Coding Techniques for Multidimensional Constrained Channels

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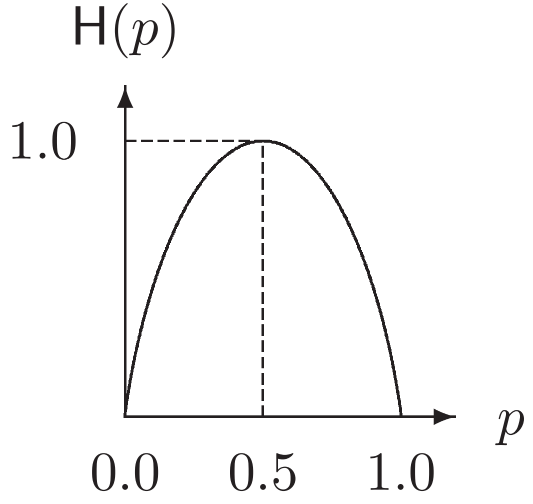
Preliminaries

- Entropy $\mathsf{H}(p) = -p \log_2 p - (1-p) \log_2 (1-p)$



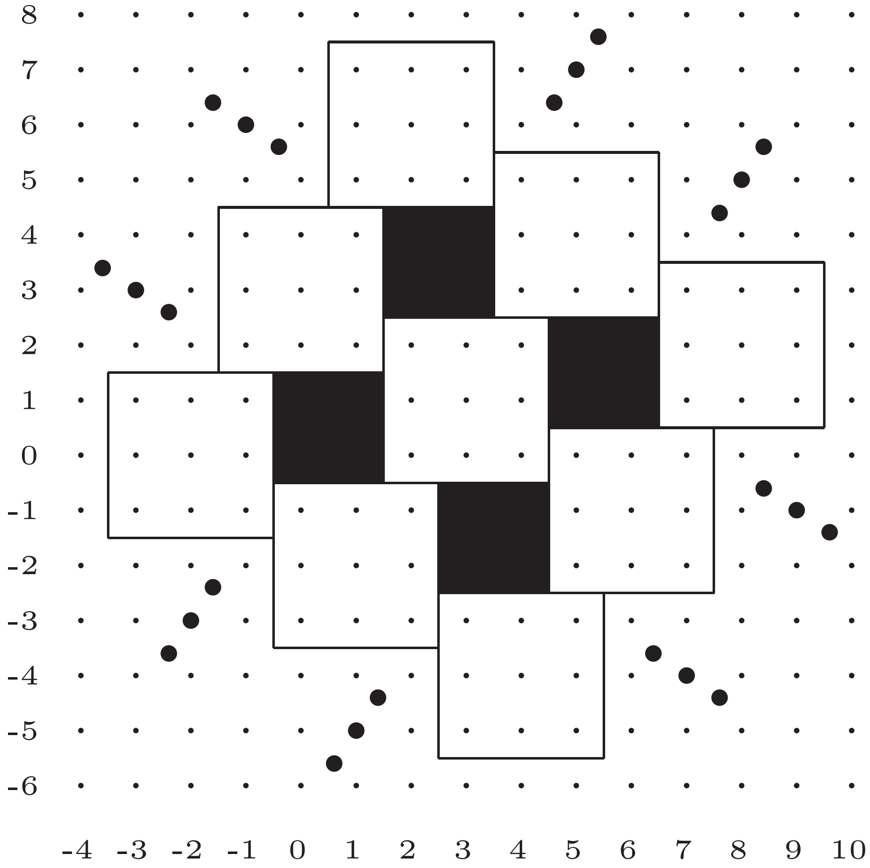
- 2-D constraint — $\mathcal{G} = (V, E_{\mathcal{G}}, L)$ and $\mathcal{H} = (V, E_{\mathcal{H}}, L)$
- Encoder, rate, capacity

Main idea

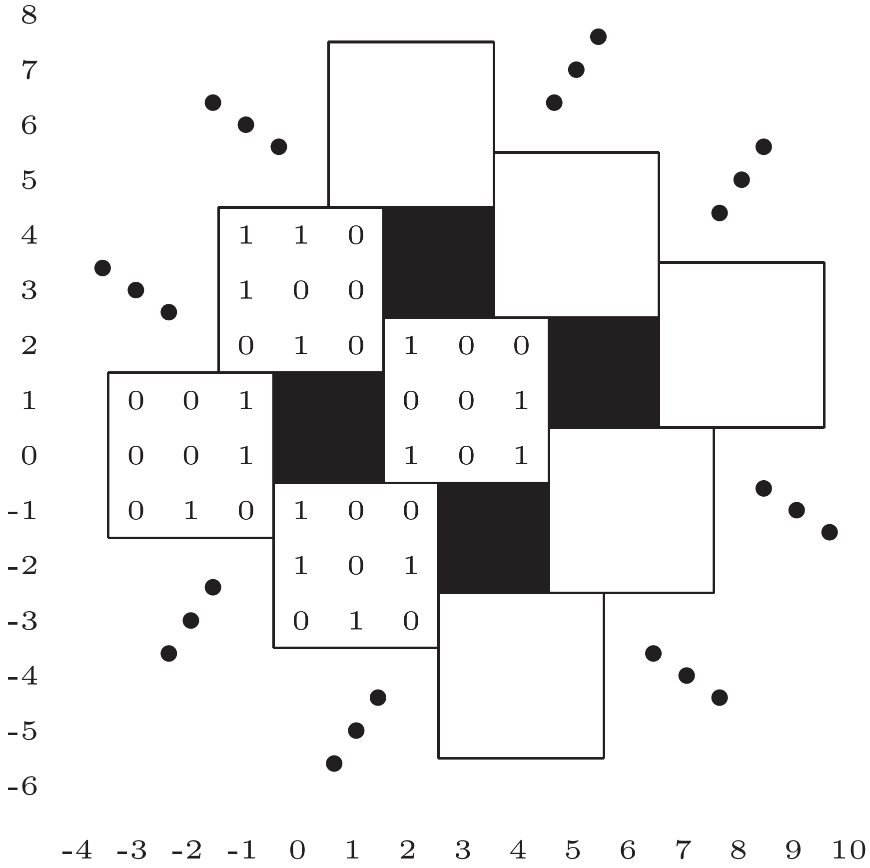
- 2-D “no isolated bits” (n.i.b.) constraint
- Restricted patterns

$$\begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \qquad \begin{array}{ccc} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{array}$$

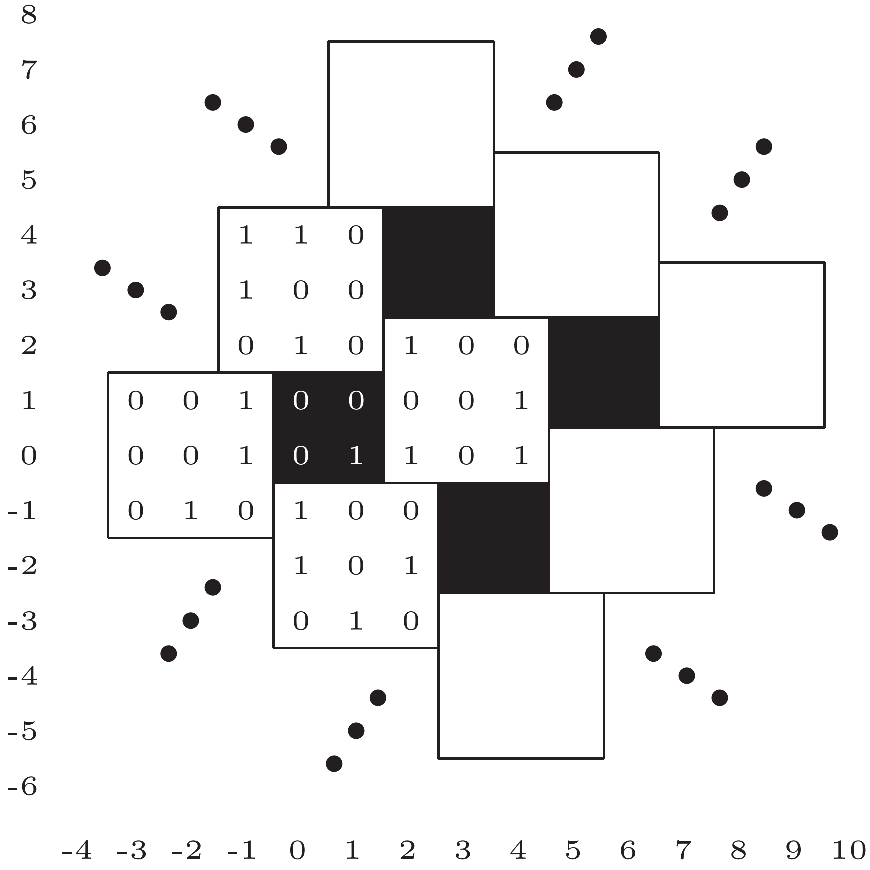
Main idea (cont.)



