Two-Dimensional Constrained Coding Based on Tiling

Artyom Sharov   Ron M. Roth

***Abstract*— A new variable-rate coding technique is presented for two-dimensional constraints. For certain constraints, such as the $(0,2)$-RLL and $(3,\infty)$-RLL constraints, the technique is shown to improve on previously-published lower bounds on the capacity of the constraint.**

***Index Terms*— Markov chain, n.i.b. constraint, Pickard random fields, runlength-limited (RLL) constraints, tiling, two-dimensional (2-D) constraints, variable-rate codes.**

I. Introduction

Constrained coding has found widespread use in optical and magnetic data storage devices [1]. The most common family of constraints appears to be that of the (d,k)-runlength-limited (RLL) constraints: the parameters (d,k) represent, respectively, the minimum and maximum admissible number of 0's separating consecutive 1's in any allowable binary sequence.

Proposals for new storage systems, such as holographic storage, and for better exploitation of current optical systems, have raised the interest in two-dimensional (2-D) and even three-dimensional (3-D) constraints, as models that describe the read–write requirements of the storage medium. Bar coding provides another example of an application where 2-D constraints can be found [2], [3].

Among the constraints of theoretical and possible practical interest are 2-D (d,k)-RLL constraints: each such constraint consists of all binary arrays in which the one-dimensional (1-D) (d,k)-RLL constraint is satisfied along each row and column.

Another example is the 2-D "no isolated bits" (in short, n.i.b.) constraint, which consists of all binary arrays that contain neither of the following two patterns:

$$\begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \qquad\qquad \begin{array}{ccc} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{array}$$

1-D constraints were extensively studied, and there are several known methodologies for designing codes for such constraints [1], [4]. On the other hand, our knowledge of 2-D constraints is much less profound. This might be attributed in part to the fact that the practical interest in those constraints has arisen relatively recently; however, it seems that the main reason for such lack of knowledge is the provable difficulty of certain problems that relate to 2-D constraints, compared to the 1-D case [5], [6].

This work was supported in part by Grant No. 2002197 from the United-States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.

Artyom Sharov and Ron M. Roth are with the Computer Science Department, Technion, Haifa 32000, Israel.

Emails: {sharov, ronny}@cs.technion.ac.il

In general, a 2-D constraint $\mathbb{S}$ over an alphabet Σ is defined by two state-labeled finite directed graphs, $\mathcal{G} = (V, E_{\mathcal{G}}, L)$ and $\mathcal{H} = (V, E_{\mathcal{H}}, L)$, with the same set of states V and the same state labeling $L : V \to \Sigma$. The constraint $\mathbb{S}$ consists of all finite rectangular arrays $x = (x_{i,j})$ over Σ for which one can associate arrays $\Gamma(x) = (v_{i,j})$ over V that satisfy the following three conditions: (a) $L(v_{i,j}) = x_{i,j}$ for all i and j, (b) each row in $\Gamma(x)$ is a path in $\mathcal{G}$, and (c) each column in $\Gamma(x)$ is a path in $\mathcal{H}$.

The capacity of a 2-D constraint $\mathbb{S}$ over Σ is defined as the growth rate of the number of $m \times n$ arrays in $\mathbb{S}$:

$$\mathsf{cap}(\mathbb{S}) = \lim_{m,n\to\infty} \frac{1}{mn} \log |\mathbb{S} \cap \Sigma^{m\times n}|$$

(hereafter all logarithms are taken to base 2). By sub-additivity [7, Lemma 8] the limit indeed exists.

In this work, we present a variable-rate coding scheme for a wide family of 2-D constraints. Similarly to other variable-rate schemes, such as bit-stuffing encoders [8], [9], [10], our coding scheme realizes a probability measure μ_n on $\mathbb{S} \cap \Sigma^{n\times n}$, for every positive integer n. The expected rate of the coding scheme is given by the (measure-theoretic) per-symbol entropy:

$$\mathsf{H}(\mu_n) = -\frac{1}{n^2} \sum_{x \in \mathbb{S}\cap\Sigma^{n\times n}} \mu_n(x) \log \mu_n(x) \ .$$

Taking the limit when $n \to \infty$, we obtain a lower bound on $\mathsf{cap}(\mathbb{S})$. While in most previous variable-rate schemes this limit is only bounded from below (one notable exception is [9]), here we will be able to compute that limit exactly.

Our coding scheme makes use of certain tilings of the plane, and we discuss their properties in Section II. In Section III, we describe the coding scheme and compute its rate, thereby obtaining a lower bound on the capacity of the constraint. Section IV presents several generalizations and improvements.

II. Tilings

We start with several definitions. Let U be a nonempty finite subset of $\mathbb{Z}^2$ and let Σ be a finite alphabet. A *U-configuration* is a mapping $\varphi : U \to \Sigma$. Given a 2-D constraint $\mathbb{S}$ over Σ, and a nonempty finite subset $U \subset \mathbb{Z}^2$, we say that a U-configuration $\varphi : U \to \Sigma$ is *compatible with* $\mathbb{S}$ (or, in short, is $\mathbb{S}$-*compatible*) if there exists an array $(x_{i,j}) \in \mathbb{S}$ such that

$$\varphi(i,j) = x_{i,j} \quad \text{for every} \quad (i,j) \in U \ ;$$

that is, the images of φ at the points $(i,j) \in U$ can be extended to an array in $\mathbb{S}$. We denote by $\mathbb{S}(U)$ the set of all $\mathbb{S}$-compatible U-configurations.

Among the subsets $U \subset \mathbb{Z}^2$ considered in this work are the squares

$$Q_n = \{(i,j) \in \mathbb{Z}^2 \ : \ 0 \le i, j < n\} \ .$$

Here, $\mathbb{S}(Q_n) = \mathbb{S} \cap \Sigma^{n \times n}$.

For a subset $U \subseteq \mathbb{Z}^2$ and a pair $(h,v) \in \mathbb{Z}^2$, denote by $\sigma_{h,v}(U)$ the shifted set

$$\{(i,j) \ : \ (i-h, j-v) \in U\} \ .$$

By the shift invariance property of 2-D constraints, we get that φ belongs to $\mathbb{S}(U)$ if and only if the composition $\varphi \circ \sigma_{-h,-v}$ belongs to $\mathbb{S}(\sigma_{h,v}(U))$, for every $(h,v) \in \mathbb{Z}^2$.

