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Shape Blending

Matching free-form surfaces

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Abstract

In this work, the matching problem between two free-form parametric surfaces, $S_0(u,v)$ and $S_1(u,v)$, has been considered in the context of a first stage of the *metamorphosis process*, which is defined as gradual and continuous transformation of one key shape into another. A method is presented to approximate the two reparametrization functions, $r(u,v)$ and $t(u,v)$, via a discrete sampled set of N^2 grid samples on the surfaces. The N^2 samples on the source surface are matched against the N^2 samples on the target surface, employing a resemblance metric between their Gauss fields as a criterion. The method preserves the connectivity of the two key surfaces as well. Thus, the resulting reparameterization functions, interpolating the discrete solution must be diffeomorphism. The algorithm works with simple open surfaces, i.e. the surfaces that are homeomorphic to a disk.

Keywords: Computer-aided geometric design; Free-form parameteric surfaces; Matching; Morphing

1. Introduction

Free-form parametric surfaces are being extensively employed in various fields such as computer graphics, computer vision, robotics and geometric modeling. The problem of finding a match between free-form surfaces (or other three-dimensional objects) attracted considerable attention in the literature during the last decade. The motivation for this problem may come from different areas. It is a useful tool for object recognition in computer vision, where an object has to be matched, and hence recognized, against a library of canonical objects [1–3]. Docking of molecules in molecular biology employs matching algorithms to find a geometric fit between parts of the boundaries of two molecules [4]. In geometric modeling, proper, self-intersection free, ruled surfaces' construction, and modeling of sweep and blending surfaces require the correspondence between the key curves [5].

In computer graphics, the matching problem is typically considered in the context of the first stage of *metamorphosis process*. Morphing (or metamorphosis) may be defined as gradual and continuous transformation of one key shape into another. The morphing problem has been investigated in the context of two-dimensional images (see, for example, [6–8]), planar polygons and polylines (i.e. piecewise linear curves) in [9,10,11], free form curves in [5,12,13], and even volumetric representations (see [14,15]). Herein, we would like to investigate the matching problem in the context of free form surfaces.

1.1. Matching criteria for the free-form surfaces

The first stage of any metamorphosis problem requires the establishment of a correspondence between the details of the given source and destination objects. When we consider the metamorphosis between free-form parametric surfaces, this stage is equivalent to finding a match between the two parameterizations of the two surfaces.

Regrettably, the proper criterion for matching the two surfaces is not readily available. When one is examining a match between two surfaces which are similar in some way, it is possible in many cases to visually evaluate the quality of the metamorphosis sequence. For example,

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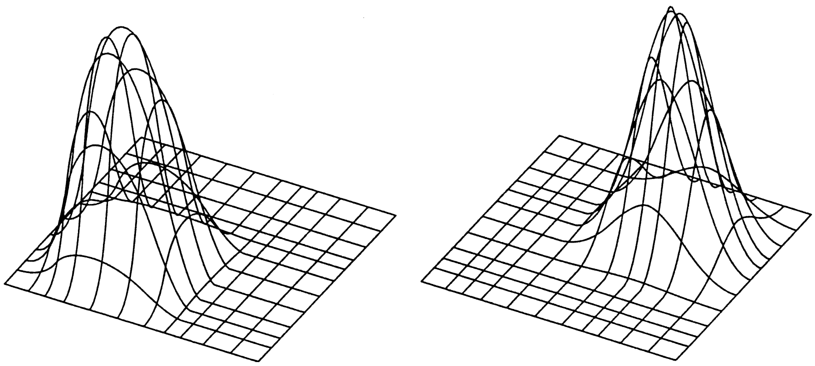

(a) (b)

Fig. 1. Two input surfaces for matching: (a) the surface $S_0(u,v)$, (b) the surface $S_1(u,v)$.

consider two animals with four legs that are to be matched. One could expect that the legs of one animal will be matched to the legs of the other, as would the heads of the two animals, etc. In contrast, if the two surfaces are significantly different, it is also difficult to visually and/or mentally evaluate the quality of the match.

The geometrical form of the surface may be described by any of its intrinsic (i.e. parameterization independent) properties. The criterion we shall consider in this work is the resemblance of the *unit normal fields* of the two surfaces. The unit normal field of the surface is translation and uniform-scaling invariant, but if one surface is the rotated version of the other, the first normal field will be the rotated version of the second. In our case, we exclude non-uniform scaling, since this operation changes the shape of the geometry (for example, from sphere to ellipsoid).

Other criteria may be used for the surface matching. Another optional criterion is the resemblance of the *principal curvatures* k_1^i and k_2^i, $i=\{1,2\}$ through all the points of the parametric domains for the matched parameterizations (see [16]). It is also possible to compare the *mean* and *Gaussian* (*or total*) *curvatures* of the given surfaces, which define the surface in a unique way up to their position and orientation in space, but not scaling. While the normal field criterion (defined in Section 1.2) is invariant to scaling, the curvature criteria are sensitive to scaling, but are immune to rotation and translation. However, the ratio of the principle curvature, k_j^0/k_j^1, represents a good criterion that resolves the scaling dependency.

Consider a match between surfaces using a normal field criterion with the aid of the *Gauss Maps*:

Definition 1.1. The Gauss Map $\mathscr{G}$ of the parametric surface $\mathscr{S}:\mathscr{D}\to\mathbb{R}^3$ is a mapping to the unit sphere $\mathscr{S}^2$, $\mathscr{G}:\mathscr{D}\to\mathscr{S}^2$, comprising the following points:

$$(x,y,z)\in\mathscr{S}^2 : \exists(u,v)\in\mathscr{D}$$

$$(x,y,z)=\mathbf{N}(u,v) = \frac{(\partial S/\partial u)(u,v)\times(\partial S/\partial v)(u,v)}{\|(\partial S/\partial u)(u,v)\times(\partial S/\partial u)(u,v)\|}, \quad (1.1)$$

where

$$\mathscr{S}^2=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1\}.$$

Given *Gauss Map* of a region of the first surface, $S_0(u,v)$, we seek the best possible resemblance to the *Gauss Map* of the matched region of the second surface, $S_1(u,v)$.

