

# Bounds and Constructions for Granular Media Coding

Artyom Sharov      Ron M. Roth

Computer Science Department, Technion, Haifa 32000, Israel.

{ronny, sharov}@cs.technion.ac.il

**Abstract**—Bounds on the rate of grain-correcting codes are presented. The lower bounds are Gilbert–Varshamov-like ones, whereas the upper bounds improve on the previously known result by Mazumdar *et al.*. Constructions of  $t$ -grain-correcting codes of length  $n$  for certain values of  $n$  and  $t$  are discussed.

## I. INTRODUCTION

Conventional magnetic recording media are composed of two-dimensional arbitrarily-shaped basic units called *grains* that can be magnetized to take on one of two possible types of polarity. Current technologies divide the writing medium into cells, typically larger in size than the grains, hence setting a value to a cell boils down to magnetizing grains within the boundaries of this cell. Recently, Wood *et al.* [8] suggested a mechanism that enables magnetizing areas as small as the size of grains, thereby creating a different type of medium where the grain polarity is determined by the last bit written into the grain. Iyengar *et al.* [2] modeled the one-dimensional version of the medium as a write channel and studied its information theoretic properties.

Mazumdar *et al.* [6] considered a combinatorial error model describing this one-dimensional granular medium. In what follows, we will define a somewhat generalized version of the model. Let  $[s]$  denote the set  $\{0, 1, \dots, s-1\}$  for any positive integer  $s$ . Let  $q$  be a positive integer, and let  $\Sigma = [q]$  be an alphabet. A *grain* (of length 2) ending at location  $e$  in a word  $\mathbf{x} = (x_i)_{i \in [n]}$  of length  $n$  over  $\Sigma$  causes the value of  $x_e$  to equal that of  $x_{e-1}$ . Given  $n$  consecutive positions on the medium (where words of length  $n$  over  $\Sigma$  are to be written), define a *grain pattern* as a set  $\mathcal{S} \subseteq [n] \setminus \{0\}$  containing all the grain locations in these  $n$  positions. We will commonly refer to the elements of  $\mathcal{S}$  (which indicate grain locations) simply as grains. Thus, a grain pattern  $\mathcal{S}$  inflicts errors to a word  $\mathbf{x} = (x_i)_{i \in [n]}$  over  $\Sigma$  by means of the smearing operator  $\sigma = \sigma_{\mathcal{S}}$  that yields an output word  $\mathbf{y} = (y_i)_{i \in [n]} = \sigma(\mathbf{x})$  over  $\Sigma$  in the following way: for any index  $e \in [n] \setminus \{0\}$ ,  $y_e = x_{e-1}$  if  $e \in \mathcal{S}$  and  $y_e = x_e$  otherwise. We will say that a grain pattern has *overlaps* if there exist two grains  $e, e' \in \mathcal{S}$  such that  $e' = e+1$ ; otherwise the grain pattern will be called *nonoverlapping*.

*Example 1.1:* Let  $\Sigma = [3]$  ( $q = 3$ ),  $n = 6$ ,  $\mathbf{x} = 102022$ ,  $\mathcal{S} = \{1, 3, 5\}$  and  $\mathcal{S}' = \{1, 2\}$ . Then  $\sigma_{\mathcal{S}}(\mathbf{x}) = 112222$  and  $\sigma_{\mathcal{S}'}(\mathbf{x}) = 110022$ . The grain pattern  $\mathcal{S}$  is nonoverlapping whereas the grain pattern  $\mathcal{S}'$  has overlaps.  $\square$

For a positive integer  $t$  and  $\mathbf{x} \in \Sigma^n$ , let  $\mathcal{R}_t(\mathbf{x})$  be defined as the set of all words  $\mathbf{y} \in \Sigma^n$  such that there exist grain patterns  $\mathcal{S}, \mathcal{S}'$  of size  $t$  at most for which  $\sigma_{\mathcal{S}}(\mathbf{x}) = \sigma_{\mathcal{S}'}(\mathbf{y})$ . Two words  $\mathbf{x}, \mathbf{y} \in \Sigma^n$  are  $t$ -*confusable* if  $\mathbf{y} \in \mathcal{R}_t(\mathbf{x})$  (and therefore  $\mathbf{x} \in \mathcal{R}_t(\mathbf{y})$ ). Words  $\mathbf{x}$  and  $\mathbf{y}$  are *finitely-confusable* if they are  $t$ -confusable for some finite  $t$ ; otherwise, we say that they are  $\infty$ -*confusable*. A code  $\mathcal{C}$  of length  $n$  over  $\Sigma$  (namely, a nonempty subset of  $\Sigma^n$ ) is called  $t$ -grain-correcting if no two distinct codewords in  $\mathcal{C}$  are  $t$ -confusable. Let  $M_q(n, t)$  denote the largest size of any  $t$ -grain-correcting code of length  $n$  over  $\Sigma$ . For  $\tau \in (0, 1)$ , define the (asymptotic) *rate* of  $\lceil \tau n \rceil$ -grain-correcting codes over  $\Sigma$  as

$$R_q(\tau) = \limsup_{n \rightarrow \infty} [\log_q M_q(n, \lceil \tau n \rceil)] / n.$$

The rest of the paper is organized as follows. In Section II, we compute asymptotic Gilbert–Varshamov-like lower bounds on  $R_q(\tau)$  for different values of  $q$ , using several results from [3] and [4]. In Section III, we find an upper bound on  $M_2(n, t)$  using a general technique from [1]. In Section IV, we present constructions of binary  $t$ -grain-correcting codes of length  $n$  for some values of  $n$  and  $t$  and show the optimality and the uniqueness of some of these codes.

## II. GILBERT–VARSHAMOV-LIKE BOUNDS

For a subset  $\mathcal{X} \subseteq \Sigma^n$ , let  $W_t(\mathcal{X}) = \sum_{\mathbf{x} \in \mathcal{X}} |\mathcal{R}_t(\mathbf{x}) \cap \mathcal{X}|$ . Namely,  $W_t(\mathcal{X})$  is the number of ordered pairs of  $t$ -confusable words in  $\mathcal{X}$ . The following lemma is essentially a reformulation of [3, Lemma 1] for grain-correcting codes.

*Lemma 2.1:* Let  $n, t$  be positive integers and let  $\mathcal{X} \subseteq \Sigma^n$ . Then  $M_{|\Sigma|}(n, t) \geq |\mathcal{X}|^2 / (4W_t(\mathcal{X}))$ .

The upcoming discussion is meant to evaluate  $W_t(\mathcal{X})$  for certain sets  $\mathcal{X}$  of words with prescribed empirical distribution of transitions.

