New Upper Bounds for Grain-Correcting and Grain-Detecting Codes

Artyom Sharov Ron M. Roth
Computer Science Department, Technion, Haifa 32000, Israel.
{sharov, ronny}@cs.technion.ac.il

***Abstract*—New upper bounds on the size and the rate of grain-correcting codes are presented. The new upper bound on the size of t-grain-correcting codes of length n improves on the best known upper bounds for certain values of n and t, whereas the new upper bound on the asymptotic rate of $\lceil \tau n \rceil$-grain-correcting codes of length n improves on the previously known upper bounds on the interval $\tau \in (0, \frac{1}{8}]$. A lower bound of $\frac{1}{2}\log_2 n$ on the minimum redundancy of ∞-grain-detecting codes of length n is presented.**

I. Introduction

In a paper by Wood *et al.* [16], a certain improvement to the writing and readback mechanisms of magnetic recording media was proposed allowing for a higher storage density due to the ability of magnetizing areas commensurate with the dimensions of basic units forming the media called *grains*. Due to the higher density of writing, one physical grain can be shared among several adjacent logical cells into which the media were partitioned, thereby introducing a new type of nonoverlapping smearing error to the information stored on these media. After the publication of [16], the granular media have been studied in several papers [1], [3], [4], [9], [12], [13]. Mazumdar *et al.* [9] described a one-dimensional model of the errors occurring in these media restricting the grains to be of lengths 1 and 2 only, and gave the first constructions and bounds on the sizes of codes that correct these so-called *grain errors*. In our earlier work [12], with a different yet conceptually similar application to shingled writing on bit-patterned media [2] in mind, we generalized the notion of grain errors to account for overlapping error patterns as well. Information-theoretic properties of the write channels representing the one-dimensional versions of both applications were studied by Iyengar *et al.* [3]. Kashyap and Zémor [4] and Gabrys *et al.* [1], using a reduction to the problem of bounding the size of packings in hypergraphs (see [8]), presented the best known upper bounds on the size and rate of codes correcting grain errors for the nonoverlapping and the overlapping cases, respectively. The best known lower bounds on the size and rate of those codes are due to another earlier work of ours [13], where we combine a construction from [1] with a general technique on improving Gilbert–Varshamov lower bounds (see the results of Kolesnik and Krachkovsky [7]). Several constructions of the codes correcting small number of grain errors were presented in [1], as well as in [13].

This work was supported in part by Grant No. 1092/12 from the Israel Science Foundation.

Let $\langle s \rangle$ denote the set $\{0, 1, \ldots, s-1\}$ for any positive integer s. Let $\Sigma = \langle 2 \rangle$ be the binary alphabet. A *grain (of length 2)* ending at location $e \in \langle n \rangle \setminus \{0\}$ in a word $\boldsymbol{x} = (x_i)_{i \in \langle n \rangle}$ of length n over Σ causes the value of x_e to equal that of x_{e-1}. Given n consecutive positions on the medium (where words of length n over Σ are to be written), define a *grain pattern* as a set $\mathcal{S} \subseteq \langle n \rangle \setminus \{0\}$ containing all the locations in these n positions where grains end. We will commonly refer to the elements of $\mathcal{S}$ (which indicate grain locations) simply as grains. Thus, a grain pattern $\mathcal{S}$ inflicts errors to a word $\boldsymbol{x} = (x_i)_{i \in \langle n \rangle}$ over Σ by means of the smearing operator $\sigma_{\mathcal{S}}$ that yields an output word $\boldsymbol{y} = (y_i)_{i \in \langle n \rangle} = \sigma_{\mathcal{S}}(\boldsymbol{x})$ over Σ in the following way: for any index $e \in \langle n \rangle \setminus \{0\}$,

$$y_e = \begin{cases} x_{e-1} & \text{if } e \in \mathcal{S} \\ x_e & \text{otherwise} \end{cases}.$$

If overlaps are disallowed in a grain pattern $\mathcal{S}$, for any two distinct grains $e, e' \in \mathcal{S}$, we require in addition that $|e' - e| > 1$.

Example 1.1: Let $n = 6$, $\boldsymbol{x} = 010111$, $\mathcal{S} = \{1, 3, 5\}$, and $\mathcal{S}' = \{1, 2\}$. Then $\sigma_{\mathcal{S}}(\boldsymbol{x}) = 000011$ and $\sigma_{\mathcal{S}'}(\boldsymbol{x}) = 001111$. The grain pattern $\mathcal{S}$ is nonoverlapping, whereas $\mathcal{S}'$ has overlaps. □

For a positive integer t and $\boldsymbol{x}, \boldsymbol{y} \in \Sigma^n$, we say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are *t-confusable* if there exist grain patterns $\mathcal{S}, \mathcal{S}'$ of size at most t for which $\sigma_{\mathcal{S}}(\boldsymbol{x}) = \sigma_{\mathcal{S}'}(\boldsymbol{y})$. A code $\mathcal{C}$ of length n over Σ (namely, a nonempty subset of Σ^n) is called *t-grain-correcting* if no two distinct codewords in $\mathcal{C}$ are t-confusable. Let $M(n, t)$ denote the largest size of any binary t-grain-correcting code of length n when overlaps are disallowed.[1] For $\tau \in [0, 1]$, define the (asymptotic) *rate* of binary $\lceil \tau n \rceil$-grain-correcting codes as

$$R(\tau) = \overline{\lim_{n \to \infty}} \frac{1}{n} \log_2 M(n, \lceil \tau n \rceil).$$

For a positive integer n, let a code $\mathcal{C}$ of length n over Σ be called *∞-grain-detecting* if for any codeword $\boldsymbol{x} \in \mathcal{C}$ and any grain pattern $\mathcal{S}$, one has $\sigma_{\mathcal{S}}(\boldsymbol{x}) \notin \mathcal{C}$. Finally, for a real $0 \le x \le 1$, let

$$\mathsf{H}(x) = -x \log_2 x - (1-x) \log_2 (1-x)$$

be the binary entropy function.