1.2. Matching by the normal fields

Every good matching criterion must capture the intrinsic geometric property of the surface, which is independent of the parameterization. For example, if we aim to match the two surfaces in Fig. 1. using the criterion described in this Section, i.e. the resemblance of the normal fields of the surfaces $S_0(u,v)$ and $S_1(u,v)$, then the humps are fully characterized by their Gauss Maps (see Definition 1.1) and should be matched.

A hump in the plane is to be metamorphed into a slightly different hump in the same plane, but in a different location. The simple metamorphosis, using a simple convex combination of the two surfaces without any reparameterization,

$$S_t(u,v)=(1-t)S_0(u,v)+tS_1(u,v),\quad t\in[0,1],$$

creates the result that is shown in Fig. 2, i.e. the first hump is continuously growing smaller while the second one is growing larger. During all the intermediate steps, we have two humps on the plane. The intent of the user

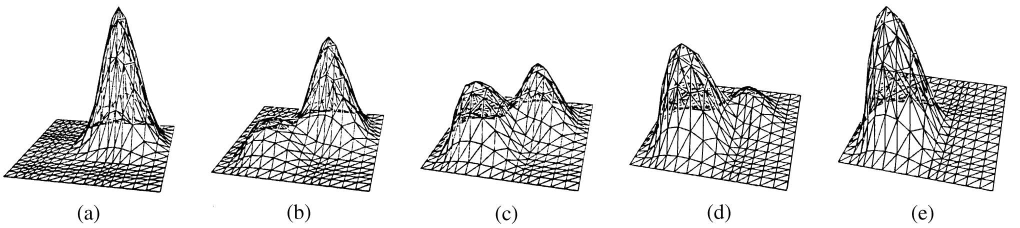

Fig. 2. Convex combination metamorphosis without the matching of the relative parameterizations of the two key surfaces.

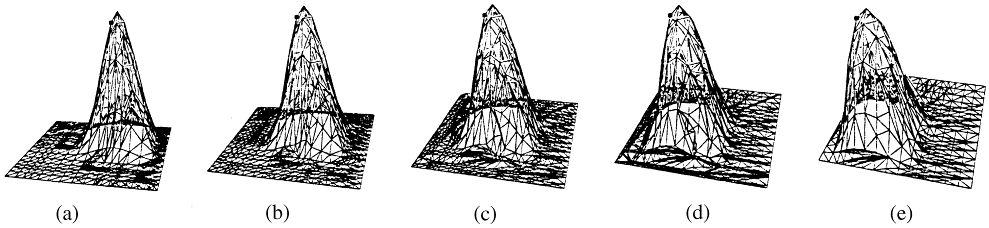

Fig. 3. Convex combination metamorphosis after the matching of the relative parameterizations of the two key surfaces. Note that the match of the particular feature shown in the picture. Compare the sequence with Fig. 2.

was probably that the hump feature would be preserved and translated from one place to the next, as is illustrated in Fig. 3. Fig. 3 is the result of a matching process that follows the discrete approach which is presented in this work.

The initial idea of matching the free form surfaces via the normal field criterion is derived from [5]. For matching of planar curves, the *tangent field* criterion works quite well. The normal field criterion may be considered as its counter-part for surfaces. In addition, the normal field is invariant to translation and scaling transformations, but, as already stated, is sensitive to rotations.

Here, we shall examine the problem of matching two simple and open free-form parametric surfaces: $S_0 : \mathscr{D} \to \mathbb{R}^3$ and $S_1 : \mathscr{D} \to \mathbb{R}^3$, where $\mathscr{D} = [0,1] \times [0,1]$ is the parametric domain for the both surfaces.

Let the unit normal field, $\mathbf{N}_i(u,v)$, of surface $S_i(u,v)$ be as defined as in Equation (1.1). Consider the similarity of the normal fields, $\mathbf{N}_0(u,v)$ and $\mathbf{N}_1(u,v)$, of surfaces $S_0(u,v)$ and $S_1(u,v)$ as the criterion for the matching of the surfaces $S_0(u,v)$ and $S_1(u,v)$. In this case, the solution to the matching problem is equivalent to the following optimization problem: find the maximum of the functional.

$$\max \quad \mathscr{F}\,(s(\cdot,\cdot), t(\cdot,\cdot))$$

$$= \max_{s(\cdot,\cdot),t(\cdot,\cdot)} \int_0^1 \int_0^1 \langle \mathbf{N}_0(u,v), \mathbf{N}_1(s(u,v),t(u,v)) \rangle \,\mathrm{d}u\,\mathrm{d}v,$$

$$\text{s.t.} \quad (s(u,v),t(u,v)) \in \mathscr{D}, \quad \forall (u,v) \in \mathscr{D},$$

$$\mathscr{J}(s,t)(u,v) = \det\begin{vmatrix} \frac{\partial s}{\partial u}(u,v) & \frac{\partial s}{\partial v}(u,v) \\ \frac{\partial t}{\partial u}(u,v) & \frac{\partial t}{\partial v}(u,v) \end{vmatrix} > 0 \quad \forall (u,v) \in \mathscr{D},$$

$$\mathscr{D} = [0,1] \times [0,1], \tag{1.2}$$

where $s(u,v), t(u,v)$ are the two reparameterization functions that we seek.