Main idea (cont.)



Main idea (cont.)



Definitions

- Squares

$$Q_n = \{(i,j) \in \mathbb{Z}^2 : 0 \leq i,j < n\}$$

- Shifted copy of U

$$\sigma_{h,v}(U) = \{(i,j) : (i-h, j-v) \in U\}$$

- Two types of nonempty finite subsets — $B, W \subset \mathbb{Z}^2$

Definitions (cont.)

- Locations of the black tiles are defined by a lattice for some 2×2 integer matrix A

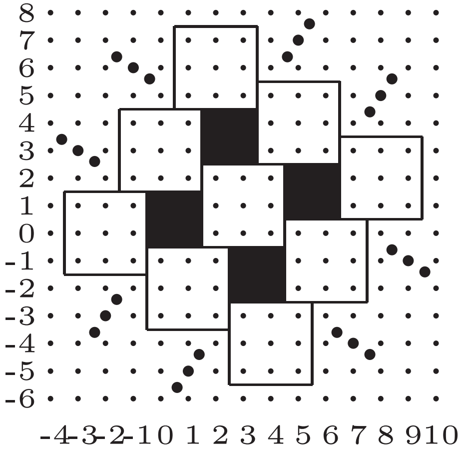
$$\mathcal{L} = \mathcal{L}(A) = \left\{ (i,j) = (t,u)A \;:\; (t,u) \in \mathbb{Z}^2 \right\}$$

- $\sigma_{h,v}(B)$, $(h,v) \in \mathcal{L}$ are all disjoint
- Locations of the white tiles — $\sigma_{h,v}(W),\ (h,v) \in \sigma_{\ell,\ell'}(\mathcal{L})$
- $[s] = \{0, 1, \ldots, s{-}1\}$

Example — n.i.b.

$$B = Q_2, \quad W = Q_3$$

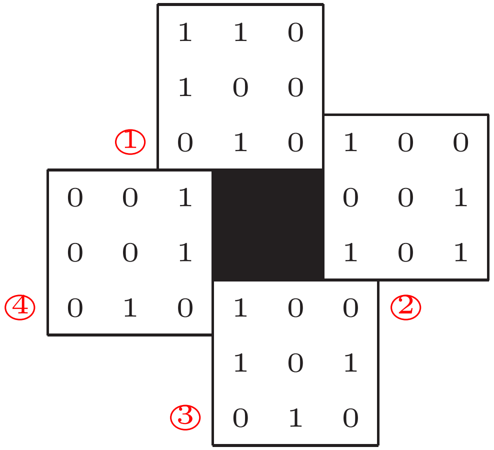
$$\mathcal{L} = \left\{ (i,j) = (t,u)\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} \;:\; (t,u) \in \mathbb{Z}^2 \right\}$$



Definitions (cont.)

- List of copies of W — $\mathcal{N} \oplus W = (\sigma_{r,s}(W))_{(r,s)\in\mathcal{N}}$
- Example of a neighborhood

$$\mathcal{N} = \{\overset{\textcircled{1}}{(-1,2)}, \overset{\textcircled{2}}{(2,0)}, \overset{\textcircled{3}}{(0,-3)}, \overset{\textcircled{4}}{(-3,-1)}\}$$



①
$$\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$

②
$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{matrix}$$

③
$$\begin{matrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$

④
$$\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$

Definitions (cont.)

- Allowed contents of a black tile with respect to a configured neighborhood

$$\mathbb{S}(B;\mathbf{y}), \mathbf{y} = (y_{r,s})_{(r,s)\in\mathcal{N}}$$

- The neighborhood from the previous example

$$\mathbf{y} = \left(\boxed{\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}} \quad \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{matrix}} \quad \boxed{\begin{matrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}} \quad \boxed{\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}} \right)$$

$$\mathbb{S}(B;\mathbf{y}) = \left\{ \boxed{\begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}}, \quad \boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}} \right\}$$

Definitions (cont.)

- Valid tiling $(B, W, \mathcal{L})$

  [CW] *White tiles are freely configurable*

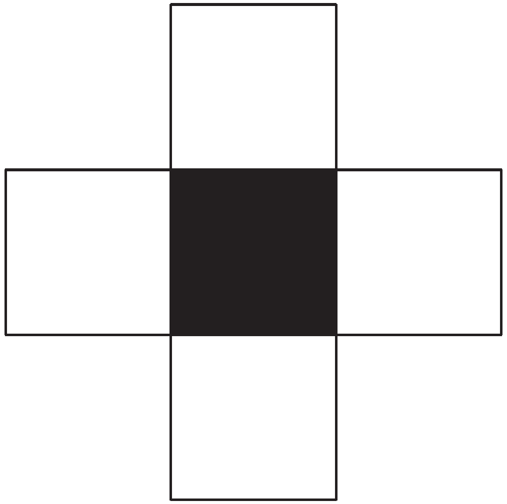
  [CB] *Black configurations are constrained only by a finite neigborhood of white tiles. There is at least one valid black configuration for any white neighborhood,* $\forall y : \mathbb{S}(B; y) \neq \varnothing$

Example — 2-D $(2, \infty)$-RLL

$$B = W = Q_2$$

$$\mathcal{L} = \left\{(2t, 2(t{+}u)) \;:\; (t, u) \in \mathbb{Z}^2\right\}$$

$$\mathcal{N} = \{(0, 2), (2, 0), (0, -2), (-2, 0)\}$$



Procedure for selecting a random array

Given a probability distribution on white configurations $\pi : \mathbb{S}(W) \to [0,1]$

1. For every white tile do

   Select a configuration $\varphi \in \mathbb{S}(W)$ according to π

2. For every black tile do
   - (a) Let $\mathbf{y} = (y_{r,s})_{(r,s)\in\mathcal{N}} \in \mathbb{S}(\mathcal{N} \oplus W)$ with entries as in Step 1
   - (b) Select a configuration $\psi \in \mathbb{S}(B; \mathbf{y})$ uniformly

Entropy

- The described procedure induces a probability measure μ_n on a $n \times n$ array
- Entropy of the measure

$$\mathsf{H}(\mu_n) = \frac{1 - o(1)}{|W|+|B|}\Big(H(\pi) + \sum_{\varphi} \mathsf{P}_\pi\{\varphi\} \log_2 |\mathbb{S}(B;\varphi)|\Big)$$

Main theorem

Let $\mathbb{S}$ be a 2-D constraint for which there is a valid tiling $(B, W, \mathcal{L})$. Then $\mathsf{cap}(\mathbb{S})$ is bounded from below by

$$\mathsf{cap}(\mathbb{S}) \geq \max_{\pi} \frac{1}{|W|+|B|} \left(H(\pi) + \sum_{\varphi} \mathsf{P}_{\pi}\{\varphi\} \log_2 |\mathbb{S}(B;\varphi)| \right)$$

Variable-rate encoder

- Configuring the white tiles — through a distribution transformer, mapping a sequence of fair coins into $\mathbb{S}(W)$, approaching π
- Configuring the black tiles — by enumerative coding

Lower bounds for 2-D constraints

Constraint	Theorem	Previous record	Reference	n
$(2, \infty)$-RLL	0.444172	0.4423	[OR]	5
$(3, \infty)$-RLL	**0.365623**	0.3641	[FL]	5
$(0, 2)$-RLL	**0.816007**	0.7736	[AM]	4
n.i.b.	0.920862	0.9156	[FL]	3

