Bounds and Constructions for Granular Media Coding

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Combinatorial model

- $[s] = \{0, 1, \ldots, s-1\}$
- An alphabet $\Sigma = [q]$
- A *grain* (of length 2) ending at location $e \in [n] \setminus \{0\}$ in a word $\mathbf{x} = (x_i)_{i \in [n]}$ smears the value of cell $e-1$ to cell e: $x_e \leftarrow x_{e-1}$
- A *grain pattern* $\mathcal{S} \subseteq [n] \setminus \{0\}$ contains all the grain locations and inflicts errors to $\mathbf{x}$ by means of operator $\sigma_{\mathcal{S}}$

Example

- $\Sigma = [3]$, $n = 6$, $\mathbf{x} = 102022$, $\mathcal{S} = \{1, 3, 5\}$ and $\mathcal{S}' = \{1, 2\}$

$\mathbf{x}$:	1	0	2	0	2	2
$\mathcal{S}$:	1	0	2	0	2	2
$\sigma_{\mathcal{S}}(\mathbf{x})$:	1	1	2	2	2	2

$\mathbf{x}$:	1	0	2	0	2	2
$\mathcal{S}'$:	1	0	2	0	2	2
$\sigma_{\mathcal{S}'}(\mathbf{x})$:	1	1	0	0	2	2

Applications

- Conventional magnetic recording media: *grains*
- Wood *et al.* suggested a new approach for magnetizing areas as small as a single grain
- Mazumdar *et al.* considered a 1-D combinatorial error model with specific substitution errors and grains of length 1 and 2
- Related to bit-patterned media recording (Iyengar *et al.*)
- $\mathcal{S}$ has *overlaps* if there exist $e, e' \in \mathcal{S}$ such that $e' = e+1$; otherwise $\mathcal{S}$ is *nonoverlapping*

Combinatorial model (cont.)

- Words $\mathbf{x}, \mathbf{y} \in \Sigma^n$ are *t-confusable* if there exist $\mathcal{S}$, $\mathcal{S}'$ of size t at most for which $\sigma_{\mathcal{S}}(\mathbf{x}) = \sigma_{\mathcal{S}'}(\mathbf{y})$
- A code $\mathcal{C} \subseteq \Sigma^n$ is *t-grain-correcting* if no two distinct codewords are t-confusable
- The largest size $M_q(n, t)$
- The *rate* $R_q(\tau) = \limsup_{n\to\infty} \frac{1}{n} \log_q M_q(n, t)$
- Asymmetry: 111 is 1-confusable with 3 words, whereas 101 is 1-confusable with 4 words

Lower bound on $M_q(n,t)$

- The number $\mathsf{W}_t(\mathcal{X})$ of ordered pairs of t-confusable words in $\mathcal{X}$

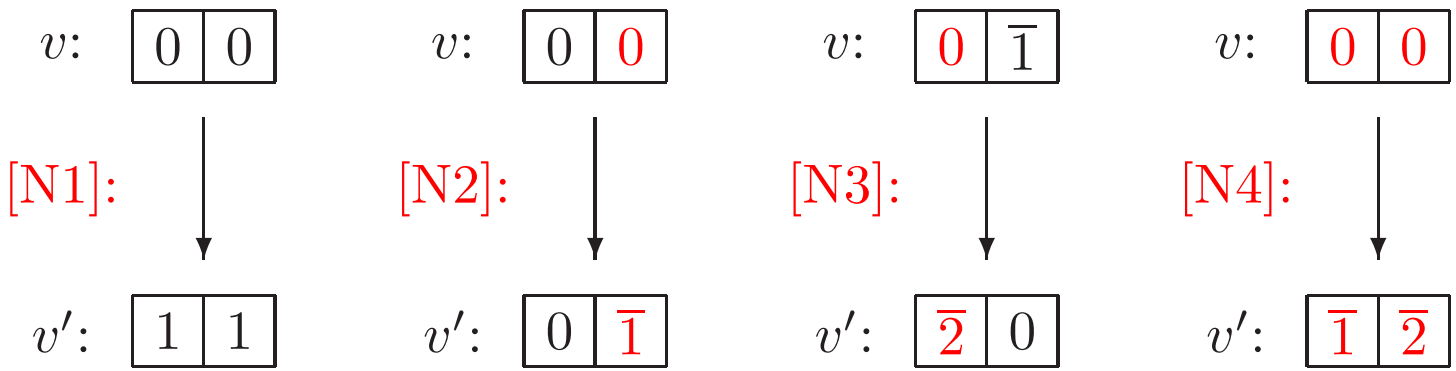
Lemma. *Let n, t be positive integers and let $\mathcal{X} \subseteq \Sigma^n$, then*

$$M_q(n,t) \geq \frac{|\mathcal{X}|^2}{4\mathsf{W}_t(\mathcal{X})}.$$

- We will evaluate $\mathsf{W}_t(\mathcal{X})$ for certain sets $\mathcal{X}$ with prescribed empirical distribution of transitions