$\mathscr{F}$ attempts to minimize the angle between the corresponding normals of the first and second surfaces, over all points in $\mathscr{D}$, and $(s,t) : \mathscr{D} \to \mathscr{D}$ is a valid reparameterization of $\mathscr{D}$, i.e. diffeomorphism. The diffeomorphism condition is equivalent to $\mathscr{J}(s,t)(u,v) > 0$, see [17], p. 224, 234.

In this work, we introduce an algorithm that approximates the solution of the problem (1.2). Our method works with the discrete version of the problem.

Even in this simplified discrete version the optimization task is not trivial. In order to find the optimal correspondence between the parameterizations of the two surfaces and satisfy all the conditions of (1.2) we need to explore nearly all possible matches. The number of such matches is exponential in the number of samples. In the one dimensional case, the problem has been solved with the aid of dynamic programming (see [5]), which is impossible in two-dimensional case due to complex connectivity conditions. Thus, our solution for the surfaces is *not a globally optimal solution*, but merely a local minimum approximization. The algorithm that is described in this work has one additional constraint: the four corners of the rectangular parametric domain $\mathscr{D}$ are fixed, i.e. our boundary conditions are

$$s(0,v)=0,\quad s(1,v)=1,$$
$$t(u,0)=0,\quad t(u,1)=1. \tag{1.3}$$

Condition (1.3) may be relaxed by allowing 90° rotations and reflections of corners to corners.

This paper is organized as follows. In Section 2 the detailed algorithmic description can be found. In Section 3 some simple but illustrative examples are introduced. Finally, we conclude in Section 4.

2. Discrete approach to matching

In practice, one could consider the matching problem in the discrete case. Any possible approach for solving the matching problem in a discrete manner will be using a discrete matching criteria, following the ones described in the Section 1, i.e. the *Gauss Maps* of the given surfaces (see Definition 1.1).

The relations between the Gauss Maps of the two given surfaces can be helpful in the different stages of matching the relative parameterizations of the two surfaces. For instance, some initial matched points on the two surfaces might be prescribed either on the boundary for open surfaces, or may be an arbitrary point on the parametric domain: $\{(u_0,v_0),(s_1,t_1)\}\in\mathscr{D}$, such that point $S_0(u_0,v_0)$ would correspond to point $S_1(s_1,t_1)$. The matching process would exploit the resemblance of the Gauss Map's locations of the neighborhoods of the feature initial points, towards a complete match between the surfaces.

Here, we shall consider the problem when both surfaces are open and simple, i.e. the following relation holds:

$$S_i(u,v)|_{\partial\mathscr{D}}\equiv\partial S_i, \tag{2.1}$$

$$S_i(u_1,v_1)=S_i(u_2,v_2)\Rightarrow(u_1,v_1)=(u_2,v_2), \tag{2.2}$$

where

$$S_i=\{(x,y,z)\in\mathbb{R}^3:\exists(u,v)\in\mathscr{D}\quad S_i(u,v)=(x,y,z)\}. \tag{2.3}$$

The input of the discrete matching problem is the square grid of the points $\mathscr{P}^0_{i,j}=(u_i,v_j)$ of size $K=2^k, k\in\mathbb{Z}^+$ in both directions of the parametric domain $\mathscr{D}$:

$$\{(u_i,v_j)\}_{i,j=0}^{K-1},\quad u_i=\frac{i}{K-1},\quad v_j=\frac{j}{K-1}. \tag{2.4}$$

The solution of the discrete problem will be $O(K^2)$ pairs of the matched points:

$$\{(u_i,v_j),(s_i,t_j)\}_{i,j=0}^{K-1},$$
$$\text{s.t.}\quad (s_i,t_j)\in\partial\mathscr{D}\quad\forall(i,j):(u_i,v_j)\in\partial\mathscr{D}\quad\text{and vice versa.} \tag{2.5}$$

The set of the points $(s_i\, t_j)$ approximates the value of the function $(s(u_i,v_j),t(u_i,v_j))$ that maximizes the object functional $\mathscr{J}(s(\cdot,\cdot),t(\cdot,\cdot))$. In addition, the mapping $(s(\cdot,\cdot),t(\cdot,\cdot))$ should satisfy conditions that are defined by Eq. (1.2).

Recall that the algorithm for free form curve matching of [5] considered the discrete version of the matching problem as well. The complexity of the straight-forward search of the optimal allowable reparameterization for the discrete versions of free-form curves is equal to the number of all possible versions of allowable reparameterziations, i.e. is exponential in the number of curve samples. Nonetheless, the problem was solved with the aid of *dynamic programming* that can find the optimal solution in $O(n^2)$ time, where n is the number of samples on each curve.

Unfortunately, this approach does not work for the search of the optimal surface reparameterization. As in the one-dimensional case (curve reparameterization), we have some constraints on the matching of the samples on the surfaces. The most severe constraint is the *connectivity* of the mapping, which in the two-dimensional case cannot be formulated in such simple way as in the one-dimensional case.

In the case of surface matching, the conditions are more complicated. If a point $\mathscr{P}^0_{i,j}=(u_i,v_j)$ is mapped to the point $\mathscr{P}^1_{i,j}=(s_i,t_j)$, the eight points adjacent to $\mathscr{P}^0_{i,j}$, i.e. the points $\mathscr{P}^0_{i-1,j-1}$, $\mathscr{P}^0_{i-1,j}$, $\mathscr{P}^0_{i-1,j+1}$, $\mathscr{P}^0_{i,j-1}$, $\mathscr{P}^0_{i,j+1}$, $\mathscr{P}^0_{i+1,j-1}$, $\mathscr{P}^0_{i+1,j}$ and $\mathscr{P}^0_{i+1,j+1}$ (see Fig. 4), are mapped to either $\mathscr{P}^1_{i,j}$, or to $\mathscr{P}^1_{i-1,j-1}$, $\mathscr{P}^1_{i-1,j}$, $\mathscr{P}^1_{i-1,j+1}$, $\mathscr{P}^1_{i,j-1}$, $\mathscr{P}^1_{i,j+1}$, $\mathscr{P}^1_{i+1,j-1}$, $\mathscr{P}^1_{i+1,j}$ and $\mathscr{P}^1_{i+1,j+1}$, and due to continuity/connectivity constraints, polygon