Define graphs  $\mathcal{G}^{(\mathcal{N})} = (V^{(\mathcal{N})}, E^{(\mathcal{N})})$ ,  $\mathcal{G}^{(\mathcal{O})} = (V^{(\mathcal{O})}, E^{(\mathcal{O})})$  corresponding to the scenarios without and with overlaps, respectively, as follows. Let  $\bar{\Sigma} = \{\bar{a} : a \in \Sigma\}$  be a set where every element  $\bar{a}$  designates a symbol whose original value  $a \in \Sigma$  was smeared by a grain error. The set of states  $V^{(\mathcal{N})} \subseteq (\Sigma \cup \bar{\Sigma})^2$  is defined as  $V^{(\mathcal{N})} = V_0 \cup V_1 \cup V_2$  where  $V_0 = \{aa : a \in \Sigma\}$ ,  $V_1 = \{a\bar{b} : ab \in \Sigma^2, a \neq b\} \cup \{\bar{a}b : ab \in \Sigma^2, a \neq b\}$ , and  $V_2 = \{\bar{a}\bar{b} : ab \in \Sigma^2, a \neq b\}$ . The set of states  $V^{(\mathcal{O})} \subseteq \Sigma^2$  is defined as  $V^{(\mathcal{O})} = V_0 \cup V_3$  where  $V_3 = \{ab : ab \in \Sigma^2, a \neq b\}$ . Specifically, for  $q = 2$ ,  $V_0 = \{00, 11\}$ ,  $V_1 = \{0\bar{1}, \bar{1}0, 1\bar{0}, \bar{0}1\}$ ,  $V_2 = \{\bar{0}\bar{1}, \bar{1}\bar{0}\}$ , and  $V_3 = \{01, 10\}$  (the states of the set  $V_2$  will have no incoming

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edges for  $q = 2$  in  $\mathcal{G}^{(\mathcal{N})}$ , so for  $q = 2$ , we will disregard  $V_2$  altogether). For  $\mathbf{x} = (x_i)_{i \in [n]} \in (\Sigma \cup \bar{\Sigma})^n$ , define the operator  $\partial(\mathbf{x}) = (\partial(x_i))_{i \in [n]}$  such that  $\partial(a) = \partial(\bar{a}) = a$  for every  $a \in \Sigma$ . There is an edge in  $E^{(\mathcal{N})}$  from  $v = \ell r$  to  $v' = \ell' r'$  if

- [N1]  $v' \in V_0$ , or
- [N2]  $\ell = \ell' \in \Sigma$  and  $rr' \in \Sigma \bar{\Sigma}$ , or  $\ell \ell' \in \Sigma \bar{\Sigma}$  and  $r = r' \in \Sigma$ , or
- [N3]  $\ell = r' \in \Sigma$  and  $\ell' r \in \Sigma^2$ , or  $\ell r' \in \Sigma^2$  and  $\ell' = r \in \Sigma$ , or
- [N4]  $v \in V_0$ ,  $v' \in V_2$ ,  $\ell \neq \partial(\ell')$ ,  $r \neq \partial(r')$ .

There is an edge in  $E^{(\mathcal{O})}$  from  $v = \ell r$  to  $v' = \ell' r'$  if

- [O1]  $v' \in V_0$ , or
- [O2]  $v \in V_0$  and  $v' \in V_3$ , or
- [O3]  $v, v' \in V_3$ ,  $\ell r \neq r' \ell'$  and either  $\ell = r'$  or  $r = \ell'$ , or
- [O4]  $v, v' \in V_3$  and  $\ell r = r' \ell'$ .

Given a path  $\gamma = (\ell_i r_i)_{i \in [n]}$  of length  $n-1$  in  $\mathcal{G}^{(\mathcal{N})}$ , define the sets  $L(\gamma) = \{i : \ell_i \in \bar{\Sigma}\}$  and  $R(\gamma) = \{i : r_i \in \bar{\Sigma}\}$ . When the path  $\gamma$  is in  $\mathcal{G}^{(\mathcal{O})}$ , let  $L(\gamma) = \{i : \ell_i \neq r_i, r_{i-1} \neq \ell_i\}$ ,  $R(\gamma) = \{i : \ell_i \neq r_i, \ell_{i-1} \neq r_i\}$ . In addition, for an edge  $e \in (\ell r, \ell' r')$  in  $\mathcal{G}^{(\mathcal{O})}$ , define the function  $\mu : E^{(\mathcal{O})} \rightarrow [2]$  that equals 1 if  $e$  satisfies Condition [O4] and 0 otherwise. Define  $\mu(\gamma) = \sum_{i \in [n-1]} \mu(\ell_i r_i, \ell_{i+1} r_{i+1})$  for a path  $\gamma = (\ell_i r_i)_{i \in [n]}$  in  $\mathcal{G}^{(\mathcal{O})}$ ; viz.,  $\mu(\gamma)$  is the number of locations where overlapping grains, if switched from  $\mathbf{x} = (\partial(\ell_i))_{i \in [n]}$  to the corresponding index of  $\mathbf{y} = (\partial(r_i))_{i \in [n]}$  (and vice versa), will still confuse  $\mathbf{x}$  and  $\mathbf{y}$ . For completeness, let  $\mu(\gamma) = 0$  when  $\gamma$  is in  $\mathcal{G}^{(\mathcal{N})}$ .

The path  $\gamma$  in  $\mathcal{G}^{(\mathcal{N})}$  starting in  $V_0$  represents a pair  $(\mathbf{x} = (\partial(\ell_i))_{i \in [n]}, \mathbf{y} = (\partial(r_i))_{i \in [n]})$  of finitely-confusable words in  $\Sigma^n$ , as well as grain patterns  $\mathcal{S} = L(\gamma)$ ,  $\mathcal{S}' = R(\gamma)$  that cause the corresponding smeared words,  $\sigma_{\mathcal{S}}(\mathbf{x})$  and  $\sigma_{\mathcal{S}'}(\mathbf{y})$ , to be equal. As for the path  $\gamma$  in  $\mathcal{G}^{(\mathcal{O})}$  starting in  $V_0$ , it represents a pair  $(\mathbf{x} = (\ell_i)_{i \in [n]}, \mathbf{y} = (r_i)_{i \in [n]})$  of finitely-confusable words in  $\Sigma^n$  and  $2^{\mu(\gamma)}$  confusing grain patterns  $\mathcal{S} = L(\gamma) \cup M(\gamma)$ ,  $\mathcal{S}' = R(\gamma) \cup M'(\gamma)$  where  $\{M(\gamma), M'(\gamma)\}$  is a partition of the indexes  $i \in [n]$  of edges of  $\gamma$  that satisfy Condition [O4].

