Technion - Israel Institute of Technology
Computer Science Department

The M^* Algorithm: Incorporating Opponent Models into Adversary Search

by

David Carmel and Shaul Markovitch

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David Carmel and Shaul Markovitch [*]

Computer Science Department
Technion, Haifa 32000
Israel
carmel@cs.technion.ac.il shaulm@cs.technion.ac.il

Abstract

While human players adjust their playing strategy according to their opponent, computer programs, which are based on the minimax algorithm, use tha same playing strategy against a novice as against an expert. This is due to the assumption of minimax that the opponent uses the same strategy as the player. This work studies the problem of opponent modelling in game playing. We recursively define a player as a pair of a strategy and an opponent model, which is also a player. A strategy can be determined by the static evaluation function and the depth of search. M^*, an algorithm for searching game-trees using an n-level modelling player that uses such a strategy, is described and analyzed. We demonstrate experimentally the benefit of using an opponent model and show the potential harm caused by the use of an inaccurate model. We then describe an algorithm, M^*_ϵ, for using uncertain models when a bound on the model error is known. Pruning in M^* is impossible in the general case. We prove a sufficient condition for pruning and present a pruning algorithm, $\alpha\beta^*$, that returns the M^* value of a tree, searching only necessary subtrees. Finally, we present a method for acquiring a model for an unknown player. First, we describe a learning algorithm that acquires a model of the opponent's depth of search by using its past moves as examples. Then, an algorithm for acquiring a model of the player's strategy, both depth and function, is described and evaluated. Experiments with this algorithm show that when a superset of the set of features used by a fixed opponent is available to the learner, few examples are sufficient for learning a model that agrees almost perfectly with the opponent.

1 Introduction

"...At the press conference, it quickly became clear that Kasparov had done his homework. He admitted that he had reviewed about fifty of DEEP THOUGHT's games and felt confident he understood the machine..." [11]

One of the most notable challenges that the Artificial Intelligence research community has been trying to face during the last five decades is the creation of a computer program that can beat the world chess champion. Most activity in the area of game-playing programs has been concerned with efficient ways of searching large game trees. However, good playing performance involves

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additional types of intelligent processes. The quote above highlights one type of such a process that is performed by expert human players: acquiring a model of their opponent's strategy.

While human players adjust their playing strategy according to their opponent, computer programs play the same against a novice as against an expert. Most playing programs use the minimax algorithm for search [19]. The main assumption of this algorithm is that the opponent uses the same strategy as the player.

There are several situations where the modelling approach has advantage over the non-modelling approach of the standard minimax procedure. Jansen [7], describes two situations in which it is important to consider the opponent's playing ability. One is a *swindle* position, where the player has reason to believe that the opponent will underestimate a good move, and will therefore play a poorer move instead. Another situation is a *trap* position, where the player expects the opponent to overestimate and therefore play a bad move. Choosing trap or swindle positions is good strategy when the player has reason to believe that its opponent searches to shallower depth than itself. Another situation, where an opponent model can be beneficial, is a losing position [2]. If all possible moves lead to a loss, minimax chooses one of them arbitrarily, in contrast to human players that can utilize their opponent model in order to select a swindle move.

Several researchers have pointed out the importance of opponent modelling [17, 2, 8, 9, 1, 18], but the acquisition and utilization of an opponent model have not received much attention in the computational game research community. Korf [9] outlined a method of generalizing the minimax algorithm for utilizing multiple-level models of evaluation functions. The work described in this paper builds on Korf's research and expands it in several ways.

The goal of this research is to study the utilization and acquisition of opponent models in game-playing. In order to do so, we will make an attempt to find answers to the following questions:

1. What is a model of opponent's strategy?
2. Assuming that we possess such a model, how can we utilize it?
3. What are the potential benefits of using opponent models?
4. How does the accuracy of the model effect its benefit?
5. How can we use an uncertain model?
6. How can a program acquire a model of its opponent?

Section 2 of this paper deals with the first two questions: Defining an opponent model and developing algorithms for using such a model. Section 3 deals with the third and the fourth questions: Measuring the potential benefits of modelling and testing the effects of modelling accuracy on its benefits. In section 4, we describe an algorithm for using uncertain models. Section 5 describes a method for incorporating pruning into the algorithms. Section 6 describes algorithms for acquiring opponent models. Finally, section 7 concludes.

2 Using opponent models

Every human player has either an explicit or an implicit model of the way that its opponent plays. In this section we give a precise definition of an opponent model and develop an algorithm for using such a model.

Definition 1 *A* player *is defined by the strategy that it uses and by the model of its opponent. The opponent model is also a player.*

1. *Given a strategy S, $P = (S, NIL)$ is a* player.
2. *Given a strategy S and a* player *O, $P = (S, O)$ is a* player.

The first element of a player is called the player's strategy *and the second element is called the* opponent model.

Definition 2 *Given a player $P = (S, O)$. The* modelling level *of a player is defined by the recurrence:*

$$ML(P) = \begin{cases} 0 & \textit{if } O = NIL \\ ML(O) + 1 & \textit{otherwise} \end{cases} \qquad (1)$$

Thus, a zero-level modelling player, (S_0, NIL), is one that does not model its opponent. A one-level modelling player, $(S_1, (S_0, NIL))$, is one that has a model of its opponent, but assumes that its opponent is a zero-level modelling player. A two-level modelling player, $(S_2, (S_1, (S_0, NIL)))$, is one that uses a strategy S_2, and has a model of its opponent, $(S_1, (S_0, NIL))$. The opponent's model uses a strategy S_1 and has a model, (S_0, NIL), of the player. The recursive definition of a player is in the spirit of the Recursive Modelling Method (RMM) by Gmytrasiewicz, Durfee and Wehe [15].

2.1 The M^* algorithm

Most of the game-playing programs use a minimax search procedure in which the player evaluates boards by a function f, and believes that the opponent evaluates boards by the function $-f$. Assume that the player uses a function f_1, but believes that the opponent uses a different function f_0. What is a good search strategy that incorporates this belief? What is a good search strategy when the player uses a function f_2, believes that the opponent uses function f_1, and also believes that the opponent believes that the player evaluates boards using a third function f_0?

We have developed an algorithm, M^*, shown in figure 1, that can handle such modelling to any level[1]. Let us assume that a playing strategy is determined by a static evaluation function f. A *player* is then, according to definition 1, a pair (f, MODEL) where MODEL is also a player. The input for the algorithm is a position, a depth limit, and a player, and the output is the move selected by the player and its value.

The algorithm generates the successor boards and simulates the opponent's search from each of them in order to anticipate its choice. This simulation is done by applying the algorithm recursively with the opponent model as the player. The player then evaluates each of its optional moves by evaluating the outcome of its opponent's reaction by applying the algorithm recursively using its own strategy.

Figure 2 shows an example for a search tree spanned by $M^*(a, 3, f_2(f_1, f_0))$. The numbers at the bottom are the static values of the leaves. The recursive calls applied to each node are listed next to the node. The dashed lines indicate what move is selected by each recursive call.

[1]We have presented an earlier version of the M^* algorithm in a previous publication [4]. However, the earlier version was limited to one-level modelling players.

```
Procedure $M^*\,(pos, depth, (f_{pl}, \mathrm{OPP\_MODEL}))$
  if $depth = 0$
    return $\langle NIL, f_{pl}(pos)\rangle$
  else
    $SUCC \leftarrow MoveGen(pos)$
    for each $succ \in SUCC$
      if $depth = 1$
        $player\_value \leftarrow f_{pl}(succ)$
      else
        $\langle opp\_board, opp\_value\rangle \leftarrow M^*\,(succ, depth-1, \mathrm{OPP\_MODEL})$
        $\langle player\_board, player\_value\rangle \leftarrow M^*\,(opp\_board, depth-2, (f_{pl}, \mathrm{OPP\_MODEL}))$
      if $player\_value > max\_value$
        $max\_value \leftarrow player\_value$
        $max\_board \leftarrow succ$
    return $\langle max\_board, max\_value\rangle$
```

Figure 1: The M^* algorithm

The player simulates its opponent's search from nodes b and c. The opponent simulates the player by using its own model of the player from nodes d and e. At node d the model of the player used by the opponent (f_0) selects node h. The opponent then applies its f_1 function to node h and concludes that node h, and therefore node d, are worth -6. The opponent then applies the player's model (f_0) to node e, concludes that the player will select node j, applies its own function (f_1) to node j, and decides that node j, and therefore node e, are worth -8. Therefore, the opponent model, when applied to node b, selects the move that leads to node d. The player then tests how much node d is worth according to its criterion (f_2). It applies M^* to node d and concludes that node d, and therefore node b, are worth 8. Using a similar search from node c yields 10. Therefore, the player selects the move that leads to c with a value of 10. Note that using a regular minimax search with f_2 would have resulted in selecting the move leads to node b with a value of 7.

How does minimax fit into our new algorithm? It turns out that a minimax search to depth d with evaluation function f is in fact equivalent to the M^* algorithm called with the d-level modelling player $\overbrace{(f,(-f,(f,\ldots)))}^{d}$.

There is one potential complication that deserves special handling. Each simulation of the opponent's search involves a call to M^* with a lower level modelling player. What will the procedure do if the modelling level of the original player n is smaller than the depth of the search tree d? In such a case, we assume that the 0-level modelling player f_0 is playing minimax, and therefore replace it by $\overbrace{(f_0,(-f_0,(f_0,\ldots)))}^{d-n}$.