The rest of the paper is organized as follows. In Section II, we develop new upper bounds on $M(n, t)$ and $R(\tau)$, whereas

[1] Therefore the upper bound on $M(n, t)$ that we are about to present is also an upper bound on the largest size of any binary t-grain-correcting code of length n when overlaps are allowed.

Section III presents an upper bound on the size of ∞-grain-detecting codes of length n.

II. Upper bounds on $M(n,t)$ and $R(\tau)$

For positive integers n and t, define $M_Z(n,t)$ to be the size of the largest code of length n correcting t asymmetric errors $1 \to 0$. We start off by establishing a correspondence between $M(n,t)$ and $M_Z(n,t)$.

Lemma 2.1: Let n be a positive integer and $t \leq n/2$ be an integer. Then

$$M(n,t) \leq 2^{\lceil n/2 \rceil} \cdot M_Z(\lfloor n/2 \rfloor, t) .$$

Proof: Let $\mathcal{C}$ be a largest binary t-grain-correcting code of length n. For a word $\boldsymbol{x} = (x_i)_{i \in \langle \lceil n/2 \rceil \rangle}$, define $\mathcal{C}(\boldsymbol{x})$ as a subcode of $\mathcal{C}$ with codewords containing $\boldsymbol{x}$ as a substring on the even-indexed positions, namely,

$$\mathcal{C}(\boldsymbol{x}) = \left\{ \boldsymbol{c} = (c_i)_{i \in \langle n \rangle} \in \mathcal{C} : \text{for all } i \in \langle \lceil n/2 \rceil \rangle , c_{2i} = x_i \right\} .$$

By an averaging argument, there exists a word $\boldsymbol{x}$ of length $\lceil n/2 \rceil$, such that

$$|\mathcal{C}(\boldsymbol{x})| \geq |\mathcal{C}| / 2^{\lceil n/2 \rceil} . \tag{1}$$

Notice that a grain ending at position e can introduce an error to a word $\boldsymbol{c} = (c_i)_{i \in \langle n \rangle}$ only if $c_{e-1} \oplus c_e = 1$, where $\oplus$ is the addition modulo 2, and that the value of $c_{e-1} \oplus c_e$ changes to a 0, as a consequence. Therefore, if we restrict the grain patterns to the subset $\{e \in \langle n \rangle : e \text{ is odd}\}$ of odd-indexed locations only, the code $\mathcal{C}(\boldsymbol{x})$ will be equivalent to the following code $\mathcal{C}^{\oplus}$ of length $\lfloor n/2 \rfloor$ and of size $|\mathcal{C}(\boldsymbol{x})|$ correcting t asymmetric errors $1 \to 0$:

$$\mathcal{C}^{\oplus} = \left\{ \boldsymbol{y} = (c_{2i} \oplus c_{2i+1})_{i \in \langle \lfloor n/2 \rfloor \rangle} : \boldsymbol{c} = (c_i)_{i \in \langle n \rangle} \in \mathcal{C}(\boldsymbol{x}) \right\} .$$

This, along with (1), implies that

$$\begin{aligned} M(n,t) = |\mathcal{C}| &\leq 2^{\lceil n/2 \rceil} |\mathcal{C}(\boldsymbol{x})| = 2^{\lceil n/2 \rceil} \left| \mathcal{C}^{\oplus} \right| \\ &\leq 2^{\lceil n/2 \rceil} M_Z(\lfloor n/2 \rfloor, t). \end{aligned}$$

□

Using the best known bounds on $M_Z(\lfloor n/2 \rfloor, t)$ from [15, Table 10] results[2] in improvements on the best known upper bounds on $M(n,t)$, as shown in Table I, which contains the best known upper bounds (with the corresponding best known lower bounds in parenthesis) on $M(n,t)$ for small values of n and t. Therein, the best upper bounds due to Lemma 2.1 are marked in bold, whereas the best upper bounds on $M(n,1)$ due to [4, Cor. 3] are marked with stars; the best lower bounds on $M(n,1)$ due to [1, Constr. A] are marked with daggers, the best lower bounds on $M(n,2)$ and $M(n,3)$ due to [1, Ex. 4] (or variations thereof) are marked with diamonds, and the rest of the values are derived from [12, Table 2], [13, Table 3], and variations thereof. Tight upper bounds are marked in italics.

For a positive integer n, define the *asymmetric distance* $\Delta(\boldsymbol{c}, \boldsymbol{c}')$ between two words $\boldsymbol{c} = (c_i)_{i \in \langle n \rangle}$ and $\boldsymbol{c}' = (c'_i)_{i \in \langle n \rangle}$ over Σ as

$$\Delta(\boldsymbol{c}, \boldsymbol{c}') \triangleq \max \left\{ \Delta^{\star}(\boldsymbol{c}, \boldsymbol{c}'), \Delta^{\star}(\boldsymbol{c}', \boldsymbol{c}) \right\} ,$$

[2] Our definition of $M_Z(n,t)$ is equivalent to $Z(n,t+1)$ in [15].

TABLE I
Bounds on the sizes $M(n,t)$ of the largest known t-grain-correcting codes of length n.

t \ n	2	3	4	5	6	7	8	9	10	11	12	13
1	*2*	*4*	*6*	*8*	*16*	*26*	*44*	88(72)	176(112)	352(210†)	682*(372)	1260*(702†)
2			*4*	*8*	*10*	*16*	*22*	*32*	**64** (44)	**128** (68$^{\diamond}$)	**256** (88)	**512** (136$^{\diamond}$)
3					*8*	*16*	*18*	*32*	**64** (38)	**128** (64)	**128** (76)	**256** (128)

t \ n	14	15	16	17	18
1	**2304**(1272)	4368*(2400†)	8190*(4522)	15420*(8428)	29126*(15348)
2	**512** (176)	**1024** (312$^{\diamond}$)	**1792** (418$^{\diamond}$)	**3584** (836$^{\diamond}$)	**6144** (1318$^{\diamond}$)
3	**256** (152)	**512** (260$^{\diamond}$)	**1024** (304)	**2048** (520$^{\diamond}$)	**2048** (608)

where

$$\Delta^{\star}(\boldsymbol{c}, \boldsymbol{c}') = \left| \{ i \in \langle n \rangle : c_i = 0, c'_i = 1 \} \right| ,$$

and the *minimum asymmetric distance* of a code $\mathcal{C} \subseteq \Sigma^n$ as

$$\Delta(\mathcal{C}) \triangleq \min_{\boldsymbol{c}, \boldsymbol{c}' \in \mathcal{C} : \boldsymbol{c} \neq \boldsymbol{c}'} \left\{ \Delta(\boldsymbol{c}, \boldsymbol{c}') \right\} .$$