*Example 2.1:* For  $\Sigma = [3]$  and  $n = 6$ , consider the path

$$\gamma = (v_i)_{i \in [n]} = 11 \quad 22 \quad 2\bar{0} \quad \bar{1}2 \quad 00 \quad \bar{1}\bar{2}$$

in  $\mathcal{G}^{(\mathcal{N})}$ . This path corresponds to the smeared words  $122\bar{1}0\bar{1}$  and  $12\bar{0}20\bar{2}$  (the grain-free words are  $\mathbf{x} = 122101$  and  $\mathbf{y} = 120202$ ), and the overlines indicate the grain patterns  $\mathcal{S} = L(\gamma) = \{3, 5\}$ ,  $\mathcal{S}' = R(\gamma) = \{2, 5\}$  that make  $\mathbf{x}$  and  $\mathbf{y}$  confusable. The edges  $(v_i, v_{i+1})$  for  $i = 0, 1, 2$  correspond to Conditions [N1]–[N3], respectively; the edge  $(v_4, v_5)$  corresponds to Condition [N4]. Now, for the same alphabet  $\Sigma$  and  $n = 7$ , consider the path

$$\gamma = (v_i)_{i \in [n]} = 11 \quad 22 \quad 20 \quad 12 \quad 00 \quad 12 \quad 21$$

in  $\mathcal{G}^{(\mathcal{O})}$ . Here  $L(\gamma) = \{3, 5\}$ ,  $R(\gamma) = \{2, 5\}$  and  $\mu(\gamma) = 1$ . The edges  $(v_i, v_{i+1})$  for  $i = 0, 1, 2$  correspond to Conditions [O1]–[O3], respectively; the edge  $(v_5, v_6)$  corresponds to Condition [O4].  $\square$

For  $q = 2$ , the adjacency matrices  $A_{\mathcal{G}}^{(j)}$  of  $\mathcal{G}^{(j)}$  for  $j \in \{\mathcal{N}, \mathcal{O}\}$ , constructed as described above, are shown in Figure 1.

To make the presentation and the computation simpler, we will switch to a different criterion of confusability till the end of this section. Given a positive integer  $t$ , we will call two words  $\mathbf{x}, \mathbf{y}$  *t-confusable in a wider sense* (or *t-cws*, in short)

$A_{\mathcal{G}}^{(\mathcal{N})}$	00	01	0 $\bar{1}$	$\bar{1}0$	10	11
00	1	0	1	1	0	1
01	1	0	0	0	1	1
0 $\bar{1}$	1	0	0	1	0	1
$\bar{1}0$	1	0	1	0	0	1
10	1	1	0	0	0	1
11	1	1	0	0	1	1

$A_{\mathcal{G}}^{(\mathcal{O})}$	00	01	10	11
00	1	1	1	1
01	1	0	1	1
10	1	1	0	1
11	1	1	1	1

and

Fig. 1. Adjacency matrices  $A_{\mathcal{G}}^{(\mathcal{N})}$  and  $A_{\mathcal{G}}^{(\mathcal{O})}$  for  $q = 2$ .

if there exist grain patterns  $\mathcal{S}$  and  $\mathcal{S}'$  such that  $|\mathcal{S}| + |\mathcal{S}'| \leq 2t$  and  $\sigma_{\mathcal{S}}(\mathbf{x}) = \sigma_{\mathcal{S}'}(\mathbf{y})$ . Notice that any  $t$ -grain-correcting code in a wider sense is also  $t$ -grain-correcting in the ordinary sense. Our results will actually apply to the wider-sense notion of confusability. The following lemma (with proof omitted) establishes a correspondence between ordered pairs of  $t$ -cws words and paths in  $\mathcal{G}^{(\mathcal{N})}$  or  $\mathcal{G}^{(\mathcal{O})}$ .

*Lemma 2.2:* For  $j \in \{\mathcal{N}, \mathcal{O}\}$ , let  $\mathcal{W}_t^{(j)}$  denote the set of all  $t$ -cws (ordered) pairs  $(\mathbf{x}, \mathbf{y}) \in \Sigma^n \times \Sigma^n$  and let  $\mathcal{P}_t^{(j)}$  be the following set of paths in  $\mathcal{G}^{(j)}$ :

$$\mathcal{P}_t^{(j)} = \{\gamma = (v_i)_{i \in [n]} : v_0 \in V_0, |L(\gamma)| + |R(\gamma)| + \mu(\gamma) \leq 2t\}, \quad (1)$$

Then for each pair  $(\mathbf{x}, \mathbf{y})$  in  $\mathcal{W}_t^{(j)}$ , there is exactly one path  $\gamma = (\ell_i r_i)_{i \in [n]}$  in  $\mathcal{P}_t^{(j)}$  such that  $\mathbf{x} = (\partial(\ell_i))_{i \in [n]}$  and  $\mathbf{y} = (\partial(r_i))_{i \in [n]}$  for  $j \in \{\mathcal{N}, \mathcal{O}\}$ .

For  $j \in \{\mathcal{N}, \mathcal{O}\}$ , let  $\Gamma_n^{(j)}$  denote the set of all the cycles in  $\mathcal{G}^{(j)}$  of length  $n$  that start and terminate in the same state of  $V_0$ . Define the functions  $f^{(\mathcal{N})} : E^{(\mathcal{N})} \rightarrow [3]^2$ ,  $f^{(\mathcal{O})} : E^{(\mathcal{O})} \rightarrow [3]^2 \times [2]$  such that for any edge  $e$ ,  $f^{(\mathcal{N})}(e) = (\nu(e), \chi(e))$  and  $f^{(\mathcal{O})}(e) = (\omega(e), \chi(e), \mu(e))$  where the functions  $\nu : E^{(\mathcal{N})} \rightarrow [3]$ ,  $\omega : E^{(\mathcal{O})} \rightarrow [3]$ ,  $\chi : E^{(\mathcal{N})} \cup E^{(\mathcal{O})} \rightarrow [3]$  are defined next. For an edge  $e = (\ell r, \ell' r')$ :

- $\nu(e)$  equals 2 if  $e$  satisfies Condition [N4], 1 if  $e$  satisfies Conditions [N2] or [N3], and 0 otherwise.
- $\omega(e)$  equals 2 if  $e$  satisfies Condition [O2] for  $\ell = r \notin \{\ell', r'\}$ , 1 if  $e$  satisfies either Condition [O2] for  $\ell = r \in \{\ell', r'\}$  or Condition [O3], and 0 otherwise.
- $\chi(e)$  equals 2 if  $\partial(\ell) \neq \partial(\ell')$  and  $\partial(r) \neq \partial(r')$ , 0 if  $\partial(\ell) = \partial(\ell')$  and  $\partial(r) = \partial(r')$ , and 1 otherwise.