It is obvious that the M^* algorithm does multiple expansions of parts of the search tree. For analyzing the complexity of M^*, let us assume that the search tree has a uniform branching factor b. The number of leaves spanned by M^* for such a tree with depth d can be measured by the

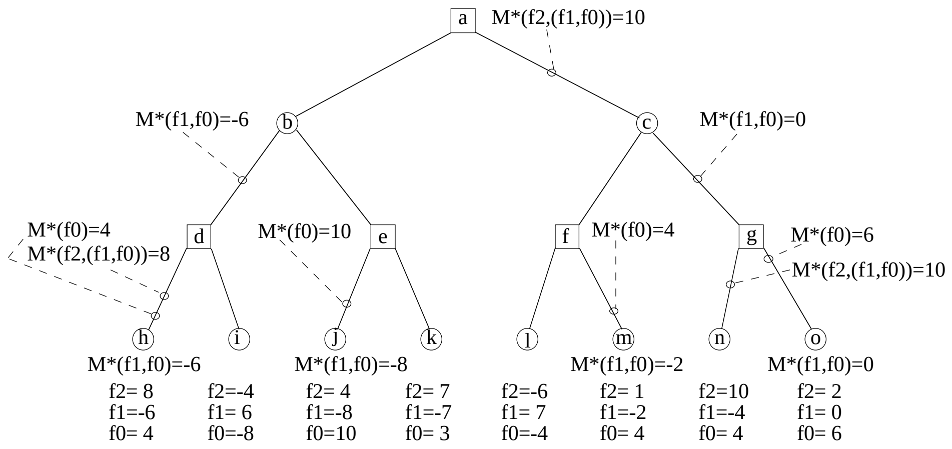


Figure 2: The set of recursive calls generated by calling $M^*(a, 3, (f_2(f_1, f_0)))$. For sake of clarity we have included only the player in the parameter list of the calls. Each call is written next to the node it is called from. The dashed lines show what move is selected by each call.

following recurrence :

$$(2) \qquad T_b(d) = \begin{cases} 1 & \text{if } d = 0 \\ b & \text{if } d = 1 \\ b\left[T_b(d-1) + T_b(d-2)\right] & \text{otherwise} \end{cases}$$

Lemma 1 *1. The number of leaves spanned by M^* for a tree with uniform branching factor b and depth d is*

$$(3) \qquad T_b(d) = \frac{\phi_b^{d+1} - \overline{\phi_b}^{d+1}}{\sqrt{b^2+4b}}$$

where $\phi_b = \frac{b+\sqrt{b^2+4b}}{2}$, and $\overline{\phi_b} = \frac{b-\sqrt{b^2+4b}}{2}$ [2]

2. $T_b(d) \leq \phi_b^d \leq (b+1)^d$

Proof:

By induction on d: It is easy to verify equation 3 for $d = 0$ and $d = 1$. Assume it's correctness for depths less than d.

$$(4) \qquad T_b(d) = b\left[T_b(d-1) + T_b(d-2)\right] = b\left[\frac{\phi_b^d - \overline{\phi_b}^d}{\sqrt{b^2+4b}} + \frac{\phi_b^{d-1} - \overline{\phi_b}^{d-1}}{\sqrt{b^2+4b}}\right] =$$

$$= \frac{b}{\sqrt{b^2+4b}}\left[(\phi_b+1)\phi_b^{d-1} - (\overline{\phi_b}+1)\overline{\phi_b}^{d-1}\right]$$

It is also easy to verify that $b(\phi_b + 1) = \phi_b^2$ and $b(\overline{\phi_b} + 1) = \overline{\phi_b}^2$. Placing the two equalities in equation 4 completes the proof of the first statement. For the second statement, the first inequality will be proved by induction on d. It is easy to verify the inequality for $d = 0$ and $d = 1$. Assume its correctness for *depth* $< d$.

$$(5) \qquad T_b(d) = b\left[T_b(d-1) + T_b(d-2)\right] \leq b\left[\phi_b^{d-1} + \phi_b^{d-2}\right] = b\left[(\phi_b+1)\phi_b^{d-2}\right] = \phi_b^d$$

[2]For $b = 1$ we get the Binet formula for Fibonacci sequence.

The second inequality follows from the inequality $\forall b[\phi_b \leq b+1]$. □

Therefore, the number of leaves spanned by M^* is bounded by $(b+1)^d$, while minimax spans b^d leaves. For games with large branching factors, the overhead of using M^* (compared to minimax) is therefore neglectable, while for games with small branching factors it is quite significant.

2.2 A one-pass version of M^*

We have developed another version of the M^* algorithm, called M^*_{1-pass}, that expands the tree one time only, just as minimax does. The algorithm expands the search tree in the same manner as minimax. However, node values are propagated differently. Whereas minimax propagates only one value, M^* propagates $n+1$ values, $(V_n, \ldots, V_0)$. The value V_i represents the merit of the current node according to the i-level model, f_i. M^* passes values to V differently for values associated with the player and values associated with the opponent. In a player's node (a node where it is the player's turn to play)[3], for values associated with the player $(V_n, V_{n-2}, \ldots)$, V_i gets the maximal V_i value among its children. For values associated with the opponent $(V_{n-1}, V_{n-3}, \ldots)$, V_i gets the V_i value of the child that gave the maximal value to V_{i-1}. For example, the opponent believes (according to the model) that the player evaluates nodes by V_{n-2}. At a player's node, the opponent assumes that the player will select the child with maximal V_{n-2} value. Therefore, the value of the current node for the opponent is the V_{n-1} value of the selected child with the maximal V_{n-2} value. At an opponent's node, we do the same but the roles of the opponent and the player are switched. Figure 3 lists the M^*_{1-pass} algorithm.

```
Procedure $M^*_{1-pass}(pos, depth, (f_n, (f_{n-1}, (\ldots, f_0)\ldots)))$
   if $depth = 0$
      return $\langle f_n(pos), \ldots, f_0(pos)\rangle$
   else $SUCC \leftarrow MoveGen(pos)$
        for each $succ \in SUCC$
            succ_V $\leftarrow M^*_{1-pass}(succ, depth-1, (f_n, (f_{n-1}, \ldots, f_0)\ldots))$
            for each $i$ associated with current player
                if succ_V[$i$] $> V[i]$
                   $V[i] \leftarrow$ succ_V[$i$]
                   if $i < n$
                      $V[i+1] \leftarrow$ succ_V[$i+1$]
        return $\langle V[n], \ldots, V[depth]\rangle$
```

Figure 3: M^*_{1-pass}: A version of the M^* algorithm that performs only one pass over the search tree

The *maxn* algorithm [13] also propagates vectors of values up the game tree. However, M^*_{1-pass} and *maxn* are targeted at different goals. *Maxn* is an extension of minimax that can handle n players while M^*_{1-pass} is an extension of minimax that can handle n-level modelling. While all the values of a vector in *maxn* come from the same leaf, the values of a vector in M^*_{1-pass} come from different leaves. It should be relatively easy to combine the two algorithms. Figure 4 shows an example for a tree spanned by M^*. Note that the values in the vectors correspond to the results of the recursive calls in figure 2.

[3]Traditionally such a node is called a MAX node. However, we assume that both players are maximizers

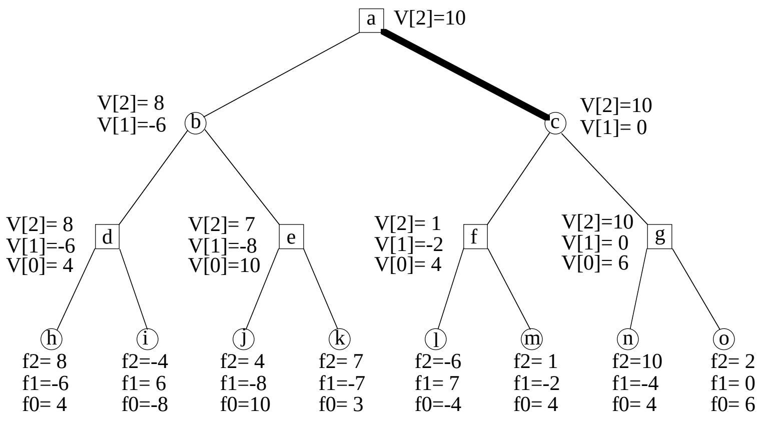


Figure 4: The value vectors propagated by M^*_{1-pass}.

Lemma 2 *Assume that PLAYER is a n-level modelling player. Let* $\langle v, b\rangle = M^*(pos, depth, PLAYER)$*, and let* $\langle V[n]\rangle = M^*_{1-pass}(pos, depth, PLAYER)$*. Then* $v = V[n]$*.*

Proof:

For $d = 0$ and $d = 1$ the proof is immediate.

Assume that M^* and M^*_{1-pass} return the same value for a tree of depth $d \leq depth - 1$. We will prove that they return the same value for $d = depth$.

1. For each successor at level 1, M^* first determines the board at level 2 selected by its opponent. By the induction hypothesis,

$$M^*(succ, d-1, \text{OPP\_MODEL}) = M^*_{1-pass}(succ, d-1, \text{OPP\_MODEL}) = V[n-1]$$

and therefore these will be the boards with the maximal $V[n-1]$.

2. M^* determines the value of each successor by calling itself on the board selected by the opponent. By the induction hypothesis

$$M^*(b, d-2, PLAYER) = M^*_{1-pass}(b, d-2, PLAYER) = V[n]$$

and therefore these M^* values will be equal to the $V[n]$ values of these nodes.

3. M^* associates with each successor the player's value of the board selected by the opponent. M^*_{1-pass} assigns to each successor the $V[n]$ value associated with the board with maximal $V[n-1]$ value. Thus, the value assigned to each successor by M^* is equal to its $V[n]$ value.

4. M^* returns the maximal value for its successors, while M^*_{1-pass} returns the maximal $V[n]$ of its successors. These are the same values.

□

2.3 The relationship between M^* and minimax

An interesting property of M^* is that it always selects a move with a value greater or equal to the value of the move selected by minimax that searches the same tree with the same strategy.

Lemma 3 *Assume that M^* and Minimax use the same f_{player}. Then*

$$(6) \qquad Minimax(pos, depth, f_{player}) \leq M^*(pos, depth, (f_{player}, OPP\_MODEL))$$

for any OPP_MODEL.

Proof:

We will prove that this property exists for every node in the tree spanned by the two algorithms by induction on the depth of search. For convenience we will prove it for the version of M^*_{1-pass}.

For $depth = 0$, for any OPP_MODEL

$$(7) \qquad Minimax(pos, depth) = f_{player}(pos) = M^*_{1-pass}(pos, depth, (f_{player}, \mathrm{OPP\_MODEL})$$

Let us assume correctness for $d \leq depth$ and prove it for $d = depth + 1$
If pos is a player's node,

$$(8) \qquad Minimax(pos, depth+1) = \max\{Minimax(succ, depth) \mid succ \in \mathrm{SUCC}(pos)\}$$

According to the inductive assumption,

$$\leq \max\left\{M^*_{1-pass}(succ, depth, (f_{player}, \mathrm{OPP\_MODEL})) \mid succ \in \mathrm{SUCC}(pos)\right\}$$
$$= M^*_{1-pass}(pos, depth+1, (f_{player}, \mathrm{OPP\_MODEL}))$$

If pos is an opponent's node,
let s be the successor with the maximal value according to the opponent's model.

$$(9) \qquad Minimax(pos, depth+1) = \min\{Minimax(succ, depth) \mid succ \in \mathrm{SUCC}(pos)\}$$

$$\leq Minimax(s, depth)$$

According to the inductive assumption,

$$\leq M^*_{1-pass}(s, depth, (f_{player}, \mathrm{OPP\_MODEL}))$$
$$= M^*_{1-pass}(pos, depth+1, (f_{player}, \mathrm{OPP\_MODEL})) \quad \Box$$

The intuition behind the above lemma is that minimax assumes an adversary model of the opponent. It assumes that the opponent knows the player's strategy, and its only interest is to select moves that are worst according to the player's strategy. The lemma says, that if you have a *good* reason to believe that your opponent's model is different than that, you could only benefit by using M^* instead of minimax. The reader should note that the above lemma does not mean that M^* selects a move that is better according to some objective criterion, but rather a subjectively better move (from the player's point of view, according to its strategy). If the player does not have a reliable model of its opponent, then playing minimax is a good cautious strategy.