Let $\mathsf{d}(\boldsymbol{c}, \boldsymbol{c}')$ denote the *Hamming distance* between two words $\boldsymbol{c}, \boldsymbol{c}' \in \Sigma^n$ and $\mathsf{d}(\mathcal{C})$ denote the *minimum Hamming distance* of the code $\mathcal{C} \subseteq \Sigma^n$. In the following theorem, which is the main result of this section, we prove a new upper bound on $R(\tau)$.

Theorem 2.2: Let $\tau \in [0, \frac{1}{8}]$. Then

$$R(\tau) \leq \rho(\tau) \triangleq \frac{1}{2} \Big(1 + \min_{0 < x \leq 1 - 8\tau} \{ b(x) \} \Big) , \tag{2}$$

where

$$b(x) = 1 + h(x^2) - h(x^2 + 8\tau x + 8\tau)$$

and

$$h(x) = \mathsf{H}\left(0.5 (1 - \sqrt{1 - x}) \right) .$$

Proof: Let n be a positive integer and let $\mathcal{C}$ be a code of length n correcting $\lceil \tau n \rceil$ asymmetric errors of size $M_Z(n, \lceil \tau n \rceil)$. Its asymmetric distance $\Delta(\mathcal{C})$ is therefore at least $\lceil \tau n \rceil + 1$ (see [5, Th. 2.1]). By an averaging argument, there exists a constant-weight subcode $\mathcal{C}(w)$ of $\mathcal{C}$ whose codewords are of Hamming weight $w \in \langle n \rangle \setminus \{0\}$, whose size is at least $(|\mathcal{C}| - 2)/(n-1)$, and whose asymmetric distance is clearly at least $\lceil \tau n \rceil + 1$. Since, by [5, Lemma 2.1], $\mathsf{d}(\boldsymbol{c}, \boldsymbol{c}') = 2\Delta(\boldsymbol{c}, \boldsymbol{c}')$ for any two codewords $\boldsymbol{c}, \boldsymbol{c}' \in \mathcal{C}(w)$, one has $\mathsf{d}(\mathcal{C}(w)) \geq 2(\lceil \tau n \rceil + 1)$, therefore $\mathcal{C}(w)$ can correct at least $\lceil \tau n \rceil$ (Hamming) errors.

Let $M_H(n,t)$ denote the size of a largest binary code of length n correcting t (Hamming) errors and

$$R_H(\tau) = \overline{\lim_{n \to \infty}} \frac{1}{n} \log_2 M_H(n, \lceil \tau n \rceil)$$

denote the (asymptotic) *rate* of the binary codes of length n correcting $\lceil \tau n \rceil$ (Hamming) errors. The above discussion implies

$$\begin{aligned} M_Z(n, \lceil \tau n \rceil) = |\mathcal{C}| &\leq (n-1) \left| \mathcal{C}(w) \right| + 2 \\ &\leq (n-1) M_H(n, \lceil \tau n \rceil) + 2, \end{aligned}$$

which, combined with the result of Lemma 2.1, yields

$$M(n, \lceil \tau n \rceil) \leq 2^{\lceil n/2 \rceil} \cdot \Big((\lfloor n/2 \rfloor - 1) \cdot M_H(\lfloor n/2 \rfloor, \lceil \tau n \rceil) + 2 \Big). \tag{3}$$

Asymptotically, the inequality (3) implies

$$R(\tau) \le \frac{1}{2}(1+R_H(2\tau)) \ .$$

Finally, to obtain the upper bound (2), we use the second MRRW upper bound [10, Ch. 17, Th. 37] on $R_H(2\tau)$. □

Figure 1 depicts the upper bound $\rho(\tau)$ of Theorem 2.2 along with two previously best known upper bounds $\rho_1(\tau)$ (see [4, Th. 5]) and $\rho_2(\tau)$ (see [13, Th. 3.3]) obtained using information-theoretic and sphere-packing arguments, respectively. The best known lower bound $\varrho(\tau)$ (see [13, Th. 2.4]), which is essentially a modification of the Gilbert–Varshamov bound

$$\varrho_4(\tau) \triangleq 1-\mathsf{H}(2\tau) \ ,$$

is plotted therein for comparison (along with $\varrho_4(\tau)$). Also for comparison, in a dotted line we plot the Gilbert–Varshamov lower bound

$$\varrho_5(\tau) \triangleq 1-\frac{1}{2}\mathsf{H}(4\tau)$$

on the rate of the largest $\lceil \tau n \rceil$-grain-correcting codes of length n when the grain patterns are restricted to the subset $\{e \in \langle n \rangle : e \text{ is odd}\}$. The upper bound $\rho(\tau)$ improves on $\rho_1(\tau)$ and on $\rho_2(\tau)$ on the entire interval $(0, \frac{1}{8}]$, and at $\tau = \frac{1}{8}$, it coincides with the lower bound of $\frac{1}{2}$ on $R(\tau)$ obtained by a simple construction from [9, Sec. 2]. The upper bound $\rho(\tau)$ also improves on the entire interval $(0, \frac{1}{8}]$ on the upper bound $\rho_3(\tau)$ derived from [1, Th. 1] on the rate of $\lceil \tau n \rceil$-grain-correcting codes of length n when overlaps are allowed.