The function  $\nu(e)$  counts the smallest number of grains making  $\ell \ell'$  and  $rr'$  confusable for  $j = \mathcal{N}$ . The function  $\omega(e)$  counts the smallest number of nonoverlapping grains making  $\ell \ell'$  and  $rr'$  confusable for  $j = \mathcal{O}$ . The function  $\chi(e)$  counts the number of crossovers in  $\ell \ell'$  or  $rr'$ , namely, we add 1 if  $\partial(\ell) \neq \partial(\ell')$  and another 1 if  $\partial(r) \neq \partial(r')$ .

For a stationary Markov chain  $P : E \rightarrow [0, 1]$ , denote by  $E_P \{f\}$  the expected value of  $f$  with respect to  $P$ , that is,  $E_P \{f\} = \sum_{e \in E} P(e) f(e)$ . For a cycle  $\gamma = (v_i)_{i \in [n+1]}$  (where  $v_0 = v_n$ ) of length  $n$  in  $\mathcal{G}$ , let  $P_\gamma : E \rightarrow [0, 1]$  be the empirical probability distribution of  $\gamma$ , namely, for  $e \in E$ ,

$$P_\gamma(e) = \frac{1}{n} |\{i \in [n] : (v_i, v_{i+1}) = e\}|.$$

Now, set  $\tau, p \in (0, 1)$ , let  $\epsilon > 0$  and define

$$U_{\tau, p, \epsilon}^{(\mathcal{N})} = \{(u_1, u_2) : -\epsilon < u_1 < 2\tau + \epsilon, |u_2 - 2p| < 2\epsilon\},$$

$$U_{\tau,p,\epsilon}^{(\mathcal{O})} = \{(u_1, u_2, u_3) :$$

$$u_1, u_3 > -\epsilon, u_1 + u_3 < 2\tau + \epsilon, |u_2 - 2p| < 2\epsilon\}.$$

For  $j \in \{\mathcal{N}, \mathcal{O}\}$ , let  $\Gamma_{\tau,p,\epsilon}^{(j)} = \{\gamma \in \Gamma_n^{(j)} : \mathbb{E}_{P_\gamma}\{f^{(j)}\} \in U_{\tau,p,\epsilon}^{(j)}\}$ . Additionally, for  $j \in \{\mathcal{N}, \mathcal{O}\}$  and the same  $\tau, p, \epsilon$ , let

$$\mathcal{P}_{\tau,p,\epsilon}^{(j)} = \{\gamma \in \mathcal{P}_{\lceil \tau(n-1) \rceil}^{(j)} : |\mathbb{E}_{P_\gamma}\{\chi\} - 2p| \leq \epsilon\}.$$

The following lemma (with proof omitted) holds for sufficiently large values of  $n$ .

**Lemma 2.3:** Let  $\tau, p \in (0, 1)$  and  $\epsilon > 0$ . Then  $|\mathcal{P}_{\tau,p,\epsilon}^{(j)}| \leq |\Gamma_{\tau,p,\epsilon}^{(j)}|$  for  $j \in \{\mathcal{N}, \mathcal{O}\}$ .

Let  $k$  be a positive integer. For a graph  $G = (V_G, E_G)$ , a vector of indeterminates  $\mathbf{z} = (z_i)_{i \in [k]} \in (0, \infty)^k$ , and a function  $f = (f_i)_{i \in [k]} : E_G \rightarrow \mathbb{R}^k$ , define the matrix function  $A_G(\mathbf{z}) : (0, \infty)^k \rightarrow \mathbb{R}^{|V_G| \times |V_G|}$  (whose rows and columns are indexed by  $V_G$ ) as

$$[A_G(\mathbf{z})]_{v,v' \in V_G} = \begin{cases} \mathbf{z}^{f(e)} = \prod_{i \in [k]} z_i^{f_i(e)} & e \in E_G \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

where  $e = (v, v')$ . We proceed by citing special cases of [4, Lemma 2] and [4, Lemma 5] which we are going to employ next. In both lemmas,  $\mathcal{M}(f; U)$  denotes the set of all stationary Markov chains  $P$  on  $G$  such that  $\mathbb{E}_P\{f\} \in U \subseteq \mathbb{R}^k$ .

**Lemma 2.4:** Let  $G = (V_G, E_G)$  be a primitive graph and  $f : E_G \rightarrow \mathbb{R}^k$  be a function. Let  $U$  be an open rectangular parallelepiped  $\prod_{i \in [k]} (\tilde{s}_i, s_i)$  and let  $\Gamma_n$  be a set of all cycles of length  $n$  in  $G$ . Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log_q |\{\gamma \in \Gamma_n : \mathbb{E}_{P_\gamma}\{f\} \in U\}| = \sup_{P \in \mathcal{M}(f; U)} H_q(P)$$

where  $H_q(P) = \sum_{v \in V_G} \sum_{e=(v,v') \in E_G} P(e) \log_q [\pi(v)/P(e)]$  and  $\pi(v) = \sum_{e=(v,v') \in E_G} P(e)$ .

**Lemma 2.5:** Let  $\mathbf{p} = (p_i)_{i \in [k']} \in [0, 1]^{k'}$  and let  $G = (V_G, E_G)$  be a graph. Let  $f : E_G \rightarrow \mathbb{R}^k$  and  $f' : E_G \rightarrow \mathbb{R}^{k'}$  be functions. Let  $U$  be a closed rectangular parallelepiped  $\prod_{i \in [k]} [0, s_i]$ . Then  $\sup_{\substack{P \in \mathcal{M}(f; U) \\ \mathbb{E}_P\{f'\} = \mathbf{p}}} H_q(P)$  equals

$$\inf_{\mathbf{z}, \mathbf{h}} \{\log_q \lambda(A_G(\mathbf{z}, \mathbf{h})) - \sum_{i \in [k]} s_i \log_q z_i - \sum_{i \in [k']} p_i \log_q h_i\},$$

where  $\lambda(\cdot)$  denotes the spectral radius of a square real matrix,  $\mathbf{z} = (z_i)_{i \in [k]}$  ranges over  $(0, 1]^k$  and  $\mathbf{h} = (h_i)_{i \in [k']}$  ranges over  $(0, \infty)^{k'}$ .