Generalizations — multicolor tiling

- Valid tiling revised $(B, G, W, \mathcal{L})$

[CW] White tiles are freely configurable

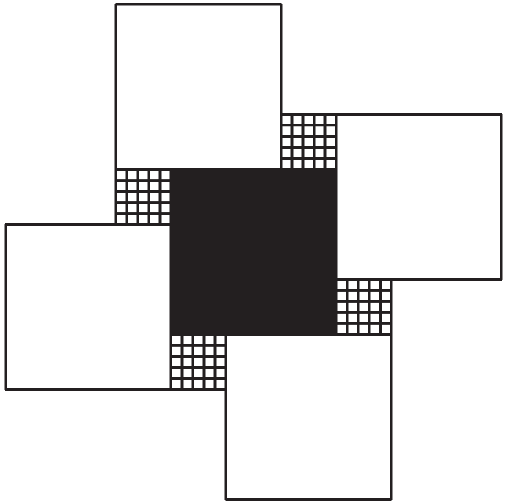
[CG] Gray configurations are constrained only by a finite neighborhood of white tiles. There is at least one valid gray configuration for any white neighborhood

[CB] Black configurations are constrained only by a finite neigborhood of white and gray tiles. There is at least one valid black configuration for any white and gray neighborhood

Example — n.a.k.

$$B = W = Q_3 \ , \qquad G = Q_1 \cup \sigma_{-1,3}(Q_1)$$

$$\mathcal{L} = \left\{ (i,j) = (t,u)\begin{pmatrix} 2 & 4 \\ 4 & -2 \end{pmatrix} \ : \ (t,u) \in \mathbb{Z}^2 \right\}$$



Lower bounds for n.a.k. constraint

n	Lower bound	No. of variables
3	0.423076	10
4	0.423955	33
5	0.424350	217

$$\mathsf{cap}(\mathbb{S}_{\mathrm{n.a.k.}}) \geq 0.425029$$

Generalizations — dependence among white tiles

- Example 2-D $(1,\infty)$-RLL

n	Independent	Markov	Pickard field	BMRF field
1	0.566144	0.574094	0.584387	0.584418
2	0.582075	0.584798	**0.587855**	
3	0.585350	0.586485		
4	0.586459			
5	0.586974			
4	$\mathbb{S}_{\text{n.a.k.}}$	0.424558		

Generalizations — 3-D

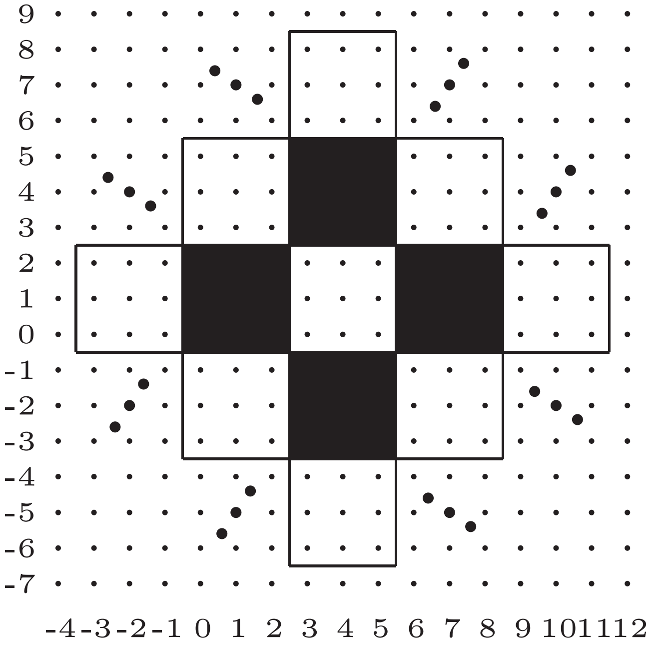
- For λ-D $(1,\infty)$-RLL

$$\mathcal{L} = \left\{ (t_1, t_2, \ldots, t_{\lambda-1}, 2t_\lambda - \sum_{i=1}^{\lambda-1} t_i) : \forall 1 \le i \le \lambda : t_i \in \mathbb{Z} \right\}$$

- Rate $\quad \max_{p\in[0,1]} \frac{1}{2}\left(H(p) + (1-p)^{2\lambda}\right)$
- For $\lambda = 3$ equals **0.513864**
- Improves to at least **0.521270** when a 3-D Pickard field is assumed
- Still short of the best known lower bound of **0.5225** on the capacity by Nagy & Zeger

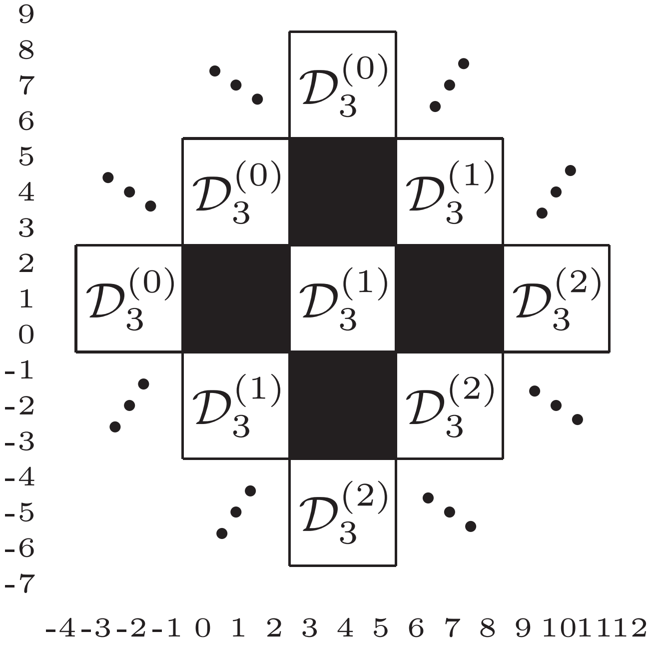
Fixed-rate encoders

- Motivation
- Running example: 2-D $(1, \infty)$-RLL



Valid partitioned tiling

- $\mathcal{D}=\{(mz-m, mz) : z \in \mathbb{Z}\},\ \mathcal{D}^{(b)}=\sigma_{mb,-mb}(\mathcal{D})$ for $b \in \mathbb{Z}$
- $\mathcal{D}_N=\{(mz-m, mz) : z \in [N]\},\ \mathcal{D}^{(b)}=\sigma_{mb,-mb}(\mathcal{D}_N)$ for $b \in \mathbb{Z}$
- $(B, W, \mathcal{L}, \mathcal{D})$



Markov chain

- Conditional probability distribution $\rho : S(W) \times S(W) \to [0,1]$
- Stationary probability $\pi : S(W) \to [0,1]$
- Per-tile entropy of ρ $\displaystyle\sum_{i \in \mathbb{S}(W)} -\pi(i) \sum_{j \in \mathbb{S}(W)} \rho(j \mid i) \log_2 \rho(j \mid i)$
- Probability of an assignment
  $\varphi = (\varphi_{(-m,0)}, \varphi_{(m,0)}, \varphi_{(0,-m)}, \varphi_{(0,m)}) \in \mathbb{S}(\mathcal{N} \oplus W)$

$$\mathsf{P}_\rho\{\varphi\} = \mathsf{P}_\rho\{\varphi_{-m,0},\ \varphi_{0,m}\}\mathsf{P}_\rho\{\varphi_{0,-m},\ \varphi_{m,0}\}$$
$$= \pi(\varphi_{-m,0} \circ \sigma_{-m,0})\rho(\varphi_{0,m} \circ \sigma_{0,m} \mid \varphi_{-m,0} \circ \sigma_{-m,0})$$
$$\cdot\ \pi(\varphi_{0,-m} \circ \sigma_{0,-m})\rho(\varphi_{m,0} \circ \sigma_{m,0} \mid \varphi_{0,-m} \circ \sigma_{0,-m})$$

Main theorem revised

Let $\mathbb{S}$ be a 2-D constraint for which $(B, W, \mathcal{L}, \mathcal{D})$ is a valid partitioned tiling. Then $\mathsf{cap}(\mathbb{S})$ is bounded from below by

$$\mathsf{cap}(\mathbb{S}) \geq \max_{\rho} \frac{1}{|W|+|B|}\Big(\mathsf{H}(\rho) + \sum_{\varphi} \mathsf{P}_{\rho}\{\varphi\} \log_2 |\mathbb{S}(B;\varphi)|\Big),$$

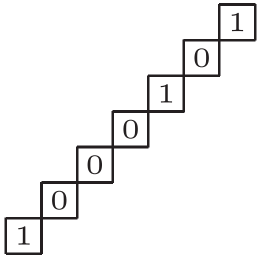
where the maximization is taken over all distributions
$\rho : \mathbb{S}(W) \times \mathbb{S}(W) \to [0,1]$.