Our coding scheme will be defined through a periodic tiling (i.e., a partition with a certain regular pattern) of the plane $\mathbb{Z}^2$ using shifted copies of (generally) two types of nonempty finite subsets $B, W \subset \mathbb{Z}^2$, and we refer hereafter to these copies, respectively, as "black" and "white" tiles. The set of locations of the black tiles is defined by a lattice

$$\mathcal{L} = \mathcal{L}(A) = \left\{(i,j) = (t,u)A \ : \ (t,u) \in \mathbb{Z}^2\right\}$$

for some 2×2 integer matrix A, so that the black tiles

$$\sigma_{h,v}(B) \ , \quad (h,v) \in \mathcal{L} \ ,$$

are all disjoint. The set of locations of the white tiles is defined by some shifted copy of $\mathcal{L}$: for some $(\ell, \ell') \in \mathbb{Z}^2$, these tiles are given by

$$\sigma_{h,v}(W) \ , \quad (h,v) \in \sigma_{\ell,\ell'}(\mathcal{L}) \ ,$$

where all the tiles (black and white) are disjoint and cover $\mathbb{Z}^2$. Thus, a tiling is characterized by the triple $(B, W, \mathcal{L})$, and the shift (ℓ, ℓ') can be taken as any pair that yields a covering of $\mathbb{Z}^2$ (for our purposes, the particular pair selected is immaterial).

Example 2.1: Figure 1 shows a tiling of $\mathbb{Z}^2$ with

$$B = Q_m, \quad W = Q_{m+1},$$

and

$$\mathcal{L} = \left\{(i,j) = (t,u)\begin{pmatrix} m & m+1 \\ 2m+1 & 1 \end{pmatrix} \ : \ (t,u) \in \mathbb{Z}^2\right\},$$

with the shift $(\ell, \ell') = (-1, m)$ (the figure is drawn to scale for $m = 2$). □

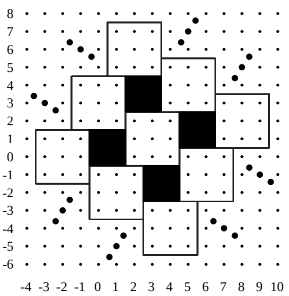


Fig. 1. Example of a tiling of $\mathbb{Z}^2$.

For two nonempty finite subsets $\mathcal{N}, T \subset \mathbb{Z}^2$, we let $\mathcal{N} \oplus T$ denote the following list of $|\mathcal{N}|$ shifted copies of T:

$$\mathcal{N} \oplus T = (\sigma_{r,s}(T))_{(r,s) \in \mathcal{N}}$$

(typically, we will take T to be W). Given a 2-D constraint $\mathbb{S}$, we define accordingly the following set:

$$\mathbb{S}(\mathcal{N} \oplus T) = \left\{\boldsymbol{y} = (y_{r,s})_{(r,s) \in \mathcal{N}} \ : \ y_{r,s} \in \mathbb{S}(\sigma_{r,s}(T))\right\} \ .$$

That is, each element $\boldsymbol{y} \in \mathbb{S}(\mathcal{N} \oplus T)$ is a list of $\mathbb{S}$-compatible configurations which are defined on $|\mathcal{N}|$ shifted copies of T, where the shifts are determined by $\mathcal{N}$.

Given now a list $\boldsymbol{y} = (y_{r,s})_{(r,s) \in \mathcal{N}}$ in $\mathbb{S}(\mathcal{N} \oplus T)$ and an additional nonempty finite subset $U \subset \mathbb{Z}^2$, we define $\mathbb{S}(U; \boldsymbol{y})$ to be the set of all configurations $\psi \in \mathbb{S}(U)$ such that ψ can be extended to an array $x = x(\psi) = (x_{i,j})$ in $\mathbb{S}$ that agrees (with ψ and) with each entry in the list $\boldsymbol{y}$; namely, for every $(i,j) \in U$,

$$\psi(i,j) = x_{i,j} \ ,$$

and for every $(r,s) \in \mathcal{N}$ and $(i,j) \in \sigma_{r,s}(T)$,

$$y_{r,s}(i,j) = x_{i,j}$$

(typically, U will be taken as a black tile).

Example 2.2: Figure 2 shows a portion of the tiling described in Example 2.1, including the black tile $B = Q_m$ (namely, the black tile in Figure 1 that contains the point $(0,0)$) and a surrounding neighborhood of four white tiles. These white tiles are given by $\sigma_{h,v}(W)$, where $W = Q_{m+1}$

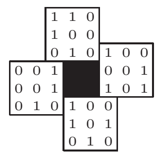


Fig. 2. Example of a neighborhood of B.

and (h,v) is any of the four elements in

$$\mathcal{N} = \{(-1,m), (m,0), (0,-m-1), (-m-1,-1)\} \ . \qquad (1)$$

Hence, in this case, the list $\mathcal{N} \oplus W$ is the quadruple

$$\Big(\sigma_{-1,m}(W) \ \ \sigma_{m,0}(W) \ \ \sigma_{0,-m-1}(W) \ \ \sigma_{-m-1,-1}(W)\Big) \ .$$

Figure 2 also shows examples of assignments of configurations to the white tiles (assuming that $m = 2$). All these configurations can be verified to be compatible with respect to the n.i.b. constraint; thus, if $\mathbb{S}$ stands for this constraint, then the following list of configurations

$$\boldsymbol{y} = \left(\boxed{\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}} \ \ \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{matrix}} \ \ \boxed{\begin{matrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}} \ \ \boxed{\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}} \right)$$

belongs to $\mathbb{S}(\mathcal{N} \oplus W)$. One can check that for this list of configurations we have

$$\mathbb{S}(B; \boldsymbol{y}) = \left\{ \boxed{\begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}}, \ \boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}} \right\} \ ;$$

namely, there are (exactly) two B-configurations that can be assigned to the black tile in Figure 2 so that the bit pattern in the figure can be extended into an array in $\mathbb{S}$. □

We call a tiling $(B, W, \mathcal{L})$ *valid* for a given 2-D constraint $\mathbb{S}$, if the following two conditions hold:

[CW] *White tiles are freely configurable.* For every two nonempty finite subsets $\mathcal{M} \subset \sigma_{\ell,\ell'}(\mathcal{L})$ and $U \subset \mathbb{Z}^2$ and every list $\boldsymbol{y} \in \mathbb{S}(\mathcal{M} \oplus W)$ one has $\mathbb{S}(U; \boldsymbol{y}) \neq \emptyset$.

