Coding for New Applications in Storage Media

Doctoral Thesis Research Proposal

Artyom Sharov
Advisor: Ron M. Roth
Computer Science Department
Technion – Israel Institute of Technology

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1 Background

Conventional magnetic recording media are composed of basic magnetizable units called *grains* which might be random in size and shape. Information is stored on the medium through a write mechanism that sets the magnetic polarities of the grains. Each grain can be magnetized to take on one of the two possible types of magnetic polarity. Thus, each grain represents at most one bit of information. If the boundaries of the grains had been known to the write and the readback apparatuses, then, theoretically, it would have been possible to attain the storage capacity of one bit per grain.

However, even if the write and the readback apparatuses were aware of the shapes and locations of specific grains in the medium, it would still be impossible to attain the capacity of one bit per grain since the existing technologies are still incapable of setting magnetic polarities of regions as small as a single grain. In current technologies, the writing in a magnetic medium is carried out by dividing the medium into periodically partitioned *cells* and writing one bit of data into these cells. Since a cell is typically larger in size than a grain, the writing of a bit into a cell boils down to uniformly magnetizing all the grains inside the cell (whereas the grains straddling the boundary between cells can be neglected).

Sometimes it is useful to view the recording medium as a *channel* that is a "black box" with inputs and outputs where output can differ from the input due to the adversary distortions of the medium. On the one hand, the ever-growing demand for storage forces the designers of such channels to write more data per unit area, thereby making the system less reliable. On the other hand, current recording applications require storage devices to have very high immunity against errors: to handle this problem, *error-correcting codes*, *constrained codes* or a combination of the two can be used. An extensive treatment of the constrained coding in recording channels is given, for instance, in [3],[5]. For a general background on the subject of magnetic recording the reader can refer to [7].

2 Known results

2.1 Wood *et al.*

Recently, Wood *et al.* [8] proposed a new mechanism that can magnetize areas as small as individual grains. With such a mechanism in place, the only bottleneck to attaining the maximum capacity is the absence of knowledge of grain boundaries. To estimate the information loss caused by this, Wood *et al.* considered a medium subdivided into a 14×14 grid of equally-sized cells which were composed of 100 randomly-shaped grains. The cells were written in a raster-scan fashion, where a grain took on the polarity of the written cell if it had at least 30 percent of its area lying in that cell. In that simulation it turned out that 31 percent of the 196 cells were reported with wrong polarity.

Another simulation from [8] resulted in modeling a one-dimensional granular medium as a channel and computing a lower bound on the capacity of the channel. The one-dimensional medium was divided into cells whose boundaries coincided with the boundaries of grains of lengths 1, 2, or 3. The polarity of a grain was set by the last value that was written into its area, namely, the last value written into a grain overwrote all the values that were previously written therein.

2.2 Mazumdar *et al.*

In another recent paper, Mazumdar *et al.* [6] considered the following combinatorial error model, which corresponds to the one-dimensional granular medium mentioned above. They assumed that the medium is comprised of n equally-shaped cells indexed from 1 through n. A partition into grains (or a *grains layout*) is defined by s integers $1 \leq j_1 < j_2 < \cdots < j_s \leq n$ such that j_i denotes the cell index of the beginning of the i-th grain. In other words, Mazumdar *et al.* tacitly assumed that the grains in the one-dimensional medium are *non-overlapping*. Additionally, the model assumes that the first bit written within a grain sets the polarity of the entire grain (this corresponds to setting the bits in reverse order). Formally, if $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ is the input data to be written to the medium with the grains layout defined by $j_1, j_2, \ldots, j_s$ and the actual (output) data written is $\mathbf{y} = (y_1, y_2, \ldots, y_n) = \varphi(\mathbf{x}) \in \{0,1\}^n$, then for any $1 \le i \le s$ and $j_i \le j < j_{i+1}$ one has $y_j = x_{j_i}$.