Count matrix

- $\mathbf{d} = (d_{r,s})_{(r,s)\in\mathcal{D}_N} \in \mathbb{S}(\mathcal{D}_N \oplus W)$
- Reindexing $\mathbf{d}$ as $(d_z)_{z\in[N]}$ with $d_z = d_{m(z+b-1),m(z-b)}$
- Setting $n = |\mathbb{S}(W)|$ and renaming $\mathbb{S}(W)$ as $[n]$
- $f_{i,j} = |\{z \in [N] : (d_z, d_{z+1}) = (i,j)\}|$
- $f_{i\cdot} = \sum_{j\in[n]} f_{i,j},\ f_{\cdot j} = \sum_{i\in[n]} f_{i,j}$
- $\sum_{i,j\in[n]} f_{i,j} = \sum_{i\in[n]} f_{i\cdot} = \sum_{j\in[n]} f_{\cdot j} = N$
- $f_{i\cdot} = f_{\cdot i}$ for all $i \in [n]$
- *Count matrix* $\mathcal{F}(\mathbf{d}) = (f_{i,j})_{i,j\in[n]}$

Example – 2-D $(1,\infty)$-RLL

$$\mathcal{F}(\mathbf{d}) = \begin{pmatrix} f_{0,0} & f_{0,1} \\ f_{1,0} & f_{1,1} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$



Finding a good count matrix

- "On row-by-row coding for 2-D constraints" by Tal and Roth
- Given $P = (p_{i,j})_{i,j\in[n]} = (\pi(i)\rho(j|i))_{i,j\in[n]}$ and N', returns an integer matrix $F \geq 0$, s.t.

  (P1) $N' - \lfloor \frac{n}{2} \rfloor \leq \sum_{i,j\in[n]} f_{i,j} \leq N'$

  (P2) $f_{i\cdot} = f_{\cdot i}$ for any $i \in [n]$

  (P3) $(N' - \lfloor \frac{n}{2} \rfloor) p_{i,j} - f_{i,j} \leq 1$ for any $i, j \in [n]$

  (P4) $\mathsf{H}(\rho) = \frac{1}{N'} \left\lfloor \log_2 \frac{\prod_{i\in[n]} f_{i\cdot}!}{\prod_{i,j\in[n]} f_{i,j}!} \right\rfloor - O\left(\frac{n^2 \log N'}{N'}\right)$
- Set $N = \sum_{i,j\in[n]} f_{i,j}$
- F is a count matrix of assignments to $\mathcal{D}_N \oplus W$

Counting Eulerian paths

- $A^* = (a^*_{i,j})_{i,j\in[n]},\ a^*_{i,j} = \begin{cases} \delta_{i,j} - \frac{a_{i,j}}{a_{i\cdot}} & \text{if } a_{i\cdot} > 0 \\ \delta_{i,j} & \text{if } a_{i\cdot} = 0 \end{cases}$
- Multinomial $\mathsf{M}(A) = \frac{\prod_{i\in[n]} a_{i\cdot}!}{\prod_{i,j\in[n]} a_{i,j}!}$
- The (v,u)-th cofactor of A^* — $\mathsf{C}_{v,u}(A)$, $\quad \mathsf{C}_u(A) \triangleq \mathsf{C}_{o,u}(A)$
- Whittle's formula

$$\mathsf{T}^{(w)}_{u,v}(F) = \mathsf{M}(F)\mathsf{C}_{v,u}(F),\ \mathsf{T}^{(w)}_{u}(F) \triangleq \mathsf{T}^{(w)}_{u,o}(F)$$

- Counts assignments to $\mathcal{D}_w \oplus W$ with $d_0 = u, d_w = v, \mathcal{F}(\mathbf{d}) = F$

Enumerative coding of $\mathbf{d} \in \mathbb{S}(\mathcal{D}_N \oplus W)$

```
1. $A = (a_{i,j}) \leftarrow F$; $d_0 \leftarrow o$;
2. for $z \in [N-1]$ {
   (a) $t \leftarrow d_z$;
   (b) for $u \in \mathcal{J}_t(A)$ {
         i. $\Gamma \leftarrow \partial_{t,u}(A)$;
        ii. if $\varsigma < \mathsf{T}_u^{(N-z-1)}(\Gamma)$ { $A \leftarrow \Gamma$; $d_{z+1} \leftarrow u$; break; }
       iii. else { $\varsigma \leftarrow \varsigma - \mathsf{T}_u^{(N-z-1)}(\Gamma)$; }
       }
   }
```

Contribution of white tiles

- F is rational and $\mathsf{C}_o(F) > 0$, therefore $\mathsf{C}_o(F) \geq \frac{1}{\prod_{i\in[n]} f_{i\cdot}} \geq \frac{1}{N^n}$
- $R(F) = \frac{\lfloor \log_2 \mathsf{T}_o^{(N)}(F) \rfloor}{N}$
- $\mathsf{H}(\rho) = R(F) + O(\frac{n^2 \log N}{N}) + O(\frac{n \log N}{N})$
- $\mathsf{H}(\rho) = \lim_{N\to\infty} R(F)$

Black diagonals

- Two neighboring white diagonals $\mathcal{D}_N^{(b)}$ and $\mathcal{D}_N^{(b+1)}$
- $\mathbf{d} \in \mathbb{S}(\mathcal{D}_N^{(b)} \oplus W), \mathbf{d}' \in \mathbb{S}(\mathcal{D}_N^{(b+1)} \oplus W)$
- $f_{i,j,k,\ell}(\mathbf{d}, \mathbf{d}') = |\{z : (d_z, d_{z+1}, d'_z, d'_{z+1}) = (i, j, k, \ell)\}|$
- $\mathsf{R}(\mathbf{d}, \mathbf{d}') = \sum_{i,j,k,\ell \in [n]} \frac{f_{i,j,k,\ell}(\mathbf{d},\mathbf{d}')}{N} \log_2 |\mathbb{S}(B; (i, j, k, \ell))|$
- Cyclic right shift $\theta_r(\mathbf{d}') = (e_{-r}, e_{-r+1}, \ldots, e_{-r+N-1})$

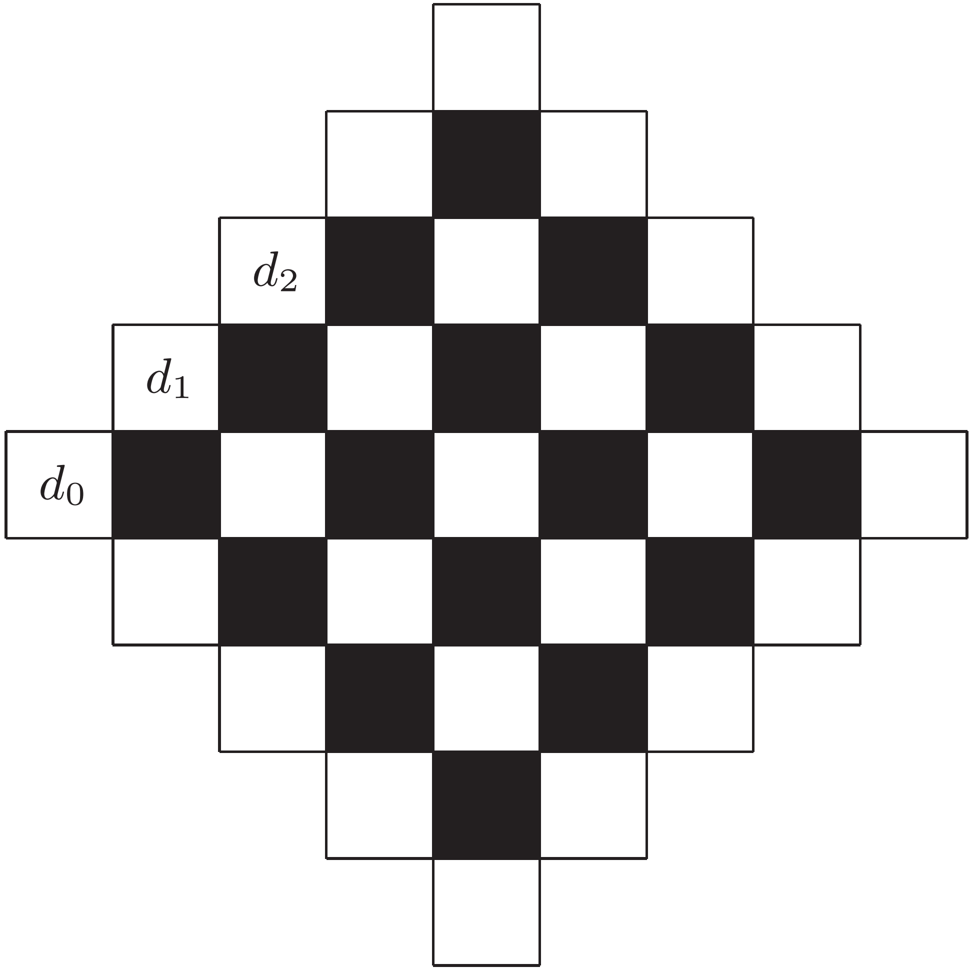
Proposition. *There exists at least one* $g \in [N]$ *such that*

$$\mathsf{R}(\mathbf{d}, \theta_g(\mathbf{d}')) \geq \frac{1}{N^2} \sum_{i,j,k,\ell \in [n]} f_{i,j} f_{k,\ell} \log_2 |\mathbb{S}(B; (i, j, k, \ell))|.$$