[CB] *Black configurations are constrained only by a finite neighborhood of white tiles.* There is a nonempty finite subset $\mathcal{N} \subset \sigma_{\ell,\ell'}(\mathcal{L})$ such that for every list $\boldsymbol{y} \in \mathbb{S}(\mathcal{N} \oplus W)$ and every nonempty finite subset $U \subset \mathbb{Z}^2 \setminus B$,

$$|\mathbb{S}(B \cup U; \boldsymbol{y})| = |\mathbb{S}(B; \boldsymbol{y})| \cdot |\mathbb{S}(U; \boldsymbol{y})| \ .$$

Remark 2.1: Condition [CB] means that given any list $\boldsymbol{y} \in \mathbb{S}(\mathcal{N} \oplus W)$, if we extend a configuration that already agrees with $\boldsymbol{y}$ on $\mathbb{S}(\mathcal{N} \oplus W)$ into an $\mathbb{S}$-compatible configuration $x = (x_{i,j})$, the values $x_{i,j}$ at points (i,j) in B can be set independently of the values that are set at points outside B (and $\cup_{(r,s)\in\mathcal{N}} \sigma_{r,s}(W)$). □

Remark 2.2: By the shift invariance property of $\mathbb{S}$, Condition [CB] remains unaffected if we replace B and $\mathcal{N}$ therein by $\sigma_{h,v}(B)$ and $\sigma_{h,v}(\mathcal{N})$, for any $(h,v) \in \mathcal{L}$. □

In its basic setting, our coding technique will apply to constraints for which valid tilings exist. Here are several examples of such constraints.

Example 2.3: For the 2-D (d,∞)-RLL constraint, we get a valid tiling $(B, W, \mathcal{L})$ by taking

$$B = W = Q_m \quad \text{and} \quad \mathcal{L} = \left\{ (mt, m(t{+}2u)) \ : \ (t,u) \in \mathbb{Z}^2 \right\}, \tag{2}$$

for every integer $m \geq d$ (here the tiles form an infinite checkerboard, where each square has order $m \times m$). To see why Condition [CW] holds, suppose that each white tile that intersects with a given finite subset $U \subset \mathbb{Z}^2$ is assigned a configuration that is compatible with the constraint. Then a compatible U-configuration is obtained if each of the black tiles that intersects with U is assigned the all-zero configuration. As for Condition [CB], we take the neighborhood $\mathcal{N}$ as

$$\mathcal{N} = \{(0,m), (m,0), (0,-m), (-m,0)\} \tag{3}$$

(see Figure 3, which is drawn to scale for $m = 2$). The condition then holds since we assume m to be at least the memory, d, of the 1-D (d,∞)-RLL constraint satisfied by every row and column.

For similar reasons the same tiling is valid also for the 2-D $(0,k)$-RLL constraint, provided that $m \geq k$. □

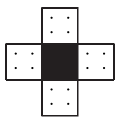


Fig. 3. Neighborhood for the valid tiling for the 2-D (d,∞)-RLL and $(0,k)$-RLL constraints, with $m \geq k, d$.

Example 2.4: The tiling described in Example 2.1 is valid for the n.i.b. constraint, for every integer $m \geq 2$. We leave the formal proof to the reader. □

Given a 2-D constraint, our coding scheme, to be presented in the next section, will be based on a tiling that is valid for the constraint. For the tilings in Examples 2.2 and 2.3, the value m therein will be one of the design parameters for the scheme: larger values of m are expected to yield higher coding rates, yet with higher encoding and decoding complexity.

1) *(Configuring the white tiles)* For every $(h,v) \in \mathcal{L}'_n$ do:
   a) Select a configuration $\varphi_{h,v}$ at random from $\mathbb{S}(\sigma_{h,v}(W))$, according to the probability distribution
   $$\mathsf{P}_\pi \left\{ \varphi_{h,v} = \alpha \right\} = \pi(\alpha \circ \sigma_{h,v}) \ , \quad \alpha \in \mathbb{S}(\sigma_{h,v}(W))$$
   (the choice for distinct pairs (h,v) is carried out independently).
   b) For every $(i,j) \in \sigma_{h,v}(W)$, set
   $$x_{i,j} = \varphi_{h,v}(i,j) \ .$$
2) *(Configuring the black tiles)* For every $(h,v) \in \mathcal{L}''_n$ do:
   a) Let
   $$\boldsymbol{\varphi}_{h,v} = (\varphi_{r,s})_{(r,s)\in\sigma_{h,v}(\mathcal{N})}$$
   be the list in $\mathbb{S}(\sigma_{h,v}(\mathcal{N}) \oplus W)$ whose entries are as selected in Step 1. Select a configuration $\psi_{h,v}$ uniformly at random from $\mathbb{S}(\sigma_{h,v}(B); \boldsymbol{\varphi}_{h,v})$.
   b) For every $(i,j) \in \sigma_{h,v}(B)$, set
   $$x_{i,j} = \psi_{h,v}(i,j) \ .$$
3) *(Setting the boundary values)* For all $(i,j) \in Q_n \setminus \hat{Q}_n$, set $x_{i,j}$ to take the smallest values (according to some lexicographic ordering) so that $(x_{i,j})$ becomes an element of $\mathbb{S}(Q_n)$.

Fig. 4. Procedure for selecting at random an array $(x_{i,j})$ from $\mathbb{S}(Q_n)$.

III. Lower bound on the capacity

Given a 2-D constraint $\mathbb{S}$ for which a valid tiling $(B, W, \mathcal{L})$ exists, we next define a probability measure on $\mathbb{S}(Q_n)$. Our coding scheme will then realize this probability measure.