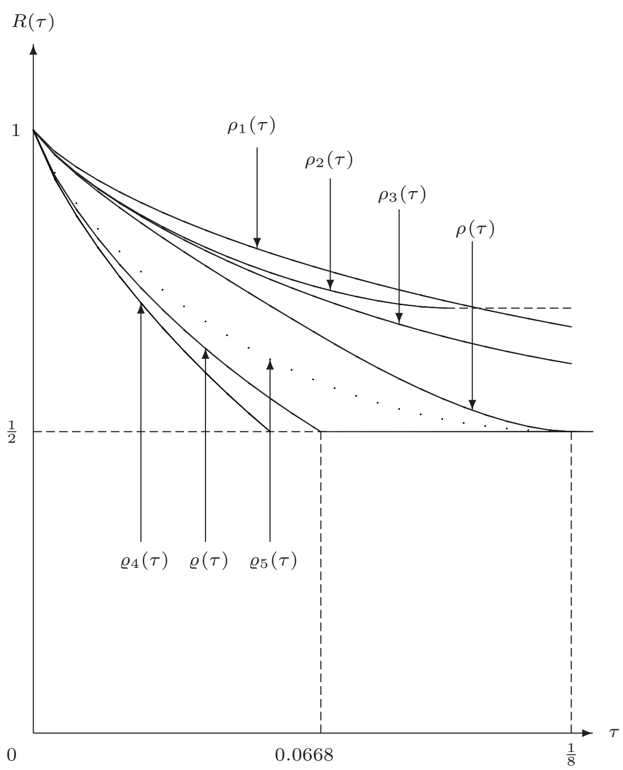

Fig. 1. Upper bound $\rho(\tau)$ along with upper bounds $\rho_1(\tau)$, $\rho_2(\tau)$ and $\rho_3(\tau)$ and lower bounds $\varrho(\tau)$, $\varrho_4(\tau)$, $\varrho_5(\tau)$.

The fact that the new upper bound $\rho(\tau)$ meets the lower bound of $\frac{1}{2}$ at $\tau = \frac{1}{8}$ implies a very slow decrease in the size $M(n, \lceil \tau n \rceil)$ of a largest $\lceil \tau n \rceil$-grain-correcting code of length n when τ runs from $\frac{1}{8}$ to $\frac{1}{2}$, which is evidenced in the following example. Let t be a positive integer and let $n = 4t$. Since a largest code of length $n/2 = 2t$ correcting $t = n/4$ asymmetric errors is of size 2, by Lemma 2.1, the size $M(n, n/4)$ of a largest $n/4$-grain-correcting code of length n is at most $2^{n/2+1}$. As, due to [9, Prop. 1], $M(n, n/2) = 2^{n/2}$, when t runs from $\frac{n}{4}$ to $\frac{n}{2}$, the largest code size $M(n,t)$ decreases only by at most a factor of 2.

III. Grain detection

In [12, Prop. 5.1], we have proved the existence of ∞-grain-detecting codes $\mathcal{C}$ (that is, codes capable of detecting any number of grain errors) of length n over Σ with *redundancy*

$$n - \log_2 |\mathcal{C}| \le 1.5 \log_2 n + O\left(\tfrac{1}{n}\right)$$

for the overlapping and nonoverlapping scenarios. Employing arguments similar to those used in the proof of Lemma 2.1, we conclude that the size of a largest ∞-grain-detecting code of length n over Σ is bounded from above by $2^{\lceil n/2 \rceil}$ times the size of a largest code of length $\lfloor n/2 \rfloor$ over Σ capable of detecting any number of asymmetric errors, which is known to be $\binom{\lfloor n/2 \rfloor}{\lfloor n/4 \rfloor}$ [14]. Altogether, this implies a lower bound of $\frac{1}{2}\log_2 n + O(1)$ on the *minimum redundancy*

$$r_n \triangleq n - \max_{\substack{\mathcal{C} \subseteq \Sigma^n \text{ is an} \\ \infty\text{-grain-detecting code}}} \{\log_2 |\mathcal{C}|\}$$

of ∞-grain-detecting codes of length n when overlaps are allowed or disallowed.

For the overlapping scenario, the upper bound on the size of a largest ∞-grain-detecting code of length n can be improved by a constant factor (namely, by an additive constant term in the redundancy). In what follows, we will show how to obtain such an upper bound; the proof technique is inspired by the Christmas tree pattern [6, Sec. 7.2.1.6] of arranging 2^n binary strings into chains of subsets.

Define the following (partial) order relation $\preceq$ between two words $\boldsymbol{x}$ and $\boldsymbol{y}$ of the same length over Σ. The word $\boldsymbol{x}$ will be *dominated* by the word $\boldsymbol{y}$, $\boldsymbol{x} \preceq \boldsymbol{y}$, if there exists a grain pattern $\mathcal{S}$ such that $\sigma_{\mathcal{S}}(\boldsymbol{y}) = \boldsymbol{x}$. Our construction will be by induction on the value of ℓ where at step ℓ we will create s_ℓ new sets $\mathsf{C}_{\ell;j}$ of words of length ℓ for $j \in \langle \mathsf{s}_\ell \rangle$ out of $\mathsf{s}_{\ell-1}$ sets $\mathsf{C}_{\ell-1;j}$ of words of length $\ell-1$ for $j \in \langle \mathsf{s}_{\ell-1} \rangle$. Each one of the sets $\mathsf{C}_{\ell;j}$ will be shown in Lemma 3.4 to be totally ordered with respect to $\preceq$, and the "biggest" and the "smallest" words in $\mathsf{C}_{\ell;j}$ will be denoted by $\mathsf{F}(\mathsf{C}_{\ell;j})$ and $\mathsf{f}(\mathsf{C}_{\ell;j})$, respectively. The value of $2\mathsf{s}_n$ will then determine an improved upper bound on the size of a largest ∞-grain-detecting code of length n when overlaps are allowed, as will be explained shortly.