Moreover, the authors of [6] assumed that the lengths of grains are 1 (1-*grains*) and 2 (2-*grains*) only, implying that the errors can be introduced by the 2-grains only. The set of operators φ corresponding to the media with n cells where the number of 2-grains is bounded from above by t is denoted by $\Phi_{n,t}$. Given $\mathbf{x} \in \{0,1\}^n$, $\Phi_{n,t}(\mathbf{x}) = \{\varphi(\mathbf{x}) \mid \varphi \in \Phi_{n,t}\}$ denotes the set of all the output vectors that can be obtained by writing $\mathbf{x}$ into a granular media from $\Phi_{n,t}$. A binary code $\mathcal{C}$ is said to be *t-grain-correcting* if no two distinct codewords $\mathbf{c_1}, \mathbf{c_2}$ are *t-confusable*, namely, $\Phi_{n,t}(\mathbf{c_1}) \cap \Phi_{n,t}(\mathbf{c_2}) = \emptyset$ for any $\mathbf{c_1} \neq \mathbf{c_2} \in \mathcal{C}$. Let $M(n,t)$ denote the maximum size of a t-grain-correcting binary code of length n and let the asymptotic *rate* of grain-correcting codes with the constant ratio $\tau = \frac{t}{n}$ be defined as

$$R(\tau) = \limsup_{n\to\infty} \frac{\log_2 M(n, n\tau)}{n}.$$

2.2.1 Lower bounds on $M(n,t)$

The immediate lower bound [6, Sec. 2] on $M(n,t)$ that Mazumdar *et al.* obtained was

$$M(n,t) \ge \begin{cases} \max\left(2^{\lceil \frac{n}{2} \rceil}, \frac{2^n}{n^t}\right) & n \text{ is a power of } 2 \\ 2^{\lceil \frac{n}{2} \rceil} & \text{otherwise} \end{cases}. \tag{1}$$

The lower bound of $2^{\lceil \frac{n}{2} \rceil}$ is implied by the following simple construction:

$$\mathcal{C}_n = \left\{(c_1, c_2, \ldots, c_n) \mid \forall 1 \le i \le \frac{n}{2},\ x_{2i-1} = x_{2i}\right\}$$

when n is even and

$$\mathcal{C}_{n+1} = (0|\mathcal{C}_n) \cup (1|\mathcal{C}_n)$$

when n is odd. The lower bound of $\frac{2^n}{n^t}$ when n is a power of 2 is obtained when $\mathcal{C}$ is taken to be a binary BCH code of length $n-1$ that corrects t bit-flips.

2.2.2 Upper bounds on $M(n,t)$ and $R(\tau)$

Mazumdar *et al.* also derived several general upper bounds on $M(n,t)$ using the count of runs of identical symbols in a binary vector and using the clique partitioning of the so-called confusability graph of $\{0,1\}^n$ (defined below). The first method yields [6, Th. 2] an upper bound on $M(n,t)$ for a fixed value of t:

$$M(n,t) \le \frac{2^{n+t}t!}{n^t}(1+o(1)),$$

where the $o(1)$ term goes to 0 as n goes to ∞. A similar idea [6, Prop. 3] might be used to obtain an upper bound on the asymptotic rate $R(\tau)$ of a t-grain-correcting code:

$$R(\tau) \le \mathsf{H}\left(\frac{1-\tilde{x}}{2}\right), \tag{2}$$

where $\tilde{x}$ is the smallest positive solution of the equation

$$\mathsf{H}\left(\frac{1-x}{2}\right) + \frac{1-x}{4}\mathsf{H}\left(\frac{4\tau}{1-x}\right) = 1,$$

and $\mathsf{H}(\cdot)$ is the binary entropy function. The upper bound (2) is only applicable for $\tau \le 0.072$.

The second method employs the smallest number of components $\chi_{n,t} = \chi(G(n,t))$ in any clique partition of the *confusability graph* $G = G(n,t) = (V,E) = (\{0,1\}^n, E)$ defined as follows. Two states (vertices from $\{0,1\}^n$) are connected by an edge from E if and only if they are t-confusable. An obtained upper bound [6, Cor. 5] states that for two integers $\tilde{t}$ and $\tilde{n}$ satisfying $\tau = \frac{t}{n} \le \frac{\tilde{t}}{\tilde{n}}$, one has

$$M(n,t) \le \chi_{\tilde{n},\tilde{t}}^{\lfloor t/\tilde{t} \rfloor} 2^{n-\tilde{n}\lfloor t/\tilde{t} \rfloor}. \tag{3}$$

By plugging $\tilde{t} = 1, \tilde{n} = 2$ into (3), noticing that $\chi_{2,1} = 2$ and combining the result with (1), one can readily obtain that $M(n, \lfloor \frac{n}{2} \rfloor) = 2^{\lceil \frac{n}{2} \rceil}$ for any positive integer n.