To handle boundary effects, we introduce the set

$$\hat{Q}_n = \left(\bigcup\nolimits_{(h,v)\in\mathcal{L}'_n} \sigma_{h,v}(W) \right) \cup \left(\bigcup\nolimits_{(h,v)\in\mathcal{L}''_n} \sigma_{h,v}(B) \right) \tag{4}$$

where

$$\mathcal{L}'_n = \left\{ (h,v) \in \sigma_{\ell,\ell'}(\mathcal{L}) \ : \ \sigma_{h,v}(W) \subseteq Q_n \right\} \tag{5}$$

and

$$\mathcal{L}''_n = \left\{ (h,v) \in \mathcal{L} \ : \ \sigma_{h,v}\big(B \cup \bigcup\nolimits_{(r,s)\in\mathcal{N}} \sigma_{r,s}(W)\big) \subseteq Q_n \right\} ;$$

namely, $\hat{Q}_n$ is the largest subset of Q_n that contains full white tiles as well as full black tiles along with their whole neighborhoods of white tiles (as specified by $\mathcal{N}$). Note that as n goes to infinity, the ratios $n^2/|\mathcal{L}'_n|$ and $n^2/|\mathcal{L}''_n|$ converge to $|W| + |B|$.

We start off with a prescribed probability distribution π which is defined on the $\mathbb{S}$-compatible W-configurations:

$$\pi : \mathbb{S}(W) \to [0,1] \ .$$

Next, we define the probability measure $\mu_n : \mathbb{S}(Q_n) \to [0,1]$ through the procedure shown in Figure 4, which describes how one selects at random an array $(x_{i,j})$ from $\mathbb{S}(Q_n)$.

We now compute the per-symbol entropy $\mathsf{H}(\mu_n)$. Let Φ denote the random $(\mathcal{L}'_n \oplus W)$-configuration formed by the selections made in Step 1 in Figure 4; namely, for every $(h,v) \in \mathcal{L}'_n$ and $(i,j) \in W$,

$$\Phi(i{+}h, j{+}v) = \varphi_{h,v}(i,j) \ .$$

It follows from Condition [CW] that Φ is $\mathbb{S}$-compatible. From Step 1 it is easy to see that the entropy of Φ (per $n \times n$ array) is given by

$$\mathsf{H}_n(\Phi) = |\mathcal{L}'_n| \cdot H(\pi) \;, \tag{6}$$

where $H(\pi)$ is the entropy (per tile) of a compatible W-configuration:

$$H(\pi) = -\sum_{y \in \mathbb{S}(W)} \pi(y) \log \pi(y) \;.$$

Denote by Ψ the random $(\mathcal{L}''_n \oplus B)$-configuration formed by the selections made in Step 2 in Figure 4: for every $(h,v) \in \mathcal{L}''_n$ and $(i,j) \in B$,

$$\Psi(i{+}h, j{+}v) = \psi_{h,v}(i,j) \;.$$

(From Condition [CB] we get that the selections made in both Steps 1 and 2 form a combined configuration which is in $\mathbb{S}(\hat{Q}_n)$. Conditions [CW]–[CB] also guarantee that Step 3 will always be successful in extending that configuration into one in $\mathbb{S}(Q_n)$.) The entropy of Ψ conditioned on Φ is readily given by

$$\mathsf{H}_n(\Psi|\Phi) = |\mathcal{L}''_n| \cdot \sum_{\boldsymbol{\varphi}} \mathsf{P}_\pi\{\boldsymbol{\varphi}\} \log |\mathbb{S}(B; \boldsymbol{\varphi})| \;, \tag{7}$$

where $\boldsymbol{\varphi} = (\varphi_{r,s})_{(r,s)\in\mathcal{N}}$ ranges over all lists in $\mathbb{S}(\mathcal{N} \oplus W)$ and

$$\mathsf{P}_\pi\{\boldsymbol{\varphi}\} = \prod_{(r,s)\in\mathcal{N}} \mathsf{P}_\pi\{\varphi_{r,s}\} = \prod_{(r,s)\in\mathcal{N}} \pi(\varphi_{r,s} \circ \sigma_{r,s}) \;. \tag{8}$$

Combining (6) and (7), the per-symbol entropy $\mathsf{H}(\mu_n)$ is given by

$$\begin{aligned} \mathsf{H}(\mu_n) &= \frac{1}{n^2}\Big(\mathsf{H}_n(\Phi) + \mathsf{H}_n(\Psi|\Phi)\Big) \\ &= \frac{|\mathcal{L}'_n|}{n^2} \cdot H(\pi) + \frac{|\mathcal{L}''_n|}{n^2} \cdot \sum_{\boldsymbol{\varphi}} \mathsf{P}_\pi\{\boldsymbol{\varphi}\} \log |\mathbb{S}(B;\boldsymbol{\varphi})| \\ &= \frac{1-o(1)}{|W|{+}|B|}\Big(H(\pi) + \sum_{\boldsymbol{\varphi}} \mathsf{P}_\pi\{\boldsymbol{\varphi}\} \log |\mathbb{S}(B;\boldsymbol{\varphi})|\Big), \end{aligned}$$

where $\boldsymbol{\varphi}$ ranges over $\mathbb{S}(\mathcal{N} \oplus W)$ and $o(1)$ stands for an expression that goes to zero as n goes to infinity.

We thus get the following result.

Theorem 3.1: Let $\mathbb{S}$ be a 2-D constraint for which there is a valid tiling $(B, W, \mathcal{L})$. Then $\mathsf{cap}(\mathbb{S})$ is bounded from below by

$$\mathsf{cap}(\mathbb{S}) \geq \max_\pi \frac{1}{|W|{+}|B|}\Big(H(\pi) + \sum_{\boldsymbol{\varphi}} \mathsf{P}_\pi\{\boldsymbol{\varphi}\} \log |\mathbb{S}(B;\boldsymbol{\varphi})|\Big), \tag{9}$$

where $\boldsymbol{\varphi} = (\varphi_{r,s})_{(r,s)\in\mathcal{N}}$ ranges over $\mathbb{S}(\mathcal{N} \oplus W)$, the probabilities $\mathsf{P}_\pi\{\boldsymbol{\varphi}\}$ are defined by (8), and the maximization is taken over all distributions $\pi : \mathbb{S}(W) \to [0,1]$.