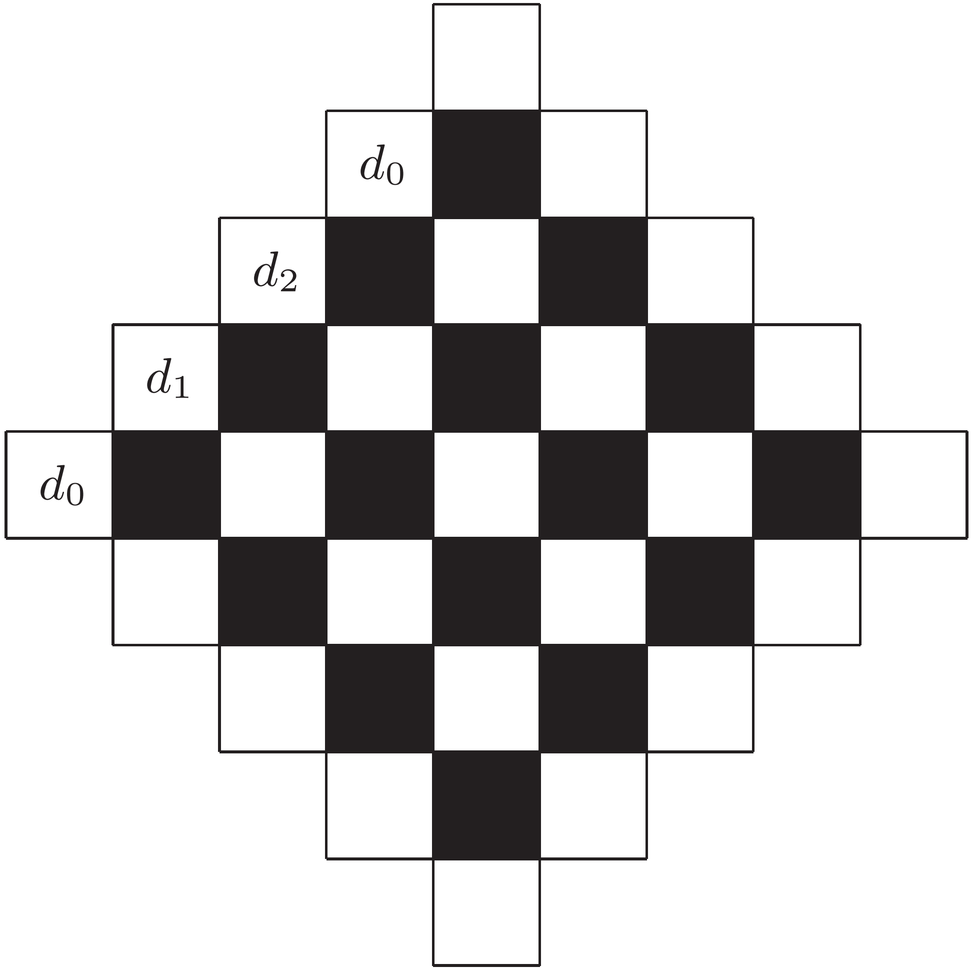
Black diagonals (cont.)

- Due to (P3), $\sum_{\varphi} \mathsf{P}_{\rho}\{\varphi\} \log_2 |\mathbb{S}(B;\varphi)| = \frac{1}{N^2} \sum_{i,j,k,\ell \in [n]} f_{i,j} f_{k,\ell} \log_2 |\mathbb{S}(B;(i,j,k,\ell))| - O(\frac{n^2}{N})$

