting morph. Consider a morphing of two shapes—two animals: a horse and a dog, for example. If the correspondence between these shapes is such that the horse’s tail is matched to the dog’s head, clearly, any morph with this correspondence will look very unpleasing. Regrettably, a formal definition of a successful correspondence does not exist, as well as a definition of a successful morph. Usually, the correspondence problem is solved manually or at least requires human hints as a given partial correspondence. Sometimes, heuristic algorithms are used to ease the process. In [5], differential properties of curves are exploited as a heuristic to find a natural correspondence. A dynamic programming technique is used in order to closely approximate the optimal (in sense of the differential curve properties) matching between two (or even $n$) given curves. In [19], a correspondence between two polygons is found based on a physical model, in which one of the polygons is viewed as being constructed of wire. The resulting correspondence minimizes the amount of work required to bend/stretch the wire of the first polygon into the shape of the second polygon. However, it is possible to construct counter examples, which confound all existing heuristics. Hence, the correspondence problem can only be assisted by heuristic algorithms, and in general, requires human intervention.

The second problem of morphing is the trajectory problem, once a correspondence has been established. The naive approach to solve this problem is to choose the trajectories to be straight lines, where every feature of the first shape travels with a constant velocity towards the corresponding feature of the second shape. A morph generated by this approach is said to be a linear morph. Unfortunately, this simple approach can lead to some undesirable results. The intermediate shapes can vanish, namely, degenerate into a single point, see Figure 1.1. Moreover, intermediate shapes
may have self-intersections, even though the two initial shapes are self-intersection free, see Figure 1.2. Even if the linear morph is free from self-intersections and degenerate regions, its intermediate shapes may have areas or distances between the shape elements far from those of the given shapes, resulting in a ‘misbehaved’ looking morph.

Most of the research on solving the trajectory problem for morphing concentrates on the elimination of self-intersections and on the preservation of geometrical properties of the intermediate shapes. In [18], piecewise linear curves (polylines) are morphed using a heuristic algorithm which interpolates the angles between adjacent
edges as well as the length of the edges. In [20], the geometry of both closed poly-lines (polygons) is preprocessed into an intermediate representation called a *skeleton* which takes into account the interiors of the polygons as well as the boundaries. The work in [17] concentrates on self-intersection elimination using two methods. The first method builds a 3D homotopy of two planar curves and projects it back into the plane to obtain a sequence of planar curves. The second method flips segments of the curves involved in self-intersection to eliminate it. In [13], *multiresolution representations* of two polygons, based on *curve evolutions*, are precomputed. A morphing sequence is reconstructed from the intermediate representations in the different resolutions of the curves. All of the above methods achieve good results for many inputs. However, none guarantees that the resulting morph is always self-intersection free.

Image morphing is a computer animation technique, which aims to continuously transform one image into another. Image *warping*, being closely related to image morphing, deals with continuous deformations of images in such a way that specific parts of the images move relative to each other. Image morphing has the inherent problems discussed above. In [3], an image morphing approach is based on attaching a set of corresponding line segments to images such that image pixels have coordinates with respect to this set. This approach performs linear interpolation on the line segments and determines pixel values using their line coordinates. Since each line segment is morphed linearly, the method may induce self-intersection of the line segments, resulting in foldover of image regions. In [12], an image warping method is presented which exploits a time-varying triangulation to obtain a foldover-free warp. The method permits points, line-segments and even polygons as corresponding features of the image. However it requires as input also trajectories that the image features traverse during the warping process.
In [11], an innovative approach for morphing planar triangulations has been introduced. Two triangulations with a predefined correspondence are said to be compatible (isomorphic), if they are topologically equivalent. This work presents a robust technique for morphing compatible triangulations. The approach is based on convex representation of triangulations. It is shown that the method, applied to compatible triangulations with an identical boundary, always yields a valid morph constructed from a continuous sequence of valid triangulations, namely, the morph is self-intersection free.

1.2 This Work

This work is based on the approach introduced in [11], where only the basic concept is presented. Here we analyze this approach in depth and investigate its properties and capabilities. In addition to analysis of the basic global scheme, we present several extensions that allow more local control over the morphs, namely, self-intersection free morphs with prescribed properties may be obtained. Two linear-reducible schemes are proposed. Both schemes produce linear trajectory morphs if possible, or a morph with close to linear trajectories otherwise. These schemes may also be combined to obtain a morph, which seems to be as close as possible to the linear trajectory morph. Moreover, we introduce an efficient method for verification of the validity of the linear morph between two triangulations. We also show how to obtain a morph through a predefined intermediate triangulation. As a another demonstration of the extensibility of the basic approach, this work presents a method that generates a morph with a natural (almost uniform) evolution of triangle areas.
Chapter 2

Background

In this chapter we define some terms and describe some techniques that will be used throughout this work.

2.1 Triangulations

First, some standard definitions from graph theory: a simple graph $G = G(V, E)$ is a set of vertices $V = \{1, \ldots, |V|\}$ and a set of edges $E$, such that $E$ is subset of the set of all unordered pairs of vertices $\{i, j\}$, when $i \neq j$. We consider only finite graphs, namely $V$ and $E$ are always finite sets. Two graphs $G_0$ and $G_1$ are isomorphic if there is a 1–1 correspondence between their vertices and edges in such a way that corresponding edges link corresponding vertices.

A simple graph $G$ is said to be planar if it can be drawn in the plane in such a way that the following holds:

(i) there exists a 1–1 mapping $\varphi$ of vertices $V$ onto a set of distinct points of the plane;
(ii) each edge \( \{i, j\} \in E \) corresponds to a simple curve with endpoints \( \varphi(i) \) and \( \varphi(j) \);

(iii) the only intersections between curves are at common endpoints.

The above representation of a planar graph \( G \) is called a plane graph, denoted by an ordered pair \((G, \varphi)\). A mapping \( \varphi : V \rightarrow \mathbb{R}^2 \) may also be viewed as a point sequence \( \{i \mapsto (x_i, y_i) \mid i \in V\} \). The following notations are equivalent: \( \varphi(i), p_i, (x_i, y_i) \).

A plane graph divides the plane into connected regions called faces. Obviously, a plane graph has a single unbounded face, denoted by the outer face. The set of vertices and the set of edges adjoining to the outer face form a subgraph \( \partial G \), called the boundary of a plane graph \((G, \varphi)\). It should be noted that different representations of a planar graph in the plane may result in different boundaries. Thus different plane graphs of the same planar graph may subdivide the plane topologically in different ways.

A plane graph is said to be triangulated if all its bounded faces have exactly three edges. A planar triangulation \( \mathcal{T} = \mathcal{T}(G, \varphi) \) is a simple triangulated plane graph such that its edges are represented by straight lines. In this work, we will deal only with planar triangulations, and they will be simply called triangulations.

Triangulations are frequently used in computer graphics as a parameterization for images. In some applications it is necessary to find a mapping between the pixels of two images. Consider two triangulations; the boundary edges of each triangulation define a polygon in the plane. We would like to find a mapping between the two regions bounded by the two corresponding polygons. If there exists a certain conformality between the two triangulations, such a mapping can be described by a piecewise affine mapping from the first triangulation to the second. In this case, we say that two triangulations are isomorphic.
CHAPTER 2. BACKGROUND

Intuitively, two (legal) triangulations are isomorphic, if they are topologically equivalent. We will define this more formally using a notion of orientation. A face $f$ is an ordered triplet of its vertices $(i, j, k)$. An orientation of face $f$ is said to be counterclockwise if the face vertices $i, j, k$, in this specific order, lie in the counterclockwise direction relative to the centroid of the face. Analogously, an orientation is said to be clockwise if the face vertices lie in the clockwise direction. Since the boundary $\partial G$ of $G$ is a simple closed polyline in the plane, it also has well-defined orientation in specific direction relative to the interior faces.

Definition 1 Two triangulations $T_0 = T(G_0, \varphi_0)$ and $T_1 = T(G_1, \varphi_1)$ are isomorphic if the following conditions hold.

1. The two graphs $G_0$ and $G_1$ are isomorphic.
2. There is a 1–1 correspondence between the bounded faces of $T_0$ and $T_1$ such that corresponding faces join corresponding vertices.
3. The boundaries $\partial G_0$ and $\partial G_1$ have the same orientation, i.e. there exist two sequences of vertices of $\partial G_0$ and $\partial G_1$ in counterclockwise direction relatively to the interior faces such that the sequences correspond.

Note that (3) is equivalent to (3').

3'. All corresponding bounded faces have the same orientation.

From the above definition it follows that the isomorphism relation is transitive.

Two sequences of $N$ points $\varphi_0$ and $\varphi_1$ are said to be compatible if there exists a planar graph $G$ such that two (valid) triangulations $T_0 = T(G, \varphi_0)$ and $T_1 = T(G, \varphi_1)$ are isomorphic. A sequence of $N$ points $\varphi_0$ is said to be compatible with a triangulation $T_1 = T(G, \varphi_1)$ if $T_0 = T(G, \varphi_0)$ is a (valid) triangulation and $T_0$ and $T_1$ are isomorphic.

It is easy to verify (decide) whether two triangulations are isomorphic or whether
a point set is compatible with a given triangulation. The problem arises when it is necessary to find isomorphic triangulations over corresponding point sets. The decision problem determining whether two point sequences are compatible has been investigated [16, 1] and is believed to be NP-hard. It is shown [21] that two sequences of $N$ points may be ensured to be compatible by adding $O(N^2)$ Steiner (extraneous) points. Moreover, there exist two $N$-points sequences requiring $\Omega(N^2)$ points to be added to obtain compatibility.

2.2 Barycentric Coordinates

This section describes methods for determining barycentric coordinates of a point $p$ in the plane with respect to vertices of a polygon. The case when a point $p$ lies in the kernel of a star-shaped polygon is considered.

Given a polygon with $k$ vertices $p_1, p_2, \ldots, p_k$, $k \geq 3$, any point $p$ in the plane can be expressed as

$$p = \sum_{i=1}^{k} \lambda_i \cdot p_i, \quad \sum_{i=1}^{k} \lambda_i = 1. \quad (2.1)$$

The coefficients $\lambda_1, \ldots, \lambda_k$ from the above equations are said to be barycentric coordinates of $p$ relative to $p_1, \ldots, p_k$. When $p$ lies in the convex hull of a polygon, it can be expressed as a convex combination of the polygon vertices, namely, all barycentric coordinates have positive values.

We are now interested in finding barycentric coordinates of $p$ with respect to vertices of a polygon. A special case is when the polygon has three vertices. We denote by $S(a, b, c)$ the area of a triangle with vertices $a, b, c$. It is well known that
\( S(a, b, c) \) may be expressed using coordinates of the vertices as follows:

\[
S(a, b, c) = \frac{1}{2} \begin{vmatrix}
1 & 1 & 1 \\
x_a & x_b & x_c \\
y_a & y_b & y_c
\end{vmatrix}
\]  

If \( p \) lies in a triangle \( \triangle(p_1, p_2, p_3) \), its barycentric coordinates with respect to the vertices \( p_1, p_2, p_3 \) are unique:

\[
\lambda_1 = \frac{S(p, p_2, p_3)}{S(p_1, p_2, p_3)}, \quad \lambda_2 = \frac{S(p, p_3, p_1)}{S(p_1, p_2, p_3)}, \quad \lambda_3 = \frac{S(p, p_1, p_2)}{S(p_1, p_2, p_3)}.
\]

A point \( p \) in a star-shaped polygon with more than three vertices has non-unique barycentric coordinates. A simple solution is to choose any triangle containing \( p \) whose vertices are vertices of the polygon. Barycentric coordinates of \( p \) now may be defined as non-zero coordinates with respect to the triangle vertices, and zeros for all other vertices.

More uniform barycentric coordinates may be defined by the following scheme [10]. This scheme combines barycentric coordinates generated using the above simple scheme applied to a number of triangles. For each vertex \( p_i \) from the \( k \) vertices of the polygon there exists a triangle \( \triangle_i \) such that \( p_i \) is one of the triangle vertices, and \( \triangle_i \) contains \( p \). It should be noted that if \( p \) lies on an edge of \( \triangle_i \), the choice of \( \triangle_i \) is not unique. In this case an arbitrary triangle may be chosen, since barycentric coordinates with respect to any triangle \( \triangle_i \) are defined solely by the vertices of this edge. These \( k \) different barycentric coordinates of \( p \), not necessary distinct, can be averaged to obtain other barycentric coordinates. It is easy to see [10] that the result is legal.