Construction 3.1: **Basis** ($\ell = 1$). Let $\mathsf{C}_{1;0} = \{0\}$.
Step ($\ell \ge 2$). For $j \in \langle \mathsf{s}_{\ell-1} \rangle$, from a set $\mathsf{C}_{\ell-1;j}$ of size 1, we derive a new set

(C1) $\mathsf{C}_{\ell-1;j} \times \Sigma$.

From a set $\mathsf{C}_{\ell-1;j}$ of size at least 2 whose words all end with $a \in \Sigma$, we derive two new sets

(C2) $(\mathsf{C}_{\ell-1;j} \times \{\overline{a}\}) \cup \{\mathsf{f}(\mathsf{C}_{\ell-1;j})a\}$,
(C3) $(\mathsf{C}_{\ell-1;j} \times \{a\}) \setminus \{\mathsf{f}(\mathsf{C}_{\ell-1;j})a\}$,

where $\overline{a}$ denotes the *binary complement* of the symbol $a \in \Sigma$.

From a set $\mathsf{C}_{\ell-1;j}$ of size at least 2 such that there exists only one word $\boldsymbol{c} \in \mathsf{C}_{\ell-1;j}$, $\boldsymbol{c} \neq \mathsf{F}(\mathsf{C}_{\ell-1;j})$, that ends with a and the rest of the words end with $\overline{a}$, we derive two new sets

(C4) $(\mathsf{C}_{\ell-1;j} \times \{a\}) \cup \{\boldsymbol{c}\overline{a}\}$,

(C5) $(\mathsf{C}_{\ell-1;j} \times \{\overline{a}\}) \setminus \{\boldsymbol{c}\overline{a}\}$. □

Remark 3.2: Notice that for all sets $\mathsf{C}_{\ell;j}$, either the last symbols of the words of $\mathsf{C}_{\ell;j}$ are the same or there is only one word $\boldsymbol{c} \neq \mathsf{F}(\mathsf{C}_{\ell;j})$ whose last symbol is different from the last symbols of the rest of the words of $\mathsf{C}_{\ell;j}$. □

Example 3.3: The first four rounds of Construction 3.1 yield $\mathsf{C}_{1;0} = \{0\}$, $\mathsf{C}_{2;0} = \{00, 01\}$, $\mathsf{C}_{3;0} = \{000, 001, 010\}$, $\mathsf{C}_{3;1} = \{011\}$, $\mathsf{C}_{4;0} = \{0001, 0011, 0010, 0101\}$, $\mathsf{C}_{4;1} = \{0000, 0100\}$, $\mathsf{C}_{4;2} = \{0111, 0110\}$. □

The following lemma proves by induction on ℓ that each set $\mathsf{C}_{\ell;j}$ is totally ordered with respect to $\preceq$ which justifies the use of the operator $\mathsf{f}(\cdot)$ in Construction 3.1.

Lemma 3.4: For any positive integer ℓ and any $j \in \langle \mathsf{s}_\ell \rangle$, the set $\mathsf{C}_{\ell;j}$ is totally ordered with respect to $\preceq$.

Proof: Readily, the set $\mathsf{C}_{1;0} = \{0\}$ is totally ordered, which is the basis of our induction proof. As for the induction step, let us assume that each one of the sets $\mathsf{C}_{\ell-1;j}$ is totally ordered for every $j \in \langle \mathsf{s}_{\ell-1} \rangle$. To prove the statement of the lemma, it will suffice to take two words $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{C}_{\ell-1;j}$ such that $\boldsymbol{x} \preceq \boldsymbol{y}$ and show the order between all the words in $\mathsf{C}_{\ell;j'}$ whose prefixes of length $\ell-1$ are $\boldsymbol{x}$ and $\boldsymbol{y}$, for each one of the cases (C1)–(C5).

(C1) In this case, $\boldsymbol{x} = \boldsymbol{y}$. When $\boldsymbol{x}$ ends with a 0, the order between $\boldsymbol{x}0$ and $\boldsymbol{x}1$ is $\boldsymbol{x}0 \preceq \boldsymbol{x}1$, whereas when $\boldsymbol{x}$ ends with a 1, the order is $\boldsymbol{x}1 \preceq \boldsymbol{x}0$.

(C2) When $\boldsymbol{x} \neq \mathsf{f}(\mathsf{C}_{i-1;j})$, the order between $\boldsymbol{x}\overline{a}$ and $\boldsymbol{y}\overline{a}$ is $\boldsymbol{x}\overline{a} \preceq \boldsymbol{y}\overline{a}$; when $\boldsymbol{x} = \mathsf{f}(\mathsf{C}_{i-1;j})$, the order between $\boldsymbol{x}a$, $\boldsymbol{x}\overline{a}$, and $\boldsymbol{y}\overline{a}$ is $\boldsymbol{x}a \preceq \boldsymbol{x}\overline{a}$, $\boldsymbol{x}\overline{a} \preceq \boldsymbol{y}\overline{a}$, and $\boldsymbol{x}a \preceq \boldsymbol{y}\overline{a}$.

(C3) The order between $\boldsymbol{x}a$ and $\boldsymbol{y}a$ is $\boldsymbol{x}a \preceq \boldsymbol{y}a$.

(C4) When $\boldsymbol{x}, \boldsymbol{y} \neq \boldsymbol{c}$, the order between $\boldsymbol{x}a$ and $\boldsymbol{y}a$ is $\boldsymbol{x}a \preceq \boldsymbol{y}a$; when $\boldsymbol{x} = \boldsymbol{c}$, the order between $\boldsymbol{x}a$, $\boldsymbol{x}\overline{a}$, and $\boldsymbol{y}a$ is $\boldsymbol{x}a \preceq \boldsymbol{x}\overline{a}$, $\boldsymbol{x}\overline{a} \preceq \boldsymbol{y}a$, and $\boldsymbol{x}a \preceq \boldsymbol{y}a$; when $\boldsymbol{y} = \boldsymbol{c}$, the order between $\boldsymbol{x}a$, $\boldsymbol{y}a$, and $\boldsymbol{y}\overline{a}$ is $\boldsymbol{x}a \preceq \boldsymbol{y}a$, $\boldsymbol{y}a \preceq \boldsymbol{y}\overline{a}$, and $\boldsymbol{x}a \preceq \boldsymbol{y}\overline{a}$.