$$\mathscr{P}=\{\mathscr{P}^1_{i-1,j-1}\,\mathscr{P}^1_{i-1,j}\,\mathscr{P}^1_{i-1,j+1}\,\mathscr{P}^1_{i,j+1}$$
$$\mathscr{P}^1_{i+1,j+1}\,\mathscr{P}^1_{i+1,j}\,\mathscr{P}^1_{i+1,j-1}\,\mathscr{P}^1_{i,j-1}\},$$

must contain the point $\mathscr{P}^1_{i,j}$ but none of the points $\mathscr{P}^1_{k,l}$, where $k\notin\{i,i\pm1\}$, or $l\notin\{j,j\pm1\}$, so that the isomorphism condition is satisfied (see Eq. (1.2)).

Let us introduce some new notations. Denote by $\mathscr{A}$, a convex quadrilateral that is a subset of the parametric domain, $\mathscr{D}$, of surface $S(u,v)$.

If the set $\mathscr{A}$ coincides with the entire parametric domain, $\mathscr{D}$, then we define the line $\gamma_l(\mathscr{A})$ to be the left edge, $\gamma_r(\mathscr{A})$ to be the right edge, $\gamma_u(\mathscr{A})$ to be the upper edge, and $\gamma_d(\mathscr{D})$ to be the lower edge (down from $\mathscr{A}$), on the boundary of $\mathscr{A}$, in the following way (see Fig. 5):

$$\gamma_l(\mathscr{D})=(0,v),\ v\in[0,1],$$

$$\gamma_r(\mathscr{D})=(1,v),\ v\in[0,1],$$

$$\gamma_u(\mathscr{D})=(u,0),\ u\in[0,1],$$

$$\gamma_d(\mathscr{D})=(u,1),\ u\in[0,1]. \tag{2.6}$$

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two straight lines, each intersecting two non-adjacent edges of the boundary of $\mathscr{A}$: the line

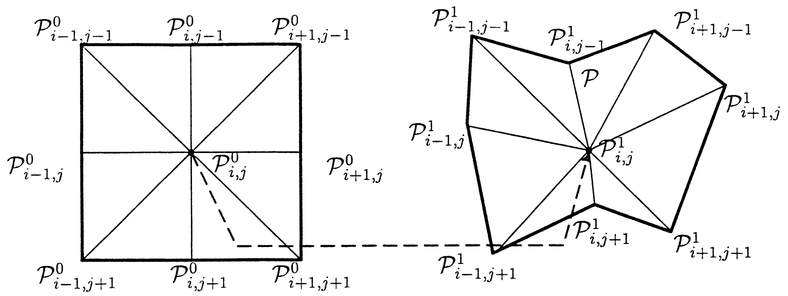

Fig. 4. The matching of the neighborhood of the point (u_i, v_j) in the parametric domain $\mathscr{D}_0$ of the first surface with polygon $\wp$ on the parametric domain $\mathscr{D}_1$ of the second surface.

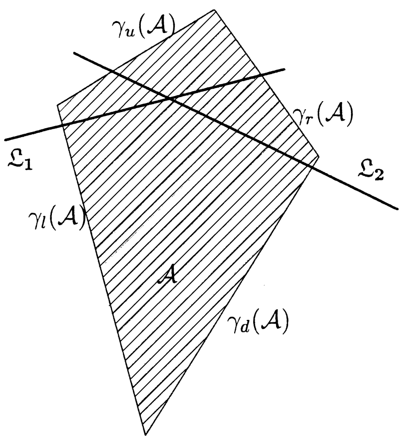

Fig. 5. The convex quadrangle $\mathscr{A}$ divided into four areas by the line $\mathscr{L}_1$ and $\mathscr{L}_2$.

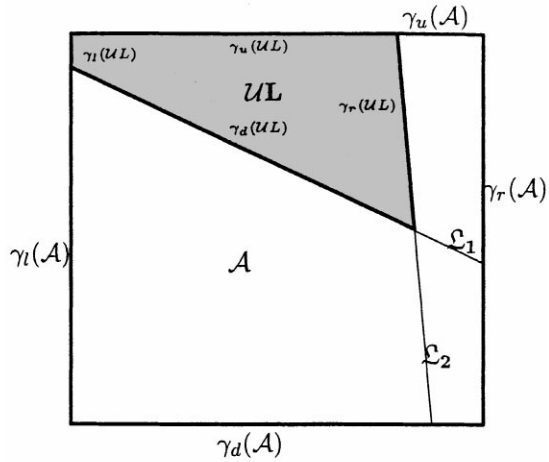

Fig. 6. The edges of one quarter $\mathscr{UL} = \{Upper(\mathscr{A}) \cap Left(\mathscr{A})\}$ of the convex quadrangle $\mathscr{A}$.

$\mathscr{L}_1$ intersects the left edge $\gamma_l(\mathscr{A})$ as well as the right edge $\gamma_r(\mathscr{A})$; similarly, the line $\mathscr{L}_2$ intersects the upper and lower edges of the set $\mathscr{A}$, i.e., $\gamma_u(\mathscr{A})$ and $\gamma_d(\mathscr{A})$ (see Fig. 5). Note that $\mathscr{L}_1$ and $\mathscr{L}_2$ divide $\mathscr{A}$ into four convex quadrilateral areas, each containing precisely one of the vertices of the original quadrilateral $\partial\mathscr{A}$.