We introduce a new scheme that maintains some good properties. The scheme finds barycentric coordinates \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of a point \( p \) with respect to a polygon...
with \( k \) vertices \( p_1, \ldots, p_k \) satisfying the following condition. The variance between values of the barycentric coordinates is minimal. We can calculate this by solving the following problem:

\[
\begin{align*}
\text{minimize} & \quad f(\lambda_1, \ldots, \lambda_k) = \sum_{i=1}^{k} \left( \frac{1}{k} - \lambda_i \right)^2, \\
\text{subject to} & \quad p = \sum_{i=1}^{k} \lambda_i \cdot p_i, \\
& \quad \sum_{i=1}^{k} \lambda_i = 1.
\end{align*}
\]

This problem can be solved by Lagrange multipliers method. The equations (2.4) and (2.5) are written as

\[
\begin{align*}
g_1(\lambda_1, \ldots, \lambda_k) &= \sum_{i=1}^{k} \lambda_i \cdot x_i - x = 0, \\
g_2(\lambda_1, \ldots, \lambda_k) &= \sum_{i=1}^{k} \lambda_i \cdot y_i - y = 0, \\
g_3(\lambda_1, \ldots, \lambda_k) &= \sum_{i=1}^{k} \lambda_i - 1 = 0.
\end{align*}
\]

We can now define the function of the Lagrange multipliers method as follows:

\[
F(\lambda_1, \ldots, \lambda_k, \mu_1, \mu_2, \mu_3) = f(\lambda_1, \ldots, \lambda_k) + \mu_1 g_1 + \mu_2 g_2 + \mu_3 g_3. \tag{2.6}
\]

\( F \) has a unique extremum, since the vertices \( p_1, \ldots, p_k \) are distinct, and can be found by satisfying \( \Delta F = 0 \). Namely, we need to solve the following system of linear equations:

\[
\begin{align*}
\frac{\partial F}{\partial \lambda_i} &= 0, & i &= 1, \ldots, k, \\
\frac{\partial F}{\partial \mu_i} &= 0, & i &= 1, \ldots, 3.
\end{align*}
\]

The derived solution \( \lambda = \lambda^{\text{min}} \) is the minimum of \( F \).

But since \( p \) lies in the kernel of the star-shaped polygon \( p_1, \ldots, p_k \), we wish to find \( \lambda = (\lambda_1, \ldots, \lambda_k) \) to be non-negative barycentric coordinates, namely, \( \lambda \) lies in
a domain \( \mathcal{D}^\lambda = \{0 \leq \lambda_i \leq 1, i = 1, \ldots, k\} \). However \( \lambda^{\text{min}} \) does not necessarily lie in \( \mathcal{D}^\lambda \). We need to add the following constraints to (2.4) and (2.5):

\[
\lambda_i \geq 0, \quad i = 1, \ldots, k.
\]

(2.8)

This can be solved by the Kuhn-Tucker method [14], which extends the Lagrange multipliers method by adding inequality constraints.

It should be noted that the above scheme is stable, namely, small changes in the coordinates of the polygon vertices or in the location of \( p \) may cause only small changes in barycentric coordinates of \( p \). We conclude this because \( F \) is continuous and \( \mathcal{D}^\lambda \) is connected and convex. Moreover, the dependence of the solution on the input is \( C^1 \) continuous in the case when \( \lambda^{\text{min}} \in \mathcal{D}^\lambda \). In cases when \( \lambda^{\text{min}} \notin \mathcal{D}^\lambda \), the negative values of few \( \lambda_i \)'s are very close to zero and, in general, almost negligible. We will return to the properties of this scheme later in Section 4.2.

2.3 Drawing Triangulated Graphs

It has been shown [8] that every planar graph has a straight line representation, namely it can be drawn in the plane in such a way that its edges are straight line segments. In this section we consider simple triangulated plane graphs. We wish to find a straight line representation of a simple triangulated graph \( \mathcal{G} \), that is, to find a point sequence \( \varphi \) such that \( \mathcal{T} = \mathcal{T}(\mathcal{G}, \varphi) \) is a (valid) triangulation. A method that finds such a triangulation was introduced by Tutte [22]. This method builds a triangulation of a graph \( \mathcal{G} \) in the following manner. The boundary vertices of \( \mathcal{G} \) are mapped to an arbitrary convex polygon with the same number of vertices and the same vertex order. Then, the interior vertices are placed in such a way that every vertex is the centroid of the polygon of its neighboring vertices, see Figure 2.1(b).
Figure 2.1: Constructing a straight line representation of a triangulated plane graph: (a) a triangulated plane graph; (b) a corresponding triangulation—the boundary is a convex polygon, the interior vertices are centroids of their neighboring vertices; (c) each interior vertex is an arbitrary convex combination of its neighboring vertices.

This scheme was extended by Floater [10]. Each interior vertex can be any convex combination of its neighbors, see Figure 2.1(c). In terms of barycentric coordinates, any positive barycentric coordinates for each interior vertex may be chosen with respect to its neighbors.

To compute $\phi$, we use the following method. Let $\mathcal{G} = \mathcal{G}(V, E)$ be a simple triangulated graph, with $|V| = N$. We assume that boundary vertices of $\mathcal{G}$ are defined. These boundary vertices correspond to some planar representation of $\mathcal{G}$. Let $V_I$ be a set of the interior vertices and $V_B$ be a set of the boundary vertices, while $|V_I| = n$ and $|V_B| = N - n = k$. We can define without loss of generality $V_I = \{1, \ldots, n\}$ and $V_B = \{n + 1, \ldots, N\}$. Now we wish to find coordinates of the graph vertices, namely, for each vertex $i \in V$ to find $\phi(i) = (x_i, y_i)$. We define $\phi$ for each vertex $i \in V_B$ to be coordinates of the vertices of a $k$-sided convex polygon with the same vertex order as of $\partial \mathcal{G}$.

For each interior vertex we choose arbitrary non-negative barycentric coordinates relative to its neighbors. Now for each vertex $i \in V_I$ we define a set of scalars $\lambda_{i,j}$
for \( j = 1, \ldots, N \) such that

\[
\begin{cases}
\lambda_{i,j} = 0, & \{i, j\} \notin E, \\
\lambda_{i,j} \geq 0, & \{i, j\} \in E, \quad \text{barycentric coordinate of } i \text{ relatively to } j.
\end{cases}
\]  \hspace{1cm} (2.9)

Since for each vertex \( i \), a set of \( \lambda_{i,j} \) where \( \{i, j\} \in E \) is legal barycentric coordinates of vertex \( i \) with respect to its neighbors, it holds that \( \sum_{i=1}^{N} \lambda_{i,j} = 1 \). We define \( p_1, \ldots, p_n \) to be the solution of the linear system of the following equations

\[
p_i = \sum_{i=1}^{N} \lambda_{i,j} \cdot p_j, \quad i = 1, \ldots, n.
\]  \hspace{1cm} (2.10)

The above linear system has \( n \) equations with \( n \) variables. It was shown [10] that the matrix corresponding to these equations is non-singular. Therefore, a unique solution always exists. These equations may be solved analytically, or iteratively, due to the spectral properties of the corresponding matrix.

The problem of finding a straight line representation of a triangulated plane graph may conceivably be obtained as a solution to a set of constraints. Using (2.2) we can define constraints for the orientation of the triangles within the plane graph. For a fixed orientation of a specific triangle, its area defined by (2.2) is always positive or always negative. Thus, for each triangle \( \triangle_{i}(a, b, c) \) the constraint for its orientation may be written as:

\[
\begin{vmatrix}
1 & 1 & 1 \\
x_a & x_b & x_c \\
y_a & y_b & y_c
\end{vmatrix} > 0
\]  \hspace{1cm} (2.11)

This is a inequality, which is quadratic in its variables. Beyond finding a straight line representation, we may optimize the vertex positions by choosing a cost function that describes some geometric properties, e.g., triangle areas are as close as possible
to some required values, or edges have lengths as close as possible to prescribed. However, this minimization problem with quadratic constraints is difficult in general, and existing numerical methods can usually find only a local minimum.

This approach, which might be suitable for finding a straight line representation, is unsuitable for morphing between two triangulations (by solving the problem for all intermediate triangulations). We would have to find a morph between two triangulations in such a way that a cost function is gradually moved from that defining the initial triangulation to another defining the final one. Since solutions to the corresponding minimization problems are not global (only some local minima), the resulting morph might even be not $C^0$ continuous, namely, not gradual but jerky.
Chapter 3

Morphing

using Neighborhood Matrices

This chapter presents the basic scheme for morphing isomorphic triangulations. We investigate its characteristics and study properties of the morphs generated using this scheme.

3.1 The Neighborhood Matrix

We define the neighborhood matrix for a given triangulation $\mathcal{T} = \mathcal{T}(\mathcal{G}, \phi)$ in such a way that it describes the connectivity of the graph $\mathcal{G}$ as well as the mutual disposition of the interior vertices of $\mathcal{G}$ using barycentric coordinates. We use the definitions of Section 2.3. Let $A = A(\mathcal{T})$ be a $N \times N$ matrix. With the barycentric coordinates of
Each interior vertex $v_i \in V_I$ defined as in (2.9), the matrix $A$ is defined as follows:

$$A(i, j) = \begin{cases} 
\lambda_{i,j}, & i = 1, \ldots, n, \quad j = 1, \ldots, N \\
1, & i > n, \quad j = i, \\
0, & i > n, \quad j \neq i. 
\end{cases} \quad (3.1)$$

The general form of $A$ is:

$$A = \begin{bmatrix} \lambda_{1,1} & \cdots & \lambda_{1,n} & \lambda_{1,n+1} & \cdots & \lambda_{1,N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\lambda_{n,1} & \cdots & \lambda_{n,n} & \lambda_{n,n+1} & \cdots & \lambda_{n,N} \\
0 & \cdots & 0 & I 
\end{bmatrix} = \begin{bmatrix} A_I & A_B \\
0 & I 
\end{bmatrix} \quad (3.2)$$

It should be noted that all rows of $A$ sum to unity, since each row is barycentric coordinates of vertex $i$ with respect to all vertices in $\mathcal{G}$ (zeros stand for non-neighbor components).

Let $x = x(\mathcal{T}) = (x_1, \ldots, x_N)$ be a column vector of $x$-components of the triangulation vertex coordinates $\varphi(\mathcal{T})$, and $y = y(\mathcal{T}) = (y_1, \ldots, y_N)$ the vector of $y$-components respectively. Every row $i$ of $A$ satisfies the following equation: $p_i = \sum_{j=1}^{N} \lambda_{i,j} \cdot p_j$. Putting these equations together for $i = 1, \ldots, N$ we obtain:

$$A \cdot x = x, \quad A \cdot y = y. \quad (3.3)$$

Namely, $x$ and $y$ are eigenvectors of $A$.

The matrix $A$ may be constructed for a given triangulation $\mathcal{T}$ by calculating barycentric coordinates for the interior vertices of $\mathcal{T}$. It is also possible to build a triangulation given a matrix $A$ and coordinates of the boundary vertices. The zero solution for $A \cdot x = x$ is impossible due to the given (non-zero) boundary vertices. We show how this can be reduced to the solution of the linear system (2.10).
A vector $x$ can be partitioned into two vectors $x_I = (x_1, \ldots, x_n)$ and $x_B = (x_{n+1}, \ldots, x_N)$. $y$ is partitioned in the same manner. $x_I$ and $y_I$ stand for vectors of coordinates of the interior vertices, while $x_B$ and $y_B$ — for the boundary vertices. We can now write $A \cdot x = x$, namely, $(A - I) \cdot x = 0$ as follows:

$$
\begin{bmatrix}
A_I - I & A_B \\
0 & 0
\end{bmatrix} \cdot \begin{bmatrix}
x_I \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
x_B
\end{bmatrix} = 0
$$

(3.4)

$$(A_I - I) \cdot x_I + A_B \cdot x_B = 0
$$

(3.5)

Equation (3.5) has $n$ variables in $x_I$. So, we get another form of (2.10). It was proven [10] that the $n \times n$ matrix $(A_I - I)$ is not singular, and thus we always have a unique solution for $x_I$.

There is a one-to-many correspondence between a triangulation and neighborhood matrices. Given a triangulation, it is possible to build its neighborhood matrix in different ways. This follows from the existence of non-unique barycentric coordinates of the triangulation interior vertices. But given a neighborhood matrix and the corresponding boundary points, the interior points are uniquely determined. This property of the correspondence between a triangulation and its neighborhood matrices will be exploited further, in morphing of triangulations.

### 3.2 Morphing Triangulations

Morphing of two isomorphic triangulations $\mathcal{T}_0 = \mathcal{T}(\mathcal{G}, \varphi_0)$ and $\mathcal{T}_1 = \mathcal{T}(\mathcal{G}, \varphi_1)$ is a gradual transformation of $\mathcal{T}_0$ into $\mathcal{T}_1$. This transformation may be viewed as a continuous function $\mathcal{T}(t)$, where $0 \leq t \leq 1$ and $\mathcal{T}(0) = \mathcal{T}_0$, $\mathcal{T}(1) = \mathcal{T}_1$. In this work we consider triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$ such that the boundaries of the triangulations
CHAPTER 3. MORPHING USING NEIGHBORHOOD MATRICES

coincide. Hence, it is required that the boundaries of \( \mathcal{T}(t) \) for \( 0 \leq t \leq 1 \) also coincide. The above condition is written as \( p_i(t) = p_i(0) = p_i(1) \) for \( n < i \leq N \) and \( 0 \leq t \leq 1 \). To find \( \mathcal{T}(t) \) means to find \( p_i(t) \) for \( 1 \leq i \leq n \) in such a way that the point sequence \( \varphi_t \) is compatible with \( \mathcal{T}_0 \) (and \( \mathcal{T}_1 \)).

We show how to find \( \mathcal{T}(t) \) using neighborhood matrices. Neighborhood matrices \( A_0 \) and \( A_1 \) are generated corresponding to \( \mathcal{T}_0 \) and \( \mathcal{T}_1 \). The next step is to find a continuous function \( A(t) \) such that \( A(0) = A_0, \ A(1) = A_1 \) and for all \( t, \ 0 \leq t \leq 1 \), \( A(t) \) is a legal neighborhood matrix describing the same connectivity as that of \( A_0 \) and \( A_1 \). We can now find \( \varphi_t \) by solving \( A(t) \cdot x(t) = x(t) \) and \( A(t) \cdot y(t) = y(t) \). \( \mathcal{T}(t) = \mathcal{T}(\mathcal{G}, \varphi_t) \) is a legal triangulation since \( A(t) \) is a legal neighborhood matrix with the connectivity of \( \mathcal{G} \). \( \mathcal{T}(0), \mathcal{T}(1) \) and \( \mathcal{T}(t) \) are isomorphic due to the common fixed boundary.