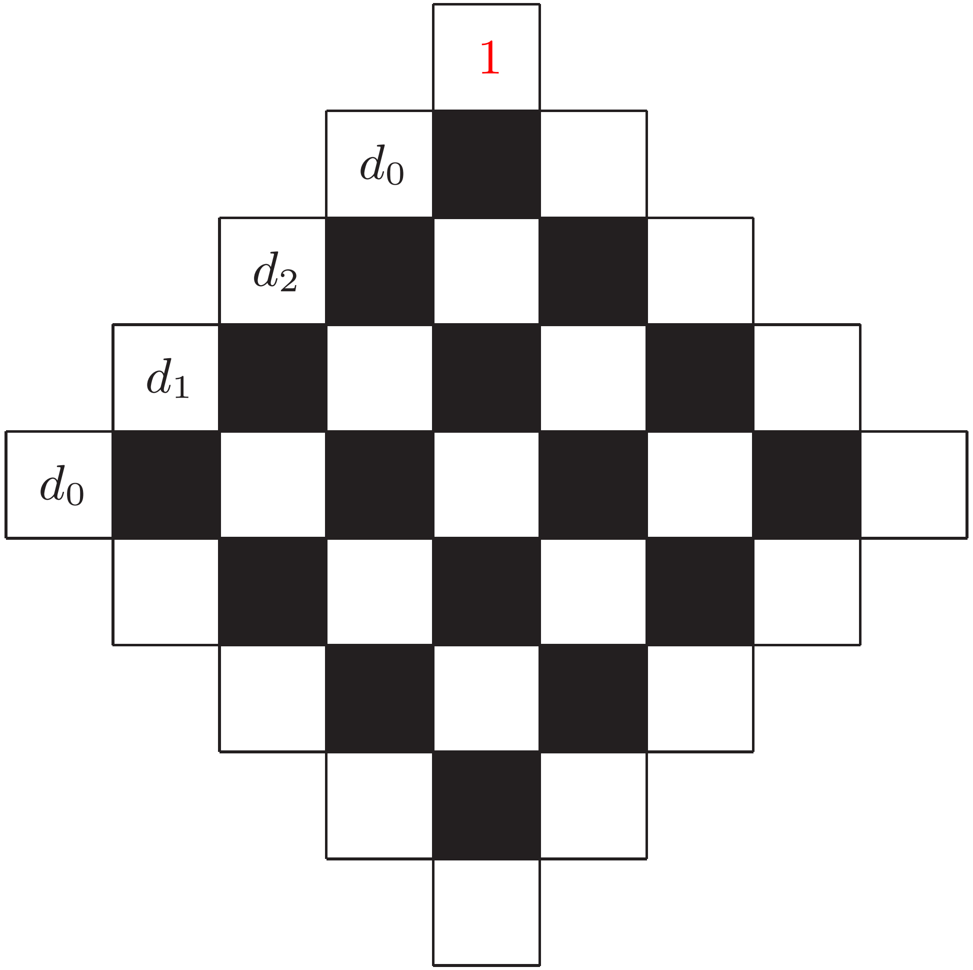
Fixed-rate encoder



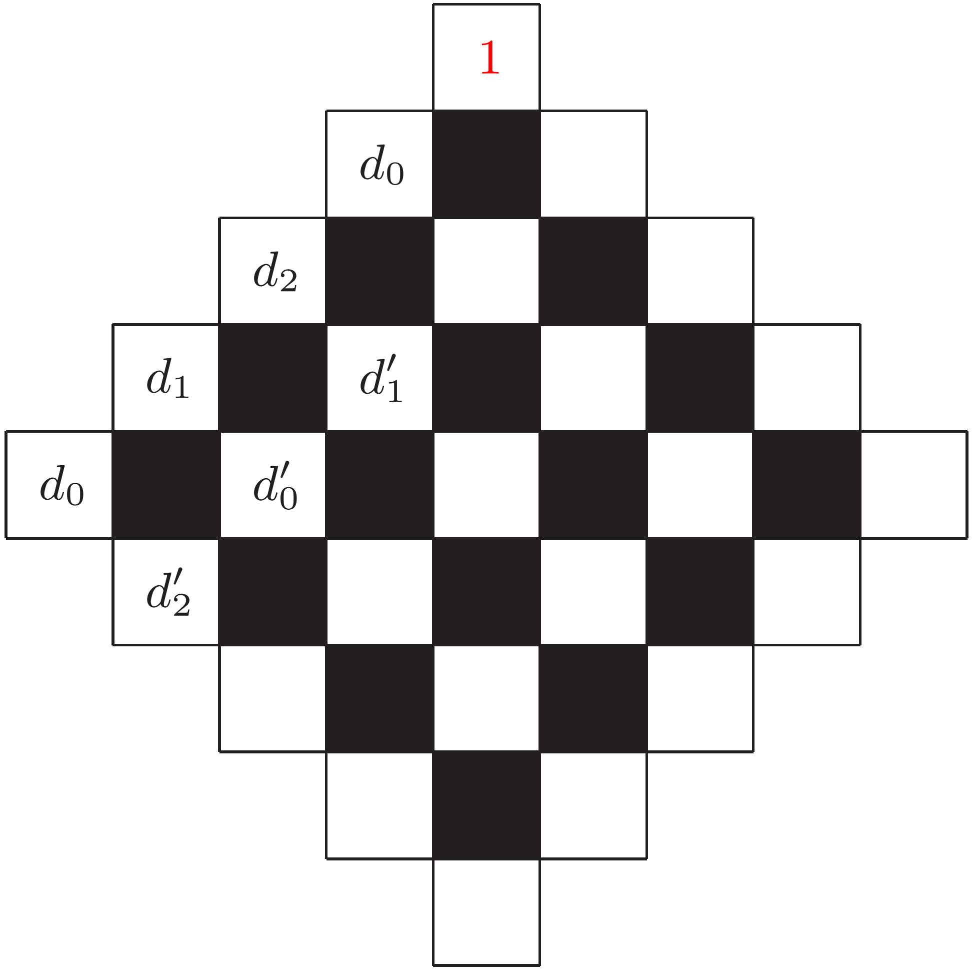
Fixed-rate encoder (cont.)



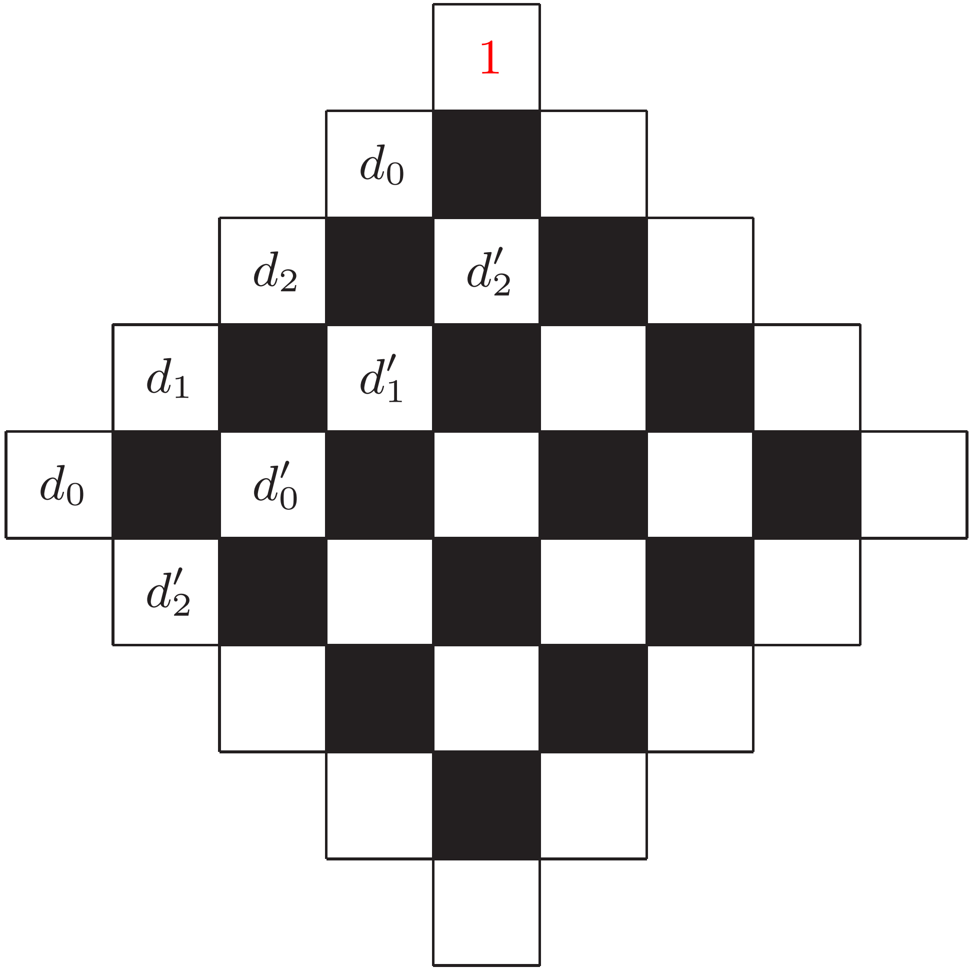
Fixed-rate encoder (cont.)



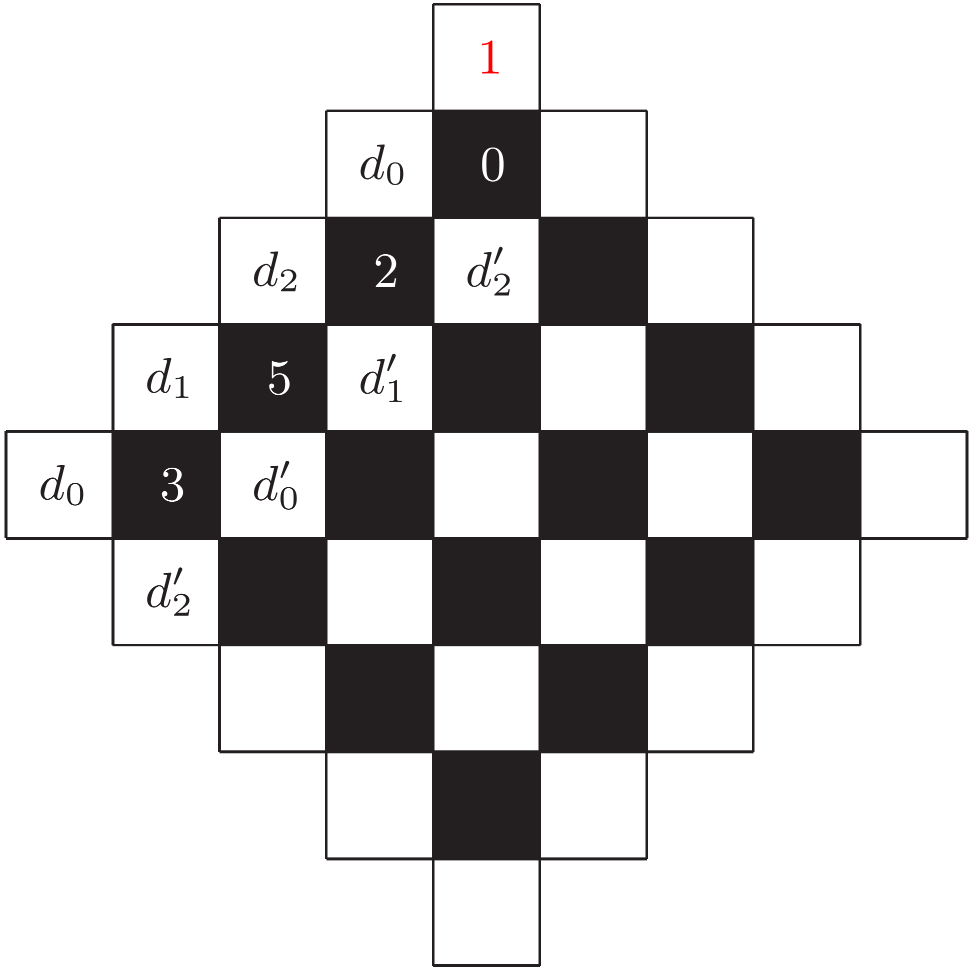
Fixed-rate encoder (cont.)