Another immediate implication of (3) yields another upper bound [6, Cor. 6] on $R(\tau)$:

$$R(\tau) \le 1 - \tau\left(\frac{\tilde{n}}{\tilde{t}} - \frac{1}{\tilde{t}}\log_2 \chi_{\tilde{n},\tilde{t}}\right),$$

where, again, $\tau = \frac{t}{n}$.

2.2.3 Lower bounds when the grains layout is known

Additional lower bounds were obtained in [6, Sec. 4] on the maximal size $M_d(n,t)$ of a t-grain-correcting code of length n and the asymptotic rate

$R_d(\tau)$ when the grains layout is known to the decoder, and on the respective expressions $M_e(n,t)$ and $R_e(\tau)$ in case when the grains layout is known to the encoder.

A simple argument [6, Prop. 7] shows that

$$M_d(n,t) \geq \frac{2^n}{\sum_{i=0}^{t} \binom{n-i}{i}},$$

hence

$$R_d(\tau) \geq 1 - (1-\tau)\mathsf{H}\left(\frac{\tau}{1-\tau}\right).$$

When the encoder knows the grains layout, a variation of the technique by Basalygo *et al.* [1] can be employed [6, Prop. 8] to yield

$$M_d(n,t) \geq \frac{1}{2n} \frac{2^n}{\sum_{i=0}^{t} \binom{n-i}{i}},$$

implying

$$R_e(\tau) \geq 1 - (1-\tau)\mathsf{H}\left(\frac{\tau}{1-\tau}\right).$$

3 Research goals

In the preliminary stages of our research we examined an application of the technique by Marcus and Roth [4] (which is an improvement on the basic Gilbert-Varshamov bound) to the combinatorial error model suggested in [6] in order to bound $R(\tau)$ from below. It turns out that in the range $\tau \leq 0.05664$ it yields an improvement on the trivial lower bound $R(\tau) \geq 0.5$. The technique relies on the constructive claim that given two binary t-confusable vectors $\mathbf{x_1}, \mathbf{x_2} \in \{0,1\}^n$, we can always find operators $\varphi_1, \varphi_2 \in \Phi_{n,t}$ such that $\varphi_1(\mathbf{x_1}) = \varphi_2(\mathbf{x_2})$ and the number of grains in both φ_1 and φ_2 is minimal.

We also investigated a generalization of the combinatorial model from [6] by allowing 2-grains to *overlap*. Surprisingly, the lower bound we obtained in the case where the grains did not overlap applies also for this generalized model. This result is attained by making reductions from both combinatorial models to the same optimization problem.

We propose the following directions as tentative continuation for our research:

- Trying to derive an explicit construction of a t-grain-correcting code that attains the lower bound on $R(\tau)$ that we have discovered. It would be even more desirable if the code were linear.

- Generalizing the combinatorial model from [6] further by assuming the existence of grains of length greater than 2 and trying to apply the technique of Marcus and Roth in this new model.
- Understanding whether in this generalized model with grains of various lengths we can still circumvent the overlapping grains the way we managed to do it in the model with 1-grains and 2-grains only.
- Finding a practical extension of the model from [6] and the bounding technique from [4] to multi-dimensionsional media.
- There is a certain similarity between the model of grains and the binary asymmetric channel (the Z-channel) [2, Def. 2.1]. The theory of asymmetric binary channels might be a source of inspiration for discovering new constructions and/or bounding techniques.
- Trying to improve on the upper bounds on $R(\tau)$ discovered by Mazumdar *et al.* employing the similarity with the Z-channel.

References

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