3.3 Matrix Space

For any triangulation \( \mathcal{T} \) its corresponding matrix \( A(\mathcal{T}) \) may be generated in different ways. Let \( \mathcal{A}(\mathcal{T}) \) be a set of all possible neighborhood matrices \( A(\mathcal{T}) \). We define \( \mathcal{A}(\mathcal{T}) \) to be a set of all neighborhood matrices corresponding to different triangulations isomorphic with \( \mathcal{T} \). \( \mathcal{A}(\mathcal{T}) \) is said to be the \( A\text{-space} \) of \( \mathcal{T} \). If \( \mathcal{T}_0 \) and \( \mathcal{T}_1 \) are isomorphic, then \( \mathcal{A}(\mathcal{T}_0) = \mathcal{A}(\mathcal{T}_1) \). While the \( A\text{-space} \) of \( \mathcal{T} \) describes only the “topology”, \( \mathcal{A}(\mathcal{T}) \) describes the “geometry” as well. From the above definitions follows: \( A(\mathcal{T}) \in \mathcal{A}(\mathcal{T}) \), \( \mathcal{A}(\mathcal{T}) \subset \mathcal{A}(\mathcal{T}) \). \( \mathcal{A}(\mathcal{T}) \) is the equivalence class in \( \mathcal{A}(\mathcal{T}) \) that consists of all neighborhood matrices describing the “geometry” of \( \mathcal{T} \).

**Proposition 1** The set \( \mathcal{A}(\mathcal{T}) \) does not contain interior points, namely, if \( A \in \mathcal{A}(\mathcal{T}) \) then \( A \in \partial \mathcal{A}(\mathcal{T}) \).
Figure 3.1: A line segment with endpoints $A_0$ and $A_1$ when $A_0 \in \mathcal{A}(T_0)$ and $A_1 \in \mathcal{A}(T_1)$. (a) $A_0$ is an interior point in $\mathcal{A}(T_0)$, and thus $A(\varepsilon) \in \mathcal{A}(T_0)$; (b) $\mathcal{A}(T_0)$ has no interior points, and thus $A(\varepsilon) \not\in \mathcal{A}(T_0)$.

Proof Let $T_0$ and $T_1$ be two isomorphic triangulations, and let $A_0 \in \mathcal{A}(T_0)$, $A_1 \in \mathcal{A}(T_1)$ be two arbitrary neighborhood matrices corresponding to $T_0$ and $T_1$ respectively. Let $A(s)$ be a line segment between $A_0$ and $A_1$ expressed as $A(s) = s \cdot A_1 + (1 - s) \cdot A_0$, $0 \leq s \leq 1$. We wish to show that if $\mathcal{A}(T_0) \neq \mathcal{A}(T_1)$ then $A(s) \not\in \mathcal{A}(T_0)$ for $s > 0$, see Figure 3.1. For $x_0 = x(T_0)$ and for every $A \in \mathcal{A}(T_0)$ it holds that $A \cdot x_0 = x_0$. It is necessary to show that $A(s) \cdot x_0 \neq x_0$ for $s > 0$. We assume by negation that $A(s) \cdot x_0 = x_0$ for $s > 0$. $A(s) \cdot x_0$ is expressed as:

$$A(s) \cdot x_0 = (s \cdot A_1 + (1 - s) \cdot A_0) \cdot x_0$$

$$= (s \cdot (A_1 - A_0) + A_0) \cdot x_0$$

$$= s \cdot (A_1 \cdot x_0 - A_0 \cdot x_0) + A_0 \cdot x_0$$

$$= s \cdot (A_1 \cdot x_0 - x_0) + x_0$$

The assumption is written now as:

$$s \cdot (A_1 \cdot x_0 - x_0) + x_0 = x_0$$

$$s \cdot (A_1 \cdot x_0 - x_0) = 0$$

$$A_1 \cdot x_0 = x_0 \quad \text{since } s > 0$$

For $\mathcal{A}(T_0) \neq \mathcal{A}(T_1)$ we get the contradiction. $A_1 \cdot x_0 \neq x_0$ since $A_1 \not\in \mathcal{A}(T_0)$. \qed
Figure 3.2: Any curve in $\mathcal{A}(\mathcal{T}_0)$ with endpoints in $\mathcal{A}(\mathcal{T}_0)$ and $\mathcal{A}(\mathcal{T}_1)$ defines a valid morph between $\mathcal{T}_0$ and $\mathcal{T}_1$. (a) the curve is a line segment; (b) the curve is another line segment; (c) an arbitrary curve.

**Proposition 2** The set $\mathcal{A}(\mathcal{T})$ is convex.

**Proof** Let $A_0 \in \mathcal{A}(\mathcal{T})$ and $A_1 \in \mathcal{A}(\mathcal{T})$ be arbitrary points in $\mathcal{A}(\mathcal{T})$. Let $A(s)$ be a convex combination of $A_0$ and $A_1$. We need to prove that $A(s) \in \mathcal{A}(\mathcal{T})$ for $0 \leq s \leq 1$. For $x = x(\mathcal{T})$ and for every $A \in \mathcal{A}(\mathcal{T})$ it holds that $A \cdot x = x$. Hence it suffices to show that $A(s) \cdot x = x$ for $0 \leq s \leq 1$.

\[
A(s) \cdot x = (s \cdot A_1 + (1 - s) \cdot A_0) \cdot x \\
= s \cdot A_1 \cdot x + (1 - s) \cdot A_0 \cdot x \\
= s \cdot x + (1 - s) \cdot x = x
\]

$\square$

In Section 3.2 it was shown that in order to morph two triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$ we need to find a continuous function $A(t)$, $0 \leq t \leq 1$ such that $A(0) = A(\mathcal{T}_0)$ and $A(1) = A(\mathcal{T}_1)$. In terms of this section, we need to find a curve in $\mathcal{A}(\mathcal{T}_0)$ with endpoints in $\mathcal{A}(\mathcal{T}_0)$ and $\mathcal{A}(\mathcal{T}_1)$, see Figure 3.2.
3.4 The Convex Combination Morph

The straightforward method for generating a curve \( A(t), 0 \leq t \leq 1 \) in \( \mathcal{A}(\mathcal{T}_0) \) with endpoints in \( \mathcal{A}(\mathcal{T}_0) \) and \( \mathcal{A}(\mathcal{T}_1) \) is proposed in [11]. The method chooses a straight line segment connecting arbitrary points \( A_0 \in \mathcal{A}(\mathcal{T}_0) \) and \( A_1 \in \mathcal{A}(\mathcal{T}_1) \). Thus \( A(t) = (1 - t) \cdot A_0 + t \cdot A_1 \) when \( 0 \leq t \leq 1 \). This solution is said to be the convex combination morph, since \( A(t) \) is a convex combination of \( A_0 \) and \( A_1 \).

It is necessary to show now that \( A(t) \in \mathcal{A}(\mathcal{T}_0) \) for \( 0 \leq t \leq 1 \). We clearly have the same connectivity in \( A(t) \) due to the identical connectivity of \( A_0 \) and \( A_1 \). That is \( A_0(i, j) = A_1(i, j) = A_t(i, j) = 0 \) if \( \{i, j\} \notin E(\mathcal{G}) \). Similarly, \( A_0(i, j) = A_1(i, j) = A_t(i, j) \) for \( n < i \leq N \). It remains to show that \( \sum_{i=1}^{N} A_t(i, j) = 1 \) for \( 1 \leq i \leq n 

\[
\sum_{i=1}^{N} A_t(i, j) = \sum_{i=1}^{N} [(1 - t) \cdot A_0(i, j) + t \cdot A_1(i, j)] \\
= (1 - t) \cdot \sum_{i=1}^{N} A_0(i, j) + t \cdot \sum_{i=1}^{N} A_1(i, j) \\
= (1 - t) \cdot 1 + t \cdot 1 = 1
\] (3.6)

We now start to investigate properties of the convex combination morph. We wish to show that trajectories traversed by the interior vertices during the morph are continuous and smooth, namely, that \( p_i(t), 1 \leq i < n \) are \( C^0 \) and \( C^1 \) continuous. Since \( A(t) \) depends continuously and smoothly on \( t \), the exact solution of the linear system (2.10) is continuous and smooth. More specifically, the \( i \)th component \( x_i \) of the solution has the form:

\[
x_i(t) = \frac{P_i^x(t)}{P_\Delta(t)},
\] (3.7)

where \( P_i^x(t) \) and \( P_\Delta(t) \) are polynomials of degree \( n \) on \( t \); \( P_\Delta(t) \neq 0 \) being the determinant of the corresponding non-singular [10] matrix for the linear system (2.10).
It is essential to check that $\mathcal{T}(t)$ never passes through the same triangulation $\mathcal{T}'$ twice; and even a small advance of $t$ from 0 to 1 achieves a non-zero change in $\mathcal{T}(t)$ towards $\mathcal{T}$. The following proposition states this fact.

**Proposition 3** $\mathcal{T}(t)$ obtained by the convex combination morph is injective.

**Proof** We assume by contraposition that there exist $\mathcal{T}(t_1)$ and $\mathcal{T}(t_2)$ such that $\mathcal{T}(t_1) = \mathcal{T}(t_2)$ and $t_1 \neq t_2$. Suppose, without loss of generality, that $t_1 < t_2$. Let $\mathcal{T}' = \mathcal{T}(t_1) = \mathcal{T}(t_2)$. Since $\mathcal{T}(t_1) = \mathcal{T}(t_2)$, $A(t_1) \in \mathcal{A}(\mathcal{T}')$ and $A(t_2) \in \mathcal{A}(\mathcal{T}')$. Obviously, $A(t_1) \neq A(t_2)$, because $A(t)$ is a convex combination of $A_0$ and $A_1$ when $A_0 \neq A_1$. Due to Proposition 2, $\mathcal{A}(\mathcal{T}')$ is convex. Therefore $A(t) \in \mathcal{A}(\mathcal{T}')$ for $t_1 \leq t \leq t_2$. $\mathcal{A}(\mathcal{T}_0)$ and $\mathcal{A}(\mathcal{T}(t_2))$ have no internal points by Proposition 1. Consequently, for $A(t)$, being a line segment between $A_0$ and $A(t_2)$, it holds that $A(t) \notin \mathcal{A}(\mathcal{T}_0)$ and $A(t) \notin \mathcal{A}(\mathcal{T}(t_2))$ when $0 < t < t_2$. This is a contradiction, since $A(t_1) \in [\mathcal{A}(\mathcal{T}(t_2)) = \mathcal{A}(\mathcal{T}')]$ when $t_1 < t_2$. \qed

An important property of morphing schemes is invariance to affine transformations. Consequently, this property should be verified for the convex combination morph. Clearly, the schemes for choosing barycentric coordinates in Section 2.2 are invariant to affine transformations due to the inherent nature of barycentric coordinates. Thus $A_0$ and $A_1$ do not change under an affine transformation. It remains to show that the generated $\mathcal{T}(t)$, $0 \leq t \leq 1$ is invariant to affine transformations. Let $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ be an affine transformation. Let $x_t = x(T(t))$ and $y_t = y(T(t))$. It is necessary to verify whether $A(t) \cdot \phi_x(x_t, y_t) = \phi_x(x_t, y_t)$. Note that $\phi_y(x_t, y_t)$ is treated analogously. Let $\phi_x(x_t, y_t) = a \cdot x_t + b \cdot y_t + c \cdot \mathbf{1}$, where $a, b, c$ are scalars and $\mathbf{1}$ is a
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column vector of \( N \) ones, \((1, \ldots, 1)\). Hence:

\[
A(t) \cdot \phi_x(x_t, y_t) = A(t) \cdot [a \cdot x_t + b \cdot y_t + c \cdot \mathbf{1}]
\]

\[
= a \cdot (A(t) \cdot x_t) + b \cdot (A(t) \cdot y_t) + c \cdot (A(t) \cdot \mathbf{1})
\]

(since sum of each row in \( A(t) \) is 1, \( A(t) \cdot \mathbf{1} = \mathbf{1} \))

\[
= a \cdot x_t + b \cdot y_t + c \cdot \mathbf{1}
\]

\[
= \phi_x(x_t, y_t)
\]

A morph of two isomorphic triangulations is said to be linear if the interior vertices traverse linear trajectories with constant velocities. Formally, it is expressed as \( p_i(t) = (1 - t) \cdot p_i(0) + t \cdot p_i(1), 1 \leq i \leq N \). The next trait of the convex combination morph emerges when the triangulations have a single interior point. In this case the convex combination morph is linear. This follows from the dependency of the interior vertex solely on the fixed boundary vertices, \( p_i(0) = p_i(1) = p_i, 1 < i \leq N \):

\[
p_i(t) = \sum_{j=2}^{N} \lambda_{i,j}(t) \cdot p_j
\]

\[
= \sum_{j=2}^{N} [(1 - t) \cdot \lambda_{i,j}(0) + t \cdot \lambda_{i,j}(1)] \cdot p_j
\]

\[
= (1 - t) \cdot \sum_{j=2}^{N} \lambda_{i,j}(0) \cdot p_j + t \cdot \sum_{j=2}^{N} \lambda_{i,j}(1) \cdot p_j
\]

\[
= (1 - t) \cdot p_i(0) + t \cdot p_i(1)
\]

(3.8)

In contrast, the convex combination morph of two triangulations with more than one interior vertices is never linear. This follows from the fact that equation (3.7) may not be reduced to the form \((1 - t) \cdot x_i(0) + t \cdot x_i(1)\) for \( n > 2 \), since the polynomials \( P^x_i(t) \) and \( P^\Delta(t) \) are of full degree \( n \) in \( t \). This can be demonstrated by the example in Figure 3.3.
Figure 3.3: Trajectories of the convex combination morph between triangulations with more than one interior vertex are not linear. The lower interior vertex has a fixed position in both the initial and the final triangulations, however, it has a ‘to and fro’ trajectory (instead of staying put).