(C5) The order between $\boldsymbol{x}\overline{a}$ and $\boldsymbol{y}\overline{a}$ is $\boldsymbol{x}\overline{a} \preceq \boldsymbol{y}\overline{a}$. □

In light of the result of Lemma 3.4 and by the simple observation that $\{\mathsf{C}_{n;j} : j \in \langle \mathsf{s}_n \rangle\}$ is a partition of $0\Sigma^{n-1}$ for a positive integer n, each set $\mathsf{C}_{n;j}$ for $j \in \langle \mathsf{s}_n \rangle$ can contribute at most one word to an ∞-grain-detecting code of length n. Therefore, by extending the above argument to all the words of length n that start with a 1, we obtain an upper bound of $2\mathsf{s}_n$ on the size of a largest ∞-grain-detecting code of length n. It is left to find the value of s_n; to that end, let us first observe the values of s_ℓ for small values of ℓ, as shown in Table II.

TABLE II
VALUES OF s_ℓ FOR SMALL VALUES OF ℓ.

ℓ	1	2	3	4	5	6	7	8	9	10
s_ℓ	1	1	2	3	6	10	20	35	70	126

The sequence of the values of s_ℓ appearing in Table II matches the beginning of the sequence [11] which equals the number $\binom{\ell-1}{\lfloor(\ell-1)/2\rfloor}$ of walks of length $\ell-1$ on the square lattice from the origin $(0, 0)$ by moving down or moving right, all the while staying on the points (x, y) satisfying $\mathsf{x}+\mathsf{y} \geq 0$. This observation gives rise to the following lemma.

Lemma 3.5: Let ℓ be a positive integer and let $\boldsymbol{x} = (x_i)_{i \in \langle \ell \rangle}$ be a word of length ℓ over Σ. For a positive integer $k \in \langle \ell \rangle \setminus \{0\}$, let

$$\begin{aligned} \mathsf{p}_k(\boldsymbol{x}) &= 2\,|\{s \in \langle k \rangle : x_s \neq x_{s+1}\}| - k \\ &= \ |\{s \in \langle k \rangle : x_s \neq x_{s+1}\}| - |\{s \in \langle k \rangle : x_s = x_{s+1}\}| \end{aligned}$$

be the difference between the number of symbol alternations and the number of symbol repetitions in the prefix of length $k+1$ of $\boldsymbol{x}$. Then for any $j \in \langle \mathsf{s}_\ell \rangle$, the only word $\boldsymbol{x}$ in $\mathsf{C}_{\ell;j}$ which satisfies $\mathsf{p}_k(\boldsymbol{x}) \geq 0$ for all $k \in \langle \ell \rangle \setminus \{0\}$ is $\mathsf{F}(\mathsf{C}_{\ell;j})$.

Proof: One can readily see that for any positive integer $\ell \geq 2$ and any $j \in \langle \mathsf{s}_\ell \rangle$, one has

$$|\mathsf{C}_{\ell;j}| = \mathsf{p}_{\ell-1}(\mathsf{F}(\mathsf{C}_{\ell;j}))+1\ . \tag{4}$$

We will prove the claim of the lemma by induction on ℓ. Clearly, the claim holds for $\ell = 2$, namely, the only word $\boldsymbol{x}$ in $\mathsf{C}_{2;0}$ that satisfies $\mathsf{p}_1(\boldsymbol{x}) \geq 0$ is $\mathsf{F}(\mathsf{C}_{2;0}) = 01$. As for the induction step, let us assume that for $\ell \geq 3$, the only word $\boldsymbol{x}$ in each one of the sets $\mathsf{C}_{\ell-1;j}$ which satisfies $\mathsf{p}_k(\boldsymbol{x}) \geq 0$ for all $k \in \langle \ell-1 \rangle \setminus \{0\}$ is $\mathsf{F}(\mathsf{C}_{\ell-1;j})$. To prove the claim of the lemma, it will suffice to take $\boldsymbol{x} = \mathsf{F}(\mathsf{C}_{\ell-1;j})$ and, for each one of the cases (C1)–(C5), show that the word $\boldsymbol{y}$ in $\mathsf{C}_{\ell;j'}$, whose prefix of length $\ell-1$ is $\boldsymbol{x}$, satisfies $\mathsf{p}_k(\boldsymbol{y}) \geq 0$ for all $k \in \langle \ell \rangle \setminus \{0\}$ if and only if $\boldsymbol{y} = \mathsf{F}(\mathsf{C}_{\ell;j'})$.

(C1) Without loss of generality, $\boldsymbol{x}$ ends with a 0 and $\boldsymbol{x}1 = \mathsf{F}(\mathsf{C}_{\ell;j'})$. Since $\mathsf{p}_{\ell-1}(\boldsymbol{x}1) = 1 + \mathsf{p}_{\ell-2}(\boldsymbol{x}) \geq 1$ by the induction hypothesis and $\mathsf{p}_k(\boldsymbol{x}1) = \mathsf{p}_k(\boldsymbol{x})$ for $k \in \langle \ell-1 \rangle \setminus \{0\}$, the word $\boldsymbol{x}1$ satisfies $\mathsf{p}_k(\boldsymbol{x}1) \geq 0$ for all $k \in \langle \ell \rangle \setminus \{0\}$. Moreover, by (4), $\mathsf{p}_{\ell-2}(\boldsymbol{x}) = 0$, therefore $\mathsf{p}_{\ell-1}(\boldsymbol{x}0) = -1$ implying that $\boldsymbol{x}1$ is the only word $\boldsymbol{y}$ in $\mathsf{C}_{\ell;j'}$ satisfying $\mathsf{p}_k(\boldsymbol{y}) \geq 0$ for all $k \in \langle \ell \rangle \setminus \{0\}$.

