Coding Techniques for
Multidimensional Constrained Channels
A Thesis Proposal

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December 14, 2007

1 Background

Constrained coding has found widespread use in optical and magnetic data storage devices [9]. Proposals for new storage systems such as holographic storage have caused interest in two-dimensional (2D) and three-dimensional (3D) constrained systems as models of data storage on a surface.

One-dimensional constraints were extensively studied and there are several known methodologies for designing codes for such constraints [8], [10], [11]. On the other hand, our knowledge of 2D constraints is much less profound. This might be attributed in part to the fact that the practical interest in those constraints has been risen relatively recently. However, it seems that the main reason for such lack of knowledge is the provable difficulty of certain problems that relate to 2D constraints compared to the one-dimensional (1D) case [1], [15].

1.1 Constrained systems

Many data storage systems, such as those based upon magnetic or optical recording technologies, require the use of constrained modulation codes. These codes transform, in a lossless manner, streams of arbitrary binary data into binary sequences that satisfy certain prescribed constraints. The set of words from which the code sequences may be drawn is referred to as a *constrained system* or simply a *constraint.* A constraint is characterized by a finite directed labeled graph, the paths of which generate the words in the set. The study of these systems was initiated by Shannon [18] who defined the capacity of a constrained system S as

$$\mathrm{cap}(S) = \limsup_{n\to\infty} \frac{\log_2 |S(n)|}{n},$$

where $S(n)$ denotes the number of sequences of S of length exactly n. Shannon showed that whenever there is a rate $p:q$ encoder (that is, an encoder which generates on average q output symbols for every p input bits) of a constrained system, then we must have $\frac{p}{q} \leq \mathrm{cap}(S)$. Furthermore, this bound is tight. The definition of constraint and capacity generalizes to higher-dimensional constraints. If S denotes a 2D constraint and $S(m,n)$ denotes the set of $m \times n$ arrays in S, then

$$\mathrm{cap}(S) = \limsup_{m,n\to\infty} \frac{\log_2 |S(m,n)|}{mn}.$$

1.2 (d,k)-RLL constrained systems

Arguably, the most famous family of constrained systems belongs to the run-length limited (RLL) type. A (d,k)-RLL constrained system consists of all binary words, in which every two 1's are separated by at least d 0's, and there are no more than k consecutive 0's anywhere in the word. The motivation for introducing this constraint comes from the fact that certain sensor characteristics restrict the minimum time between adjacent 1's or else the two will be merged in the receiver, while a clock drift between the transmitter and the receiver may cause spurious 0's or missing 0's at the receiver if too many appear consecutively.

The definition of a (d,k)-RLL constrained system can be easily extended to higher dimensions. For instance, a 2D (d,k)-RLL constrained system consists of 2D arrays, each row of which and each column of which satisfy the 1D (d,k)-RLL constraint. Let us denote the D-dimensional (d,k)-RLL constrained system as $S^D_{d,k}$. Among the multidimensional constraints, perhaps, one of the most widely studied is $S^2_{0,1}$, also known as the *hard-square model.* This model was introduced in statistical physics to capture some of the behavior of a gas whose particles have nonnegligible radii and cannot overlap; here 1's represent particles and 0's represent empty locations. The model has also been used in telecommunications for modelling situations where an occupied node disables all its neighboring nodes. It can be easily verified that, up to bit complementation, $S^2_{0,1}$ (and more generally, $S^D_{0,1}$) is essentially the same as $S^2_{1,\infty}$ (respectively, $S^D_{1,\infty}$).

1.3 The NIB constraint

Another important 2D constrained system is the "no isolated bits" (NIB) 2D constraint, which consists of all binary arrays that do not contain any of the following patterns:

$$\begin{array}{ccc} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{array} \quad \text{and} \quad \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array}.$$

1.4 Bit-stuffing encoders

Various techniques were suggested for encoding of constrained systems. One of them is the so-called *bit-stuffing algorithm*. This is a technique for coding constrained sequences by the insertion of constrained bits into an arbitrary data sequence. This approach was previously introduced and mainly applied to (d,k)-RLL constrained systems.

The bit-stuffing encoder for a (d,k)-RLL constrained system consists of two components: a distribution transformer and a constrained encoder. The distribution transformer converts an unbiased independent and identically distributed binary input sequence into a biased sequence of independent random bits, whose probability of a 0 is some $p \in (0,1)$. This conversion can be implemented in a one-to-one manner. Hence we can apply the reverse transformation to recover the unbiased data. The constrained encoder writes the biased sequence while keeping track of the number of consecutive zeros in the sequence, called the run length. Once the run length equals k, the encoder inserts (stuffs) a 1 followed by d 0's. Whenever encountering a biased 1, the encoder inserts d 0's. To decode, one keeps track of the run length in the constrained sequence, identifying the stuffed bits and discarding them. The resulting sequence is then fed into the inverse distribution transformer, so as to obtain the original unbiased data. Finally, we optimize over p to obtain the maximum average rate.