(a), (b) the initial and the final triangulations; (c) trajectories.

3.5 Back to the Linear Morph

We have seen in the previous section that while the convex combination morph has many positive properties, it may result in a strange-looking morph, even when a simple linear morph exists. Thus it is suboptimal. Consequently, in cases similar to the above, we would like to obtain a morph that is as close as possible to the linear one. In the context of our morphing scheme we wish to find a continuous matrix function $A(t)$ for $0 \leq t \leq 1$, such that $A(0) = A_0$, $A(1) = A_1$. Consider the following definition:

$$A(t) = (1 - t) \cdot A_0^m + t \cdot A_1^m, \quad m \in \mathbb{N}. \quad (3.9)$$

This equation constructs a morph that approaches the linear morph as the power $m$ increases. For $m$ approaching infinity the generated morph is linear (but might be invalid).

To see why this is true, observe that any triangulation can be viewed as a graph of states with transitions. Given a triangulation $\mathcal{T} = \mathcal{T}(\mathcal{G}, \varphi)$, define a corresponding
transition graph $G = G(V, E)$ that has the same vertices as $\mathcal{G}(\mathcal{T})$, namely, $V(G) = V(\mathcal{G}(\mathcal{T}))$. $G$ is defined as a directed graph, see Figure 3.4. For each undirected edge of $\mathcal{G}(\mathcal{T})$ which links two interior vertices we define two directed edges in $E(G)$ connecting these vertices in the opposite directions. Every edge between a boundary vertex $v_B$ and an interior vertex $v_I$ corresponds to a directed edge $(v_I \rightarrow v_B)$. In addition, we attach loop edges to vertices corresponding to the boundary vertices of $\mathcal{G}(\mathcal{T})$. Formally, we have

$$E(G) = \{(i \rightarrow j), (j \rightarrow i) \mid \{i, j\} \in E(\mathcal{G}(\mathcal{T})) \land i \notin \partial \mathcal{G}(\mathcal{T}) \land j \notin \partial \mathcal{G}(\mathcal{T})\}$$

$$\cup \{(i \rightarrow j) \mid \{i, j\} \in E(\mathcal{G}(\mathcal{T})) \land i \notin \partial \mathcal{G}(\mathcal{T}) \land j \in \partial \mathcal{G}(\mathcal{T})\}$$

$$\cup \{(i \rightarrow i) \mid i \in \partial \mathcal{G}(\mathcal{T})\}. \quad (3.10)$$

The barycentric coordinates of the interior vertices of $\mathcal{T}$ may be viewed as probabilities of transitions in $G$. With each directed edge $(i \rightarrow j) \in E(G)$, $i \neq j$, associate a probability of transition from state $i$ to state $j$ as the corresponding barycentric coordinate of vertex $i$ with respect to vertex $j$ in $\mathcal{T}$. For the loop edges $(i \rightarrow i) \in E(G)$ we define the probability to stay in $i$ to be equal to 1. Since barycentric coordinates
sum to unity, $G$ has legal transition probabilities.

The transition probabilities defined for $G$ are described in an exact manner by the neighborhood matrix of triangulation $T$. Consequently, the neighborhood matrix of $T$ is a stochastic matrix describing the transition process in $G$. The process is known as a Markov chain [9, 15]. Let $q = (q_1, \ldots, q_N)$ be a vector of probabilities, where $q_i$ is a probability to be in state $i$. The neighborhood matrix $A$, being a stochastic matrix, is used to determine the probabilities to be in one of the $N$ states after a single transition. Namely, $A \cdot q$ is a vector of probabilities after a single transition. Consequently, the stochastic matrices $A^2, A^3, \ldots$ may be used to determine the probabilities after a number of transitions.

We are interested in $A^m$ when $m$ approaches infinity. For states corresponding to the boundary vertices, our transition graph $G$ has probability of 1 to stay in these states. Such states are called absorbing states of the Markov chain. From every state in $G$ corresponding to an interior vertex there exists a path to the absorbing states due to (3.10). Therefore $\lim_{m\to\infty} A^m$ exists and the probability to be in a non-absorbing state is zero (after $m \to \infty$ transitions from any state). Denote $\lim_{m\to\infty} A^m$ to be $A^\infty$. The elements $A^\infty$ are defined as:

$$A^\infty(i, j) = \begin{cases} a_{i,j} > 0, & \text{if } n < j \leq N, \quad 1 \leq i \leq n, \\ 0, & \text{if } 1 \leq j \leq n, \quad 1 \leq i \leq n, \\ A(i, j), & \text{otherwise.} \end{cases}$$

Thus, the matrix $A^\infty$ has the following form:

$$A^\infty = \begin{bmatrix} 0 & A^\infty_B \\ 0 & I \end{bmatrix}$$

(3.11)
Now we wish to show that a morph generated using (3.9) is linear when $m$ approaches infinity. First note that $A(t)$ has the form of (3.11), since $A_0^\infty$ and $A_1^\infty$ have this form. The equation $A(t) \cdot x(t) = x(t)$ may be written as:

$$(A_I(t) - I) \cdot x_I(t) + A_B(t) \cdot x_B = 0,$$

see (3.5). Since $A(t)$ has the form of (3.11), $A_I(t) = 0$. Thus we have $x_I(t) = A_B(t) \cdot x_B$ for $0 \leq t \leq 1$.

$$x_I(t) = A_B(t) \cdot x_B$$

$$= [(1 - t) \cdot A_B(0) + t \cdot A_B(1)] \cdot x_B$$

$$= (1 - t) \cdot A_B(0) \cdot x_B + t \cdot A_B(1) \cdot x_B$$

$$= (1 - t) \cdot x_I(0) + t \cdot x_I(1)$$

So, the generated morph is linear when $m$ approaches infinity. But it is known that the linear morph is not always a valid morph. Actually, while $A_0$ and $A_1$ are neighborhood matrices, $A_0^\infty$ and $A_1^\infty$ are not. In general, for $m > 1$, $A^m$ is not a valid neighborhood matrix. Therefore the point sequence $\varphi_t$ generated using $A(t)$ is not necessarily compatible with $\mathcal{T}_0$ (and $\mathcal{T}_1$). Nevertheless, the following two conjectures allow to obtain a morph that is closer to the linear morph than the convex combination morph.

**Conjecture 1** If the linear morph is valid, namely, all $\varphi(t)$, $0 \leq t \leq 1$, are compatible with $\mathcal{T}_0$ (and $\mathcal{T}_1$), then for all odd $m \in \mathbb{N}$ $A(t)$ in (3.9) defines a valid morph, which approaches the linear morph as $m$ increases.

**Conjecture 2** If the linear morph is invalid, then there exists a $M \geq 1$ such that for all odd $m \leq M$ $A(t)$ in (3.9) defines a valid morph, which approaches the linear morph as $m$ increases; and for all $m > M$ the resulting morph is invalid.
Figure 3.5: Trajectories of vertex $i$ during morphs generated with different powers $m$ of neighborhood matrices.

In order to obtain a morph that is closer to the linear morph than the convex combination morph, odd powers of neighborhood matrices are used. Figure 3.5 demonstrates trajectories of an interior vertex for various powers of neighborhood matrices, see also Figures 3.9 and 3.10. Conjecture 2 allows to choose the maximum $m$, namely $m = M$ that guarantees a valid morph. A simple algorithm which finds $M$, sequentially check morphs for every $m > 1$, incrementing $m$ by 2. The number of verified morphs may be significantly reduced to $O(\log M)$. First, we find an upper bound $m_{\text{max}}$ by doubling $m$ until the morph is invalid. Then, the resulting $M$ is found by binary search in the interval $[\frac{m_{\text{max}}}{2}, m_{\text{max}}]$.

Furthermore, a morph that is even closer to the linear morph than a morph defined by $m = M$ may be obtained. Consider the following definition of $A(t)$:

$$A(t) = (1 - t) \cdot [(1 - d) \cdot A_0^m + d \cdot A_0^{m+2}] + t \cdot [(1 - d) \cdot A_1^m + d \cdot A_1^{m+2}] \quad (3.12)$$

The above equation may be used to average the neighborhood matrices of the valid
<table>
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<th>Triangulation $\mathcal{T}(t = 0)$</th>
<th>Triangulation $\mathcal{T}(t = \frac{1}{4})$</th>
<th>Triangulation $\mathcal{T}(t = \frac{1}{2})$</th>
<th>Triangulation $\mathcal{T}(t = \frac{3}{4})$</th>
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</thead>
<tbody>
<tr>
<td>Trajectories of the interior vertices</td>
<td>Figure 3.6: Figures with morph examples have the displayed layout.</td>
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morph defined by $m = M$ with the neighborhood matrices of the invalid morph, defined by $m = M + 2$. Using power $m$ for neighborhood matrices and a parameter $d$, equation (3.12) may be viewed as a morph with a real power for neighborhood matrices $m + d \in \mathbb{R} (m + d \geq 1)$. In order to obtain a morph, which is the closest possible to the linear morph, the maximal parameter $d$ may be chosen by binary search in the interval $[0, 1]$, verifying validity of morphs for every step, see Figures 3.11 and 3.12.
Figure 3.7: The convex combination morph.
See also Figure 3.6.

Figure 3.8: The linear morph is invalid.
Figure 3.9: An invalid morph is generated by raising neighborhood matrices to power $m = 2$.

Figure 3.10: An invalid morph is generated by raising neighborhood matrices to power $m = 3$. 
Figure 3.11: A valid morph is generated by raising neighborhood matrices to power $m = 1.2$. This morph has trajectories closer to straight lines than the convex combination morph.

Figure 3.12: A valid morph is generated by raising neighborhood matrices to power $m = 1.3$. This morph has trajectories more closer to straight lines than the convex combination morph. However further advancing to the linear morph will result in an invalid morph.
3.6 Morphing with an Intermediate Triangulation

This section demonstrates how to find a morph between two triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$ in such a way that at a given time $t = t_m$ the morph interpolates a given triangulation $\mathcal{T}_m$. The triangulations $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_m$ are isomorphic and with identical boundaries. A naive solution is to find two convex combination morphs independently: the first one is between $\mathcal{T}_0$ and $\mathcal{T}_m$, and the second morph—between $\mathcal{T}_m$ and $\mathcal{T}_1$. The disadvantage of this solution that while the two independent morphs are continuous and smooth, the whole morph will usually have a $C^1$ discontinuity at the intermediate vertices. The reason for this is that the path in $\mathcal{A}(\mathcal{T}_0)$ from $A_0 \in \mathcal{A}(\mathcal{T}_0)$ to $A_1 \in \mathcal{A}(\mathcal{T}_1)$ through $A_m \in \mathcal{A}(\mathcal{T}_m)$ is not smooth. Specifically, the path consists of two line segments $[A_0, A_m]$ and $[A_m, A_1]$.

In order to find a smooth morph, it is necessary to smoothly interpolate $A_0$, $A_m$ and $A_1$ in $\mathcal{A}(\mathcal{T}_0)$. Consequently, the corresponding elements of the three matrices should be smoothly interpolated. Given three points $(0, \lambda_{i,j}(0))$, $(t_m, \lambda_{i,j}(t_m))$ and $(1, \lambda_{i,j}(1))$ in $\mathbb{R}^2$, it is necessary to find an interpolation $\lambda_{i,j}(t)$ for all $t \in [0,1]$, see Figure 3.13. Since the entries of the matrices are barycentric coordinates, the interpolation must satisfy $0 \leq \lambda_{i,j}(t) \leq 1$. It will be shown later in this section how to find such an interpolation within the bounded region $[0,1] \times [0,1]$.

An important point is that interpolations for the matrix entries are performed independently. But every row $i$, $1 \leq i \leq n$, of $A(t)$, being barycentric coordinates of the interior vertex $i$, should sum to unity. Due to the independent interpolations, this might not be the case. Normalizing the elements of each row can solve this problem. The normalized entry $\overline{\lambda}_{i,j}(t)$ is defined as follows:

$$
\overline{\lambda}_{i,j}(t) = \frac{\lambda_{i,j}(t)}{\sum_{k=1}^{N} \lambda_{i,k}(t)} 
$$

(3.13)
Since $\lambda_{i,j}(t)$ is smooth, the sum of $\lambda_{i,j}(t)$’s is also smooth. Therefore the normalized $\overline{\lambda}_{i,j}(t)$ is a smooth interpolation. See an example demonstrating a smooth morph in Figure 3.14 and 3.15.