2.4 Incorporating depth of search into the model

The M^* algorithm as well as minimax (which is a special case of M^*), incorporates an implicit assumption about the depth to which the opponent searches. M^* that searches a tree of depth d, assumes that its opponent searches to level $d-1$, and then assumes that its opponent assumes that the player searches to level $d-2$, etc. This is a potentially wrong assumption, but it is a good defensive mechanism. The player assumes that its opponent searches as deep as the player can simulate the opponent's search. It is meaningless for the player to assume that its opponent searches to a deeper level since it can not simulate a search to such depth. However, as was discussed in the introduction, if the player has reason to believe that its opponent searches to a lesser depth, then it can utilize this belief against the opponent (to set traps, for example).

But why does the player have to search to level d_1 if it knows that its opponent searches to level $d_0 \ll d_1$? In order to predict the opponent's selection, it is indeed enough to simulate its search to level d_0. However, in order to *evaluate* the merit of the opponent's selection for the player, it searches as deep as it can.

We have extended M^* to handle models of depth of search. We define a strategy to be a pair (f_{player}, d_{player}). A *player* is then a pair $((f_{player}, d_{player}), \mathrm{MODEL})$ where MODEL is also a player. Figure 5 lists the extended M^* algorithm.

```
**Procedure** $M^*(pos, depth, ((f_{pl}, d_{pl}), \mathrm{OPP\_MODEL}))$
    **if** $depth = 0$
        **return** $\langle NIL, f_{pl}(pos)\rangle$
    **else**
        $SUCC \leftarrow MoveGen(pos)$
        **for** each $succ \in SUCC$
            **if** $depth = 1$
                $player\_value \leftarrow f_{pl}(succ)$
            **else**
                $\langle opp\_board, opp\_value\rangle \leftarrow M^*(succ, \min(depth-1, d_{opp}), \mathrm{OPP\_MODEL})$
                $\langle player\_board, player\_value\rangle \leftarrow M^*(opp\_board, depth-2, ((f_{pl}, d_{pl}), \mathrm{OPP\_MODEL}))$
            **if** $player\_value > max\_value$
                $max\_value \leftarrow player\_value$
                $max\_board \leftarrow succ$
        **return** $\langle max\_board, max\_value\rangle$
```

Figure 5: An extended version of the M^* algorithm that can handle a model of depth

The extended algorithm is different from its simpler version on one point. Whereas the simple algorithm allocates as many resources as possible to the call that simulates the opponent's search (which is the current depth allocation minus one), the extended algorithm allocates the simulation as maby resources as it believes that the opponent would have used. Of course, regardless of the model of its opponent's depth of search, the player should never exceed the total depth limit allocated for the procedure (hence the *min* in the recursive call). Figure 6 shows an example of applying M^* with a player that searches to depth 3 and an opponent model that searches to depth 1.

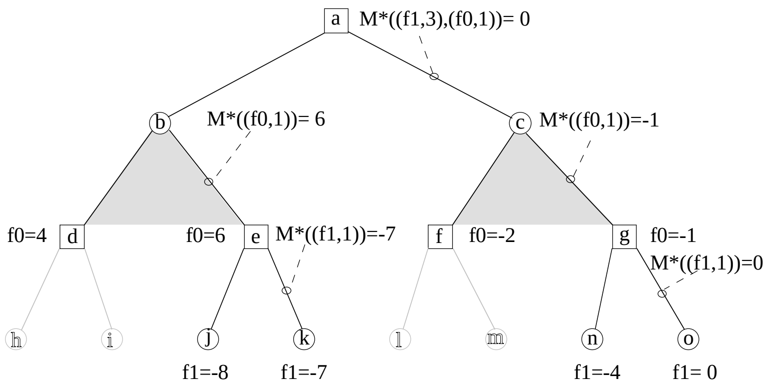


Figure 6: The set of recursive calls generated by calling $M^*(a, 3, ((f_1, 3), (f_0, 1)))$. For sake of clarity we have included only the player in the parameter list of the calls. Each call is written next to the node it is called from. The dashed lines show what move is selected by each call.

As in the former case, this algorithm can also be written in a one-pass form so that the game tree will be expanded one time only. The extended one-pass version is shown in figure 7.

```
**Procedure** $M^*_{1-pass}(pos, depth, ((f_n, d_n), ((f_{n-1}, d_{n-1}), (\ldots, (f_0, d_0))\ldots)))$
  **loop for** $i = n$ downto 0
      $V[i][0] \leftarrow f_i(pos)$
  **if** $depth = 0$
    **return** $\langle V[n], \ldots, V[0] \rangle$
  **else** $SUCC \leftarrow MoveGen(pos)$
      **for** each $succ \in SUCC$
        $succ\_V \leftarrow M^*_{1-pass}(succ, depth-1, ((f_n, d_n), ((f_{n-1}, d_{n-1}), (\ldots, (f_0, d_0))\ldots)))$
        **for** each $i$ associated with current player
          $d = \min(d_i, depth)$
          **for** $j = 1$ to $d$
            **if** $succ\_V[i][j-1] > V[i][j]$
              $V[i][j] \leftarrow succ\_V[i][j-1]$
              **if** $j = d$ and $i < n$
                **for** $k = 1$ to $\min(d_{i+1}, depth)$
                  $V[i+1][k] \leftarrow succ\_V[i+1][k-1]$
      **return** $\langle V[n], \ldots, V[depth] \rangle$
```

Figure 7: An extended non-recursive version of the M^* algorithm that can handle a model of depth

The algorithm propagates vectors of values where each entry is associated with one of the models as before. But while the simplified non-recursive algorithm carries one value per model, the extended algorithm carries a list of values for each model. The $j'th$ element of the $i'th$ list is the value associated with a search to depth j by the $i'th$ model. In the original M^*, there was only one search frontier. Therefore, the non-recursive version propagated only one value for every model. However, the extended M^* can call itself recursively from any node to any depth. Hence, every level in the search tree is potentially a search frontier of one of those recursive calls. That is the reason why the extended non-recursive algorithm carries a list of values for each model. Figure 8 shows the value matrices propagated up the tree when calling M^*_{1-pass} on the same tree as in figure 6.

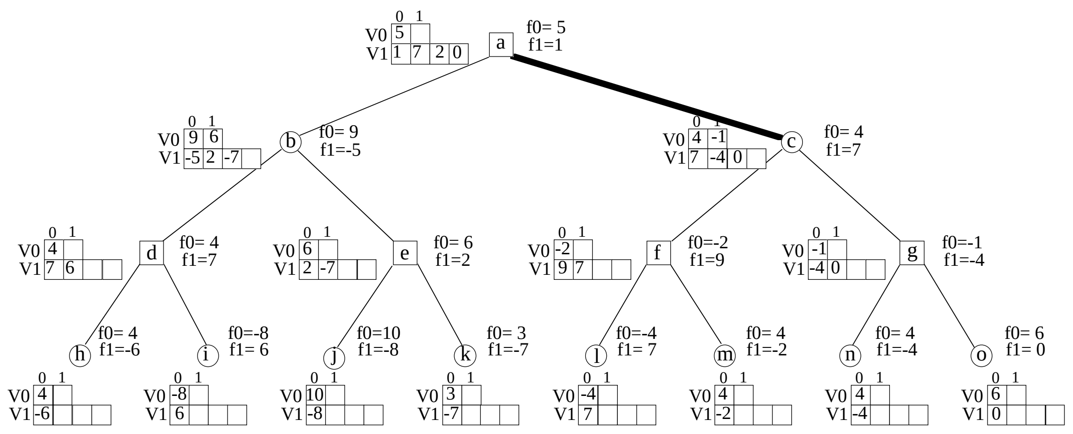


Figure 8: The value matrices propagated by M^*_{1-pass}.

The original M^* algorithm, called with $(f_n, (f_{n-1}, (\ldots, f_0)\ldots))$, is equivalent to calling the extended M^* with $((f_n, d), ((f_{n-1}, d-1)\ldots, (f_0, 0)\ldots))$ (recall that if n, the level of modelling is less than the depth of search, we replace the zero-level player by a minimax player to make the top-level player a d-level modeller). In particular, we can get the minimax algorithm by calling the extended algorithm with $((f, d), ((-f, d-1), ((f, d-2), \ldots)\ldots))$. There is one difference in the way that we extend the zero-level player for the case where $n < d$. In the case of the extended algorithm, we replace the zero-level model by $((f_0, d_0), ((-f_0, d_0 - 1)\ldots, (f_0, 0)\ldots))$.

3 Experimental study: The potential benefit of using opponent models

Now that we have developed an algorithm for using opponent models, we would like to evaluate the potential benefit of using this algorithm. We conducted a set of experiments with the M^* algorithm in order to test the effect of various parameters on the benefit of using an opponent model.

In order to make the experimentation more feasible, we limited the experiments to one-level modelling players, such that the player possesses a model of its opponent's strategy (an evaluation function and a depth of search), and assumes that its opponent is a standard minimax player. Furthermore, the actual opponent used for our experimentation was indeed a minimax player. In order to evaluate the algorithms, we also allowed a regular minimax player to play against the same opponent.

Therefore, the experiments described in this section involve three players[4]:

$$\begin{array}{lcl} MSTAR & = & ((f_{player}, d_{player}), (f_{model}, d_{model})) \\ MINIMAX & = & ((f_{player}, d_{player}), NIL) \\ OPPONENT & = & ((f_{opponent}, d_{opponent}), NIL) \end{array}$$

[4]Remember that M^* expands these players to d-level modelling players using the opposite function and a decreasing series of depth.

3.1 Experimentation methodology

In the experiments described in this section, we studied the effect of the following independent variables:

depth-difference The difference between the depth of search used by the player and that used by *OPPONENT*.

function-difference The difference in quality between the evaluation function used by the player and that used by *OPPONENT*.

depth-error The difference between the depth of search used by *MSTAR* for the model of its opponent, and that actually used by *OPPONENT*.

function-error The difference between the function used by *MSTAR* for the model of its opponent, and that actually used by *OPPONENT*.

In order to be able to measure the function quality and the function error, we defined the following functions:

$$\mathcal{F}_{f,a}(x) = \begin{cases} f(x) & \text{if } |(f(x)| < a \\ 0 & \text{otherwise} \end{cases} \qquad (10)$$

For a given evaluation function f, we can create a sequence of functions by varying a. As a increases, the quality of the function increases. Therefore, we measure the distance in quality between $\mathcal{F}_{f,a_1}$ and $\mathcal{F}_{f,a_2}$ by $a_1 - a_2$.