Fixed-rate encoder (cont.)



Fixed-rate encoder (cont.)



Main theorem for fixed-rate encoders

Theorem. *Let* $\mathbb{S}$ *be a 2-D constraint for which* $(B, W, \mathcal{L}, \mathcal{D})$ *is a valid partitioned tiling. Then the coding rate of the presented fixed-rate encoder is*

$$\frac{1}{|W|+|B|}\Bigg(R(F)+\frac{1}{N^2}\sum_{i,j,k,\ell\in[n]} f_{i,j}f_{k,\ell}\log_2|\mathbb{S}(B;(i,j,k,\ell))|\Bigg) - O\Big(\frac{n^2\log N}{N\log n}\Big),$$

which, for $N \to \infty$*, is at least*

$$\max_\rho \frac{1}{|W|+|B|}\Big(\mathsf{H}(\rho)+\textstyle\sum_\varphi \mathsf{P}_\rho\{\varphi\}\log_2|\mathbb{S}(B;\varphi)|\Big).$$

Lower bounds on the rates of 2-D fixed-rate encoders

Constraint	Lower bound	Reference
$(1,\infty)$-RLL	**0.586485**	[RSW]
$(2,\infty)$-RLL	0.444172	[OR]
$(3,\infty)$-RLL	**0.365623**	
$(0,2)$-RLL	**0.816007**	
n.i.b.	**0.920862**	

Problem

- N might be a big number
- Enumerative coding of white diagonals
  - At least $O(N \log N)$ per tile encoding naively
- Shifting white diagonals
  - At least $O(N)$ operations per tile encoding naively

Efficient enumerative coding

- Approximate enumerative coding, “A practical method for approaching the channel capacity of constrained channels” by Immink
- α is approximated with $\overline{\alpha} = c \cdot 2^h$
- $2^\mu \leq c < 2^{\mu+1}, -2^{\epsilon-1} \leq h < 2^{\epsilon-1}$
- $2^\mu 2^{-2^{\epsilon-1}} \leq \alpha \leq (2^{\mu+1} - 1)2^{2^{\epsilon-1}-1}$
- Relative precision $\left|\frac{\overline{\alpha}}{\alpha} - 1\right| \leq \frac{1}{2^\mu}$
- $\epsilon = (1 + o(1)) \log_2 N, \; \mu = \log_2 (N^2 + N\eta(n))$
- $\eta(n) = 2(n^2 + n + 1)$

Lookup tables

- For $1!, 2!, \ldots, (\max_i f_{i\cdot})!$
- For $\kappa(1), \kappa(2), \ldots, \kappa(N)$
- $\kappa(1) = (1 - \frac{1}{2^\mu+1})^{\eta(n)+2(N-1)}$
- For $1 \leq w \leq N{-}1$,

$$\frac{\kappa(w+1)}{\kappa(w)} = (1 - \frac{2}{2^\mu + 1})^{\eta(n)+2(N-w)}(1 + \frac{1}{2^\mu})^2.$$

- Storage complexity $O(N(\mu{+}\epsilon))$

Approximation of $\mathsf{T}_u^{(N-z-1)}(\Gamma)$

- Compute $\widetilde{\mathsf{T}}_u^{(N-z-1)}(\Gamma)$ instead of $\mathsf{T}_u^{(N-z-1)}(\Gamma)$
- $\widetilde{\mathsf{M}}(\Gamma) = \left(\prod_{i\in[n]} \overline{\gamma_{i\cdot}!}\right) \div \left(\prod_{i,j\in[n]} \overline{\gamma_{i,j}!}\right)$
- $\widetilde{\mathsf{T}}_u^{(N-z-1)}(\Gamma) = \left\lceil \overline{\kappa}(N-z-1) \times \widetilde{\mathsf{M}}(\Gamma) \times \mathsf{C}_u(\Gamma) \right\rceil$
- Computation of $\widetilde{\mathsf{M}}(F)$ takes $O(n^2)$ operations
- Computation of $\widetilde{\mathsf{M}}(\Gamma)$ takes $O(1)$ operations
- Input index ς is taken from $[\widetilde{\mathsf{T}}_o^{(N)}(F)]$ rather than $[\mathsf{T}_o^{(N)}(F)]$

Validity of approximation

- $\partial_{t,u}(A)$ — A with the entry $a_{t,u}$ decremented by 1
- $\sum_{u\in\mathcal{J}_t(A)} \mathsf{T}_u^{(N-z-1)}(\partial_{t,u}(A)) = \mathsf{T}_t^{(N-z)}(A)$

Lemma. *For* $z \in [N-1]$,

$$\sum_{u\in\mathcal{J}_t(A)} \widetilde{\mathsf{T}}_u^{(N-z-1)}(\partial_{t,u}(A)) \geq \widetilde{\mathsf{T}}_t^{(N-z)}(A),$$

and for $z \in [N]$,

$$\widetilde{\mathsf{T}}_t^{(N-z)}(A) \leq \mathsf{T}_t^{(N-z)}(A).$$

Price of the approximation

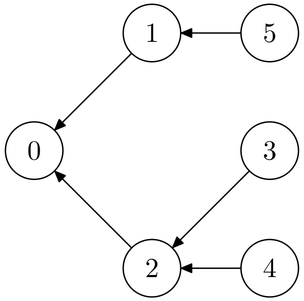
- Coding rate penalty $\frac{1}{N\log n}\left(\lfloor \log_2 \mathsf{T}_o^{(N)}(F)\rfloor - \lfloor \log_2 \widetilde{\mathsf{T}}_o^{(N)}(F)\rfloor\right) = O\left(-\frac{\log \kappa(N)}{N\log n}\right) = O\left(\frac{1}{N\log n}\right)$
- Number of floating-point operations for multinomial computations per tile encoding $O(n)$

Cofactors

- n cofactors $\mathsf{C}_u(\Gamma)$
- Naively, $O(n^4)$ per tile encoding
- Can be improved to $O(n^{3.2})$ using "On the complexity of computing determinants" by Kaltofen and Villard
- Can be further improved to $o(1)$ (expected), if the following lower bound is used

Lower bound on $\mathsf{T}_u^{(N-z-1)}(\Gamma)$

- Oriented tree $\mathcal{T}$



- $\widehat{\mathsf{T}}_u^{(N-z-1)}(\Gamma) = \mathsf{M}(\Gamma) \cdot \widehat{\mathsf{C}}(\Gamma)$, $\widehat{\mathsf{C}}(\Gamma) = \prod_{i \in \mathbb{S}(W) \setminus \{o\},\ \gamma_{i,*} > 0} \frac{\gamma_{i,\mathrm{P}_{\mathcal{T}}(i)}}{\gamma_{i,*}}$
- $\mathcal{T} = \arg\max_{\mathcal{T}'} \left(\mathsf{M}(F) \cdot \prod_{i \in \mathbb{S}(W) \setminus \{o\}} \frac{f_{i,\mathrm{P}_{\mathcal{T}'}(i)}}{f_{i,*}} \right)$

Validity of approximation with $\widehat{\mathsf{T}}_u^{(N-z-1)}(\Gamma)$

Lemma. *For* $z \in [N-1]$,

$$\sum_{u \in \mathcal{J}_t(A)} \widehat{\mathsf{T}}_u^{(N-z-1)}(\partial_{t,u}(A)) \geq \widehat{\mathsf{T}}_t^{(N-z)}(A),$$

and for $z \in [N]$,

$$\widehat{\mathsf{T}}_t^{(N-z)}(A) \leq \mathsf{T}_t^{(N-z)}(A).$$

Integration with the floating-point approximation

- $\overline{\mathsf{M}}(\Gamma) = \left(\overline{\gamma_{o,*}!} \times \prod_{\substack{i\in\mathbb{S}(W)\setminus\{o\}\\ \gamma_{i,*}>0}} \overline{(\gamma_{i,*}-1)!}\right) \div \left(\prod_{\substack{i,j\in\mathbb{S}(W)\\ j\neq\mathtt{P}_{\mathcal{T}}(i)}} \overline{\gamma_{i,j}!} \times \prod_{\substack{i\in\mathbb{S}(W)\setminus\{o\}\\ \gamma_{i,*}>0}} \overline{(\gamma_{i,\mathtt{P}_{\mathcal{T}}(i)}-1)!}\right)$
- $\overline{\mathsf{T}}_u^{(N-z-1)}(\Gamma) = \left\lceil \overline{\kappa}(N-z-1) \times \overline{\mathsf{M}}(\Gamma) \right\rceil$

Lemma. *For* $z \in [N-1]$,

$$\sum_{u\in\mathcal{J}_t(A)} \overline{\mathsf{T}}_u^{(N-z-1)}(\partial_{t,u}(A)) \geq \overline{\mathsf{T}}_t^{(N-z)}(A),$$

and for $z \in [N]$, $\overline{\mathsf{T}}_t^{(N-z)}(A) \leq \mathsf{T}_t^{(N-z)}(A)$.