The rest of this section describes a method for finding an interpolation of three points $(0, \lambda_{i,j}(0))$, $(t_m, \lambda_{i,j}(t_m))$ and $(1, \lambda_{i,j}(1))$ within the bounded region $[0, 1] \times [0, 1]$ when $0 \leq t_m \leq 1$. Note that any point of the interpolation should be located within $[0, 1] \times [0, 1]$. For notational simplicity we denote:

$$a_0 = (0, \lambda_{i,j}(0)), \quad a_m = (t_m, \lambda_{i,j}(t_m)), \quad a_1 = (1, \lambda_{i,j}(1)).$$

The piecewise Bézier interpolation is useful for finding an interpolation within the bounded region, since the Bézier curve is located in the convex hull of the Bézier control points. In order to interpolate three points we use two Bézier functions, both with three control points. Specifically, the first Bézier function $B_0(s)$ is defined by $(a_0, h_0, a_m)$ and the second Bézier function $B_1(s)$ by $(a_m, h_1, a_1)$, where $h_0$ and $h_1$ are two auxiliary points. The Bézier function $B_0(s)$ and $B_1(s)$ are defined on $[0, 1]$. Since
Figure 3.14: The convex combination morph. See also Figure 3.6.

Figure 3.15: A smooth morph interpolates an intermediate triangulation at $t = \frac{1}{2}$.
the piecewise interpolation must be smooth, points $h_0$, $h_1$ and $a_m$ should be located on the straight line, which in turn will be the tangent line of the interpolation curve for $t = t_m$. In addition, $h_0$ and $h_1$ must be chosen within $[0,1] \times [0,1]$ to keep the resulting interpolation within the bounded region, see Figure 3.16.

The first step is to choose the tangent line for $t = t_m$. If the intermediate point has the maximal or minimal $\lambda$-value between the three points, the tangent line is horizontal, see Figure 3.17(a). Otherwise, the slope of the tangent line is the same as of the line passing through the points $a_0$ and $a_1$, see Figure 3.17(b).

The next step is to choose auxiliary points $h_0$ and $h_1$ on the tangent line. Point $h_0$ is chosen on the tangent line in such a way that the distances $\|a_0 - h_0\|$ and $\|a_m - h_0\|$ are equal. In the same manner, point $h_1$ satisfies $\|a_1 - h_1\| = \|a_m - h_1\|$. If in order to satisfy the above condition $h_0$ or $h_1$ may be located outside the region $[0,1] \times [0,1]$. In this case the following correction is performed. An invalid point, say $h_0$, is shifted to the intersection point between the boundary of the region and the tangent line,
Figure 3.17: The choice of a tangent line: (a) if \( a_m \) has the maximal (or minimal) \( \lambda_{i,j} \)-component among \( a_0, a_m \) and \( a_1 \), a tangent line is horizontal; (b) otherwise, a tangent line has the same slope as the line through \( a_0 \) and \( a_1 \).

see Figure 3.18. Namely, \( h_0 \), being outside the region, is replaced by the point within the region closest to \( h_0 \) and on the tangent line.

The resulting interpolation is the piecewise Bézier function defined as:

\[
(t, \lambda_{i,j}(t)) = \begin{cases} 
B_0(s_0), & 0 \leq t \leq t_m, \quad s_0 = \frac{t}{t_m} \\
B_1(s_1), & t_m \leq t \leq 1, \quad s_1 = \frac{t-t_m}{1-t_m}
\end{cases}
\]

Note that for simplicity of implementation the following approximation may be used:

\[
\lambda_{i,j}(t) = \begin{cases} 
y(B_0(s_0)), & 0 \leq t \leq t_m, \quad s_0 = \frac{t}{t_m} \\
y(B_1(s_1)), & t_m \leq t \leq 1, \quad s_1 = \frac{t-t_m}{1-t_m}
\end{cases}
\]

This approximation preserves smoothness but differs slightly in curvature.
Figure 3.18: Control points $h_0$ and $h_1$ are located on the tangent line. $h_1$ is an equidistant point from $a_m$ and $a_1$; $h_0$ first is located as an equidistant point from $a_0$ and $a_m$ but then is shifted to the boundary of $[0, 1] \times [0, 1]$. 
Chapter 4

Local Schemes

This chapter extends the set of properties that morphs should possess, and introduces a new method which allows control over these properties.

4.1 Controlling the Morph

A well-behaved morphing scheme should preserve some common sense properties like those shown in Section 3.4. Trajectories traversed by the interior vertices should be smooth and even (not jerky and not bumpy). It would also be useful if the morphing scheme would generate the linear morph if it is valid. Such a scheme is said to be linear-reducible. Note that the convex combination scheme is not linear-reducible. A trivial scheme that first verifies whether the linear morph is valid and generates the linear morph or the convex combination morph according to the result, is linear-reducible. However, when the linear morph is invalid, this scheme would not necessarily yield a morph as close as possible to the linear one, which is the next natural requirement.
In Section 3.5 it was shown how to approach the linear morph. This scheme generates a morph that is close to the border between valid morphs and invalid morphs in the sequence of morphs leading to the linear morph. The disadvantage of the scheme is that the morphs of this sequence are *globally* transformed to the linear morph. Namely, if the morph is closer to the linear morph, then the trajectories of its interior vertices are closer to straight lines. Therefore a small local region may invalidate the morph due to a single ‘too straight’ vertex trajectory, and thus may prevent all other interior vertices from approaching the linear morph.

Another major disadvantage of this scheme is that the resulting morph, being quite close to the invalid linear morph, may have triangles close to degenerate. Hence, the following may be added to the list of good properties. It would be useful to be able to control triangle areas in such a way that they are transformed naturally (uniformly) during the morph. This may help to prevent cases such as collapsing (squeezing) of triangles.

From the above, we can conclude that to obtain well-behaved morphs it is necessary to control trajectories of the interior vertices, triangle areas etc. in a local manner. The method presented in this chapter allows a local treatment of geometric features of triangulations.

### 4.2 The Basic Scheme

The method is based on neighborhood matrices. Specifically, we need to find a curve $A(t)$ for $0 \leq t \leq 1$ in $A(T_0)$ with endpoints in $A(T_0)$ and $A(T_1)$. This scheme does not find $A(t)$ as a function $f(A_0, A_1, t)$, as this treats local geometric features and trajectories of the interior vertices as depending solely on the matrices $A_0$ and
$A_1$, which is not the case. Each row of $A(t)$ corresponding to an interior vertex is constructed separately. We define $\mathcal{T}'(G, \varphi)$ to be a subtriangulation of a triangulation $\mathcal{T}(G, \varphi)$ if $\mathcal{T}'$ is a valid triangulation, $G'$ is a subgraph of $G$ and the coordinates of the corresponding vertices of $\mathcal{T}'$ and $\mathcal{T}$ are equal. The triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$ are decomposed into $n$ subtriangulations in the following manner. Each interior vertex $i$, $1 \leq i \leq n$ corresponds to a subtriangulation that consists of the interior vertex, its neighbors and edges connecting these vertices, see Figure 4.1. A subtriangulation defined above is said to be a star denoted by $Z_i$. Namely, every star $Z_i$ corresponds to the interior vertex $i$, and $\bigcup_{i=1}^{n} Z_i = T$.

Let $Z_i(0)$ when $1 \leq i \leq n$ be stars of the triangulation $\mathcal{T}_0$; stars $Z_i(1)$ are defined analogously for $\mathcal{T}_1$. Clearly, $Z_i(0)$ and $Z_i(1)$ are two isomorphic triangulations, since they are the same subgraph of two isomorphic triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$. Barycentric coordinates of the interior vertex in star $Z_i$ with respect to the boundary vertices of that star are barycentric coordinates of the interior vertex $i$ in a triangulation $\mathcal{T}$ with respect to its neighbors. Thus all $Z_i(0)$'s when $1 \leq i \leq n$ together define a neighborhood matrix $A_0$ in $A(\mathcal{T}_0)$; $Z_i(1)$ for $1 \leq i \leq n$ define $A_1$ respectively. In the
same manner, we would like to define $A(t)$ for a specific $t$ using stars $Z_i(t)$, $1 \leq i \leq n$. The question is how to find $Z_i(t)$ for $0 < t < 1$ such that it will define barycentric coordinates with some intermediate values between barycentric coordinates of $Z_i(0)$ and $Z_i(1)$. Barycentric coordinates defined by $Z_i(t)$ must vary gradually from that of $Z_i(0)$ to $Z_i(1)$. Obviously, a morph of two stars $Z_i(0)$ and $Z_i(1)$ should suffice to obtain this. It remains to show that morphing of two stars is not as difficult and intricate as morphing of two general triangulations. If so $A(t)$ is generated by morphing separately stars $Z_i$ for $1 \leq i \leq n$. A morph of $Z_i(t)$ when $0 \leq t \leq 1$ defines a single row $i$ of a neighborhood matrix function $A(t)$. The scheme that generates $A(t)$ by morphing the stars of two triangulations is said to be the local scheme.

It is also important that the scheme, which is used to calculate barycentric coordinates of the interior vertex, should be stable. We use the scheme from Section 2.2 to calculate barycentric coordinates. Due to the stability of this scheme, each barycentric coordinate of the interior vertex is a $C^1$ continuous function (only in some very rare cases it is only $C^0$ continuous, see Section 2.2). Consequently, the resulting morph is also continuous and smooth.

We show a simple way to morph two stars $Z(0)$ and $Z(1)$. Let $v_0$ be the interior vertex of the stars with degree $d$. The boundary vertices are indexed without loss of generality as $v_1, \ldots, v_d$ in a counterclockwise order with respect to the interior vertex. First, we translate the two stars in such a way that the interior vertices are placed at the origin. Let $q_0(0), q_1(0), \ldots, q_d(0)$ and $q_0(1), q_1(1), \ldots, q_d(1)$ be translated vertex coordinates of $Z(0)$ and $Z(1)$ respectively. $q_0(0) = q_0(1) = 0$ due to the translation. To find a morph of the two stars means now to find functions $q_0(t), q_1(t), \ldots, q_d(t)$ when $0 \leq t \leq 1$ such that the point sequence $(q_0(t), q_1(t), \ldots, q_d(t))$ is compatible with $Z(0)$ (and $Z(1)$) for any $t \in [0, 1]$. We choose $q_i(t)$ to be the convex combination
of \( q_i(0) \) and \( q_i(1) \) in polar coordinates. Namely, if \((\rho_i(t), \varphi_i(t))\) are polar coordinates of \( q_i(t) \) then:

\[
\rho_i(t) = (1 - t) \cdot \rho_i(0) + t \cdot \rho_i(1) \\
\varphi_i(t) = (1 - t) \cdot \varphi_i(0) + t \cdot \varphi_i(1) 
\]

(4.1)

when \( i = 0, \ldots, d, \quad 0 \leq t \leq 1. \)

Note that the choice of polar coordinates \((\rho_i, \varphi_i)\) of \( q_i \) is not unique due to the \( \varphi \)-component. Not every choice results in a valid morph. For example, by adding \(2\pi\) to one of two polar coordinates \((\rho_i(0), \varphi_i(0))\) and \((\rho_i(1), \varphi_i(1))\) the equation (4.1) causes the vertex \( i \) to traverse in the opposite direction round the origin. The following simple heuristics to choose \( \varphi_i(0) \) and \( \varphi_i(1) \) do not guarantee a valid morph:

- all vertices of the star traverse in the same direction round the origin;
- all vertices traverse less than a semicircle, namely, \(|\varphi_i(0) - \varphi_i(1)| < \pi\).

We use the convex combination morph \( T(t) \) between \( T_0 \) and \( T_1 \) to solve this problem. The stars \( Z_i(t), \ 1 \leq i \leq n \) of \( T(t) \) for \( 0 \leq t \leq 1 \), translated such that the interior vertex is placed into the origin, define valid morphs between the stars \( Z_i(0) \) and \( Z_i(1) \). We choose \( \varphi_i(0) \) and \( \varphi_i(1) \) such that the vertex \( i \) traverses in the same direction and the same number of the convolutions as in the morph of the star defined by the convex combination morph. See Figure 4.2 for an example of a morph generated by this scheme.

A morph between two triangulations is valid if the orientations of all triangles do not change. In general, an orientation of a triangle may change during a continuous transformation when a triangle becomes degenerate, namely, its angles becomes degenerate (zero). It remains to show now that it does not occur during the morph of the stars defined by (4.1). Let \( \theta_i(t) = \varphi_{i+1}(t) - \varphi_i(t) \) be the angle of a triangle \( i \) adjacent to the interior vertex of \( Z \). All indices are modulo \( d \). Due to (4.1) we clearly
have

\[ \theta_i(t) = (1 - t) \cdot \theta_i(0) + t \cdot \theta_i(1). \]  

(4.2)

Since \( Z(0) \) and \( Z(1) \) are isomorphic, for a specific triangle \( i \), \( \theta_i \) must satisfy

\[ 0 < \theta_i(0) < \pi \text{ and } 0 < \theta_i(1) < \pi. \]  

(4.3)

If it is not so, \( \varphi \)-components of one of the two vertices, namely, \( \varphi_j(0) \) and \( \varphi_j(1) \), \( j \in \{i, i+1\} \), may be adjusted (by adding or subtracting \( 2\pi \)) in order to obtain the consistency in orientation. From the equations (4.2) and (4.3) it follows that \( 0 < \theta_i(t) < \pi \) for \( 0 \leq t \leq 1 \). Therefore we conclude that the orientation of the star triangles does not change during the morph, namely, the morph is valid.