An experiment was conducted by varying the value of one of the independent variables and fixing the values of the rest. For each value of the tested variable, we measured the performance of the player by letting it play a set of 100 games against *OPPONENT*. The dependent variable used for performance evaluation was the mean number of points earned during the tournament, where a player gets two points for a win and one point for a draw. The experiments were conducted for two different games:

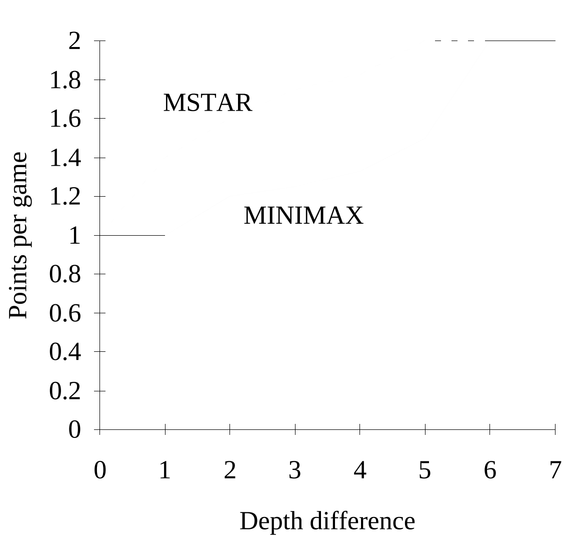
Tic-tac-toe on a 3×3 board. There is a simple winning strategy for this game. However, for the experiments described here, we ignored this strategy and used the well known "open lines advantage" evaluation function. This strategy is considered to be reasonable if used with a deep enough search [1].

Checkers Was chosen because it is complicated enough to conduct more realistic experiments. We used evaluation functions based on that used for Samuel's checkers player[16].

3.2 The effect of the difference in playing ability on the benefit of modelling

The purpose of this experiment is to test the effect of the difference between the players' ability on the benefit of modelling. The basic hypothesis tested is that as the ability difference increases (in favor of the player), the potential merit of using the minimax assumption decreases. A stronger player, who knows that its opponent is weak, can benefit from this knowledge.

We conducted two experiments. In the first one we kept the evaluation function fixed and varied the depth difference. In the second we kept the depth difference fixed and varied the difference



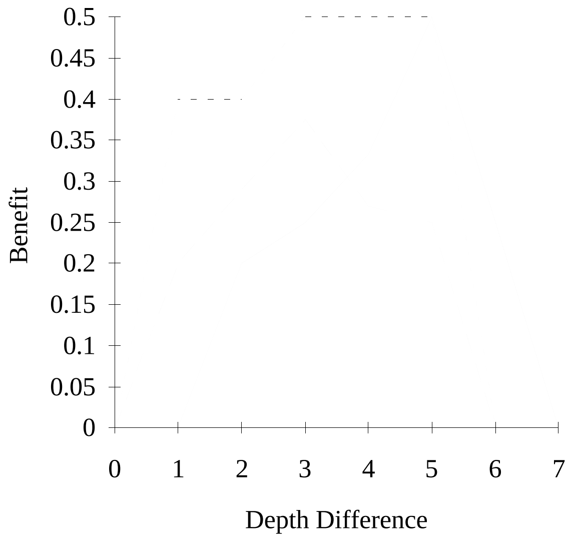


Figure 9: Tic-tac-toe: The benefit of knowing the opponent's strategy as a function of the search depth difference. The left graph shows the performance of *MSTAR* and *MINIMAX* against *OPPONENT* measured by mean points per game. The right graph shows the benefit of using M^* over using minimax against three different opponents.

between the quality of the evaluation functions. *MSTAR* knew the strategy of *OPPONENT* and used M^* while *MINIMAX* used its own strategy as a model for *OPPONENT*.

Figure 9 shows the results obtained for depth difference in the domain of tic-tac-toe while figure 10 shows the results obtained for depth difference and function difference in the domain of checkers. The benefit of using *MSTAR* over *MINIMAX*, when they both use the same player's strategy, was measured by the difference in their performance against *OPPONENT*.

Both experiments exhibited similar behavior . The advantage of *MSTAR* over *MINIMAX* increased with the difference in playing ability between their strategy and *OPPONENT*'s strategy up to a certain point where the advantage started to decline. The increase in the benefit can be explained by the observation that *MINIMAX* was too careful in predicting its opponent's moves, while *MSTAR* utilized its model and exploited the weaknesses of its opponent to its advantage. The decline in higher differences can be explained by the observation that when the difference in playing ability becomes larger, *MINIMAX*, as well as *MSTAR*, win in almost all games. In such a case there is little place for improvement by modelling.

3.3 The effect of the modelling error on the player's performance

The previous experiments demonstrated the benefit of using opponent model. In the following experiments we tested the risk of using a wrong model. We conducted two experiments in the domain of checkers. For the first experiment, we fixed f_{player}, f_{model} and $f_{opponent}$ and varied the values of d_{player}, d_{model} and $d_{opponent}$. For the second experiment, we fixed the depth parameters of the strategies and varied the functions f_{player}, f_{model} and $f_{opponent}$ by setting the a parameter of the $\mathcal{F}_{f,a}$ (see eq. 10) functions.

A tournament of 100 games was conducted for each combination of values. Figure 11 shows the results obtained. Both experiments show similar behavior. The benefit of using an opponent model is maximal for the case of no error. When the error increases, the benefit of using the model decreases. Overestimation (negative error) is less harmful than underestimation.

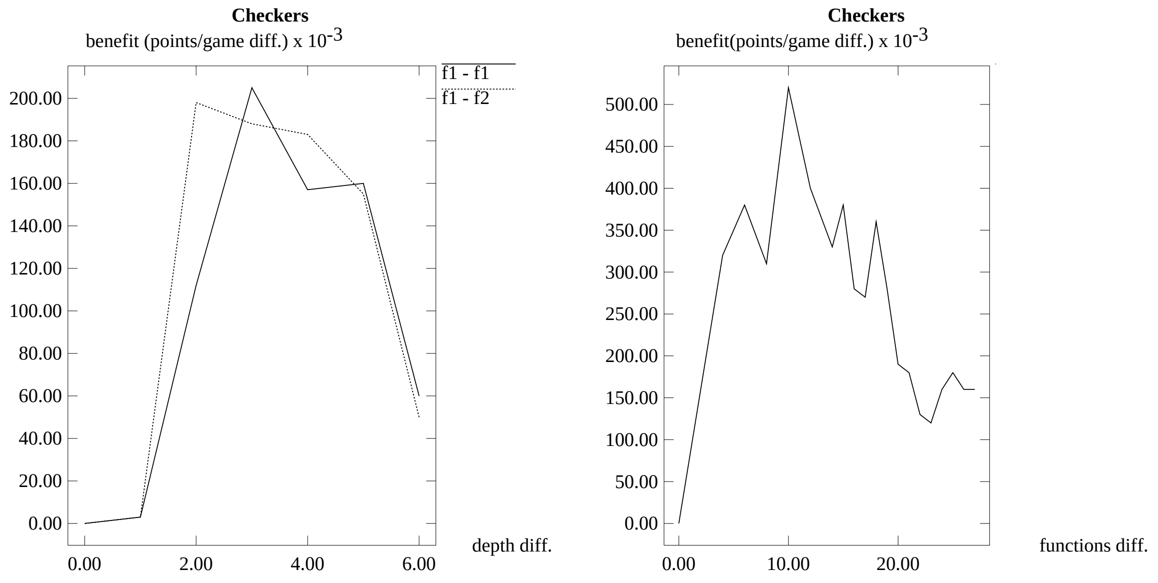


Figure 10: Checkers: The benefit of knowing the opponent strategy as a function of the playing ability difference. Measured by difference in mean points per game. The left graph shows the benefit of usingM^* over minimax as a function of the difference in depth search between the player and the opponent. The right graph shows the benefit as a function of the difference in the quality of the evaluation function.

3.4 Summary

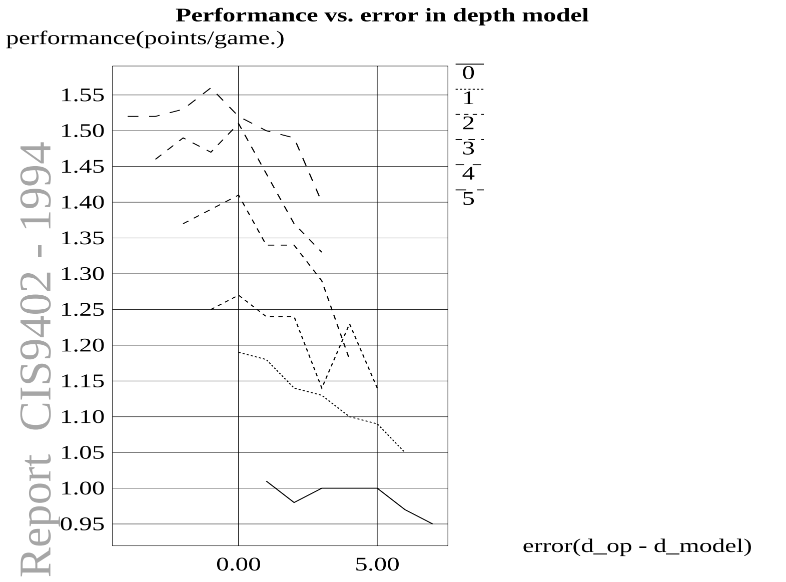
The experiments described in this section confirmed the hypothesis that it is beneficial to use an opponent model, and that the benefit is greater against weaker opponents. It also demonstrates the harmfulness of using wrong model.

Our experiments failed to test the situation where the two opponents have similar playing ability but use different evaluation functions. Such a situation is quite common in game playing. We predict that in such a case, M^* with an appropriate model would have a significant advantage over minimax.

4 Using uncertain models

In the previous sections, we assumed that the player is certain about its opponent model. However, it is quite possible that a player is uncertain about its opponent model, especially when the model is learned by the player. In this section, we generalize M^* to a new algorithm, M^*_ϵ that can handle uncertain models.

The algorithm assumes that the player possesses, in addition to a model of its opponent's function, an upper bound, ϵ, on the distance between the model function and the actual opponent's function. Therefore, a player is now defined as $((f,\epsilon), MODEL)$, where MODEL is also a player. This means that each evaluation function that appears in a player (the player's function, its model's function, its model's model function etc.), has an associated bound on its error. Such a player will be called an *uncertain* player, while a player with no error bound, such as the one used by M^*, will be called a *certain* player. Since the highest level player is certain about its own function, the error bound associated with the top level function will be zero.