Let us go back to our context. For  $z \in (0, 1]$  and  $h, m \in (0, \infty)$ , let the matrix functions  $A_G^{(\mathcal{N})}(z, h)$  and  $A_G^{(\mathcal{O})}(z, h, m)$ , with rows and columns indexed by the sets of states  $V^{(\mathcal{N})}$  and  $V^{(\mathcal{O})}$ , respectively, be defined as a special case of (2):

$$[A_G^{(\mathcal{N})}(z, h)]_{v,v' \in V} = \begin{cases} z^{\nu(e)} h^{\chi(e)} & e = (v, v') \in E^{(\mathcal{N})} \\ 0 & \text{otherwise} \end{cases}$$

and

$$[A_G^{(\mathcal{O})}(z, h, m)]_{v,v' \in V} = \begin{cases} z^{\omega(e)} h^{\chi(e)} m^{\mu(e)} & e=(v, v') \in E^{(\mathcal{O})} \\ 0 & \text{otherwise} \end{cases}.$$

Applying Lemma 2.4 with  $G = \mathcal{G}^{(\mathcal{N})}$ ,  $U = U_{\tau,p,\epsilon}^{(\mathcal{N})}$  and  $f = f^{(\mathcal{N})}$  and combining the result with Lemma 2.3, we conclude that  $\lim_{n \rightarrow \infty} \frac{1}{n} \log_q |\mathcal{P}_{\tau,p,\epsilon}^{(\mathcal{N})}| \leq \sup_{P \in \mathcal{M}(f^{(\mathcal{N})}; U_{\tau,p,\epsilon}^{(\mathcal{N})})} H_q(P)$ . By the continuity of the functions  $P \mapsto \mathbb{E}_P(f^{(\mathcal{N})})$  and  $P \mapsto H_q(P)$ ,

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log_q |\mathcal{P}_{\tau,p,\epsilon}^{(\mathcal{N})}| \leq \sup_{P \in \mathcal{M}(f^{(\mathcal{N})}; U_{\tau,p}^{(\mathcal{N})})} H_q(P)$$

where  $U_{\tau,p}^{(\mathcal{N})} = \{(u, 2p) : 0 \leq u \leq 2\tau\}$ . Applying Lemma 2.5 with  $G = \mathcal{G}^{(\mathcal{N})}$ ,  $f = \nu$ ,  $f' = \chi$ ,  $U = \{u : 0 \leq u \leq 2\tau\}$  and  $\mathbf{p} = (2p)$  yields

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log_q |\mathcal{P}_{\tau,p,\epsilon}^{(\mathcal{N})}| \leq K^{(\mathcal{N})} = \quad (3)$$

$$\inf_{z \in (0,1], h \in (0,\infty)} \{\log_q \lambda(A_G^{(\mathcal{N})}(z, h)) - 2\tau \log_q z - 2p \log_q h\}.$$

Turning now to  $j = \mathcal{O}$ , we define for  $\eta \in [0, 2\tau]$ ,

$$U_{\tau,p,\epsilon,\eta}^{(\mathcal{O})} = \{(u_1, u_2, u_3) : u_1, u_3 \geq 0, \\ u_1 \leq 2\tau - \eta, |u_3 - \eta| \leq 2\epsilon, |u_2 - 2p| \leq 2\epsilon\}.$$

Since  $\mu(\gamma) \in [n+1]$  for any  $\gamma \in \Gamma_{\tau,p,\epsilon,\eta}^{(\mathcal{O})}$ , there is a polynomial (in  $n$ ) number of types of cycles in  $\Gamma_{\tau,p,\epsilon,\eta}^{(\mathcal{O})}$  characterized by the same value of  $\mathbb{E}_{P_\gamma}\{\mu\}$ , therefore

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log_q |\Gamma_{\tau,p,\epsilon,\eta}^{(\mathcal{O})}| = \sup_{0 \leq \eta \leq 2\tau} \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log_q |\Gamma_{\tau,p,\epsilon,\eta}^{(\mathcal{O})}|$$

where  $\Gamma_{\tau,p,\epsilon,\eta}^{(\mathcal{O})} = \{\gamma \in \Gamma_{\tau,p,\epsilon}^{(\mathcal{O})} : |\mathbb{E}_{P_\gamma}\{\mu\} - \eta| \leq 2\epsilon\}$ . Applying Lemma 2.4 with  $G = \mathcal{G}^{(\mathcal{O})}$ ,  $U = U_{\tau,p,\epsilon,\eta}^{(\mathcal{O})}$  and  $f = f^{(\mathcal{O})}$ , then applying Lemma 2.5 with  $G = \mathcal{G}^{(\mathcal{O})}$ ,  $f = \omega$ ,  $f' = (\chi, \mu)$ ,  $U = \{u : 0 \leq u \leq 2\tau - \eta\}$  and  $\mathbf{p} = (2p, \eta)$ , and then combining the result with Lemma 2.3 yields

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log_q |\mathcal{P}_{\tau,p,\epsilon}^{(\mathcal{O})}| \leq K^{(\mathcal{O})} = \quad (4)$$

$$\sup_{0 \leq \eta \leq 2\tau} \inf_{z \in (0,1], h, m \in (0,\infty)} \{\log_q \lambda(A_G^{(\mathcal{O})}(z, h, m)) - (2\tau - \eta) \log_q z - \eta \log_q m - 2p \log_q h\}.$$

For  $j \in \{\mathcal{N}, \mathcal{O}\}$ , let  $\mathcal{W}_{\tau,p,\epsilon}^{(j)}$  be the set of pairs  $(\mathbf{x}, \mathbf{y}) \in \mathcal{W}_{\lceil \tau(n-1) \rceil}^{(j)}$  such that the average number of crossovers in both  $\mathbf{x}$  and  $\mathbf{y}$  is within  $2p \pm \epsilon$ . By Lemma 2.2, the inequalities in (3) and (4) hold if we replace  $\mathcal{P}_{\tau,p,\epsilon}^{(j)}$  therein by  $\mathcal{W}_{\tau,p,\epsilon}^{(j)}$ .

Let  $H_q : [0, 1] \rightarrow [0, 1]$  denote the  $q$ -ary entropy function:

$$H_q(p) = -p \log_q p - (1-p) \log_q (1-p) + p \log_q (q-1).$$

Define the set  $\mathcal{X} \subseteq \Sigma^n$  as the set of all words with the average number of crossovers being within  $p \pm \epsilon$  for some  $p \in (0, 1)$ . The exponential growth rate of  $\mathcal{X}$  is clearly  $H_q(p)$  when  $\epsilon \rightarrow 0$ . Using the logarithmic version of Lemma 2.1 and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log_q |\mathcal{W}_{\lceil \tau n \rceil}(\mathcal{X})| = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log_q |\mathcal{W}_{\tau,p,\epsilon}^{(j)}|$$

for  $j \in \{\mathcal{N}, \mathcal{O}\}$ , we arrive to the main theorem of the paper.