1.5 Markov random fields

Another theoretical framework that was found useful for constructing 2D encoders for constrained systems is *Markov random fields* (MRF). An MRF of order M is a measure, defined on 2D random arrays $(X_{i,j})_{i,j}$, which satisfy the following property for every (i,j):

$$\begin{aligned}&\Pr\Big(X_{i,j}=x_{i,j} \mid \textstyle\bigwedge_{i\neq k \wedge j\neq \ell}(X_{k,\ell}=x_{k,\ell})\Big)\\ &\qquad = \Pr\Big(X_{i,j}=x_{i,j} \mid \textstyle\bigwedge_{X_{k,\ell}\in N_{i,j},\, i\neq k\wedge j\neq \ell}(X_{k,\ell}=x_{k,\ell})\Big),\end{aligned}$$

where $N_{i,j}$ is the neighborhood set of $X_{i,j}$, that is

$$N_{i,j}=\{X_{k,\ell} \mid (i\neq k)\wedge(j\neq \ell)\wedge(|i-k|+|j-\ell|\leq M)\}.$$

A Markov chain can be viewed as a particular case of an MRF for 1D: recall that a Markov chain is a sequence of random variables X_1, X_2, X_3, $\ldots$ with the Markov property, namely that, given the present state, the future and past states are independent. Formally,

$$\Pr(X_{n+1}=x \mid X_n=x_n,\ldots,X_1=x_1)=\Pr(X_{n+1}=x \mid X_n=x_n).$$

1.6 Pickard fields

An actual construction of a random field with known entropy is interesting both for simulation purposes and as a method for establishing lower bounds on capacities of constrained systems. It would be very desirable to have random fields where rows and columns were described by simple Markov chains. Unfortunately this appears to be possible only in the case of the Pickard lattices [13], [14]. This is also the only case where the causal model of the field becomes a simple finite state source. Let X, Y, Z be random variables and let $X \perp Y \mid Z$ denote that X and Y are independent given Z. Using this notation, a Pickard field is defined as a measure $\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on a generic 2×2 cell $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, such that

$$((\, B \perp C \mid A \,) \wedge (\, B \perp C \mid D \,)) \vee$$
$$((\, B \perp C \mid A \,) \wedge (\, A \perp D \mid B \,) \wedge (\, A \perp D \mid C \,))$$

2 Known results

Calkin and Wilf [2] gave methods for bounding the capacity of certain very special 2D constrained systems. Using their methods, Weeks and Blahut [19] obtained the following bounds for $S^2_{0,1}$:

$$0.5878911617 \leq \mathrm{cap}\big(S^2_{0,1}\big) \leq 0.5878911618.$$

Using the same method, Nagy and Zeger [12] showed the following bounds on the capacity of the 3D $(0,1)$-RLL constrained system:

$$0.52250174138 \leq \mathrm{cap}\big(S^3_{0,1}\big) \leq 0.526880847825.$$

By exploiting Markovian properties which are peculiar to hard-square model, Roth, Siegel and Wolf [16] introduced a bit-stuffing variable-to-fixed 2D $(1,\infty)$-RLL constrained encoder, whose rate is ~0.587277 (0.1% below the capacity). They also presented a fixed-rate encoder for the same constraint, whose rate is ~0.581074 (1.2% below the capacity).

Halevy, Chen, Roth, Siegel and Wolf [6] obtained lower bounds on the rate of a bit-stuffing encoder (and therefore, lower bounds on the capacity) of the 2D (d,∞)-RLL constrained system on the square and hexagonal lattices; for the hexagonal $(1,\infty)$-RLL constrained system their bound equals ~0.480767622 (0.5% below the capacity). They also conjectured that the capacity of the NIB constraint agrees in first ten decimal places with 0.9238294367 by using first-order Richardson extrapolation from [19].

Forchhammer and Justesen [4] presented a technique, by which using different Markov models they found new entropy bounds for 2D random fields defined over arrays that satisfy certain constraints: generalizations of the hard-square model to neighborhoods of higher orders, $(d,\ k)$-RLL constraints and tilings of the plane with pieces of finitely many different types.

Forchhammer and Laursen [5] provided a lower bound on the capacity of the NIB constraint: ~0.9156 (estimated about 0.8% below the capacity). They also presented an algorithm for an iterative search for a Pickard random field.

Schwartz and Vardy [17] provided asymptotically tight bounds on the capacity of the 2D $(0,k)$-RLL constrained system:

$$\frac{\log_2(e)}{2^k}\left(\frac{1}{2}-\frac{1}{2^{k+1}}\right)(1-(k+2)2^{-(k+1)})^{k+1} \leq 1-\operatorname{cap}\left(S^2_{0,k}\right) \leq 2\left(1-\operatorname{cap}\left(S^1_{0,k}\right)\right).$$

They also showed the following upper bound on $\operatorname{cap}\left(S^D_{d,k}\right)$:

$$\operatorname{cap}\left(S^D_{d,k}\right) \leq \operatorname{cap}\left(S^D_{d,\infty}\right) + \operatorname{cap}\left(S^D_{0,k}\right) - 1.$$

3 Research goals

In the preliminary stages of our research we examined a new method for designing variable-to-fixed encoders for multi-dimensional constraints. In the few tests that we conducted it was revealed that the method gives encoders of rates close to the constraint capacity while requiring small computational cost. The method partitions a given 2D space to blocks, pertaining to two classes, which can be thought of as colors of the chess board (black and white). We find a multi-variate function, which stands for the rate of the encoder, given the contents of the black-colored blocks, which are selected randomly and independently according to some distribution, and maximize over that distribution to find the encoder with the best rate. We discovered a local maxima phenomenon, due to which the algorithm is sometimes not able to generate the best attainable distribution.

We propose the following directions to continue our research:

- Take the distribution of the black blocks to be various stationary Markov random fields (2D and higher), such as Pickard fields and the like [3], [7], and see what improvement can be obtained as compared to the original scheme.
- Understand what modifications to the number of colors and to the block form can improve the rate.
- Try to find any extension of the method that can be useful for discovering upper bounds.
- Try to find any practical way to construct a fixed-rate encoder, based on our method.
- Characterize the local maxima phenomenon and find a way to circumvent it algorithmically.
- Characterize the distributions on the constrained arrays, as induced by our method.
- Characterize the constraints that our method can be applied to.

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