Definition 2.1. Let the straight line, $\mathscr{L}_i, i \in \{1,2\}$, in the plane of the parametric domain $\mathscr{D}$, be defined by the implicit equation $a_i u + b_i v + c_i = 0$. We say that point $(u_1, v_1) \in \mathscr{A}$ is above the line $\mathscr{L}_i$, if $a_i u_1 + b_i v_1 + c_i > 0$ and that point (u_2, v_2) is below the line $\mathscr{L}_i$ if $a_i u_2 + b_i v_2 + c_i < 0$.

Denote by *Upper* $(\mathscr{A})$ the subset of $\mathscr{A}$, which is located *above* the line $\mathscr{L}_1$ and by *Lower* $(\mathscr{A})$ the subset of $\mathscr{A}$ that is *below* the line $\mathscr{L}_1$. In a similar way, the set *Right*$(\mathscr{A})$ is the subset of $\mathscr{A}$ above the line $\mathscr{L}_2$, and the set *Left*$(\mathscr{A})$ is the subset of $\mathscr{A}$ below the line $\mathscr{L}_2$.

If line $\mathscr{L}_1$ is parallel to the u-axis and passes through the centers of the edges $\gamma_l(\mathscr{A})$ and $\gamma_r(\mathscr{A})$, then the set of all points of $\mathscr{A}$ located above $\mathscr{L}_1$ will be denoted by *Upper*$(\mathscr{A})$, and the set of all points of $\mathscr{A}$ below the line $\mathscr{L}_1$ by *Lower*$(\mathscr{A})$. Analogously, if $\mathscr{L}_2$ is parallel to the v-axis and passes through the centers of the segments $\gamma_u(\mathscr{A})$ and $\gamma_d(\mathscr{A})$, then *Right*$(\mathscr{A})$ is the set of all points of $\mathscr{A}$ above the line $\mathscr{L}_2$ and *Left*$(\mathscr{A})$ is the set of all points of $\mathscr{A}$ located below the line $\mathscr{L}_2$.

The upper, lower, right and left edges of $\mathscr{A}$ serve as the upper, lower, right and left edges for the corresponding subregion of $\mathscr{A}$ (i.e. for the subsets $\{Upper(\mathscr{A}) \cap Left(\mathscr{A})\}$, $\{Upper(\mathscr{A}) \cap Right(\mathscr{A})\}$, $\{Lower(\mathscr{A}) \cap Left(\mathscr{A})\}$ and $\{Lower(\mathscr{A}) \cap Right(\mathscr{A})\}$, which are incident with these edges). The remaining edges of those areas lie on the lines $\mathscr{L}_0$ and $\mathscr{L}_1$ (see Fig. 6).

Finally, denote by $\mathscr{G}_i(\mathscr{A})$, the Gauss map of the subregion of $S_i(u,v)$, defined by the parametric domain

$\mathscr{A} \subseteq \mathscr{D}$:

$$\mathscr{G}_i(\mathscr{A}) = \{(x,y,z) \in \mathscr{S}^2\colon \ \exists (u,v) \in \mathscr{A}, \\ N_i(u,v) = (x,y,z)\}. \tag{2.7}$$

2.1. Distance between the two sets of points in the parametric domain

Consider the distance between the set of points $\mathscr{A}$ in the parametric domain $\mathscr{D}_0$ of the first surface $S_0(u,v)$ and one point (u_0,v_0) in the parametric domain $\mathscr{D}_1$ of the second surface $S_1(u,v)$ as the distance between the normal $\mathbf{N}_1(u_0,v_0)$ and the Gauss Map of $\mathscr{A}$:

$$dist((u_0,v_0),\mathscr{A}) = \inf_{(u,v)\in\mathscr{A}} \arccos(\langle \mathbf{N}_1(u_0,v_0), \mathbf{N}_0(u,v)\rangle). \tag{2.8}$$

As was shown in [13], the matching of the relative parameterizations of two free form curves, and as suggested in [5], the existence of a *valid* reparameterization between the given source and target curves can guarantee a self-intersection free metamorphosis between the two curves.

Definition 2.2. A match $s\colon [a_0,b_0] \to [a_1,b_1]$ for the curves $\gamma_0\colon [a_0,b_0] \to \mathbb{R}^3$ and $\gamma_1\colon [a_1,b_1] \to \mathbb{R}^3$ is said to be valid if for every $t \in [a_0,b_0]$ the angle between the vectors $(\mathrm{d}/\mathrm{d}t)\gamma_0(t)$ and $(\mathrm{d}/\mathrm{d}t)\gamma_1(s(t))$ is less than $\pi/2$.

Let us consider the same criterion for the normals of the source and target surfaces. If the angle between the normal $\mathbf{N}_1(u_0,v_0)$ and the normal $\mathbf{N}_0(u,v), (u,v)\in\mathscr{A}$ is greater than $\pi/2$, i.e. the match is not valid (see Definition 2.2) and one could use some (relatively large) penalty as a replacement for the distance between them. We define the distance between the two sets of points in the following way:

Definition 2.3. The *distance* between a finite set of points A in the parametric domain of the first surface $S_0(u,v)$ and a *finite* set of points B in the parametric domain of the second surface $S_1(u,v)$ is equal to

$$dist(A,B) = \frac{1}{|B|}\sum_{(u,v)\in B} dist((u,v),A). \tag{2.9}$$

Note that the *distance* defined by (2.9) is *not* symmetric, thus it is not a proper metric.

2.2. Matching algorithm

Let us introduce a method for matching the points $\mathscr{P}^0_{ij} = (u_i,v_j)$ sampled on a grid in the parametric domain of the first surface, $S_0(u,v)$, and points $\mathscr{P}^1_{kl} = (s_k,t_l), 0 \leqslant i,j,k,l \leqslant K-1$ on a grid in the parametric domain at the second surface, $S_1(u,v)$, following Eqs. (2.2)–(2.4). This method is, in fact, hierarchical and works in several resolutions, performing the steps described in Sections 2.2.1–2.2.3. We shall employ a matching criterion of the normal fields, as our example.