**Theorem 2.6:** Let  $\tau \in (0, 1)$ . Then for  $j \in \{\mathcal{N}, \mathcal{O}\}$ ,

$$R_q(\tau) \geq \varrho_q^{(j)}(\tau) = \sup_{p \in [0,1]} \{2H_q(p) - K^{(j)}\}, \quad (5)$$

where  $K^{(\mathcal{N})}$  and  $K^{(\mathcal{O})}$  are defined in (3) and (4).

To alleviate the computations, we can now merge states in  $\mathcal{G}^{(j)}$  to reduce the order of the matrix  $A_G^{(j)}$  while preserving its spectral radius for  $j \in \{\mathcal{N}, \mathcal{O}\}$ , as described in [5, Sec. 4.6]. The states of  $V_0$  can be merged into superstate 0, states of  $V_1$  in  $\mathcal{G}^{(\mathcal{N})}$  and states of  $V_3$  in  $\mathcal{G}^{(\mathcal{O})}$  — into superstate 1, whereas states of  $V_2$  — into superstate 2. Specifically, for  $q = 2$ , the merging ends up with reduced matrices  $A_G^{(\mathcal{N})}$  and  $A_G^{(\mathcal{O})}$  whose spectral radii equal those of  $A_G^{(\mathcal{N})}$  and  $A_G^{(\mathcal{O})}$ , respectively:

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1+h^2 & 2hz \\ \hline 1 & 2h & h^2m \end{array} \quad \text{and} \quad \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1+h^2 & 2hz \\ \hline 1 & 2h & h^2m \end{array}.$$

It turns out that for  $q = 2$ ,  $\varrho_2^{(\mathcal{N})}(\tau) = \varrho_2^{(\mathcal{O})}(\tau)$  for any  $\tau \in [0, 1]$ . This phenomenon is due to the fact that when

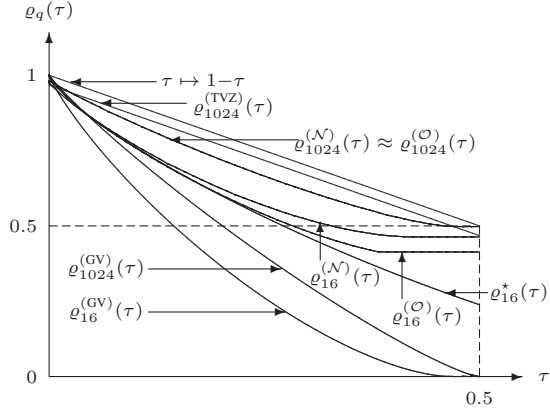


Fig. 2. Functions  $\varrho_q^{(N)}(\tau)$ ,  $\varrho_q^{(O)}(\tau)$  and  $\varrho_q^{(GV)}(\tau)$  for  $q \in \{16, 1024\}$ .

$q = 2$ , for any path  $\gamma' \in \mathcal{P}_t^{(O)}$  that confuses the  $t$ -cws pair of words  $(\mathbf{x}, \mathbf{y})$  there exists a path  $\gamma \in \mathcal{P}_t^{(N)}$  that confuses the same pair of words. The path  $\gamma$  is obtained by moving overlapping grains from  $L(\gamma')$  to  $R(\gamma')$  and vice versa until  $L(\gamma')$  and  $R(\gamma')$  do not contain consecutive numbers. Theorem 2.6 strictly improves on the traditional (Hamming-distance) Gilbert–Varshamov bound  $\varrho_2^{(GV)}(\tau) = 1 - H_2(2\tau)$  on the entire interval  $(0, 0.25]$ , however, on the interval  $[0.0566, 0.25]$  it falls short of the simple lower bound of 0.5 which is realized by Construction 4.1 from Section IV. The difference between  $\varrho_2^{(N)}(\tau) = \varrho_2^{(O)}(\tau)$  and  $\varrho_2^{(GV)}(\tau)$  on the interval  $(0, 0.0566]$  does not exceed 0.012.

For  $q > 2$ ,  $\mathcal{A}_G^{(N)}$  and  $\mathcal{A}_G^{(O)}$  are, respectively,

	0	1	2
0	$1 + (q-1)h^2$	$2(q-1)hz$	$(q-1)(q-2)h^2z^2$
1	$2h + (q-2)h^2$	$(q-1)h^2z$	0
2	$2h + (q-2)h^2$	0	0

and

	0	1
0	$1 + (q-1)h^2$	$2(q-1)hz + (q-1)(q-2)h^2z^2$
1	$2h + (q-2)h^2$	$h^2m + 2(q-2)h^2z$

Figure 2 depicts the functions  $\tau \mapsto \varrho_q^{(N)}(\tau)$  and  $\tau \mapsto \varrho_q^{(O)}(\tau)$  for  $q \in \{16, 1024\}$  along with the corresponding traditional Gilbert–Varshamov bounds  $\varrho_q^{(GV)}(\tau) : \tau \mapsto 1 - H_q(2\tau)$ . Both  $\varrho_q^{(N)}(\tau)$  and  $\varrho_q^{(O)}(\tau)$  strictly improve on  $\varrho_q^{(GV)}(\tau)$  on the entire interval  $(0, 0.5]$  (and  $\varrho_q^{(N)}(\tau)$  is strictly above  $\varrho_q^{(O)}(\tau)$ ); besides, both  $\varrho_q^{(N)}(\tau)$  and  $\varrho_q^{(O)}(\tau)$  can be shown to converge to the function  $\tau \mapsto 1 - \tau$  on that interval when  $q \rightarrow \infty$ .

For large values of  $q$  when overlaps are not allowed, the lower bound  $\varrho_q^{(N)}(\tau)$  is worse on nearly the entire interval  $(0, 0.5)$  than the following construction based on the family of linear  $[n, nR, \lceil \tau n \rceil + 1]$  codes by Tsfasman *et al.* [7] with rate  $R \geq 1 - \frac{1}{\sqrt{q-1}} - \tau - o(1)$  ( $o(1)$  goes to 0 for  $n \rightarrow \infty$ ). By an averaging argument, there exists at least one coset of an  $[n, nR, \lceil \tau n \rceil + 1]$  code of this family whose intersection  $\mathcal{C}^{(TVZ)}$  with the code  $\{\mathbf{e} = (c_i)_{i \in [n]} \in \Sigma^n : c_i \neq c_{i+1} \text{ for any } i \in [n-1]\}$  is of rate at least  $R - 1 + \log_q(q-1)$ . Since adjacent symbols in each codeword in  $\mathcal{C}^{(TVZ)}$  are different, grain errors