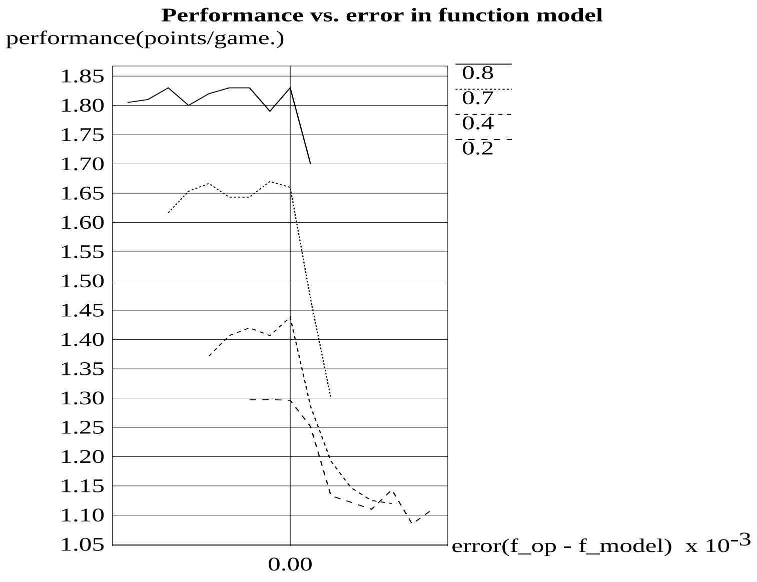


Figure 11: The performance of M^* as a function of the modelling error. The left graph shows the performance of M^* as a function of the error in depth model for various values of $d_{player} - d_{opponent}$. The right graph shows the performance of M^* as a function of the error in function model for various values of $f_{player} - f_{opponent}$.

The error bounds associated with the model's functions represent the player's uncertainty about its opponent. It is possible that the opponent is also an uncertain player. In such a case, its functions will have associated error bounds. We assume that the error bounds associated with the model's functions dominate the error bounds associated with the opponent's functions. Thus, for example, a player $((f_2, 0), ((f_1, \epsilon_1), (f_0, \epsilon_0))))$ assumes that its opponent is $((\hat{f}_1, 0), (\hat{f}_0, \hat{\epsilon_0}))$ where $f_1 - \epsilon_1 \leq \hat{f}_1 \leq f_1 + \epsilon_1$ and $[\hat{f}_0 - \hat{\epsilon_0}, \hat{f}_0 + \hat{\epsilon_0}] \subseteq [f_0 - \epsilon_0, f_0 + \epsilon_0]$.

The input for the M^*_ϵ algorithm is the same as for the M^* algorithm, but with a player that fits our extended definition. The output is different. Instead of returning a board and a value, the new algorithm returns a set of boards and a range of values. The meaning of the range is, that if we would have run M^* with any set of functions that satisfy the error constraints, we would have received a value that falls within the returned range. For every board in the set of returned boards, there is a set of functions satisfying the error constraints, for which M^* would have returned that board as the selected move.

Figure 12 shows the M^*_ϵ algorithm. The algorithm generates all the successors and calls itself recursively with the opponent model to determine which *set* of moves (boards) the opponent can choose from for each of the successors. For each board in such a set of boards, the player calls itself recursively to determine the range of values that the board is worth for the player. Since the player does not know which board of the set will actually be selected by the opponent, it takes the union of these ranges as the range of values that the successor is worth for the player. In this stage, the player has an associated range of values for each of its alternative moves. The lower bound of the range returned by the algorithm is the maximal minimums of all these ranges. The reason is that even with the worst possible set of functions satisfying the error constraints, the player is guaranteed to have at least the maximal minimums. The upper bound is the maximal maximums of all ranges since none of the boards can have a value which exceeds this maximum. The set of boards returned is the set of all boards that can have a maximal value. If the highest value of a range associated with a board is less than one of the minimums, there is no possibility that this board will be selected (the other board is guaranteed to have a higher value).

The algorithm returns a set of boards with a range of values. In order to select a move, a player can employ various selection strategies. A natural candidate is *maximin* [12, page 275-326], a strategy that selects the board with maximal minimum.

```
Procedure $M^*_\epsilon\,(pos, depth, ((f_{pl}, \epsilon_{pl}), \mathrm{OPP\_MODEL}))$
    if $depth = 0$
        return $\langle NIL, [f_{pl}(pos) - \epsilon_{pl}, f_{pl}(pos) + \epsilon_{pl}]\rangle$
    else
        $SUCC \leftarrow MoveGen(pos)$
        for each $succ \in SUCC$
            if $depth = 1$
                $succ\_range \leftarrow [f_{pl}(succ) - \epsilon_{pl}, f_{pl}(succ) + \epsilon_{pl}]$
            else
                $\langle opp\_boards, opp\_range\rangle \leftarrow M^*_\epsilon\,(succ, depth - 1, \mathrm{OPP\_MODEL})$
                for each $board \in op\_boards$
                    $\langle pl\_boards, pl\_range\rangle \leftarrow$
                        $M^*_\epsilon\,(board, depth - 2, ((f_{pl}, \epsilon_{pl}), \mathrm{OPP\_MODEL}))$
                    $succ\_ranges \leftarrow succ\_ranges \cup \{pl\_range\}$
                $succ\_range \leftarrow [\min(i), \max(j)]_{[i,j] \in succ\_ranges}$
            $root\_ranges \leftarrow root\_ranges \cup \{succ\_range\}$
        $[root_{min}, root_{max}] \leftarrow [\max(i), \max(j)]_{[i,j] \in root\_ranges}$
        $root\_boards \leftarrow \{b \in SUCC \mid b_{max} \geq root_{min}\}$
        return $\langle root\_boards, [root_{min}, root_{max}]\rangle$
```

Figure 12: The M^*_ϵ algorithm

There are other adversary search algorithms [3, 14] that return a range of values, as does M^*_ϵ. However, these algorithms were designed for different purposes. The B* algorithm [3] returns a range of values due to uncertainty associated with the player's evaluation function. Nevertheless, unlike the M^*_ϵ algorithm, B* adapts the basic zero-sum assumption and propagates values in the same manner as does minimax. The conspiracy numbers algorithm [14] also manipulates ranges of values. However, these ranges provide a heuristic measure of the accuracy of the minimax root values of incomplete subtrees.

It is easy to see that M^*_ϵ is a generalization of M^*. To get M^* we only need to call M^*_ϵ with all error bounds equal to zero[5]. Furthermore, we can prove a stronger relationship between the two algorithms.

Theorem 1 *Let P be an* uncertain *player. Let $\langle B, [i,j]\rangle = M^*_\epsilon(pos, depth, P)$. Let P_c be a* certain *player consisting of arbitrary functions that satisfy the error constraints of P. Let $\langle b, value\rangle = M^*(pos, depth, P_c)$. Then $value \in [i,j]$ and $b \in B$*

[5]In fact, M^* obtained by a specialization of M^*_ϵ has advantage over the original M^*. In the original M^*, we did not handle the case where a node has successors with equal values. M^*_ϵ called with zero error bound will correctly assume the worst case for opponent nodes with more than one possible outcome, while M^* would have un-justifiably returned the first.

Proof:

By induction on d, the depth of the search tree. For $d = 0$ and $d = 1$ the proof follows immediately.

Assume that the theorem is true for $d < k$. We will show that it is true for $d = k$. According to the first inductive assumption, for any $succ \in SUCC(pos)$, $M^*(succ, k-1, OP\_MODEL(P_c))$ returns a board b that belongs to B, the group of boards returned by $M^*_\epsilon(succ, k-1, OP\_MODEL(P))$. According to the second inductive assumption, for any $b \in B$, $M^*(b, k-2, P_c)$ returns v, a value in $[i, j]$, the range returned by $M^*_\epsilon(b, k-2, P)$. For any $succ \in SUCC(pos)$, M^*_ϵ returns a range $[a_1, a_2]$ such that a_1 is the minimal value of all the ranges of boards in B, and a_2 is the maximal value of all these ranges. Therefore, *value*, the M^* value of b that is associated with *succ*, belongs to the range that M^*_ϵ associates with *succ*. Finally, M^* returns the maximal value among the *succ* values, and in the same manner as for $depth = 1$, this value belongs to the range returned by M^*_ϵ, and all boards with this M^* value, will belong to the group of boards returned by M^*_ϵ. $\square$

It is interesting to note that if we call M^*_ϵ with a model that has an arbitrary opponent function and infinite error bounds, and with the *maximin* selection strategy, we actually get a minimax player. Thus, while in section 2 we interpreted minimax as a player who assumes that its opponent uses its own function (with opposite sign), here we interpret minimax as a player who has no model of its opponent and is therefore totally uncertain about its reactions. Whenever M^*_ϵ simulates the opponent, all successor'ss boards will be returned since none of them can be excluded. The player will then compute its values for these boards and will select the one with maximal minimum value, which is exactly what minimax does.

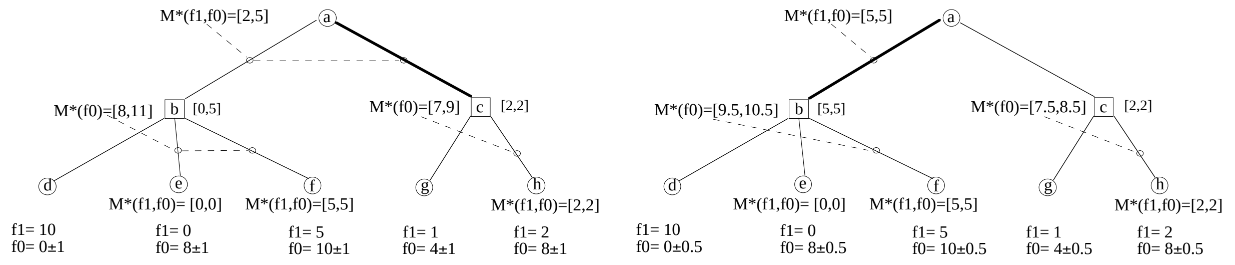


Figure 13: An example for the sequence of calls produced by M^*_ϵ. The topleft figure shows the calls for the case of $\epsilon = 1$ while the right one is for the case of $\epsilon = 0.5$.

Figure 13 shows an example of two calls for M^*_ϵ on the same search tree, once with $\epsilon = 1$ and once with $\epsilon = 0.5$. In the case of $\epsilon = 1$, M^*_ϵ selects the same move as minimax does. In the case of $\epsilon = 0.5$, M^*_ϵ selects the same move as M^* does.

5 Adding pruning to M*

One of the most significant extensions of the minimax algorithm is the $\alpha\beta$ pruning technique which can reduce the average branching factor, b, of the tree searched by the algorithm to about $\sqrt{b}$ [8]. This algorithm avoids searching subtrees that cannot effect the minimax value of the parent node.