Table I presents the lower bound of Theorem 3.1 for several 2-D constraints. For all except the n.i.b. constraint, we used the checkerboard tiling of Example 2.3, with m taken as indicated in the last column of the table (for the n.i.b. constraint, we used the tiling of Example 2.1). When computing the bound of Theorem 3.1, we took advantage of properties of the constraint (such as rotational and reflectional symmetries) in order to reduce the number of parameters that determine the distribution π over which we maximized. The values in the third column of Table I are the lower bounds on the rates of bit-stuffing encoders as presented recently in [11]: these bounds turn out to be comparable to those obtained from Theorem 3.1. The fourth column in the table presents the best lower bounds on the capacity that were known prior to this work and [11].

TABLE I

LOWER BOUND ON THE CAPACITY FOR SEVERAL 2-D CONSTRAINTS.

Constraint	Theorem 3.1	Ref. [11]	Earlier record	m
$(2,\infty)$-RLL	0.44417	0.44420	0.4423 [12]	5
$(3,\infty)$-RLL	**0.36562**	0.35973	0.3641 [9]	5
$(0,2)$-RLL	**0.81600**	0.81549	0.7736 [13]	4
n.i.b.	0.92086	0.92264	0.9156 [8]	3

The procedure in Figure 4 can be made into a variable-rate encoder for any 2-D constraint $\mathbb{S}$ that satisfies the condition of Theorem 3.1, using the method suggested in [10] (see also [14]). Specifically, we realize the random selections made in Step 1 through one distribution transformer which maps, in a one-to-one manner, a sequence of fair coins (i.e., statistically independent Bernoulli random bits, each equaling 0 with probability $1/2$), into a sequence of configurations in $\mathbb{S}(W)$ which are statistically independent and are distributed according to the maximizing π. Once the configurations on the white tiles have been determined, Step 2 in Figure 4 can be realized by enumerative coding [1, Chapter 6]. The rate of the encoder thus obtained will approach the lower bound of Theorem 3.1 (there will be a small rate penalty which results from the boundary effects caused by Step 3 in Figure 4, and an additional penalty caused by the finite accuracy by which the distribution transformer simulates the maximizing π).

IV. GENERALIZATIONS AND IMPROVEMENTS

In this section, we discuss several generalizations of the bounding (and coding) method described in Section III.

A. Multilevel tilings

The requirement that there is a valid tiling for the constraint does restrict the generality of Theorem 3.1. One possible remedy is introducing more than two types (colors) of tiles. Namely, we can allow also "gray" tiles, each being a shifted copy of some nonempty finite subset $G \subset \mathbb{Z}^2$, and all the gray tiles lie along a shifted copy of the lattice $\mathcal{L}$ so that all tiles (black, gray, and white) are disjoint and cover $\mathbb{Z}^2$. Such a (generalized) tiling will then be said to be valid for a given 2-D constraint $\mathbb{S}$ if the following three conditions hold:

- [CW] White tiles are freely configurable.
- [CG] Gray configurations are constrained only by a finite neighborhood $\mathcal{M}$ of white tiles.
- [CB'] Black configurations are constrained only by a finite neighborhood of white and gray tiles.

A probability measure μ_n can now be defined on $\mathbb{S}(Q_n)$ through a probability distribution $\pi : \mathbb{S}(W) \to [0,1]$, *and* a conditional probability distribution

$$\gamma : \mathbb{S}(G) \times \mathbb{S}(\mathcal{M} \oplus W) \to [0,1] \;,$$

where $\gamma(\psi|\boldsymbol{y})$ stands for the probability of a G-configuration ψ, given that the neighboring white tiles are configured by the list $\boldsymbol{y} \in \mathbb{S}(\mathcal{M} \oplus W)$. The distribution γ should be such that $\gamma(\psi|\boldsymbol{y}) > 0$ only when $\psi \in \mathbb{S}(G; \boldsymbol{y})$.

It is not difficult to generalize the lower bound of Theorem 3.1 to include gray tiling, and the maximization then will be on both π and γ. In fact, this generalization can be taken even further to include several gray levels, where the configurations on tiles at a given gray level are constrained only by a finite neighborhood of "lighter" levels of gray.

Example 4.1: The 2-D "non-attacking kings" (in short, n.a.k.) constraint is defined as the set of all binary arrays $(x_{i,j})$ in which all horizontally, vertically, and diagonally adjacent entries to a "1" have to be "0" (formally, if $x_{i,j} = 1$ then $x_{r,s} = 0$ for all $(r,s) \in \sigma_{i-1,j-1}(Q_3) \setminus \{(i,j)\}$; this constraint is called the "square constraint" in [15]). For this constraint, we get a valid tiling $(B, G, W, \mathcal{L})$ by taking[1]

$$B = W = Q_m \ , \quad G = Q_1 \cup \sigma_{-1,m}(Q_1) \ ,$$

and

$$\mathcal{L} = \left\{ (i,j) = (t,u) \begin{pmatrix} m-1 & m+1 \\ m+1 & -m+1 \end{pmatrix} \ : \ (t,u) \in \mathbb{Z}^2 \right\} \ ,$$

for every integer $m \geq 2$. The respective neighborhood of a black tile for $m = 3$ is shown in Figure 5. Table II shows

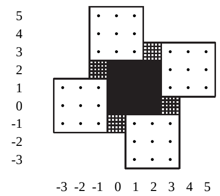


Fig. 5. Neighborhood for the valid tiling for the n.a.k. constraint.

the lower bounds on the capacity of this constraint obtained when (the generalization of) Theorem 3.1 is applied to this tiling, for $m = 3, 4, 5$. Also shown in the table is the number of variables that parametrize π (the conditional distribution γ is parametrized here by one variable only—the probability that a gray tile is assigned the value "1", provided that this value is allowed by the neighboring white tiles). While the best bound in the table is still short of the current record, 0.425029, obtained by the Calkin–Wilf method [16], [15], it is interesting to observe that a variable-rate encoder can achieve a rate greater than 0.423 using only 3×3 tiles (and two distribution transformers). □

TABLE II
LOWER BOUNDS ON THE CAPACITY OF THE N.A.K. CONSTRAINT.

m	Lower bound	No. of variables
3	0.423076	10
4	0.423955	33
5	0.424350	217

[1]Generally, a description of a tiling should also include the shift of $\mathcal{L}$ that corresponds to the gray tiles. In our example, this shift is uniquely determined.