become erasures, hence  $\mathcal{C}^{(TVZ)}$  is a  $\lceil \tau n \rceil$ -grain-correcting code of rate at least  $\varrho_q^{(TVZ)}(\tau) = \log_q(q-1) - \frac{1}{\sqrt{q-1}} - \tau$ . A similar reasoning applied to the family of linear codes guaranteed by the Gilbert–Varshamov bound yields  $\lceil \tau n \rceil$ -grain-correcting codes of rate at least  $\varrho_q^*(\tau) = \log_q(q-1) - H_q(\tau)$ . The functions  $\tau \mapsto \varrho_{1024}^{(TVZ)}(\tau)$  and  $\tau \mapsto \varrho_{16}^*(\tau)$  are shown in Figure 2 alongside the other bounds. It can be observed that  $\varrho_{16}^{(N)}(\tau)$  and  $\varrho_{16}^{(O)}(\tau)$  are strictly above  $\varrho_{16}^*(\tau)$ , whereas  $\varrho_{1024}^{(N)}(\tau)$  is above  $\varrho_{1024}^{(TVZ)}(\tau)$  only in the interval  $[0, 0.06] \cup [0.44, 0.5]$ .

### III. UPPER BOUND

Henceforth, we will restrict the discussion to  $q = 2$  only. Let  $\mathcal{C}$  be a binary  $t$ -grain-correcting code of length  $n$ . Let  $\mathcal{B}_t(\mathbf{x})$  be the set of all words  $\mathbf{w} \in [2]^n$  for which there exists a grain pattern  $\mathcal{S}$  of size  $|\mathcal{S}| \leq t$  such that  $\sigma_{\mathcal{S}}(\mathbf{x}) = \mathbf{w}$ . Suppose now that for any  $\mathbf{c} \in \mathcal{C}$  we have a lower bound  $\psi_t(r)$  on  $|\mathcal{B}_t(\mathbf{c})|$  that depends only on the number of runs  $r = r(\mathbf{c})$  in  $\mathbf{c}$  and that, as a function of  $r$ , it is nondecreasing. Hence, by sphere-packing arguments,  $\Psi = \sum_{\mathbf{c} \in \mathcal{C}} \psi_t(r(\mathbf{c})) \leq \sum_{\mathbf{c} \in \mathcal{C}} |\mathcal{B}_t(\mathbf{c})| \leq 2^n$ .

Let  $N(r)$  be the number of words  $\mathbf{x} \in [2]^n$  with  $r$  runs. Since  $\psi_t(r)$  is nondecreasing in  $r$ , we can group words with the same number of the runs till the largest integer  $\rho$  such that

$$\sum_{r=1}^{\rho} N(r) \psi_t(r) \leq \Psi, \quad (6)$$

and thus,

$$|\mathcal{C}| \leq \sum_{r=1}^{\rho} N(r) + \left\lfloor \frac{\Psi - \sum_{r=1}^{\rho} N(r) \psi_t(r)}{\psi_t(\rho+1)} \right\rfloor. \quad (7)$$

By replacing  $\Psi$  with  $2^n$  in (6) and (7) we overcount in comparison with the right-hand side of (7). Hence

$$|\mathcal{C}| \leq \sum_{r=1}^{\rho} N(r) + \left\lfloor \frac{2^n - \sum_{r=1}^{\rho} N(r) \psi_t(r)}{\psi_t(\rho+1)} \right\rfloor$$

where  $\rho$  is the largest integer such that  $\sum_{r=1}^{\rho} N(r) \psi_t(r) \leq 2^n$ .

When overlaps are not allowed, we can bound  $|\mathcal{B}_t(\mathbf{c})|$  with  $r(\mathbf{c}) = r$  from below using [6, Lemma 1]:

$$\psi_t^{(N)}(r) = 1 + \sum_{s=1}^{\min\{t, \lfloor (r-1)/3 \rfloor\}} \frac{1}{s!} \prod_{s'=0}^{s-1} (r-1-3s'). \quad (8)$$

When overlaps are allowed, we are able to calculate the size of  $\mathcal{B}_t(\mathbf{c})$  with  $r(\mathbf{c}) = r$  precisely, namely,

$$\psi_t^{(O)}(r) = \sum_{s=0}^{\min\{t, r-1\}} \binom{r-1}{s}. \quad (9)$$

Both (8) and (9) are clearly nondecreasing functions on  $r$ . In both cases,  $N(r) = 2 \binom{n-1}{r-1}$ .

The next theorem summarizes the above discussion and, in fact, reformulates the sphere-packing bound that Abdel-Ghaffar and Weber [1, Th. 5] first used in a different context (also see [5, Sec. 7.3]).

*Theorem 3.1:* For  $j \in \{\mathcal{N}, \mathcal{O}\}$ , let  $\mathcal{C}$  be a binary  $t$ -grain-correcting code of length  $n$ . Then  $|\mathcal{C}| \leq \Delta^{(j)}(n, t)$  where

$$\Delta^{(j)}(n, t) = 2 \sum_{r=1}^{\rho} \binom{n-1}{r-1} + \left\lfloor \frac{2^{n-2} \sum_{r=1}^{\rho} \binom{n-1}{r-1} \psi_t^{(j)}(r)}{\psi_t^{(j)}(\rho+1)} \right\rfloor$$

and  $\rho$  is the largest integer such that  $\sum_{r=1}^{\rho} \binom{n-1}{r-1} \psi_t^{(j)}(r) \leq 2^{n-1}$ . The formulas for  $\psi_t^{(N)}(r)$  and  $\psi_t^{(O)}(r)$  are given in (8) and (9), respectively.

Set  $\tau = t/n$  and let  $\Delta^{(j)}(\tau) = \log_2 \Delta^{(j)}(n, \tau n) / n$ . Figure 3 depicts the functions  $\tau \mapsto \Delta^{(j)}(\tau)$  for  $j \in \{\mathcal{N}, \mathcal{O}\}$  calculated for  $t \in \{1, 2, \dots, 16\}$  and  $n = 200$ .

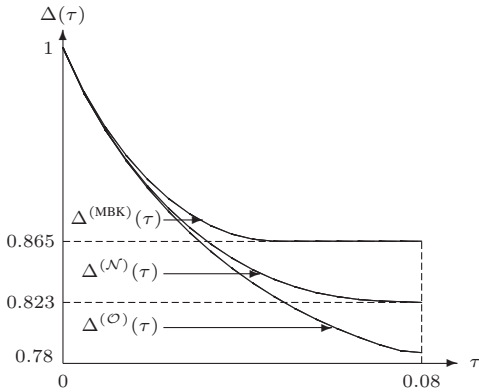


Fig. 3. Functions  $\Delta^{(\mathcal{N})}(\tau)$ ,  $\Delta^{(\mathcal{O})}(\tau)$  and  $\Delta^{(\text{MBK})}(\tau)$ .