Is it possible to add such an extension to M^* as well? Unfortunately, if we assume a total independence between f_{player} and f_{opp_model}, it is easy to show that such a procedure cannot exist. Figure 14 illustrates a situation where using minimax with evaluation function f_1 (or $M^*$with the player $(f_1, (-f_1, NIL))$) can avoid searching node g, whereas M^* that uses f_0 instead of $-f_1$, cannot

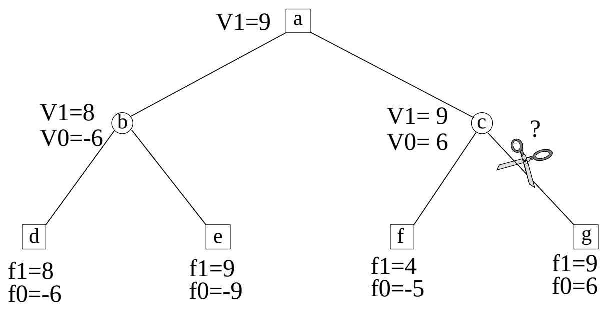


Figure 14: An example for a search tree where the standard $\alpha\beta$ would have pruned the branch leading to node g. However, such pruning would change the M^* value of the tree.

perform this pruning. Knowing that the opponent will have at least -5 for node c does not have any implications on the value of node c for the player.

A similar situation arises in multi-player game trees. Luckhardt and Irani [13] describe a search algorithm, $Maxn$, for multi-player games and conclude that pruning is impossible without further restrictions about the players' evaluation functions. Korf [10] showed that a shallow pruning for $Maxn$ is possible if we assume an upper bound on the sum of the players' functions, and a lower bound on every player's function.

The basic assumption used for the original $\alpha\beta$ algorithm is that $f_{player} + f_{opponent} = 0$ (the zero-sum assumption). This assumption is used to infer a bound on a value of a node for a player based directly on the opponent's value. A natural relaxation to this assumption is $|f_{player} + f_{opponent}| \leq b$. This assumption means that while f_{player} and $-f_{opponent}$ may evaluate a board differently, this difference is bounded. For example, the player may prefer a rook over a knight while the opponent prefers the opposite. In such a case, although the player's value is not a direct opposite of the opponent's value, we can infer a bound on the player's value based on the opponent's value and b.

The above assumption can be used in the context of the M^*_{1-pass} algorithm to determine a bound on $V_i + V_{i-1}$ at the leaves level. But in order to be able to prune using this assumption, we first need to determine how these bounds are propagated up the search tree.

Lemma 4 *Assume that A is a node in the search tree spanned by M^*_{1-pass}. Assume that $S_1, \ldots, S_k$ are its successors. If there exist non-negative bounds $B_0, \ldots, B_n$, such that for each successor S_j, and for each model i, $|V_{S_j}[i] + V_{S_j}[i-1]| \leq B_i$. Then, for each model $1 \leq i \leq n$:*

(11) $$|V_A[i] + V_A[i-1]| \leq B_i + 2 \cdot B_{i-1}.$$

Proof:

Assume $V_A[i] = V_{S_j}[i]$ and $V_A[i-1] = V_{S_k}[i]$. If $j = k$, $V_A[i]$ and $V_A[i-1]$ were propagated from the same successor, therefore,

(12) $$|V_A[i] + V_A[i-1]| \leq B_i \leq B_i + 2 \cdot B_{i-1}.$$

If $j \neq k$, A is an i'th player node and therefore $V_A[i-1]$ and $V_A[i-2]$ were propagated from the same successor S_k. It follows that

(13) $$B_{i-1} \geq V_{S_k}[i-1] + V_{S_k}[i-2] \geq V_{S_k}[i-1] + V_{S_j}[i-2].$$

It is easy to show that for any successor S,

(14) $$|V_S[i] - V_S[i-2]| \leq B_i + B_{i-1}.$$

Summing up the two inequalities for the S_j successor, we get

$$B_i + 2 \cdot B_{i-1} \geq V_{S_j}[i] + V_{S_k}[i-1] = V_A[i] + V_A[i-1]. \tag{15}$$

For the second side of the inequality,

$$V_A[i] + V_A[i-1] = V_{S_j}[i] + V_{S_k}[i-1] \geq V_{S_k}[i] + V_{S_k}[i-1] \geq -B_i \geq -B_i - 2 \cdot B_{i-1}. \square \tag{16}$$

5.1 The $\alpha\beta^*$ pruning algorithm

Based on lemma 4, we have developed an algorithm, $\alpha\beta^*$, that can perform a shallow and deep pruning assuming bounds on the absolute sum of functions of the player and its opponent model. The algorithm takes as input a position, a depth limit, and for each model i, a strategy f_i, an upper bound b_i on $|f_i + f_{i-1}|$, and a cutoff value α_i. It returns the M^* value of the root by only searching nodes that might affect this value.

```
Procedure $\alpha\beta^*(pos, depth,$
              $((f_n, b_n)((f_{n-1}, b_{n-1}), (\ldots, (f_0, b_0)) \ldots))(\alpha_n, \ldots, \alpha_0))$
  if $depth = 0$
    return $(\langle f_n(pos), \ldots, f_0(pos)\rangle, \langle b_n, \ldots, b_0\rangle)$
  else $SUCC \leftarrow MoveGen(pos)$
      for each $succ \in SUCC$
          (succ_V,succ_B ) $\leftarrow \alpha\beta^*(succ, depth-1,$
                    $((f_n, b_n)((f_{n-1}, b_{n-1}), (\ldots, (f_0, b_0)) \ldots))(\alpha_n, \ldots, \alpha_0))$
          loop for each $i$ associated with current player
              $B[i] \leftarrow succ\_B[i] + 2 \cdot succ\_B[i-1]$
              if succ_V$[i] > V[i]$
                $V[i] \leftarrow$ succ_V$[i]$
                if $i < n$
                  $V[i+1] \leftarrow$ succ_V$[i+1]$
              if $V[i] \geq \alpha_i$
                $\alpha_i \leftarrow V[i]$
          if for every $i$ not associated with current player $[\alpha_i \geq B[i] - V[i-1]]$
            return $(\langle V[n], \ldots, V[depth]\rangle, \langle B[n], \ldots, B[depth]\rangle)$
  return $(\langle V[n], \ldots, V[depth]\rangle, \langle B[n], \ldots, B[depth]\rangle)$
```

Figure 15: The $\alpha\beta^*$ algorithm

The $\alpha\beta^*$ algorithm is listed in figure 15. The algorithm works similarly to the original $\alpha\beta$ algorithm, but is much more restricted in what subtrees can be pruned. The $\alpha\beta^*$ algorithm only prunes branches that *all* models agree to prune. In regular $\alpha\beta$, the player can use the opponent's value of a node to determine whether it has a chance to get a value that is better than the current cutoff value. This is based on the opponent's value being exactly the same as the player's value (except for the sign). In $\alpha\beta^*$, the player's function and the opponent's function are not identical, but their difference is bounded. The bound on $V_i + V_{i-1}$ depends on the distance from the leaves level. At the leaves level, it can be directly computed using the input b_i. At distance d, the bound can be computed from the bounds for level $d-1$ as stated by lemma 4[6].

A cutoff value α_i for a node v is the highest current value of all the ancestors of v from the point of view of player i. α_i is modified at nodes where it is player i's turn to play, and is used for pruning where it is player $i-1$'s turn to play. At each node, for each i associated with the player whose turn it is to play, α_i is maximized any time V_i is modified. For each i such that $i-1$ is associated with the current player, the algorithm checks whether the i player wants its model (the $i-1$ player) to continue its search.

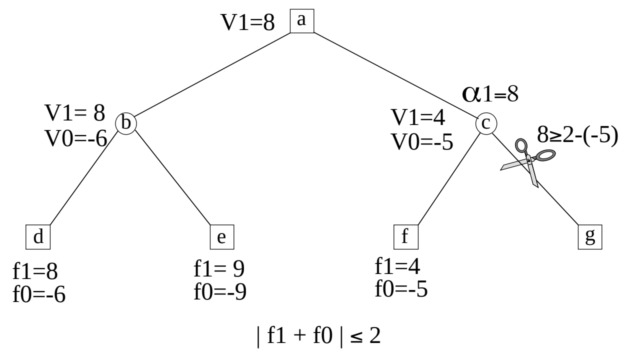


Figure 16: An example to pruning performed by $\alpha\beta^*$.

Figure 16 shows a search tree similar to the one in figure 14 with one difference: every leaf l satisfies the bound constraint $|f_1(l) + f_0(l)| \leq 2$. This bound allows the player to perform a cutoff of branch g, knowing that the value of node c for the opponent will be at least -5. Therefore, its value will be at most 7 for the player.

Lemma 5 *Let $\langle V, B\rangle = \alpha\beta^*(pos, d, PLAYER, (-\infty, \ldots, -\infty))$. Let $V' = M^*_{1-pass}(pos, d, PLAYER')$ where PLAYER' is PLAYER without the bounds. Assume that for any leaf l of the game tree spanned from position pos to depth d, $|f_i(l) + f_{i-1}(l)| \leq b_i$. Then, $V = V'$.*

Proof:
For proving that $M^*_{1-pass}(pos) = \alpha\beta^*(pos)$, it is sufficient to show that any node pruned by $\alpha\beta^*$ can have no effect on $M^*_{1-pass}(pos)$. Assume that u is an opponent's node, and after searching one of its successors, all the models associated with the player agree to prune. This means that for every model i associated with the player, $\alpha_i \geq B_i - V[i-1]$. From lemma 4, it follows that $V[i] + V[i-1] \leq B_i$ and therefore $\alpha_i \geq V[i]$. Since α_i is the current best value for $V[i]$ of the parent node, there is no way that the current $V[i]$ will affect its father's value. The same argument holds if u is a player's node. $\square$

As the player function becomes more similar to its opponent model (but with anopposite sign), the level of pruning increases up to the point where they use the same function, in which case $\alpha\beta^*$ prunes as much as $\alpha\beta$.

Finding an upper bound on the sum of the evaluation functions is an easy task for most practical evaluation functions [9]. Unfortunately, the bound on the sum of values increases with the distance

[6]For sake of clarity, the algorithm computes the bound B for each node using the bounds returned from its successors. However, for efficient implementation, a table of the B values can be computed once at the beginning of the search, since they depend only on the b_i and the depth.

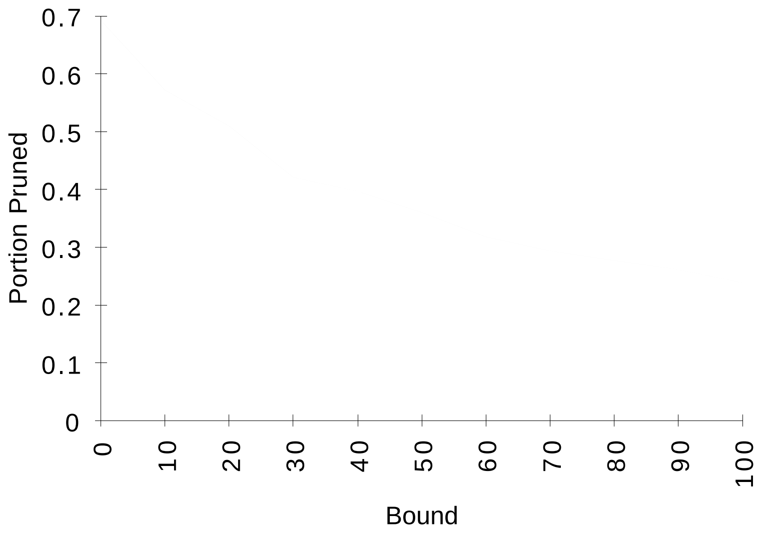


Figure 17: The portion of leaves pruned by $\alpha\beta^*$ as a function of the bound on $|f_1 + f_0|$.

from the leaves and therefore reduces the amount of pruning. Therefore, using loose bounds will probably prohibit pruning.