It should be noted that the validity of star morphs, with the interior vertex in the origin, depends only on how \( \varphi \)-components of the boundary vertices vary during the morphs. Arbitrary variations in the radial direction of the boundary vertices do not affect the validity of the morph, see Figure 4.3, as long as \( \rho \)-components are legal radii in polar coordinates, namely positive.
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Figure 4.3: Arbitrary variations in the radial direction of the boundary vertices do not affect the orientations of the star triangles.

4.3 The Linear-Reducible Scheme

The scheme also morphs separately the stars of $\mathcal{T}_0$ and $\mathcal{T}_1$ and is based on the translation of the initial and final stars to the origin. Two translated stars $\mathcal{Z}_i(0)$ and $\mathcal{Z}_i(1)$ are morphed in the following manner. If the linear morph of two stars is valid, we choose it. Otherwise, an arbitrary valid morph is taken. It can be the morph that averages the polar coordinates of the boundary vertices, as described in Section 4.2; or translated trajectories of the boundary vertices during the convex combination morph. In the last case the row corresponding to the star $\mathcal{Z}_i$ in $A(t)$ is equal to the row $i$ of $A(t)$ generated by the convex combination morph.

The following theorem states that the scheme is linear-reducible.

**Theorem 1** The linear morph of two triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$ is valid iff the linear morphs of all component stars are valid.

**Proof** Validity: If the linear morph between $\mathcal{T}_0$ and $\mathcal{T}_1$ is valid, then any triangulation $\mathcal{T}(t)$ for $0 \leq t \leq 1$ is valid and isomorphic with $\mathcal{T}_0$ (and $\mathcal{T}_1$) and thus may be
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decomposed into the valid stars. Each \( Z_i(t) \) when \( 1 \leq i \leq n \) is a valid morph between \( Z_i(0) \) and \( Z_i(1) \). If all morphs of the stars \( Z_i(t) \) are valid, then we have a legal neighborhood matrix function \( A(t) \) for \( 0 \leq t \leq 1 \), and thus \( T(t) \) is valid.

**Linearity:** Let \( p_{i,j} \) be the coordinates of the vertex \( i \) in the star \( Z_j \). First, we prove that if the morph between \( T_0 \) and \( T_1 \) is linear then the morphs of all stars are linear. We have \( p_i(t) = (1 - t) \cdot p_i(0) + t \cdot p_i(1) \) for \( 0 \leq t \leq 1 \). Every star \( Z_j(t) \) is translated in such a way that the interior vertex is placed into the origin. Thus, \( p_{i,j}(t) = p_i(t) - p_j(t) \). Putting both equations together:

\[
p_{i,j}(t) = p_i(t) - p_j(t) \\
= [(1 - t) \cdot p_i(0) + t \cdot p_i(1)] - [(1 - t) \cdot p_j(0) + t \cdot p_j(1)] \\
= (1 - t) \cdot [p_i(0) - p_j(0)] + t \cdot [p_i(1) - p_j(1)] \\
= (1 - t) \cdot p_{i,j}(0) + t \cdot p_{i,j}(1)
\]

For the second direction, we prove that if the morphs of all stars are linear then the morph between \( T_0 \) and \( T_1 \) is linear. The linear morphs of the stars mean:

\[
p_{i,j}(t) = (1 - t) \cdot p_{i,j}(0) + t \cdot p_{i,j}(1) \quad \text{for } 0 \leq t \leq 1. \tag{4.4}
\]

Let \( T(t) \) be the linear morph between \( T_0 \) and \( T_1 \), namely,

\[
p_i(t) = (1 - t) \cdot p_i(0) + t \cdot p_i(1) \quad \text{for } 0 \leq t \leq 1. \tag{4.5}
\]

It must be shown that \( A(t) \) defined by the stars \( Z_j(t) \) for \( 1 \leq j \leq n \) satisfies \( A(t) \in A(T(t)) \). We will show that every star \( Z_i(t) \) defines the same barycentric coordinates of the interior vertex \( i \) as the corresponding star \( j \) of \( T(t) \), and thus \( A(t) \in A(T(t)) \).

Clearly, \( Z_j(t) \) is isomorphic with the corresponding star \( j \) of \( T(t) \). The following states that the vertex coordinates of \( Z_j(t) \) is translated coordinates of the corresponding
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star $j$ of $\mathcal{T}(t)$. Due to the initial translations of the interior vertices into the origin:

$$p_{i,j}(0) = p_i(0) - p_j(0)$$
$$p_{i,j}(1) = p_i(1) - p_j(1)$$

(4.6)

We can now express (4.4) as:

$$p_{i,j}(t) = (1 - t) \cdot p_{i,j}(0) + t \cdot p_{i,j}(1)$$
$$= (1 - t) \cdot [p_i(0) - p_j(0)] + t \cdot [p_i(1) - p_j(1)]
= [(1 - t) \cdot p_i(0) + t \cdot p_i(1)] - [(1 - t) \cdot p_j(0) + t \cdot p_j(1)]$$

(4.7)

Hence, after the translation of $p_j(t)$ the coordinates of every vertex in the star $\mathcal{Z}_j(t)$ are equal to the coordinates of the corresponding vertex in $\mathcal{T}(t)$. Since barycentric coordinates are invariant to a translation (as a special case of affine transformations), the stars $\mathcal{Z}_j(t)$ for $1 \leq j \leq n$ define $A(t) \in \mathcal{A}(\mathcal{T}(t))$. □

This work presents two linear-reducible schemes: the scheme described in Section 3.5 and the scheme introduced in this section. It is prudent to emphasize the principal difference between these two schemes. The first scheme approaches the linear morph using neighborhood matrices raised to a specific power. This approaching significantly affects all trajectories of the interior vertices and is said to be the global approach to the linear morph. This scheme allows to choose a degree of approximation to the linear morph by specifying the power of the neighborhood matrices. However, even the morph closest to the linear morph does not improve each individual trajectory to be as ‘linear’ as possible. The global convergence may be stopped by a single problematic trajectory (which invalidates the morph), preventing the others from being straightened further, see Figures 4.4 and 4.5. On the other hand, the local
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linear-reducible scheme, morphing the component stars separately, may affect a group of trajectories of adjacent vertices almost independently on other trajectories of the morph. However, the local scheme does not attempt to approximate the linear morph for stars for which the linear morph is invalid. Thus, vertices of triangulation regions that cannot be morphed linearly have trajectories similar to those generated by the convex combination morph, and vertices of regions that may be morphed linearly have trajectories very close to straight lines, see Figure 4.6. Knowing the properties of both the linear-reducible schemes, it is possible to choose the most suitable for specific triangulations and specific requirements.

These two schemes may also be combined to obtain a morph that is closer to the linear morph than a morph generated separately by each of the schemes. First, the scheme from Section 3.5 generates a valid morph $T_P(t)$ with a maximal power for neighborhood matrices. Then we apply the scheme of this section, morphing each of the stars separately. For stars, for which the linear morph is invalid, corresponding trajectories from $T_P(t)$ are taken as a morph. For the rest of the stars, the linear morph is choosen. The resulting morph is valid, since the morphs of all component stars are valid. An example in Figure 4.7 is generated by this combined scheme.
Figure 4.4: The convex combination morph. See also Figure 3.6.
Figure 4.5: This valid morph is generated by raising neighborhood matrices to power $m = 1.3$. All trajectories are closer to straight lines than the trajectories of the convex combination morph. However, the two lower trajectories could potentially be straight lines without affecting validity of the morph.
Figure 4.6: The morph is generated using the local linear-reducible scheme. Two lower trajectories are straight lines, however, the rest are identical to the corresponding trajectories of the convex combination morph.
Figure 4.7: The morph is generated by combination of two linear-reducible schemes. The two lower trajectories are linear, the rest approach straight lines like the corresponding trajectories of the morph with power 1.3.
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4.4 Testing Validity of the Linear Morph

The linear-reducible scheme, described in the previous section, morphs a specific star linearly only in the case of the existence of the valid linear morph. The question is how to determine whether the linear morph between $Z(0)$ and $Z(1)$ is valid or not. Note that the naive test which verifies whether $Z(t)$ is a valid triangulation is not always enough, see Figure 4.8. To verify whether $Z(t)$ is a valid triangulation for all $0 \leq t \leq 1$ is impossible in practice, since $[0, 1]$ is a continuum.

We apply the following test to answer the question. The definitions from Section 4.2 are used: $(\rho_i(t), \varphi_i(t))$ are polar coordinates of the vertex $i$ with coordinates $(x_i(t), y_i(t))$, and $\theta_i(t) = \varphi_{i+1}(t) - \varphi_i(t)$ is the angle of a triangle $i$ adjacent to the interior vertex. It is assumed that the polar coordinates of the vertices are chosen using the convex combination morph, as described in the Section 4.2. Clearly, the vertices cannot traverse more than a semicircle during the linear morph. Therefore,
it is necessary to verify this first. Namely, it must hold that \(|\varphi_i(1) - \varphi_i(0)| < \pi\) for all \(1 \leq i \leq d\). The last condition is not sufficient to state that the linear morph is valid, see Figure 4.9. We must also check that the linear morph preserves the triangle orientations. For a specific \(\theta_i\) we assume that (4.3) holds, see Section 4.2. In order to complete the test it should be verified that \(0 < \theta_i(t) < \pi\) for \(0 \leq t \leq 1\). To verify this, it is sufficient to check the extrema of \(\theta_i(t)\) on \([0, 1]\). The extremum points may be found by solving \(\theta_i'(t) = 0\), when \(\theta_i(t) = \varphi_{i+1}(t) - \varphi_i(t)\).

For notational simplicity, we denote \(i = a\) and \(i + 1 = b\). Due to the linear traversals of the vertices we have:

\[
\begin{align*}
x_a(t) &= (1 - t) \cdot x_a(0) + t \cdot x_a(1) \\
x_b(t) &= (1 - t) \cdot x_b(0) + t \cdot x_b(1)
\end{align*}
\]

\[
\begin{align*}
y_a(t) &= (1 - t) \cdot y_a(0) + t \cdot y_a(1) \\
y_b(t) &= (1 - t) \cdot y_b(0) + t \cdot y_b(1)
\end{align*}
\]
Figure 4.10: Since vertex $a$ traverses less than $\pi$ round the origin, $a_0$ and $a_1$ can be rotated by the same angle $\omega_a$ round the origin such that $y(a_0) > 0$ and $y(a_1) > 0$. (a) Boundary vertices $a_0$ of star $Z_0$ and $a_1$ of $Z_1$; (b) $\tilde{a}_0$ and $\tilde{a}_1$ are corresponding rotated vertices.

The $\varphi$-component of the polar coordinates is expressed as:

$$
\varphi(t) = \text{sign}(y(t)) \cdot \arccos \left( \frac{x(t)}{\sqrt{x^2(t) + y^2(t)}} \right),
$$

where $\text{sign}(z) = \begin{cases} +1, & z \geq 0 \\ -1, & z < 0 \end{cases}$ (4.10)

The next step is to derive $\varphi'_b(t) - \varphi'_a(t)$. But the $\text{sign}(z)$ function is not convenient for the derivation. In order to overcome this problem we perform some substitutions for $\varphi(t)$. Since $|\varphi_a(1) - \varphi_a(0)| < \pi$ we can rotate both vertices $v_a(0)$ and $v_a(1)$ by the same angle $\omega_a$ round the origin such that the vertices are placed in the upper half plane, namely, the $y$-components of the vertices are positive, see Figure 4.10. We denote the rotated coordinates by $(\tilde{x}, \tilde{y})$ with the polar $\varphi$-component $\tilde{\varphi}$. Thus, we have

$$
\varphi_a(0) = \tilde{\varphi}_a(0) - \omega_a, \quad \varphi_a(1) = \tilde{\varphi}_a(1) - \omega_a.
$$

Since the rotation is an affine transformation, it is easy to see that for the line segment
defined by (4.8):

\[ \varphi_a(t) = \tilde{\varphi}_a(t) - \omega_a. \]  

(4.12)

We denote by \( \varphi((x, y)) \), \( \varphi \)-component in polar coordinates of vector \( (x, y) \). By definition of the rotation:

\[ \varphi \left( (x, y) \cdot \begin{bmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{bmatrix} \right) = \varphi((x, y)) + \omega \]  

(4.13)

It is necessary to show that (4.13) is satisfied for \( (x_a(t), y_a(t)) \), \( 0 \leq t \leq 1 \). Let \( T_R \) be a matrix of the rotation transformation:

\[ T_R = \begin{bmatrix} \cos \omega_a & \sin \omega_a \\ -\sin \omega_a & \cos \omega_a \end{bmatrix} \]  

(4.14)

\[ \varphi((\tilde{x}_a(t), \tilde{y}_a(t))) = \varphi((1 - t) \cdot (\tilde{x}_a(0), \tilde{y}_a(0)) + t \cdot (\tilde{x}_a(1), \tilde{y}_a(1))) \]

\[ = \varphi((1 - t) \cdot (x_a(0), y_a(0)) \cdot T_R + t \cdot (x_a(1), y_a(1)) \cdot T_R) \]

\[ = \varphi([[(1 - t) \cdot (x_a(0), y_a(0)) + t \cdot (x_a(1), y_a(1))] \cdot T_R) \]

\[ = \varphi((x_a(t), y_a(t)) \cdot T_R) \]

\[ = \varphi((x_a(t), y_a(t))) + \omega_a \]