Mazumdar *et al.* [6, Th. 2] obtained an upper bound on  $M_2(n, t)$  (for  $j = \mathcal{N}$ ) using a similar technique by considering a  $t$ -grain-correcting code  $\mathcal{C}$  and dividing it into two subcodes  $\mathcal{C}_1 = \{\mathbf{c} \in \mathcal{C} : |\mathbf{r}(\mathbf{c}) - n/2| \leq g(n, t)\}$  and  $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$  where  $g(n, t) = n/2 - \sqrt{nt \log_2 n}$ . The sizes of  $\mathcal{B}_t(\mathbf{c})$  for  $\mathbf{c} \in \mathcal{C}_1$  and  $\mathbf{c} \in \mathcal{C}_2$  were then bounded from below by  $\psi_t^{(\mathcal{N})}(g(n, t))$  and 1, respectively. The obtained upper bound on  $M_2(n, t)$  is

$$\Delta^{(\text{MBK})}(n, t) = \frac{2^{nt}}{(g(n, t) - 1 - 3(t-1))^t} + 4 \sum_{i=0}^{g(n, t)} \binom{n-1}{i}. \quad (10)$$

For specific values of  $n$  and  $t$ , this bound can be tightened by taking  $g(n, t)$  to minimize the right-hand side of (10). Let

$$\Delta^{(\text{MBK})}(\tau) = \frac{1}{n} \log_2 [\min_{3t-2 < g(n, t) < n/2} \Delta^{(\text{MBK})}(n, t)].$$

The function  $\tau \mapsto \Delta^{(\text{MBK})}(\tau)$  is depicted in Figure 3 alongside the other bounds calculated for  $t \in \{1, 2, \dots, 16\}$  and  $n = 200$ , as well. It can be seen that the functions  $\Delta^{(j)}(\tau)$  for  $j \in \{\mathcal{N}, \mathcal{O}\}$  improve on  $\Delta^{(\text{MBK})}(\tau)$  on the entire interval  $(0, 0.08]$ .

The asymptotic growth rate of  $\Delta^{(\mathcal{N})}(\tau)$  (as  $n \rightarrow \infty$ ) is identical to that of  $\Delta^{(\text{MBK})}(\tau)$ , characterized by [6, Prop. 3]. Similarly, the asymptotic growth rate of  $\Delta^{(\mathcal{O})}(\tau)$  for  $\tau \in [0, 0.114]$  equals  $H_2(x)$  where  $x$  is the smallest positive solution of  $H_2(x) + x \cdot H_2(\tau/x) = 1$ .

#### IV. CONSTRUCTIONS OF GRAIN-CORRECTING CODES

In this section, we restrict the discussion further by disallowing overlaps between grains. Mazumdar *et al.* pointed out a simple construction [6, Sec. 2] for grain-correcting codes and a general upper bound [6, Cor. 5] on  $M_2(n, t)$  which we cite herein.

*Construction 4.1:* For an even positive  $n$ , the binary code

$$\mathcal{C}_n = \{\mathbf{c} = (c_i)_{i \in [n]} : c_{2s} = c_{2s+1} \text{ for any } s \in [n/2]\}$$

is an  $\infty$ -grain-correcting code of length  $n$  and size  $2^{n/2}$ . For an odd positive  $n$ , the code  $\mathcal{C}_n = (0 | \mathcal{C}_{n-1}) \cup (1 | \mathcal{C}_{n-1})$  is a binary  $\infty$ -grain-correcting code of length  $n$  and size  $2^{\lceil n/2 \rceil}$ .

*Lemma 4.2:* For any  $n \in \mathbb{Z}^+$ ,  $M_2(n, \lfloor n/2 \rfloor) \leq 2^{\lceil n/2 \rceil}$ .

Construction 4.1 and Lemma 4.2 yield the following result.

*Theorem 4.3:* For any  $n \in \mathbb{Z}^+$ ,  $M_2(n, \lfloor n/2 \rfloor) = 2^{\lceil n/2 \rceil}$ .

It turns out that Construction 4.1 is the only way to construct binary  $\infty$ -grain-correcting codes of odd length  $n$  and size  $2^{\lceil n/2 \rceil}$ . We omit the details of the proof.

*Theorem 4.4:* Let  $n$  be an odd positive integer. The binary  $\infty$ -grain-correcting code of length  $n$  and size  $2^{\lceil n/2 \rceil}$  is unique.

A similar result to Theorem 4.3 (without proof) can be obtained for  $(\lfloor n/2 \rfloor - 1)$ -grain-correcting codes of odd length  $n$ .

*Theorem 4.5:* Let  $n \geq 5$  be an odd integer. Then  $M_2(n, \lfloor n/2 \rfloor - 1) = 2^{\lceil n/2 \rceil}$ .

As for the binary  $(n/2 - 1)$ -grain-correcting codes of even length  $n$ , the value of  $M_2(n, n/2 - 1)$  is realized by the augmentation of  $\mathcal{C}_n$  with the words  $(0110)^s(01)^{n-4s}$  and  $(1001)^s(10)^{n-4s}$  for  $s = \lfloor n/4 \rfloor$  (we omit the proof due to space limitations).

*Theorem 4.6:* Let  $n \geq 4$  be an even integer. Then  $M_2(n, n/2 - 1) = 2^{n/2} + 2$ .

An interesting (yet not provably optimal) construction of binary 1-grain-correcting codes can be obtained by the augmentation of a Hamming code with a subset of  $\mathcal{C}_n$ . We cite this result in the following lemma (without proof).

*Lemma 4.7:* Let  $m \geq 2$  be an integer and let  $n = 2^m - 1$ . Then  $M_2(n, 1) \geq 2^{n-m} + 2^{(n-1)/2}$ .

Table I contains the values of  $M_2(n, t)$  for small  $n$  and  $t$  obtained using computer search. Values marked in bold are guaranteed by Theorems 4.3, 4.5 or 4.6; values marked in italics are attained by unique codes due to Theorem 4.4. One can also observe that for  $(n, t) = (7, 1)$ , the construction in Lemma 4.7 gives a code of size 24 which is close to the optimum  $M_2(7, 1) = 26$ .

TABLE I  
SIZES  $M_2(n, t)$  OF LARGEST  $t$ -GRAIN-CORRECTING CODES OF LENGTH  $n$ .

$t \backslash n$	2	3	4	5	6	7	8	9
1	<b>2</b>	<b>4</b>	<b>6</b>	<b>8</b>	16	26	44	
2			<b>4</b>	<b>8</b>	<b>10</b>	<b>16</b>	22	
3					<b>8</b>	<b>16</b>	<b>18</b>	<b>32</b>

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