In order to get some idea of how much $\alpha\beta^*$ prunes, we have run a simulation applying $\alpha\beta^*$ on uniform trees of depth d, and fixed branching factor b, whose leaves were assigned two random values with a sum bounded by B (we mark such a tree by $TREE(b,d,B)$). The experiment was run with one-level modelling player, (f_1, f_0) such that $|f_1 + f_0| \leq B$. The experiments were conducted for $TREE(2,10,B)$. For each bound B we created 1000 trees and ran $\alpha\beta^*$ on them. Graph 17 shows the portion of leaves pruned as a function of the bound B. As expected, the amount of pruning decreases as the difference between the functions increases. The maximum level of pruning is achieved for $B = 0$, where $\alpha\beta^*$ is reduced to standard $\alpha\beta$.

6 Learning a model of the opponent's strategy

In the first part of this work we have presented methods for using opponent models. In this section we describe a methodology for acquiring such a model from examples. A set of boards with the opponent's decisions is given as input, and the learning procedure produces a model as output. This framework is similar to the scenario used by Kasparov as described in the opening quote.

To make the learning task more feasible, we assume that the opponent is a minimax player and its model therefore consists of two components: a depth of search and an evaluation function. Since the space of possible depth values is much smaller than the space of possible function, we start by describing a method for learning the opponent's depth-of-search, and proceed with the more complicated task of learning its function.

6.1 Learning the depth of search

Given a set of examples, each consisting of a board together with the move selected by the opponent, it is relatively easy to learn its depth of search ($d_{opponent}$) under the assumption that the opponent searches to a fixed depth and that its evaluation function is known to the learner. Since there is only a small set of plausible values for $d_{opponent}$, we can test which of them agrees best with the opponent's decisions. One possibility is to test for strict agreement, i.e., for any example $(board, move)$ we can reward any depth d for which $minimax(board, d)$ returns $move(board)$.

When $f_{model} = f_{opponent}$, such a method needs only a few examples to infer $d_{opponent}$. However, in the case where f_{model} differs from $f_{opponent}$, such an algorithm is prone to error. In order to improve the learning ability in the presence of a wrong function model, we have developed an improved algorithm that considers the relative ordering between possible moves. The algorithm rewards every depth with the number of moves that have lower minimax values, and penalizes it with the number of alternative moves with higher minimax values. The algorithm is shown in figure 18.

```
Procedure $LearnDepth(EXAMPLES)$
    for each $\langle board, move \rangle \epsilon EXAMPLES$
        $boards \leftarrow successors(board)$
        for $d$ from 1 to $MaxDepth$
            $M \leftarrow minimax(move(board), d-1)$
            $count[d] \leftarrow count[d]$
                $+ \mid \{b \,\epsilon\, boards \mid minimax(b, d-1) \leq M\} \mid$
                $- \mid \{b \,\epsilon\, boards \mid minimax(b, d-1) > M\} \mid$
    return $d$ with maximal $count[d]$
```

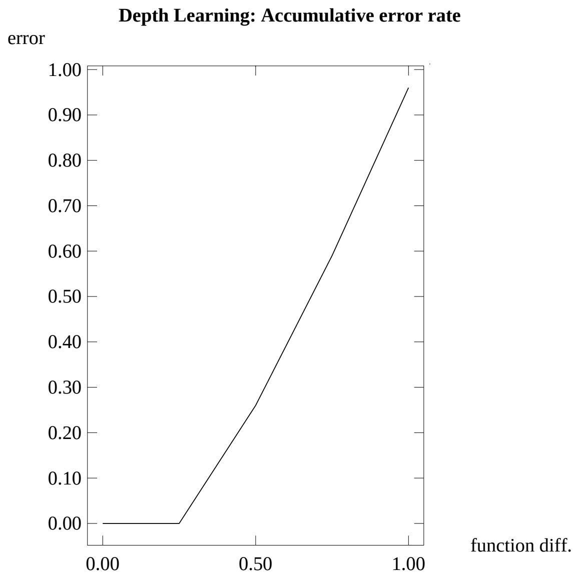
Figure 18: An algorithm for learning a model of the opponent's depth (d_{model})

In order to study the effect of the distance between the model function and the actual function on the learning ability of the algorithm, we have performed the following experiment.

1. f_{model}, used by the learning algorithm, was fixed.
2. $f_{opponent}$ was defined to return a value according to a parameter p. The function returns f_{model} with probability p and a random value with probability $1-p$.
3. A set of games between two minimax players, *PLAYER* and *OPPONENT*, was conducted. 100 boards that OPPONENT faced, together with its chosen moves, were given as examples to the learning algorithm which updated d_{model}.
4. After each move, the current d_{model} was compared against $d_{opponent}$. If they were in disagreement, the accumulative error rate was incremented.
5. The experiment was repeated for various p.

Figure 19 shows the accumulative error rate of the algorithms as a function of the distance between f_{model} and $f_{opponent}$. It also shows the d counters after 100 examples for the case of f_{model} with probability of error equals to 0.25.

The accumulative error rate is the portion of the learning session where the learner has a wrong model of its opponent's depth. The experiment shows that indeed when the function model is close enough to the opponent's function, the algorithm succeeds in learning the opponent's depth. However, when the opponent's function is significantly different from the model, the algorithm's error rate increases.



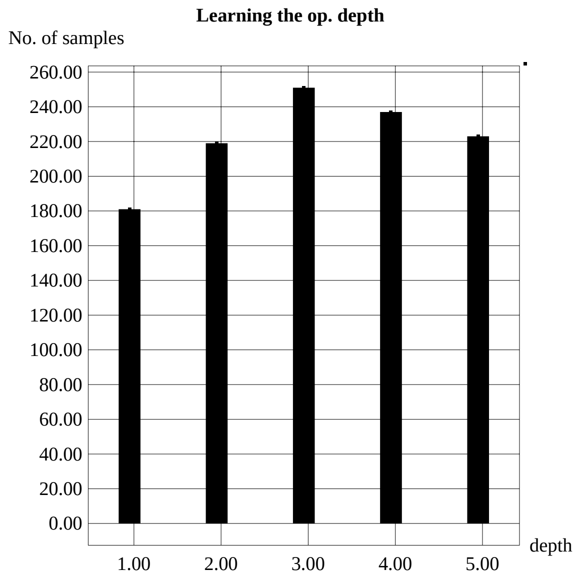


Figure 19: Learning the depth of search using 100 examples, where $d_{opponent} = 3$ and f_{model} differs from $f_{opponent}$ with probability between 0 and 1. The left graph shows the accumulative error rate as a function of the error of the function model. The right graph shows the counters of the learning algorithm for function model error of 0.25.

Learning the opponent's depth of search may provide very useful information, especially in a game against a weaker opponent. When a player is aware of the limit of its opponent's search horizon, it can lead the opponent to trap or swindle positions [7].

6.2 Learning the opponent's strategy

The performance of the previous algorithm depends on the knowledge of the opponent's evaluation function. The natural next step is to develop an algorithm for learning evaluation functions. However, learning the opponent's function will probably depend on knowing the opponent's depth of search. In order to break this circle, we have developed an algorithm for learning the function and depth of search simultaneously.

Since learning an arbitrary real function is hard, we have made the following simplifying assumptions:

1. The opponent's function is a linear combination of features. $f(b) = \overline{w} \cdot \overline{h}(b) = \sum_i w_i h_i(b)$ where b is the evaluated board and $h_i(b)$ returns the ith feature of that board.
2. A superset of the features used by the opponent is known to the learner.
3. The opponent does not change its function while playing.

Under these assumptions, the learning task is reduced to finding the pair $(\overline{w}_{model}, d_{model})$. The learning procedure listed in figure 20 computes for each possible depth d a weight vector $\overline{w}_d$, such that the strategy $(\overline{w}_d \cdot \overline{h}, d)$ most agrees with the opponent's decisions. The adapted model is the best pair found for all depths.

For each depth, the algorithm performs a hill-climbing search, improving the weight vector until no further significant improvement can be achieved. Assume that $\overline{w}_{current}$ is the best vector found

```
Procedure $LearnStrategy(examples)$
$\overline{w}_0 = \overline{1}$
for $d$ from 1 to $MaxDepth$
    $\overline{w}_d \leftarrow \overline{w}_{d-1}$
    Repeat
        $\overline{w}_{current} \leftarrow \overline{w}_d$
        $\overline{w}_d \leftarrow FindSolution(examples, \overline{w}_{current}, d)$
        $progress \leftarrow |score(\overline{w}_d, d) - score(\overline{w}_{current}, d)| \geq \epsilon$
    Until no progress
return $(\overline{w}_d, d)$ with the maximal score.

Procedure $FindSolution(EXAMPLES, \overline{w}_{current}, d)$
  $Constraints \leftarrow \phi$
  for each $\langle board, chosen\_move \rangle \epsilon EXAMPLES$
      $SUCC \leftarrow MoveGen(board)$
      for each $succ \, \epsilon \, SUCC$
          $dominant_{succ} \leftarrow Minimax(succ, \overline{w}_{current}, d-1)$
          $Constraints \leftarrow Constraints \cup \{\overline{w}(\overline{h}(dominant_{chosen\_move}) - \overline{h}(dominant_{succ})) \geq 0\}$
  return $\overline{w}$ that satisfy $Constraints$
```

Figure 20: An algorithm for learning a model of the opponent's strategy

so far for the current depth. For each of the examples, the algorithm builds a set of constraints that express the superiority of the selected move over its alternatives. The algorithm performs a minimax search using $(\overline{w}_{current} \cdot \overline{h}, d-1)$, starting from each of the successors of the example board. At the end of this stage each of the alternative moves can be associated with the "dominant" board that determines its minimax value. Assume that b_{chosen} is the dominant board of the chosen move, and $b_1, \ldots, b_n$ are the dominant boards for the alternative moves. The algorithm adds the n constraints $\{\overline{w} \cdot (\overline{h}(b_{chosen}) - \overline{h}(b_i)) \geq 0 \mid i = 1, \ldots, n\}$ to its accumulated set of constraints.

The next stage consists of solving the inequalities system, i.e., finding $\overline{w}$ that satisfies the system. The method we used is a variation of the linear programming method used by Duda and Hart [5] for pattern recognition. Before the algorithm starts its iterations, it sets aside a portion of its examples for progress monitoring. This set is not available to the procedure that builds the constraints. After solving the constraints system , the algorithm tests the solution vector by measuring its accuracy in predicting the opponent's moves for the test examples. The performance of the new vector is compared with that of the current vector. If there is no significant improvement, we assume that the current vector is the best that can be found for the current depth, and the algorithm repeats the process for the next depth using the current vector for its initial strategy.