2.2.1. Matching of the horizontal isoparametric curve

Find the straight line $\mathscr{L}_1$, connecting one point of $\gamma_1(\mathscr{D})$ with some point of $\gamma_r(\mathscr{D})$ on the parametric domain of the surface $S_1(u,v)$, such that both the values of,

$$dist(\mathscr{G}_0(\overline{Upper(\mathscr{D})}), \mathscr{G}_1(Upper(\mathscr{D})))$$

and

$$dist(\mathscr{G}_0(\overline{Lower(\mathscr{D})}), \mathscr{G}_1(Lower(\mathscr{D}))),$$

are minimized simultaneously. See Section 2.1, for the definition of the distance between the set of points in the parametric domain $\mathscr{D}_0$ of the first surface and the subset of the parametric domain $\mathscr{D}_1$ of the second surface. There are $O(K^2)$ options for such line locations, we need to calculate these distances for each such line and select the best one.

2.2.2. Matching of the vertical isoparametric curve

Find the straight line $\mathscr{L}_2$, connecting one point of $\gamma_u(\mathscr{D})$ with some point of $\gamma_d(\mathscr{D})$ on the parametric domain

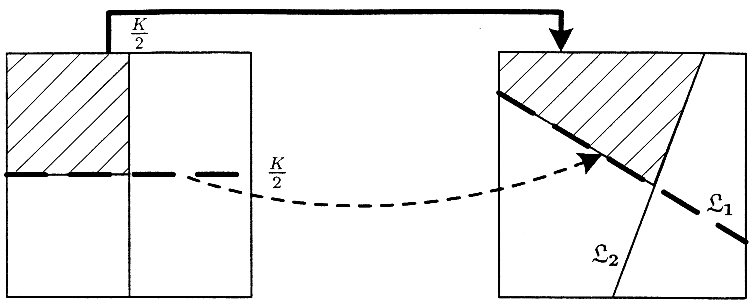

Fig. 7. Matching the quarters of the parametric domains on the first level. The dashed curve on the first domain (left) is matched with the dashed curve on the second domain (right).

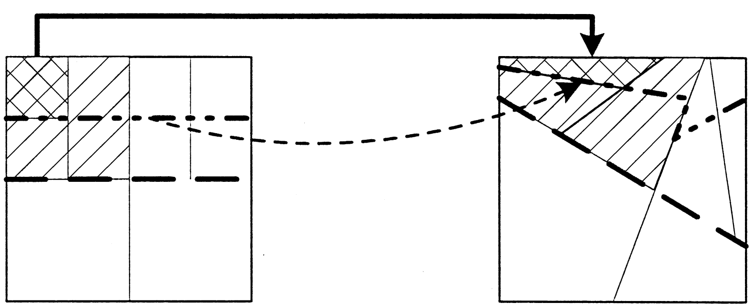

Fig. 8. Matching the quarters of the parametric domains on the second level. Note the dotted–dashed curves: the straight line on the first domain (left) is matched with the piecewise linear curve comprising *three* linear segments on the second domain (right).

of the surface $S_1(u, v)$, such that the four following distances are simultaneously minimized:

$$dist(\mathcal{G}_0(\overline{Upper(\mathcal{D})} \cap \overline{Left(\mathcal{D})}), \mathcal{G}_1(Upper(\mathcal{D}) \cap Left(\mathcal{D}))),$$

$$dist(\mathcal{G}_0(\overline{Upper(\mathcal{D})} \cap \overline{Right(\mathcal{D})}, \mathcal{G}_1(Upper(\mathcal{D}) \cap Right(\mathcal{D}))),$$

$$dist(\mathcal{G}_0(\overline{Lower(\mathcal{D})} \cap \overline{Left(\mathcal{D})}), \mathcal{G}_1(Lower(\mathcal{D}) \cap Left(\mathcal{D}))),$$

$$dist(\mathcal{G}_0(\overline{Lower(\mathcal{D})} \cap \overline{Right(\mathcal{D})}), \mathcal{G}_1(Lower(\mathcal{D}) \cap Right(\mathcal{D}))).$$

2.2.3. Recursion

Recall that K is the number of points in one of the directions (u or v) in the square parametric domain of the first surface, following Eq. (2.4).

If $K = 2$, then there is only one sample point $\mathcal{P}^0_{ij}$ in each of the four quarters of the parametric domain of the first surface $S_0(u, v)$:

$$\{\overline{Upper(\mathcal{D})} \cap \overline{Left(\mathcal{D})}\}, \quad \{\overline{Upper(\mathcal{D})} \cap \overline{Right(\mathcal{D})}\},$$
$$\{\overline{Lower(\mathcal{D})} \cap \overline{Left(\mathcal{D})}\}, \quad \{\overline{Lower(\mathcal{D})} \cap \overline{Right(\mathcal{D})}\}.$$

One should select the corresponding point $\mathcal{P}^1_{ij} = (s(u_i, v_j), t(u_i, v_j))$ for $\mathcal{P}^0_{ij}$ and associate it with a quarter in the parametric domain of the second surface $S_1(u, v)$.

For $K > 2$, recursively perform steps 2.2.1–2.2.3 on the domains

$$\{\overline{Upper(\mathcal{D})} \cap \overline{Left(\mathcal{D})}\}, \quad \{\overline{Upper(\mathcal{D})} \cap \overline{Right(\mathcal{D})}\},$$
$$\{\overline{Lower(\mathcal{D})} \cap \overline{Left(\mathcal{D})}\}, \quad \{\overline{Lower(\mathcal{D})} \cap \overline{Right(\mathcal{D})}\}$$

of surface $S_0(u, v)$ and domains

$$\{Upper(\mathcal{D}) \cap Left(\mathcal{D})\}, \quad \{Upper(\mathcal{D}) \cap Right(\mathcal{D})\},$$
$$\{Lower(\mathcal{D}) \cap Left(\mathcal{D})\}, \quad \{Lower(\mathcal{D}) \cap Right(\mathcal{D})\}$$

of surface $S_1(u, v)$.