B. Allowing statistical dependence among white tiles

We may improve the bounds obtained through Theorem 3.1 by introducing statistical dependence among the white tiles that are selected in Step 1 of Figure 4. The theorem will essentially remain the same, except that the distribution π will be defined differently (and typically it will now be parametrized by significantly more variables). With respect to the new model of π, the term $H(\pi)$ will still stand for the entropy per tile of a compatible W-configuration, and the choice for the model π needs to be such that we should be able to compute this quantity (or at least bound it from below). We demonstrate this approach in the next example.

Example 4.2: Consider the 2-D $(1, \infty)$-RLL constraint with the valid tiling of Example 2.3, and partition the lattice $\mathcal{L}$ therein into shifts of the following infinite diagonal:

$$\mathcal{D} = \{(mt, mt) \ : \ t \in \mathbb{Z}\}$$

(clearly, $\mathcal{L} = \cup_{u \in \mathbb{Z}} \sigma_{2mu,0}(\mathcal{D})$). Next, we modify Step 1 in Figure 4 so that along each (finite shifted) diagonal

$$\mathcal{D}'_n(u) = \mathcal{L}'_n \cap \sigma_{2mu+\ell,\ell'}(\mathcal{D})$$

the selected configurations $(\varphi_{h,v})_{(h,v) \in \mathcal{D}'_n(u)}$ form a stationary Markov chain π (independently for distinct u).

The lower bounds on the capacity that are obtained through this probabilistic model are presented in the third column of Table III (the second column shows the respective values for the unmodified Step 1). As expected, by introducing the statistical dependence we have gained improvements for any given m (but the number of variables that we maximize over has increased as well).

TABLE III
LOWER BOUNDS ON THE CAPACITY OF THE 2-D $(1, \infty)$-RLL CONSTRAINT.

m	Independent	Markov	Pickard field	BMRF field
1	0.566144	0.574094	0.584387	0.584418
2	0.582075	0.584798	**0.587855**	
3	0.585350	0.586485		
4	0.586459			
5	0.586974			

Further improvements can be obtained if we now assume that the white tiles are configured according to a distribution that is a stationary Markov random field. Table III shows the results for two types of random fields: the Pickard field [17] and a binary Markov random field (in short, BMRF) due to Champagnat *et al.* [18]. In our maximization procedure for the case of the Pickard field, we applied an iterative algorithm which is similar to the one presented by Forchhammer and Laursen in [8] (unlike [8], however, the random field in our case is defined over unconstrained configurations, due to Condition [CW]). The bound we obtain with the Pickard field for $m = 2$ agrees with the (true) value of the capacity up to the first four decimal places [16]. □

For the n.a.k. constraint (Example 4.1), the Markov chain approach yields, for $m = 4$, a lower bound of 0.424558.

C. Multidimensional constraints

The technique that was presented in this work can be adapted to handle λ-dimensional (λ-D) constraints in dimensions $\lambda > 2$. For example, for the λ-D $(1,\infty)$-RLL constraint one gets the lower bound

$$\max_{p\in[0,1]} \tfrac{1}{2}\left(H(p) + (1-p)^{2\lambda}\right) \;, \tag{10}$$

when taking the λ-D extension of the tiling of Example 2.3 with $m = 1$ (here π is specified by one parameter only—the probability p that a white λ-D tile is assigned the value "1"). For the case $\lambda = 3$, the bound (10) equals 0.513864: this value is smaller than the lower bound of 0.5225 obtained by Nagy and Zeger in [19], but it is higher than their lower bound on the rate of a bit-stuffing encoder [14]. The bound (10) (for $\lambda = 3$) improves to (at least) 0.521270 when we assume a distribution model which is a generalization of Pickard field to three dimensions [20]. We next describe how the model studied in [20] can be adapted to our setting.

For a positive integer n, let C_n denote the $n \times n \times n$ cube

$$C_n = \{(i,j,k) \in \mathbb{Z}^3 \;:\; 0 \le i,j,k < n\} \;.$$

We consider the 3-D tiling $(B, W, \mathcal{L})$ where $B = W = C_1$ and

$$\mathcal{L} = \{(i,j,k) \;:\; i{+}j{+}k \equiv 0 \pmod 2\} \;.$$

Figure 6 depicts the neighborhood of a $1 \times 1 \times 1$ black tile (marked as a filled circle) at some position $(i,j,k) \in \mathcal{L}$. The neighborhood consists of six tiles (marked as hollow circles) at the following positions:

$$\mathsf{L} = (i{-}1,j,k), \quad \mathsf{R} = (i{+}1,j,k), \quad \mathsf{D} = (i,j{-}1,k),$$
$$\mathsf{U} = (i,j{+}1,k), \quad \mathsf{B} = (i,j,k{-}1), \quad \text{and} \quad \mathsf{F} = (i,j,k{+}1) \;.$$

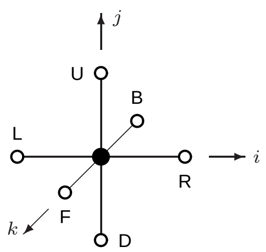


Fig. 6. Neighborhood of a 3-D black tile.