(C2) In this case, the only word in $\mathsf{C}_{\ell;j'}$ whose prefix is $\boldsymbol{x}$ is $\boldsymbol{x}\overline{a}$. Since $\boldsymbol{x}$ ends with a, by the induction hypothesis one has $\mathsf{p}_{\ell-1}(\boldsymbol{x}\overline{a}) = \mathsf{p}_{\ell-2}(\boldsymbol{x})+1 \geq 1$, so the only word $\boldsymbol{y}$ in $\mathsf{C}_{\ell;j'}$ satisfying $\mathsf{p}_k(\boldsymbol{y}) \geq 0$ for all $k \in \langle \ell \rangle \setminus \{0\}$ is $\boldsymbol{x}\overline{a}$.

(C3) In this case, the only word in $\mathsf{C}_{\ell;j'}$ whose prefix is $\boldsymbol{x}$ is $\boldsymbol{x}a$. Since $\boldsymbol{x}$ ends with a, by the induction hypothesis and by (4), one has $\mathsf{p}_{\ell-1}(\boldsymbol{x}a) = \mathsf{p}_{\ell-2}(\boldsymbol{x})-1 \geq 0$, hence the only word $\boldsymbol{y}$ in $\mathsf{C}_{\ell;j'}$ satisfying $\mathsf{p}_k(\boldsymbol{y}) \geq 0$ for all $k \in \langle \ell \rangle \setminus \{0\}$ is $\boldsymbol{x}a$.

(C4) In this case, by Remark 3.2, the only word in $\mathsf{C}_{\ell;j'}$ whose prefix is $\boldsymbol{x}$ is $\boldsymbol{x}a$. Since $\boldsymbol{x}$ ends with $\overline{a}$, by the induction hypothesis one has $\mathsf{p}_{\ell-1}(\boldsymbol{x}\overline{a}) = \mathsf{p}_{\ell-2}(\boldsymbol{x})+1 \geq 1$, so the only word $\boldsymbol{y}$ in $\mathsf{C}_{\ell;j'}$ satisfying $\mathsf{p}_k(\boldsymbol{y}) \geq 0$ for all $k \in \langle \ell \rangle \setminus \{0\}$ is $\boldsymbol{x}a$.

(C5) In this case, the only word in $\mathsf{C}_{\ell;j'}$ whose prefix is $\boldsymbol{x}$ is $\boldsymbol{x}\overline{a}$. Since $\boldsymbol{x}$ ends with $\overline{a}$, by the induction hypothesis and by (4), one has $\mathsf{p}_{\ell-1}(\boldsymbol{x}\overline{a}) = \mathsf{p}_{\ell-2}(\boldsymbol{x})-1 \geq 0$, hence the only word $\boldsymbol{y}$ in $\mathsf{C}_{\ell;j'}$ satisfying $\mathsf{p}_k(\boldsymbol{y}) \geq 0$ for all $k \in \langle \ell \rangle \setminus \{0\}$ is $\boldsymbol{x}\overline{a}$. □

Corollary 3.6: Let ℓ be a nonnegative integer. Then

$$\mathsf{s}_\ell = \binom{\ell-1}{\lfloor(\ell-1)/2\rfloor}\ .$$

Proof: Due to the result of Lemma 3.5 and the observation that $\{\mathsf{C}_{\ell;j} : j \in \langle \mathsf{s}_\ell \rangle\}$ is a partition of $0\Sigma^{\ell-1}$, instead of counting different sets $\mathsf{C}_{\ell;j}$, we can count the number of "biggest" words $\boldsymbol{x} = (x_i)_{i\in\langle\ell\rangle} \in 0\Sigma^{\ell-1}$ which satisfy $|\mathsf{p}_k(\boldsymbol{x})| \geq 0$ for all $k \in \langle\ell\rangle \setminus \{0\}$. Now, there is a natural 1-to-1 correspondence between such words and walks of length $\ell-1$ on the square lattice from the origin $(0,0)$ by moving down or moving right, all the while staying on the points (x,y) satisfying $\mathsf{x}+\mathsf{y} \geq 0$, specifically, we move right at step k of that walk if $x_{k-1} \neq x_k$ and move down otherwise. Since the number of these walks is, as we have mentioned before, $\binom{\ell-1}{\lfloor(\ell-1)/2\rfloor}$, the result of the corollary follows. $\square$

Since $\lim_{n\to\infty} 2^{\lceil n/2\rceil}\binom{\lfloor n/2\rfloor}{\lfloor n/4\rfloor}/2\binom{n-1}{\lfloor(n-1)/2\rfloor} = \sqrt{2}$, for large values of n, the upper bound on the size of ∞-grain-detecting codes of length n (with overlaps allowed) due to Corollary 3.6 is $\approx\sqrt{2}$ times smaller than the upper bound $2^{\lceil n/2\rceil}\binom{\lfloor n/2\rfloor}{\lfloor n/4\rfloor}$ on the size of ∞-grain-detecting code of length n due to Lemma 2.1 that we have mentioned at the beginning of this section.