Examine the *preimage* of the matched isoparametric curves in the parametric domains of the surfaces. The

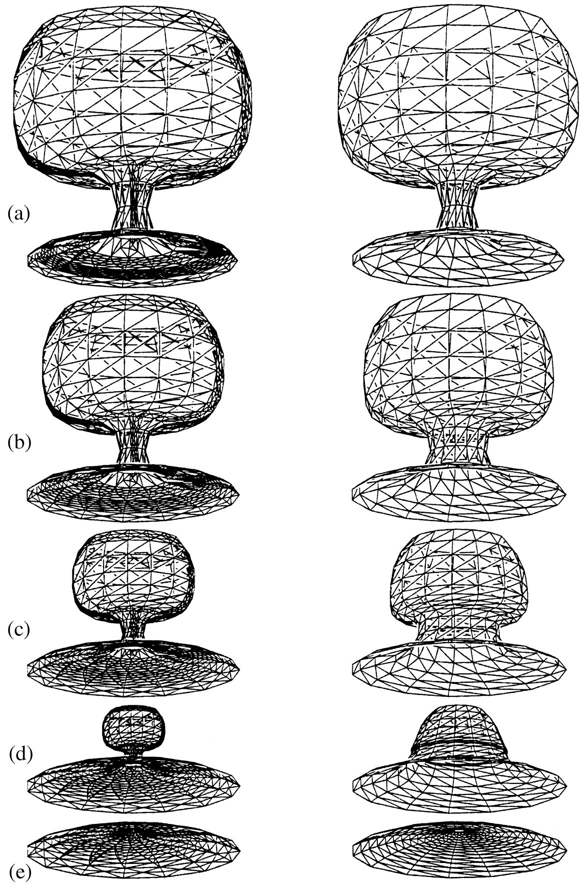

Fig. 9. Metamorphosis after matching the relative parameterizations of the two key surfaces (left column) and before matching (right column). Note that the neck of the wine glass remains slim in the left column and the lower part of the glass is matched with the dish surface, which resemble each other, while this is not true for the other version of the metamorphosis (right column). In (d), one can see that the neck of the glass has completely vanished, and the shape does not resemble neither the glass (a), nor the curved disk of (e).

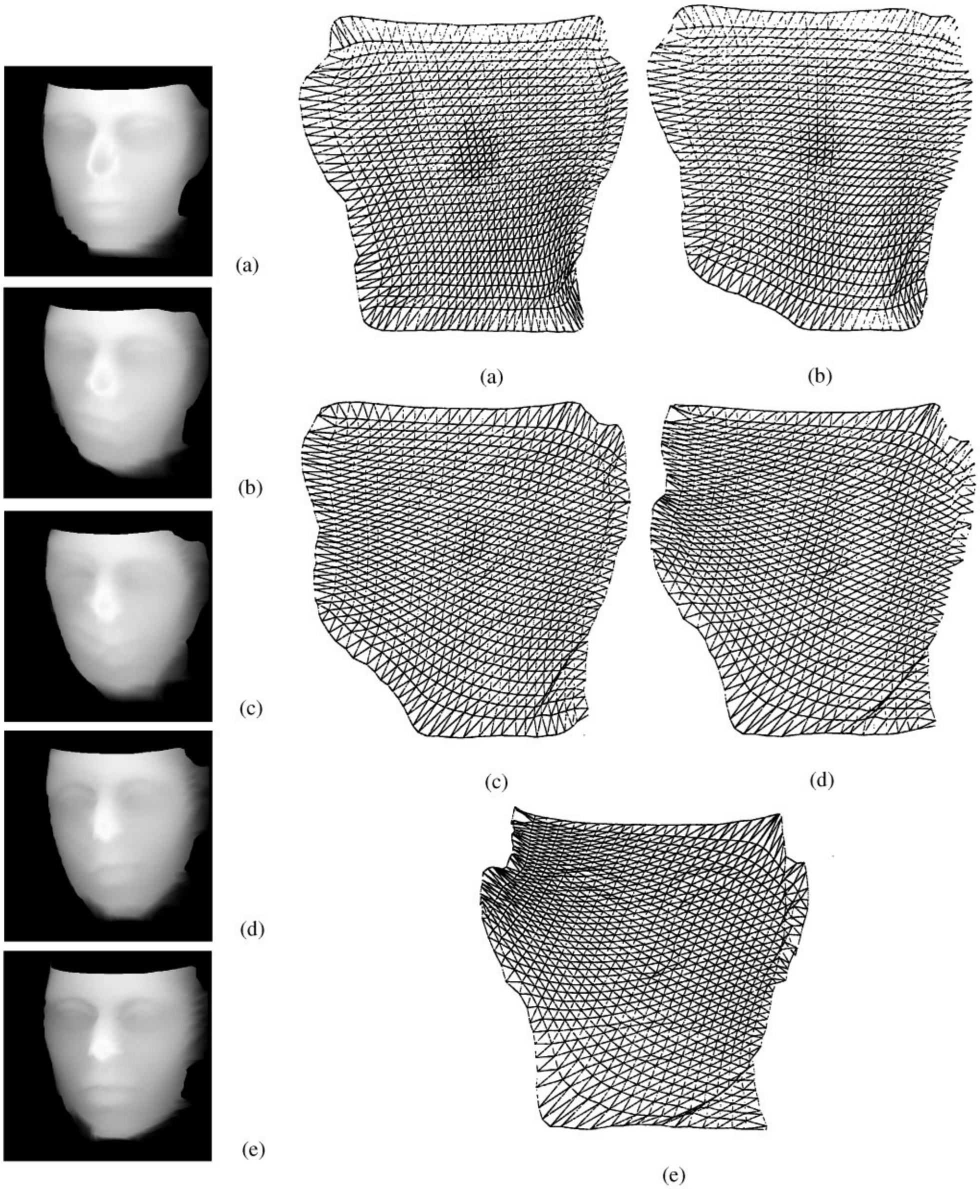

Fig. 10. Metamorphosis after matching the relative parameterizations of the two key face looking surfaces. Note the rotation of the isoparametric lines on the faces, while the face feature (nose, eyes) stay put.