Denote by $\mathcal{L}'$ the 3-D lattice of the positions of the white tiles, namely,

$$\mathcal{L}' = \sigma_{0,0,1}(\mathcal{L}) = \{(i,j,k) \;:\; i{+}j{+}k \equiv 1 \pmod 2\} \;.$$

In order to define the joint distribution π of the configurations assigned to the white tiles, we will map the lattice $\mathcal{L}'$ onto $\mathbb{Z}^3$ using the following bijection $\tau : \mathcal{L}' \to \mathbb{Z}^3$:

$$\tau(i,j,k) = \tfrac{1}{2}\left(i{-}j{+}k{+}1, i{+}j{+}k{+}1, {-}i{+}j{+}k{+}1\right) \;.$$

Figure 7 shows the images under the mapping τ of the points shown in Figure 6. As we can see, the six positions of the white tiles are mapped to six points in a shifted cube $\sigma_{h,v,z}(C_2)$, where $(h,v,z) = \tau(i,j,k{-}1)$ (the image of point B). For reference, Figure 7 also shows the image of the position (i,j,k) of the black tile in Figure 6, assuming that the domain of τ is extended to the whole set $\mathbb{Z}^3$.

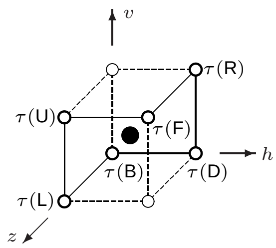


Fig. 7. Neighborhood of a 3-D black tile under the mapping τ.

Fix some (large) positive integer n, and let

$$\pi_0 : \{0,1\}^{n\times n\times n} \to [0,1]$$

be a 3-D Pickard field as defined in [20]. Namely, π_0 is a distribution over all configurations $\varphi : C_n \to \{0,1\}$ that satisfies certain stationarity and symmetry properties, and is completely characterized by the marginal distribution of C_2-configurations. In particular, due to the stationarity of π_0, we can easily compute the per-symbol entropy of π_0. For more details, see [20].

Let R_n denote the rhombohedron in $\mathbb{Z}^3$ which is the pre-image set of C_n under the mapping τ. We now define the distribution $\pi : R_n \to [0,1]$ on all unconstrained binary R_n-configurations as follows: for every R_n-configuration $x = (x_{i,j,k})_{(i,j,k)\in R_n}$, the value $\pi(x)$ equals $\pi_0(y)$, where $y = (y_{h,v,z})_{(h,v,z)\in C_n}$ is given by

$$y_{\tau(i,j,k)} = x_{i,j,k} \;, \quad (i,j,k) \in R_n \;.$$

The distribution π on the joint configuration assignment of white tiles inherits the desirable properties of the 3-D Pickard field π_0: first, we can compute the entropy of π, and, secondly, we can compute the distribution of the configuration assignment of the neighborhood of a black tile, since that neighborhood is mapped by τ into a shifted copy of C_2.

It is worthwhile pointing out that the maximization in the right-hand side of (9) is not necessarily convex, and this applies to the generalizations of (9) that are implied by the discussion of this section. As an example, it follows from the next proposition that for any dimension $\lambda \ge 3$, the expression that is maximized in the right-hand side of (10) has two local maxima in the interval $[0,1]$.

Proposition 4.1: For a positive integer n, let the function $f_n : [0,1] \to \mathbb{R}$ be defined by

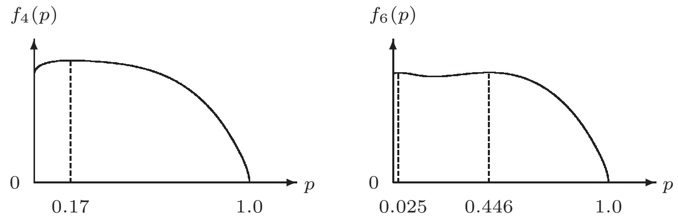
$$f_n(p) = H(p) + (1-p)^n \;.$$

Then for $n \ge 5$, this function has (exactly) two local maxima in the interval $[0,1]$.

For $n = 4$ (which corresponds to dimension $\lambda = 2$) we get only one maximum, at $p \approx 0.1702$, as shown in Figure 8(a). The function $p \mapsto f_6(p)$ is depicted in Figure 8(b).

Proof of Proposition 4.1: The first derivative of $f_n(p)$ is given by

$$f_n'(p) = \log_2\!\left(\frac{1-p}{p}\right) - n(1-p)^{n-1} \;.$$



(a) (b)

Fig. 8. Functions $p \mapsto f_n(p)$ for $n = 4, 6$.

On the one hand, we have $\lim_{p\to 0^+} f_n'(p) = \infty$. On the other hand, for $p = \varepsilon_n = 1/n$ we get

$$f_n'(\varepsilon_n) = \log_2(n-1) - n\Big(1 - \frac{1}{n}\Big)^{n-1}, \qquad (11)$$

which is negative for $n \geq 2$ (the right-hand side of (11) is smaller than $\log_2(n-1) - (n/\mathrm{e})$, where $\mathrm{e} = 2.71828\ldots$ is the base of natural logarithms). It follows that the interval $(0, \varepsilon_n)$ contains at least one local maximum of $f_n(p)$. Similarly, for $n \geq 5$ we have $f_n'((1/2)-\varepsilon_n) > 0$ and $f_n'(1/2) < 0$, which means that $f_n(p)$ has a local maximum also in the interval $((1/2)-\varepsilon_n, 1/2)$.

In order to show that there are exactly two local maxima of $f_n(p)$ in $[0,1]$, it is sufficient to prove that the second derivative

$$f_n''(p) = \frac{\log_2 \mathrm{e}}{p(p-1)} + n(n-1)(1-p)^{n-2}$$

has at most two zeros in the interval $(0,1)$. Equivalently, it suffices to show that the polynomial

$$g_n(p) = p(1-p)^{n-1} - \frac{\log_2 \mathrm{e}}{n(n-1)}$$

has at most two zeros in $(0,1)$. Indeed, this holds since the derivative

$$g_n'(p) = (1-p)^{n-2}(1-np)$$

has only one zero in that interval (at $p = 1/n$). □

Turning back to the λ-D extension of the tiling of Example 2.3 with $m = 1$, we now analyze the effect of increasing the dimension λ on the lower bound (10). As λ $(= n/2)$ goes to infinity, the abscissa of the smaller local maximum of $f_n(p)$ tends to $p = 0$: this probability value corresponds to assigning "0" (with probability 1) to each white tile, thereby allowing the black tiles to be unconstrained. The abscissa of the larger local maximum tends to $p = 1/2$: this, in turn, corresponds to an equiprobable distribution on the white tiles, thereby forcing the assignment of "0" to any given black tile with probability 1. Thus, when $\lambda \to \infty$, the lower bound (10) approaches $1/2$ and, in that limit, the maximum is attained at $p = 0$ and $p = 1/2$. In fact, it is known that in the limit, the capacity of the λ-D $(1,\infty)$-RLL constraint approaches $1/2$ [21, Theorem 1.4], [22, p. 57].