Clearly, \( \tilde{y}_a(t) > 0 \) for \( 0 \leq t \leq 1 \) due to \( \tilde{y}_a(0) > 0 \), \( \tilde{y}_a(1) > 0 \) and (4.8). Consequently, \( \varphi_a(t) \) defined as in (4.10) may now be expressed as:

\[ \varphi_a(t) = \arccos \left( \frac{\tilde{x}(t)}{\sqrt{\tilde{x}^2(t) + \tilde{y}^2(t)}} \right) - \omega_a \]  

(4.16)

Now, \( \varphi_a(t) \) may easily be derived and after the simplification we get:

\[ \varphi_a'(t) = \frac{\tilde{x}_a(0) \cdot \tilde{y}_a(1) - \tilde{x}_a(1) \cdot \tilde{y}_a(0)}{\tilde{x}_a^2(t) + \tilde{y}_a^2(t)} \]  

(4.17)
The similar procedure of a rotation may be performed for the vertex \( b \), since for \( b \) it also holds that \( |\varphi_a(t) - \varphi_b(t)| < \pi \). Therefore we can write \( \theta_b(t) = 0 \), when 
\[ \theta_b(t) = \varphi_b(t) - \varphi_a(t) \]
as:
\[
\frac{\hat{x}_b(0) - \hat{x}_a(0)}{\hat{y}_b(t) + \hat{y}_a(t)} - \frac{\hat{x}_b(1) - \hat{x}_a(1)}{\hat{y}_b(t) + \hat{y}_a(t)} = 0 \quad \text{(4.18)}
\]
The expression \( \hat{x}^2(t) + \hat{y}^2(t) \), being \( \rho \)-components of the polar coordinates, is strictly positive. Hence, the above equation is equivalent to:
\[
\begin{align*}
\left[\hat{x}_b(0) - \hat{x}_a(0)\right] \cdot \left[\hat{y}_b(t) + \hat{y}_a(t)\right] &= 0 \\
\left[\hat{x}_a(0) - \hat{x}_a(1)\right] \cdot \left[\hat{y}_b(t) + \hat{y}_a(t)\right] &= 0 
\end{align*}
\text{(4.19)}
\]
The same solution is obtained by the similar equation but with the initial non-rotated components:
\[
\begin{align*}
\left[x_b(0) - x_a(0)\right] \cdot \left[y_b(t) + y_a(t)\right] &= 0 \\
\left[x_a(0) - x_a(1)\right] \cdot \left[y_b(t) + y_a(t)\right] &= 0 
\end{align*}
\text{(4.20)}
\]
This is due to the rotational invariance of the equation multipliers:
\[
\begin{align*}
\hat{x}_a(0) \cdot \hat{y}_a(1) - \hat{x}_a(1) \cdot \hat{y}_a(0) &= 2 \cdot \mathcal{S}(\hat{a}(0), \hat{a}(1)) = \\
2 \cdot \mathcal{S}\left(a(0), 0, a(1)\right) &= x_a(0) \cdot y_a(1) - x_a(1) \cdot y_a(0) \\
\hat{x}^2_a(t) + \hat{y}^2_a(t) &= \rho^2(\hat{a}(t)) = 2^\rho(\hat{a}(t)) = x_a^2(t) + y_a^2(t) 
\end{align*}
\]
The factors corresponding to vertex \( b \) are treated similarly.

Since \( x(t) \) and \( y(t) \) are linear in \( t \), \( (4.20) \) is a quadratic equation in \( t \) and may be solved analytically.

The method that verifies whether the linear morph of a star is valid, described above, may be generalized to check the validity of linear morphs for general triangulations. According to Theorem 1 it is sufficient to check the validity of the linear...
morphs for all corresponding stars of the two triangulations. The primary test that verifies whether the boundary vertices of a star traverse less than a semicircle during the convex combination morph is omitted. The test is relevant only for the linear-reducible scheme to ensure the consistency of traversals of a vertex within the stars containing it. The complexity of this test is \(O(V(T))\). We check for each star \(Z_i\), \(1 \leq i \leq n\), the angles of every triangle adjacent to the interior vertex. In total, the complexity is twice the number of the triangulation faces.

### 4.5 Improving Triangle Area Behavior

This section describes a method for improving the behavior of the triangle areas during the morph. The triangle areas are not always transformed uniformly during the morph. In fact, the triangle areas are transformed linearly only when triangulations have a single interior vertex. For a specific triangle \(i\), we would like that its area, denoted by \(S_i\), should behave for \(0 \leq t \leq 1\) like:

\[
S_i(t) = (1 - t) \cdot S_i(0) + t \cdot S_i(1) \tag{4.21}
\]

This cannot be satisfied for all triangles of the triangulation for all \(0 < t < 1\), see Figure 4.11. Consider the following equation for triangle area with vertices \(i, j\) and \(k\):

\[
S(t) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_i(t) & x_j(t) & x_k(t) \\ y_i(t) & y_j(t) & y_k(t) \end{vmatrix} \tag{4.22}
\]

Areas of triangles with two vertices on the boundary are transformed uniformly only when the third vertex traverses linearly with a constant velocity. At the same time, areas of triangles with a single boundary vertex are quadratically (not uniformly)
Figure 4.11: Areas of triangles 1, 2, 3 and 4 are transformed uniformly (linearly) iff vertices $v_1$ and $v_2$ traverse linearly with constant velocities. However, in this case areas of triangles 5 and 6 are not transformed uniformly.

transformed, since the two non-boundary vertices traverse linearly with constant velocities, by (4.22).

The problem of the triangle area improvement may be formulated as follows. We denote by $\bar{S}_i(t)$ the desired area of a triangle $i$ that evolves linearly:

$$\bar{S}_i(t) = (1 - t) \cdot S_i(0) + t \cdot S_i(1).$$  \hspace{1cm} (4.23)

Thus, a morph between two triangulations should minimize a cost function such as:

$$\sum_i |\bar{S}_i(t) - S_i(t)|.$$ \hspace{1cm} (4.24)

In this work we show how to improve the triangle areas using the local scheme, such that the obtained morph is at least closer to (4.24) than the convex combination morph.

Since it is difficult to improve the triangle areas for the whole triangulation, the improvement may be done separately for the stars of the triangulation. The improvement of areas is performed after a morph of a specific star is defined. To preserve
the validity of the morph $\varphi$-components of the boundary vertices are kept unchanged. The improvement is done by a variation of the boundary vertices in the radial direction relatively to the origin.

First, we consider an improvement in such a way that all triangles within the star have exactly the required areas, namely, the area of each triangle is $\overline{S}_i(t)$. In fact, the solution that satisfies this exists only for an odd number of boundary vertices; for an even number of the vertices there is only an approximation. However, this approach has a serious drawback. While every triangle has a required area within the star, its shape significantly differs from the shape it may have in the whole triangulation. Furthermore, since a triangle belongs to a number of stars, its shapes in the stars contradict each other considerably. Therefore the resulting morph is very unstable. The trajectories that the interior vertices traverse are tortuous. The triangle areas are far from uniform and hardly better than those generated by the convex combination morph.

The above means that the evolution of the triangle areas within the stars must also take into account the triangle shapes. For a specific triangle $i$, one of its vertices is the interior vertex of the star, that is placed at the origin for $0 \leq t \leq 1$. The angle adjacent to the interior vertex cannot be changed, because it may affect the validity of the morph. Therefore an improvement of the triangle area is achieved by a variation of the lengths of its two edges adjacent to the interior vertex. Every edge adjacent to the interior vertex belongs to exactly two triangles. Consequently, a length of the edge after an improvement for one triangle does not always coincide with a length of the edge within the second triangle. We improve the triangle areas separately for each triangle, and the resulting length of the edge is the average of the two lengths.

We propose a simple method that improves the triangle areas with preservation of
CHAPTER 4. LOCAL SCHEMES

Figure 4.12: The morph with improved area behavior is generated using the local scheme with the method for area improvement. See Figure 3.7 for the convex combination morph of these triangulations.

the triangle shapes. This method settles the requested area of the triangle maintaining the same relation between the lengths of the edges as when the lengths vary linearly. Let \( a \) and \( b \) be the lengths of the edges, and \( \theta \) be the angle between them. The area of the triangle is defined as:

\[
S = \frac{1}{2} a b \sin(\theta) \tag{4.25}
\]

We denote by \( a(t) \) and \( b(t) \) the lengths of the edges that vary linearly:

\[
a(t) = (1 - t) \cdot a(0) + t \cdot a(1), \quad b(t) = (1 - t) \cdot b(0) + t \cdot b(1) \tag{4.26}
\]

The resulting \( \bar{a}(t) \) and \( \bar{b}(t) \) are the lengths of the edges such that the triangle area is \( \bar{S}(t) \) defined by (4.23). In order to find \( \bar{a}(t) \) and \( \bar{b}(t) \) it is necessary to solve the following system of equations with the unique solution.

\[
\begin{cases}
\bar{a}(t) = a(t) \quad \bar{b}(t) = b(t), \\
\bar{a}(t) \cdot \bar{b}(t) = \frac{2 \bar{S}(t)}{\sin(\theta(t))}, \quad \text{to satisfy } \bar{S} = \frac{1}{2} \bar{a}(t) \bar{b}(t) \sin(\theta(t)) \\
\frac{\bar{a}(t)}{\bar{b}(t)} = \frac{a(t)}{b(t)}, \quad \text{for preserving the relation between the edges.}
\end{cases} \tag{4.27}
\]

See an example of a morph generated using this method in Figure 4.12.
CHAPTER 4. LOCAL SCHEMES

4.6 Comparison between the Schemes

This section shows morphs between the shapes from Figure 1.2 generated by different schemes presented in this work. In order to find a morph using our schemes for morphing planar triangulations, the shape is embedded into two isomorphic triangulation with identical boundaries. As it was shown on Figure 1.2, the linear morph is invalid. It has self-intersection of the shape boundary and, thus, a foldover of the shape area, see Figure 4.14. The convex combination morph, being always valid, has an unpleasing behavior, see Figure 4.15. The tail part of the shape, which should make a large movement during a morph, is squeezed too much, resulting in an unnatural deformation. This squeezing of the tail part may be prevented using the local scheme with area improvement, see Figure 4.16. The morph generated by this scheme may be considered natural, taking into account the fact that the deformation of the shape preserves its area, changing only its contour. The local scheme, which averages polar coordinates of the star boundary vertices, provides good results when regions of triangulations should be rotated during a morph. See Figure 4.17 for a rather natural behavior of the shape. The scheme with an intermediate triangulation may help to obtain the desired movement of the tail part of the shape. Figure 4.18 has an intermediate triangulation at \( t = \frac{1}{2} \), such that all but one interior vertices have the same position as the corresponding interior vertices of the convex combination morph at \( t = \frac{1}{2} \). This one tail-most vertex has a significantly higher position than the corresponding vertex of the convex combination morph, resulting in a better rotation effect.
Figure 4.13: In this section, figures with morph examples have the displayed layout.
Figure 4.14: The linear morph is invalid. It has self-intersections of the shape contour and foldovers of the shape area.
Figure 4.15: The convex combination morph is valid, but has an unpleasing squeezing of the tail part of the shape.
Figure 4.16: The morph generated by the local scheme with area improvement. It behaves naturally with respect to the shape area.
Figure 4.17: The morph generated by the local scheme that averages polar coordinates of star boundary vertices. It behaves naturally, taking into account the rotational movement of the tail part of the shape.
Figure 4.18: The morph passing through an intermediate triangulation at $t = \frac{1}{2}$. All but one interior vertices have the same position as the corresponding interior vertices of the convex combination morph at $t = \frac{1}{2}$. This one tail-most vertex has a significantly higher position than the corresponding vertex of the convex combination morph.
Chapter 5

Conclusion

5.1 Summary

In this work, a robust approach for morphing planar triangulations has been considered. The approach always yields a valid self-intersection morph and, virtually, is the only known analytical method for generating guaranteed valid morphs. This work explores properties of morphs generated using this approach, and studies capabilities of the approach itself. The approach, having many degrees of freedom, may produce different morphs, and, thus, can be modified to obtain morphs with various desirable characteristics. A morph though a predefined intermediate triangulation can be derived using this technique. Two linear-reducible schemes, based on the approach, have been presented. The schemes allow generating of the linear trajectory morph if it is valid, or a morph with trajectories close to linear otherwise. This work has introduced an efficient method to verify validity of the linear trajectory morph between two triangulations. It has also demonstrated how to obtain a morph with a natural evolution of triangle areas.
5.2 Open Questions

This work has generated many questions: several of them are technicalities, but the rest require more fundamental research. One of the technical questions is to find a morph between triangulations, which interpolates \( k \) predefined intermediate triangulations at specific times \( t_1, \ldots, t_k \), see Section 3.6. In order to solve this problem it is necessary to find a smooth interpolant between \( k + 2 \) points \( (0, \lambda(0)), (t_1, \lambda_1), \ldots, (t_k, \lambda_k), (1, \lambda(1)) \), in such a way that the interpolant is located within the bounded region \([0, 1] \times [0, 1]\). An extension of the method, described in Section 3.6, may be considered.

Morphing with an intermediate triangulation also induces the following interesting problem. It would be useful to find a morph through an intermediate triangulation at a given time \( t_m \), in which a subset of the interior vertices has prescribed positions (other interior vertices may be arbitrarily positioned). Since morphing using neighborhood matrices always provides a global solution to the morphing problem, we cannot control as yet a single vertex location (or trajectory). To solve this problem means to solve the following general problem, which is not related to morphing. Given a plane graph (triangulation topology), locations of the boundary vertices (e.g., as a convex polygon) and locations of \( k \) \((1 \leq k \leq n)\) interior vertices, find locations of the rest of the interior vertices such that the resulting triangulation is valid, or answer that a valid triangulation does not exist.