matching problem of the two surfaces $S_0(u,v)$ and $S_1(u,v)$ is transformed to the problem of the matching of the two parametric domains. In the previous section, we matched the curve $(u,\frac{1}{2}), u \in [0,1]$ in the parametric domain of surface $S_0(u,v)$ with some straight line connecting a point on $\gamma_l(\mathscr{D})$ with a point on $\gamma_r(\mathscr{D})$ on the parametric domain of the surface $S_1(u,v)$ (see Fig. 7). The curve $(u,\frac{1}{4}), u \in [0,1]$ is matched with the polylines, comprising exactly three linear segments: the first one is the quarter $\{\overline{Upper(\mathscr{D})} \cap \overline{Left(\mathscr{D})}\}$, the third one in the quarter $\{\overline{Upper(\mathscr{D})} \cap \overline{Right(\mathscr{D})}\}$ and the second segment is connecting the previous two (see Fig. 8).

Several optimizations may be applied to the matching algorithm in order to decrease the computation time. In this work, we employed the following method to decrease the search time of the closest vector on the Gaussian sphere.

2.2.4. Projecting the Gaussian sphere

Consider the following problem: find the distance between one fixed normal vector and the Gauss Map of

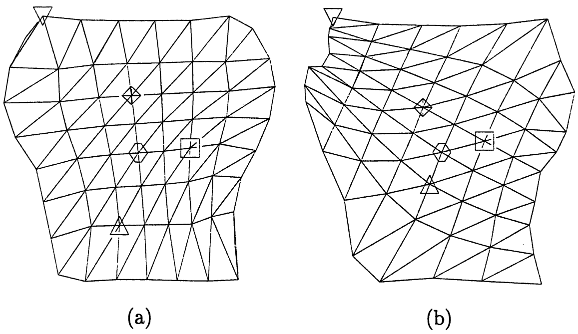

Fig. 11. Three pairs of matched points for low resolution faces.

some surface. Here, the distance can be computed as

$$dist(\mathbf{N}, \mathscr{G}(S)) = \inf_{\mathbf{n} \in \mathscr{G}(\mathscr{S})} \arccos(\langle \mathbf{N}, \mathbf{n} \rangle). \tag{2.10}$$

Note that the above metric is equivalent to the following:

$$\widetilde{dist}(\mathbf{N}, \mathscr{G}(S)) = \inf_{\mathbf{n} \in \mathscr{G}(\mathscr{S})} 1 - \langle \mathbf{N}, \mathbf{n} \rangle. \tag{2.11}$$

In order to decrease the time of this search, one may preprocess the data comprising the normals of Gauss Map into data structures that would minimize the query time. One can aim at organizing the Gaussian sphere into a two-dimensional hash table (see [3]), possibly via a central projection, with an access to the ε-neighborhood of each vector on the sphere in constant time, i.e. $O(1)$. The search of the normals which are close to a given vector on the Gaussian sphere is another operation that should be conducted efficiently. The task has been reduced to the problem of a point locating in the plane.

Such an optimization could reduce the complexity of the algorithm by the factor of K^2.

3. Examples

We present some simple examples of matching two parametric surfaces using the approach, described in Section 2.

One example has already been discussed in Section 1.2, Figs. 1–3. There, we matched the hump shaped feature of the two given surfaces.

In Fig. 9, one can find two more examples of metamorphosis sequences for surfaces with matched relative parameterizations.

One interesting example of reparameterization as computed by the presented algorithm may be found in Fig. 10. Note that the parameterization of (e) is a rotated version of the parameterization of (a). In Fig. 11, one may see several pairs of matched points on the low-resolution key surfaces.

4. Conclusions and future work

In this work, a new algorithm for matching the relative reparameterizations of two free form parametric surfaces has been defined. The algorithm deals with the simple surfaces that are topologically equivalent to disk. The method creates and uses a discrete representation of the two key surfaces using the information contained in the original data, i.e. the normal fields of the given surfaces.

We plan to continue our research in the following directions. Assume that both the source and destination surfaces are convex. In this special case and following the method of Section 2.2, one is not required to measure the resemblance of the surfaces by the distance that was presented by the Definition 2.3. It is enough to compare the *boundaries of the regions* in order to evaluate the quality of the match. As was shown in [5], for every fixed location of the lines $\mathscr{L}_1$ and $\mathscr{L}_2$, we may evaluate the best match for the boundaries of the four regions in time $O(K^2)$.

Thus, the surfaces can be decomposed into the maps of convex, concave and hyperbolic regions as a preprocessing stage, before the matching process [18]. If the two convex/concave regions were matched, then we may employ the quality of the match of the boundaries to measure the resemblance of the surfaces. Otherwise (i.e. both regions are neither convex nor concave simultaneously), the regions are to be matched by the previous algorithm that is described in Section 2.2.

Our approach does not find a global optimal solution but only approximates it using a greedy scheme. Some other improvements may be considered using a multiresolution approach. In this way, we expect to further reduce the time of the computation and improve the quality of the match.

Finally, the presented algorithm is to be extended for the general simple surfaces in the future, i.e. for the closed surfaces that are topologically equivalent to a sphere or a cylinder.

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