References

[1] K.A.S. Immink, *Codes for Mass Data Storage Systems,* Shannon Foundation Publishers, The Netherlands, 2004.

[2] "Information technology—Automatic identification and data capture techniques—Bar code symbology specifications—PDF417," ISO/IEC 15438:2001.

[3] Y.P. Wang, "System for encoding and decoding data in machine readable graphic form," US Patent 5,243,655, 1993.

[4] B.H. Marcus, R.M. Roth, P.H. Siegel, "Constrained systems and coding for recording channels," in *Handbook of Coding Theory,* V.S. Pless and W.C. Huffman (Editors), Elsevier Scientific Publishers, Amsterdam, 1998.

[5] R. Berger, *The undecidability of the Domino problem,* Mem. Amer. Math. Soc., vol. 66, 1966.

[6] R.M. Robinson, "Undecidability and nonperiodicity for tilings of the plane," *Inventiones Math.,* vol. 12, pp. 177–209, 1971.

[7] A. Kato, K. Zeger, "On the capacity of two-dimensional run-length constrained channels," *IEEE Trans. Inform. Theory,* vol. 45, pp. 1527–1540, 1999.

[8] S. Forchhammer, T.V. Laursen, "A model for the two-dimensional No Isolated Bits constraint," *Proc. IEEE Symp. Inform. Theory (ISIT'2006),* Seattle, WA, 2006, pp. 1189–1193.

[9] S. Forchhammer, T.V. Laursen, "Entropy of bit-stuffing-induced measures for two-dimensional checkerboard constraints," *IEEE Trans. Inform. Theory,* vol. 53, pp. 1537–1546, 2007.

[10] R.M. Roth, P.H. Siegel, J.K. Wolf, "Efficient coding schemes for the hard-square model," *IEEE Trans. Inform. Theory,* vol. 47, pp. 1166–1176, 2001.

[11] I. Tal, R.M. Roth, "Bounds on the rate of 2-D bit-stuffing encoders," *Proc. IEEE Symp. Inform. Theory (ISIT'2008),* Toronto, ON, 2008, pp. 1463–1467.

[12] E. Ordentlich, R.M. Roth, "Capacity lower bounds and approximate enumerative coding for 2-D constraints," *Proc. IEEE Symp. Inform. Theory (ISIT'2007),* Nice, France, 2007, pp. 1681–1685.

[13] J.J. Ashley, B.H. Marcus, "Two-dimensional low-pass filtering codes," *IEEE Trans. Commun.,* vol. 46, pp. 724–727, 1998.

[14] Z. Nagy, K. Zeger, "Bit-stuffing bounds and analysis for run-length constrained channels in two and three dimensions," *IEEE Trans. Inform. Theory,* vol. 50, pp. 3146–3169, 2004.

[15] W. Weeks IV, R.E. Blahut, "The capacity and coding gain of certain checkerboard codes," *IEEE Trans. Inform. Theory,* vol. 44, pp. 1193–1203, 1998.

[16] N.J. Calkin, H.S. Wilf, "The number of independent sets in a grid graph," *SIAM J. Discrete Math.,* vol. 11, pp. 54–60, 1998.

[17] D.K. Pickard, "Unilateral Markov fields," *Adv. Appl. Prob.,* vol. 12, pp. 655–671, 1980.

[18] F. Champagnat, J. Idier, Y. Goussard, "Stationary Markov random fields on a finite rectangular lattice," *IEEE Trans. Inform. Theory,* vol. 46, pp. 2901–2916, 2000.

[19] Z. Nagy, K. Zeger, "Capacity bounds for the 3-dimensional $(0,1)$ run-length limited channel," *IEEE Trans. Inform. Theory,* vol. 46, pp. 1030–1033, 2000.

[20] J. Idier, Y. Goussard, "Champs de Pickard tridimensionnels," Institut de génie biomédical, Université de Montréal et École Polytechnique Montréal, Tech. Rep. IGB/L2S, 1999.

[21] E. Ordentlich, R.M. Roth, "Independent sets in regular hypergraphs and multi-dimensional runlength-limited constraints," *SIAM J. Discrete Math.,* vol. 17, pp. 615–623, 2004.

[22] A.A. Sapozhenko, "The number of antichains in ranked posets," *Discrete Math. Appl.,* vol. 1, pp. 35–58, 1991.

Artyom Sharov was born in Perm, U.S.S.R., in 1982. He received the B.Sc. degree in computer science from Technion—Israel Institute of Technology, Haifa, Israel, in 2004. He is currently pursuing the M.Sc. degree in computer science at Technion. His research interests include constrained coding and coding theory.

Ron M. Roth (M'88–SM'97–F'03) was born in Ramat Gan, Israel, in 1958. He received the B.Sc. degree in computer engineering, the M.Sc. in electrical engineering and the D.Sc. in computer science from Technion—Israel Institute of Technology, Haifa, Israel, in 1980, 1984 and 1988, respectively. Since 1988 he has been with the Computer Science Department at Technion, where he now holds the General Yaakov Dori Chair in Engineering. During the academic years 1989–91 he was a Visiting Scientist at IBM Research Division, Almaden Research Center, San Jose, California, and during 1996–97 and 2004–05 he was on sabbatical leave at Hewlett–Packard Laboratories, Palo Alto, California. He is the author of the book *Introduction to Coding Theory*, published by Cambridge University Press in 2006. Dr. Roth was an associate editor for coding theory in IEEE TRANSACTIONS ON INFORMATION THEORY from 1998 till 2001, and he is now serving as an associate editor in *SIAM Journal on Discrete Mathematics*. His research interests include coding theory, information theory, and their application to the theory of complexity.