TABLE III
SIZES OF LARGEST t-GRAIN-DETECTING CODES OF LENGTH n WHEN OVERLAPS ARE DISALLOWED.

t \ n	2	3	4	5	6	7	8
1	2	4	8	16	32	64	128
2			8	10	18	34	58
3					18	32	56
4							56

TABLE IV
SIZES OF LARGEST t-GRAIN-DETECTING CODES OF LENGTH n WHEN OVERLAPS ARE ALLOWED.

t \ n	2	3	4	5	6	7	8
1	2	4	8	16	32	64	128
2		4	6	10	18	30	52
3			6	8	12	22	42
4				8	12	20	32
5					12	20	32

Tables III and IV list the sizes of largest t-grain-detecting codes of length n when overlaps are disallowed and allowed, respectively, for small values of n and t, found using a computer search.[3] It can be seen that already for length $n = 5$, there is a gap between the upper bound of $2\binom{4}{2} = 12$ on the size of ∞-grain-detecting codes of length 5 when overlaps are allowed due to Construction 3.1 and the size 8 of a largest ∞-grain-detecting code. However, using *ad hoc* arguments, it is still possible to partition the 16 words in $0\Sigma^4$ into the four sets

$$\begin{aligned}
\mathsf{C}_{5;0} &= \{00000, 00100, 01000, 01001\}\,, \\
\mathsf{C}_{5;1} &= \{00001, 00011, 00010, 00101\}\,, \\
\mathsf{C}_{5;2} &= \{00110, 01110, 01100, 01010\}\,, \\
\mathsf{C}_{5;3} &= \{00111, 01111, 01101, 01011\}
\end{aligned}$$

[3]The entries for $t = 1$ in both tables follow from simple observations that the Hamming distance between two distinct codewords with the same value in their first bit of any binary 1-grain-detecting code must be at least 2 and that binary parity code of any length is 1-grain-detecting.

of size 4, which are totally ordered with respect to $\preceq$. This, in turn, results in a tight upper bound of 8 on the size of ∞-grain-detecting codes of length 5 when overlaps are allowed.

On the other hand, using a computer search, one can establish that for $n = 6$, the smallest number of totally ordered sets $\mathsf{C}_{6;j}$ required to partition $0\Sigma^5$ is 7, which results in the upper bound of 14 on the size of a largest ∞-grain-detecting code of length 6 with overlaps; this bound is strictly less than the size 12 of a largest such code. One such partition is given by

$$\begin{aligned}
\mathsf{C}_{6;0} &= \{000000, 000001, 000010, 000101, 001010\}\,, \\
\mathsf{C}_{6;1} &= \{000110, 000100, 001100, 001101, 001010\}\,, \\
\mathsf{C}_{6;2} &= \{000011, 000111, 001011, 010111, 010101\}\,, \\
\mathsf{C}_{6;3} &= \{001000, 011000, 010000, 010001, 010010\}\,, \\
\mathsf{C}_{6;4} &= \{001001, 011001, 011011, 010011\}\,, \\
\mathsf{C}_{6;5} &= \{001111, 011111, 011110, 011101\}\,, \\
\mathsf{C}_{6;6} &= \{001110, 011100, 011010, 010100\}\,.
\end{aligned}$$

Similar phenomena occur when overlaps are disallowed: for $n = 5$ it is possible to partition $0\Sigma^4$ into 5 totally ordered sets using *ad hoc* arguments, yet for $n = 6$ it is provably impossible to partition $0\Sigma^5$ into 9 totally ordered sets.

REFERENCES

[1] R. Gabrys, E. Yaakobi, L. Dolecek, "Correcting grain-errors in magnetic media," *http://arxiv.org/pdf/1307.7087*

[2] S. Greaves, Y. Kanai, H. Muraoka, "Shingled magnetic recording on bit patterned media," *IEEE Trans. Magn.,* vol. 46, pp. 1460–1463, 2010.

[3] A.R. Iyengar, P.H. Siegel, J.K. Wolf, "Write channel model for bit-patterned media recording," *IEEE Trans. Magn.,* vol. 47, pp. 35–45, 2011.

[4] N. Kashyap, G. Zémor, "Upper bounds on the size of grain-correcting codes," *Proc. IEEE Symp. Inform. Theory (ISIT 2013),* Istanbul, Turkey, 2013.

[5] T. Kløve, "Error correcting codes for the asymmetric channel," *Tech. Rep. 18–09–07–81,* Dep. Inform., Univ. Bergen, 1981.

[6] D.E. Knuth, *The Art of Computer Programming, Vol. 4A, Combinatorial Algorithms, Part 1,* 1st edition, Addison–Wesley, Reading, MA, 2011.

[7] V.D. Kolesnik, V.Yu. Krachkovsky, "Generating functions and lower bounds on rates for limiting error-correcting codes," *IEEE Trans. Inform. Theory,* vol. 37, pp. 778–788, 1991.

[8] A.A. Kulkarni, N. Kiyavash, "Nonasymptotic upper bounds for deletion correcting codes," *IEEE Trans. Inform. Theory,* vol. 59, pp. 5115–5130, 2013.

[9] A. Mazumdar, A. Barg, N. Kashyap, "Coding for high-density recording on a 1-D granular magnetic medium," *IEEE Trans. Inform. Theory,* vol. 57, pp. 7403–7417, 2011.

[10] F.J. MacWilliams, N.J.A. Sloane, *The Theory of Error-Correcting Codes,* North-Holland Publishing Company, New York, 1977.

[11] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences,* Sequence A001405, *http://oeis.org/A001405*

[12] A. Sharov, R.M. Roth, "Bounds and constructions for granular media coding," to be published in *IEEE Trans. Inform. Theory,* available at *http://cs.technion.ac.il/~sharov/Publications/Transactions-2013.pdf*

[13] A. Sharov, R.M. Roth, "Improved bounds and constructions for granular media coding," to be published in *Proc. Allerton Conf. Commun., Control Comput.,* Allerton Retreat Center, Monticello, IL, 2013, available at *http://cs.technion.ac.il/~sharov/Publications/Allerton-2013.pdf*

[14] E. Sperner, "Ein satz über untermengen einer endlichen menge," *Mathematische Zeitschrift* (in German), vol. 27, pp. 544–548, 1928.

[15] J.H. Weber, C. de Vroedt, D.E. Boekee, "Bounds and constructions for binary codes of length less than 24 and asymmetric distance less than 6," *IEEE Trans. Inform. Theory,* vol. 34, pp. 1321–1331, 1988.

[16] R. Wood, M. Williams, A. Kavčić, J. Miles, "The feasibility of magnetic recording at 10 terabits per square inch on conventional media," *IEEE Trans. Magn.,* vol. 45, pp. 917–923, 2009.