In Section 3.5, two conjectures (1 and 2) are used to generate a morph that approaches the linear morph. Numerous and different examples motivate these conjectures, but a proof still eludes us. Since matrices used to generate morphs in this
section are not legal neighborhood matrices, the proof requires more a profound comprehension of the method. Another problem is to prove the following conjecture. Each interior vertex trajectory of the convex combination morph has a curvature that does not change its sign along the entire trajectory.

Section 4.5 presents two heuristics for improving the evolution of triangle areas. Further analysis of the correlation between vertex trajectories as well as triangle area behavior within the stars and behavior of these elements in the triangulation, may provide insight to achieve more successful heuristics, perhaps even some optimal approximation to the required triangle areas.

5.3 Future Work

Obviously, the problems described in the previous section are significant and interesting research challenges. Furthermore, it is important to make the techniques presented in this work more applicable to real-world scenarios. The problem considered herein is morphing of planar triangulations with identical boundaries. Numerous works deal with morphing of freeform curves, polylines, polygons. A sequence of contiguous edges within a triangulation, namely, a path in the graph of triangulation, may be viewed as a piecewise linear curve; a simple closed path forms a polygon. Hence, a morph of two corresponding polylines or polygons embedded into two corresponding triangulations may be found by morphing these triangulations. However, first, it is necessary to find embeddings that allow this. A problem similar to the following should be solved: given two polygons with a correspondence between their vertices, find two corresponding triangulations with identical boundaries such that the two
given polygons are embedded into the corresponding triangulations and these embeddings correspond. In [1], an algorithm that constructs corresponding isomorphic triangulations of simple polygons is presented. A subsequent work [2] constructs isomorphic triangulations of polygons with holes. Both these works might allow us to build the required embeddings of polygons into corresponding isomorphic triangulations. So, it remains to verify this experimentally.

Another challenging research subject would be to extend the techniques of this work to 3D, certainly, starting from an extension of [22]. Furthermore, it would be interesting to address the problems in [1, 2, 21, 7] for 3D.
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REFERENCES


מרפאות של שלושים ומשרידיים

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חיבור על מחקה

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מנסער עד מועד
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בֶּפֶלְקְלוּטֶה לַמִּדִּיעַ המֶחֶשֶב

הכרת鎴

ברצוניןلاحודות ד”ר חיימ גוסטמן על הנהנית המשוללה, הונימין העזרה

בכל שבילו המאבק.

אצוי מודה לטכניו על הונימעה הבספינית הגדולה בחברותינו
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רשימת אורות

מרץ ליאור הוא אורות בינוני מוגזת
מרץ ליאור הוא אורות יומי
בנירת י espaço של גור מקוון ש"י קהיר פיטור
כבודה של מקום נקודת פיטורית
עкарונות במרוחק מתמרונות שב널ות אחרי מגדירתי מבית חקיע
המשלוליםvla שירה של הקדחותי הפיטוריים של מזרחי הפיתורים הקדומים
בי שילושי שין מקודקדו פנימי גן
orgia המהבהבה של שילוש
מסלקים של נקודת ערב החודש שנות של מתורפים שולת
הנבית של אריזים וזרמים של מופשים
זוטאמול המרופר הפריך הקבר
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תכשיות

所需要的变形，以及变形的具体方法和步骤，应该依据文献和相关资料，以确保其正确性和有效性。
מותן מבטיחתו מפורזת ואחרת,دلלה במעל יבשה בניות_href גוונ viper עובד במצבי הבניאדנאטריה בין יבשה обла الكويיש. בחקו
שה遏יך גיא Devin העכשורים וחלק מתונירה לא המליצה ושניים,וזה גיא Devin

ישטנית הבכייתים מבוטסות על תוספת החמקה לפלירה,גרף מישורי על קויוס וירוס
כששקוחות שנגריך מחזחת ואותו: השכיקובה פעלת בראון הנה: צומתית בשפה של
גרף מעצב-pills מבוטסות על פלירון קוקר עז אוות מיסטר עד שזנקדית
כזל יצמיים מפיכי מפורזת Каיאורב קוקר קלושרות של שבלות ייחו עם מיקום הידיד
שילוש מיירי מתכף קוטרحو. את מתרגמר Каיאורוב של שבלות ייחו עם מיקום הידיד
של האמוסיים. מתﺰאות כשניה גבירות בריגטרווס בריגטרווס בניבל קואידנטווג בריגטרווס
של האמוסיים בשניה גבירות בלשוניות קואידנטווג בריגטרווס בניבל קואידנטווג בלשוניות קואידנטווג
מי מזורת גיון של האמוסיים קוקר של שבלות. לכן, הבחנה על הטכיקובה קלוצר
גרף מישורי על קויוס וירוס. מתכף קוטרחור ייחו עם מיקום בצמיות של השבלות
מדימיית בראון ת-משמעי את השבלות כולה.

שהאלות הטאה היא אט בלוט מתכף בקופיות בגרות שלש楂. על מזג לנשחט וגדת
איבר המאצום מפיים של תხב בריגטרווס בריגטרווס על יבשה שלשבן קוטרחור
בריגטרווס של קוטרחור בציר ייבשות לשלשה קוקרות ארוזים מזרקה בראון חומרים-
אולפים קואידנטווג בריגטרווס בציר ייבשות לשלשה קוקרות ארוזים חומרים-
קימומול מפיכי לחיונים קואידנטווג בריגטרווס. הרגולציה מיינפורה טסיקקה נספה, וען
ה蟆לות המזרקות ייחוד הפרדר גיון מתכפים קוטרחור שלושה מזרקות חמקה של שבלות.
כעט ואילו לדאר את האסימיות המזרקות מורק בוצי שילושם התالمعים
- $T_0$ איבר $A_0$ $A_1$, 너 עב $T_1$ וRgb: מתפקדות A(t) $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$ ורגוב $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$ $A(0) = A_0$ $A_0$ $A_1$ $A_1$- $A(t)$ $A(t)$. שילושם הבכייתם זאל היו אנוייסכניסוז רה עכבר השפה המזרקות הגיא Devin הקופיהית.
שלמותולחמטים המתקבלים מחודש מורח חולק בין

דרפ פשיטה אתחолод פונקציית השינויו \( A(t) \) של מטרפוץ שבטיים היא לחלק יירית הקマー

ביר \( A_1 \) \( A_0 \) \( 0 \) \( 1 \) \( T(0) \) \( T(t) \) \( t \) \( 0 \) \( 1 \) \( 0 \) \( 1 \) \( A_0 \) \( A_1 \) \( T(0) \) \( T(t) \) \( t \) \( 0 \) \( 1 \) \( 0 \) \( 1 \) \( T(0) \) \( T(t) \) \( t \) \( 0 \) \( 1 \) \( 0 \) \( 1 \)

הנה הידיעה העדכנית על מטרפוץ שבטיים וחומרים נוספים המ"><

האם מטרפוץ שבטיים חומרים נוספים ופלמידים מחודש מורח קマー

مهر אטימה הפופולרי גם על מטרפוץ שבטיים וחומרים נוספים המルー<br>
מה כאשר שואף לאגף והומר חינ חלוף. אבל, חזרה, מב产业扶贫 שפיצוף החית גורם
תקפה. זה negerמק שמטרויות שנוצרות בחית גורם 1 < m Cabr את המטרויות שפיצוף החית.
הנה ומצוי לבחר תקף המר את בשתיית מרכז חית, יותר מרכז מרכז jegע מצוללים שיזכרו בלגיורי עד
כמה השתיים מתמשכות.

השיטה המסיבית והирующועת בillery תור şiיוית ל录音ח biên של שילוחים מושרי-

אם נגעל תור_IPV ב)^ önce genomes למסת מרכז אשר בשתיית מוסיסי עובר דריך שילוש
m טור^ - טור 0 יש ביחס נמר m ה-
พรรณנים והאובי רחלק את מרכזים שיוולי קומר כרויים בחלק חול: טור m
המורquez והושק ב 0 טור 0 - 0 להלך והשיץ של הסדרים משלמים: מסלול
ורהמויות הפולמוסgien אוג ההליך צאש והЁרביס את האמתים במרחבי הבניין
t-t = t יזהו שפגיוןים פורסום שיתופי (A(t)) יאייה תור קורו 
A(t) = \sum_i \rho_i A_i \cdot A(t) \cdot A(t) \cdot (t_m, A_m) \cdot A_m, A_0 \cdot A_0.
ןוגז לבעים עשויים תפיסיה תור קורו כרוח של א xnxxים את ה-
A(t) \cdot A(t) \cdot (t_m, A_m) \cdot A_m, A_0 \cdot A_0
פונים. בלע אנטיצאפרפיליז של כל מיסיה תפיסיה עשויה באמות בחלק חול: וככ קריא-
ديث בusterityות עלולות להתקבך לא חיתות (סיכום אי 1). כדי לפרק את הנבעה
מספיקים לנמר את כל מיסיות השתיות. לכל תור קורו כרוח השתיות
בה היא מתאמה. ככ קובר פשקינו מסייפת שCGRectים חיתות אוש מזריחי מרכז חית
הוונגרזכר שילוש הבניין הלה.

הארון קודה צף לகבל מרכז שיוויי קובר ללגיוורי מואר מרכז מרכז.kafka.
יוסף של הסכמה האהה זה כי נוספים החות הכתיבת וקרובות אית כל המסложений
לא롼יויר. מתכונה מקי לייזה קור (כולל אמתיות שעיסים) עלל ליפור והיתרVOKE לሙיק
ולאᛐ. geschriتش all המסלווים בשתיות למשל. כיוון ששתית שקועות של האיון החתומות
וז const. ואת זה תضرورة של הסכמה היא לא mostrar מסלולו קובר בכי מכלה שampionships.

משולעים זהב על ידי פשליי בייגי עלולים להיתק פנדי קובר זה מחודש
דרישת טבעת נ昔日 להכניסה של מעופיו היא ודרישה בהגנה: שטחי משולשים.
משתנים בפים לאירונ על עמנית. הדרישה לשיא_hand נahrungfections בקבצים שarhus תקנות ומדברים המושלים קורבים מצאורים._profes
ענבים הממקור משמר של שויתות בחרזים המקדמיiale המוחה עילה המוחה.

הפרטאים והכותרים ואנטירטוז של שויתות, הטכניקה שמדborah על מופרעת וב-
וזז מפורעת שמירית. אaleb רפכי מפורעת שילושיה לאישה פורמטות ביוות 진ינות נבינה בפר.
A(t) של התרשים של המ디אמה Парשא עם שויתות מקומיות צולם שילושיה. וכל פהני פונסי המ디אמה הניה.
על ממצה ערפת ישועות הפרקיקים צולד-שלישיים. כל פהני פונסי המידאים הניה.
שלוש הפרקיקים מקומיים, שמינווכל הקשקשות ביניהם. השילושיה צאל כدرك מכבד.
ожет הפרקיקים בורבוד וחוזות ההפונים של הרכזות לחולאנדריקוז או הפרטאיות.
שההפונים המר蒺ים בחרזים של הרכזות בנייה של הרכזות בניתי מתור. T_0 של הרכזות המרפיסות בחרזים ונחלת שילושיה מי. כל הרכזות בניתי של הרכזות בנייה T_1, A_1, A_0 של צל של הרכזות מצירוף שופן.
הרצף שופן של המרכזים של המרפיסות בנייה A(t) של הרכזות המרפיסות בנייה T_0, T_0 של המרכזים בנייה A(t) של הרכזות המרפיסות בנייה T_0, T_1, A_1, A_0 של צל של הרכזות מצירוף שגן.
בכל כוכב גידי לחרזografía מתפורעת שופןית בנייה A(t) של הרכזות המרפיסות בנייה T_0, T_0 של המרכזים בנייה A(t) של הרכזות המרפיסות בנייה T_0, T_1, A_1, A_0 של צל של הרכזות מצירוף שגן.

בִּמְזַיְשׂ הַשִּׁלָּשָׁה בָּעֵר הַפְּרָק הַשִּׁלָּשָׁה לְכֹבָּכָה קְרָאת סכינה המקומית.

כּוֹנֶה, מַזָּאֵת מַרְכַּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מַצָּאר הַמְּרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְch

בריצפה והפרים הטורפים שעון החודש הוא: את הריתוך הלשון ניוו

כּוֹנֶה, מַזָּאֵת מַרְכַּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְחָה וּתְךֶלֶת מְרָכָּז בִּי צֵלַּשְׁאָה אֲשֶׁר עַזָּא הַבִּרְch
הנכם מחויבים לשלב סדר כינון מדיניות שיתוף פעולה עם החברות המתאימות.

一緒にработаем над созданием политики сотрудничества с подходящими компаниями.

互いに協力するための政策立案に携わる必要があります。

合伙による協力のための政策作成が必要です。

Shinzena hōkō no shokuryō de shinmoku sharoni kahe. Hōkō no shin'yō de

必要な効果の生産を盛んに説明しています。

Es necesario explicar en profundidad el impacto positive.

It's necessary to explain in depth the positive impact.

有効な結果の生産を盛んに説明しています。

重要な影響の生産を強調して説明しています。

It's necessary to emphasize the significant impact.