Approximate enumerative coding

```
1. $A = (a_{i,j}) \leftarrow F$; $d_0 \leftarrow o$;
2. for $z \in [N-1]$
   (a) $t \leftarrow d_z$;
   (b) for $u$ in $\mathcal{J}_t(A)$ do    /* neighbors of $t$ */
         i. if $u = \mathsf{P}_{\mathcal{T}}(t)$ and $a_{t,u} = 1$ and $a_{t,*} > 1$ { continue; }
        ii. $\Gamma \leftarrow \partial_{t,u}(A)$;
       iii. if $\varsigma < \overline{\mathsf{T}}_u^{(N-z-1)}(\Gamma)$ { $A \leftarrow \Gamma$; $d_{z+1} \leftarrow u$; break; }
        iv. else $\varsigma \leftarrow \varsigma - \overline{\mathsf{T}}_u^{(N-z-1)}(\Gamma)$;
```

Price of the approximation

- Coding rate penalty

  $$\frac{1}{N\log_2 n}(\lfloor\log_2 \mathsf{T}_o^{(N)}(F)\rfloor - \lfloor\log_2 \overline{\overline{\mathsf{T}}}_o^{(N)}(F)\rfloor) = O(-\frac{\log\kappa(N)}{N\log n})+$$

  $$O(\frac{1}{N\log n}\log\frac{1}{\mathsf{C}_o(F)}) = O(\frac{1}{N\log n})+O(\frac{n\log N}{N\log n}) = O(\frac{n\log N}{N\log n})$$

Shifting white diagonals

- Deterministic DFT-based method — $O(n^4 N \log N)$ time complexity
- Probabilistic method — $O(N)$ expected time complexity
- Both take $\mathbf{d} = (d_z)_{z \in [N]}$ and $\mathbf{d}' = (d'_z)_{z \in [N]}$ as arguments
- Both return a good shift of $\mathbf{d}'$ with respect to $\mathbf{d}$
- The probabilistic method assumes a relaxation of a good shift

Cyclic convolution

- Given $\mathbf{a} = (a_r)_{r\in[N]}$ and $\mathbf{b} = (b_r)_{r\in[N]}$
- $\mathbf{c} = (c_z)_{z\in[N]} = \mathbf{a} * \mathbf{b}$
- $c_z = \sum_{r\in[N]} a_r b_{(z-r) \bmod N}$
- In fact, $c_z = |\{r : r \in [N],\ a_r = h_{r-z} = 1\}|$ where $h_r = b_{-r}$ for all $r \in [N]$
- Takes $O(N \log N)$ operations with known implementations of DFT

Back to the context

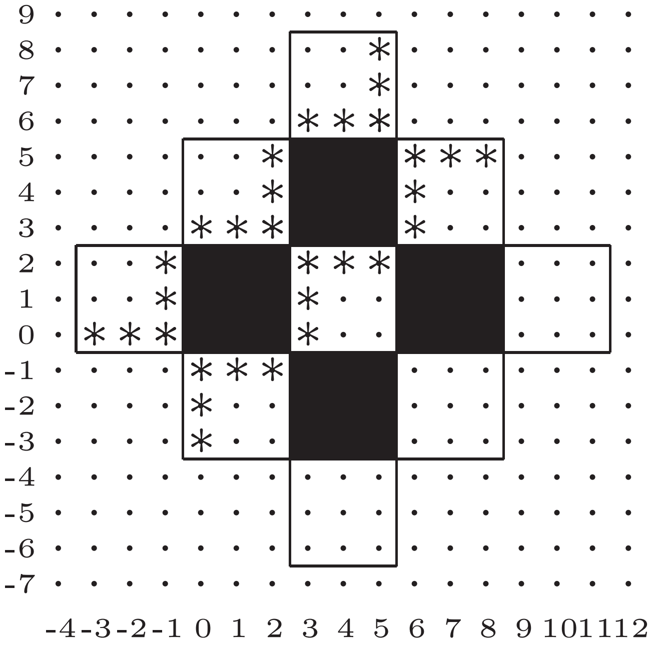
- $\mathbf{e} = \mathbf{e}_{i,j} = (e_z)_{z \in [N]}$
- $\mathbf{e}' = \mathbf{e}'_{k,\ell} = (e'_z)_{z \in [N]}$
- $e_z = \delta(d_z, i)\delta(d_{z+1}, j)$ for all $z \in [N]$
- $e'_z = \delta(d'_{-z}, k)\delta(d'_{-z+1}, \ell)$ for all $z \in [N]$
- $\mathcal{X}_{i,j,k,\ell}{=}(\chi_z)_{z \in [N]}{=}\mathbf{e} * \mathbf{e}'$, $\chi_z = f_{i,j,k,\ell}(\mathbf{d}, \theta_z(\mathbf{d}'))$ for all $z \in [N]$
- The (i, j, k, ℓ)-th row of X is $\mathcal{X}_{i,j,k,\ell}$
- $\mathbf{v} = (\log_2 |\mathbb{S}(B; (i, j, k, \ell))|)_{i,j,k,\ell \in [n]}$
- $\mathbf{y} = \mathbf{v}X = (N\mathsf{R}(\mathbf{d}, \theta_r(\mathbf{d}')))_{r \in [N]}$

Deterministic algorithm

```
1. $\mathbf{Y} = (Y_z)_{z\in[N]} \leftarrow \mathbf{0}$;
2. for $(i, j, k, \ell) \in (\mathbb{S}(W))^4$ { /* $O(n^4 N \log N)$ */
    (a) $\mathbf{e} \leftarrow (\delta(d_z, i))_{z\in[N]} \odot (\delta(d_{z+1}, j))_{z\in[N]}$;
    (b) $\mathbf{e}' \leftarrow (\delta(d'_{-z}, k))_{z\in[N]} \odot (\delta(d'_{1-z}, \ell))_{z\in[N]}$;
    (c) $\mathbf{X} \leftarrow \mathsf{DFT}(\mathbf{e}) \odot \mathsf{DFT}(\mathbf{e}')$;
    (d) $\mathbf{Y} \leftarrow \mathbf{Y} + \log_2 |\mathbb{S}(B; (i, j, k, \ell))| \cdot \mathbf{X}$;
    }
3. $(y_z)_{z\in[N]} \leftarrow \mathsf{DFT}^{-1}(\mathbf{Y})$; /* $O(N \log N)$ */
4. return $\arg\max_{z\in[N]} y_z$; /* $O(N)$ */
```

Remark

Specific properties can be exploited: finite memory, symmetries, etc. For 2-D $(1,\infty)$-RLL, the finite memory helps reducing the complexity to $O(2^{\sqrt{\log n}} N \log N) < O(nN \log N)$.



Probabilistic algorithm

- $0 < \vartheta < 1$
- *Almost good shift* z — $\mathsf{R}(\mathbf{d}, \theta_z(\mathbf{d}')) \geq (1 - \vartheta)\alpha$
- $\alpha = \frac{1}{N^2} \sum_{i,j,k,\ell \in [n]} f_{i,j} f_{k,\ell} \log_2 |\mathbb{S}(B; (i,j,k,\ell))|$

Lemma. *There are at least* $\frac{\alpha\vartheta}{n-(1-\vartheta)\alpha} N$ *almost good shifts among the shifts of* $\mathbf{d}'$.

- Algorithm: keep choosing a random number z uniformly from $[N]$ and return the first z for which $\mathsf{R}(\mathbf{d}, \theta_z(\mathbf{d}')) \geq (1 - \vartheta)\alpha$.
- Expected time complexity: $O(\frac{n-(1-\vartheta)\alpha}{\alpha\vartheta} N) = O(\frac{N}{\vartheta})$

Conclusion

- Using the approximate enumerative coding
- Using the DFT-based algorithm for shifting white diagonals
- Time complexity $O(n^4 \log N)$
- Rate penalty $O(\frac{n^2 \log N}{N \log n})$