The inner loop of the algorithm, that searches for the best function for a given depth, is similar to the method used by DEEP THOUGHT [6] and by CHINOOK [18] for tuning their evaluation function from book moves. However, these programs assume a fixed small depth for their search. Meulen [20] used a set of inequalities for book learning, but his program assumes only one level depth of search.

The strategy learning algorithm was tested by the following experiment. Three fixed strategies $(f1, 8)$, $(f2, 8)$ and $(f1, 6)$, were used as opponents, where $f1$ and $f2$ are two variations of Samuel's function. Each strategy was used to play games until 1600 examples were generated and given to the learning algorithm. The algorithm was also given a set of ten features, including the six features actually used by the strategies.

The algorithm was run with a depth limit of 11. The examples were divided by the algorithm into a training set and a testing set of size 800. For each of the eleven depth values, the program performed 2-3 iterations before moving to the next depth. Each iteration included using the linear programing method for a set of several thousand constraints[7].

The results of the experiment for the three strategies is shown in figure 21. The algorithm succeeded for the three cases, achieving an accuracy of 100% for two strategies and 93% for the third. Furthermore, the highest accuracy was achieved for the actual depth used by the strategies.

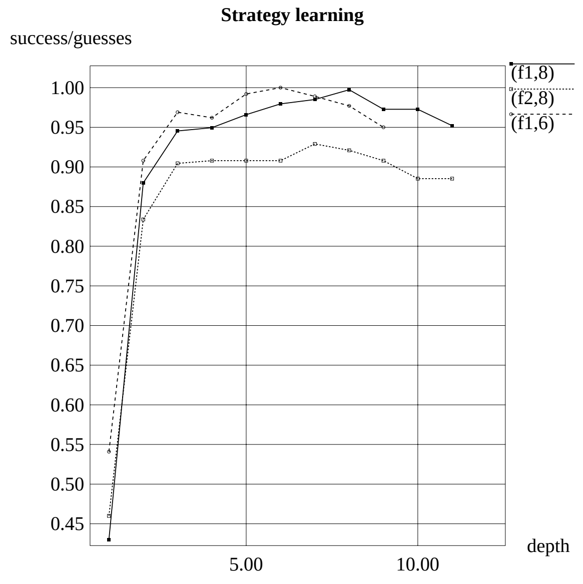


Figure 21: Learning opponent's strategy

6.3 OLSY: An Opponent Learning System

In order to test how well the M^* algorithm and the learning algorithm can be integrated, we have built a game playing program, OLSY, that is able to acquire and maintain a model of its opponent and use it to its advantage. The system consists of two main components, a game-playing program and a learning program. The learning program accumulates the moves performed by the opponent. After playing a small number of moves (10 in our experiments), the system updates its model of opponent.

The OLSY system was tested in a realistic situation by letting it play a sequence of games against regular minimax players. OLSY and its two opponents each used a different strategy but with a roughly equivalent playing ability. We stopped the games several times to test how well OLSY performs against its opponents. The test was conducted by turning off the learning mechanism and performing a tournament of 100 games between the two.

Figure 22 shows the results of this experiment. The players start with almost equivalent ability. However, after only few examples, the learning program becomes significantly stronger than its

[7]We have used the very efficient lp_solve program, written by M.R.C.M. Berkelaar, for solving the constraints system

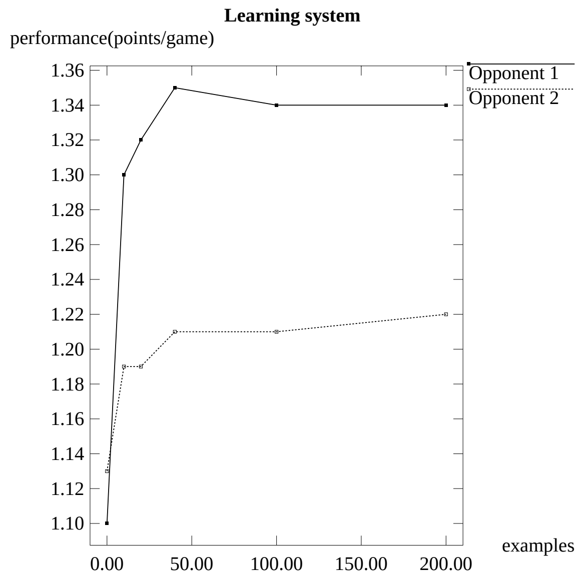


Figure 22: The performance of the learning system as a function of the number of examples. Measured by mean points per game.

non-learning opponents. The performance of OLSY kept increasing until about 50 examples were processed. After that, the learning curve was levelled. We inspected the accuracy of the models as determined by the learning procedure. After 100 examples, OLSY succeeded in acquiring models for its two opponents with a quality of 95% for the first and 98% for the second. Learning more examples did not cause any further modification of the models.

7 Conclusions

The minimax algorithm assumes that the opponent uses the same strategy as does the player. In this paper we presented a generalized version of the minimax algorithm that can utilize different opponent models. We started by defining a player as a pair of a strategy and a model, where the model is also a player. We proceeded with the M^* algorithm, which simulates the opponent's search to determine its expected decision for the next move, and evaluates the resulted board by searching its associated subtree using its own strategy. We then presented the M^*_{1-pass} algorithm, which is an efficient version M^* that expands the tree only once, but propagates all the necessary values. The M^* algorithm (as well as Minimax) assumes that the opponent searches as deep as the player's search horizon. We extended the M^* and M^*_{1-pass} algorithms to enable the utilization of models of the opponent's depth of search.

Experiments performed in the domains of checkers and tic-tac-toe demonstrated the advantage of M^* over minimax but also showed that an error in the model can deteriorate performance significantly. We developed an algorithm, M^*_ϵ, for the cases where the player knows a bound on the error. The algorithm converges to M^* when the error bound approaches zero, and converges to minimax when the bound goes to infinity.

One of the most important techniques in searching game trees is $\alpha\beta$ pruning. We explored the possibility of applying similar pruning techniques to M^*. Unfortunately, pruning is impossible in the general case since the zero-sum assumption does not hold. However, we developed a pruning algorithm, $\alpha\beta^*$, that utilizes a relaxed version of the zero-sum assumption to allow pruning. Pruning is allowed given a tight bound on the sum of the player's and model's functions. When the bound approaches zero, the amount of pruning approaches that of $\alpha\beta$.

In the second part of the paper we tackled the problem of learning an opponent's model by using its moves as examples. We developed an algorithm for learning an opponent model, both depth and evaluation function. The algorithm works by iteratively increasing the model depth and learning a function that best predicts the opponent's moves for that depth. For testing the algorithm, we tried to learn minimax players that search to a fixed depth and use an evaluation function based on a linear combination of features, known to the learner. The results show that few examples are needed for learning a model that agrees almost perfectly with such a player. In the future, we mean to investigate the algorithm's ability to model more sophisticated players. We tested the algorithm in a game-playing learning system, named OLSY. The system was tested in a realistic situation, learning its opponent while playing against it. The system indeed showed an improvement as more examples of opponent's moves became available.

We believe that this work presents significant progress in the area of *using* opponent models. The M^* family of algorithms presented in this paper are all generalizations of minimax that allow us to use n-level opponent models together with bounds on their errors. This work also presents initial steps in the area of *learning* opponent models. The task of learning n-level models is extremely difficult and deserves further research.

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References

[1] B. Abramson. Expected outcome: A general model of static evaluation. *IEEE Trans. on Pattern Analysis and Machine Intelligence 12,182-193*, 1990.

[2] H. Berliner. Search and knowledge. In *Proceeding of the International Joint Conference on Artifical Intelligence (IJCAI 77)*, pages 975–979, 1977.

[3] H. Berliner. The b* tree search algorithm: A best-first proof procedure. *Artifical Intelegence 12, 23-40*, 1979.

[4] D. Carmel and S. Markovitch. Learning models of opponent's strategies in game playing. In *Proceedings of the AAAI Fall Symposium on Games: Planning and Learning*, pages 140–147, Raleigh, NC, Oct 1993.

[5] R. O. Duda and P. Hart. *Pattern Classification and Scene Analysis.* New York: Wiley and Sons, 1973.

[6] F.-H. Hsu, T. Ananthraman, M. Campbell, and A. Nowatzyk. Deep thought. In T. Marsland and J. Schaeffer, editors, *Computers, Chess and Cognition*, pages 55–78. Springer New York, 1990.

[7] P. Jansen. Problematic positions and speculative play. In T. Marsland and J. Schaeffer, editors, *Computers, Chess and Cognition*, pages 169–182. Springer New York, 1990.

[8] D. Knuth and R. Moore. An analysis of alpha-beta pruning. *Artifical Intelligence 6, no.4, 293-326*, 1975.

[9] R. E. Korf. Generalized game trees. In *Proceeding of the International Joint Conference on Artifical Intelligence (IJCAI 89)*, pages 328–333, Detroit, MI, Aug. 1989.

[10] R. E. Korf. Multy-player alpha-beta pruning. *Artifical Intelegence 48, 99-111*, 1991.

[11] D. Levy and M. Newborn. *How Computers Play Chess.* W.H. Freeman, 1991.

[12] R. D. Luce and H. Raiffa. *Games and Decisions.* New York: Wiley and Sons, 1957.

[13] C. A. Luckhardt and K. B. Irani. An algorithmic solution of n-person games. In *Proceeding of the Ninth National Conference on Artifical Intelligence (AAAI-86)*, pages 158–162, August 1986.

[14] D. A. McAllester. Conspiracy numbers for min-max search. *Artifical Intelegence 35, 287-310*, 1988.

[15] E. H. D. P. J. Gmytrasiewicz and D. K. Wehe. A decision theoretic approach to coordinating multiagent interactions. In *Proceedings of the International Joint Conference on Artifical Intelligence (IJCAI 91)*, pages 62–68, 1991.

[16] A. Samuel. Some studies in machine learning using the game of checkers. *IBM Journal, 3, 211-229*, 1959.

[17] A. Samuel. Some studies in machine learning using the game of checkers ii–recent progress. *IBM Journal, 11, 601-617*, 1967.

[18] J. Schaeffer, J. Culberson, N. Treloar, B. Knight, P. Lu, and D. Szafron. A world championship caliber checkers program. *Artifical Intelegence 53, 273-289*, 1992.

[19] C. E. Shannon. Programming a computer for playing chess. *Philosophical Magazine, 41, 256-275*, 1950.

[20] M. van der Meulen. Weight assessment in evaluation functions. In D. Beal, editor, *Advances in Computer Chess 5*, pages 81–89. Elsevier Science Publishers, Amsterdam, 1989.