Graph $\mathcal{G}^{(\mathcal{N})} = (V^{(\mathcal{N})}, E^{(\mathcal{N})})$

- Set of states $V^{(\mathcal{N})} = V_0 \cup V_1 \cup V_2$ where $V_0 = \{aa : a \in \Sigma\}$, $V_1 = \{a\bar{b} : ab \in \Sigma^2, a \neq b\}$ and $V_2 = \{\bar{a}\bar{b} : ab \in \Sigma^2, a \neq b\}$
- For $q = 2$, $V_0 = \{00, 11\}$, $V_1 = \{\bar{0}1, 0\bar{1}, \bar{1}0, 1\bar{0}\}$, $V_2 = \{\overline{01}, \overline{10}\}$
- There is an edge in $E^{(\mathcal{N})}$ from v to v':



Example

- $\Sigma = [3]$, $n = 6$, $\gamma = (v_i)_{i \in [n]} = 11 \quad 22 \quad 2\overline{0} \quad \overline{1}2 \quad 00 \quad \overline{12}$

$$\begin{array}{c}
\boxed{1\,|\,1} \\
[\mathrm{N1}]: \downarrow \\
\boxed{2\,|\,2} \\
[\mathrm{N2}]: \downarrow \\
\boxed{2\,|\,\overline{0}} \\
[\mathrm{N3}]: \downarrow \\
\boxed{\overline{1}\,|\,2} \\
[\mathrm{N1}]: \downarrow \\
\boxed{0\,|\,0} \\
[\mathrm{N4}]: \downarrow \\
\boxed{\overline{1}\,|\,\overline{2}}
\end{array}$$

- The patterns $\mathcal{S} = \{3, 5\}$ and $\mathcal{S}' = \{2, 5\}$ make $\mathbf{x} = 122101$ (the left path) and $\mathbf{y} = 120202$ (the right path) confusable

Adjacency matrix $A_{\mathcal{G}}^{(\mathcal{N})}$ for $q = 2$

$$A_{\mathcal{G}}^{(\mathcal{N})} = \begin{array}{c|cccccc} & 00 & \overline{0}1 & 0\overline{1} & \overline{1}0 & 1\overline{0} & 11 \\ \hline 00 & 1 & 0 & 1 & 1 & 0 & 1 \\ \overline{0}1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0\overline{1} & 1 & 0 & 0 & 1 & 0 & 1 \\ \overline{1}0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1\overline{0} & 1 & 1 & 0 & 0 & 0 & 1 \\ 11 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}$$

Correspondence between pairs of words and paths

Lemma. *For each t-confusable (ordered) pair $(\mathbf{x}, \mathbf{y}) \in \Sigma^n \times \Sigma^n$ there is at least one path $\gamma = (v_i)_{i \in [n]} = (\ell_i r_i)_{i \in [n]}$ in $\mathcal{G}^{(\mathcal{N})}$ such that*

1. $v_0 \in V_0$,
2. $\mathbf{x} = (\partial(\ell_i))_{i \in [n]}$,
3. $\mathbf{y} = (\partial(r_i))_{i \in [n]}$ *and*
4. *The total number of bars in γ is at most $2t$.*

Function $f^{(\mathcal{N})}$

- Function $f^{(\mathcal{N})} : E^{(\mathcal{N})} \to [3]^2$
- For an edge $\mathsf{e} = (\ell r, \ell' r') \in E^{(\mathcal{N})}$, $f^{(\mathcal{N})}(\mathsf{e}) = (\nu(\mathsf{e}), \chi(\mathsf{e}))$
- $\nu(\mathsf{e})$ counts the smallest number of grains confusing $\ell\ell'$ and rr'; $\chi(\mathsf{e})$ counts the number of crossovers

	ℓr:	$\ell' r'$:
$\nu(\mathsf{e})=0$, $\chi(\mathsf{e})=0$	$0 \; 0$	$0 \; 0$
$\nu(\mathsf{e})=1$, $\chi(\mathsf{e})=1$	$0 \; 0$	$0 \; \overline{1}$
$\nu(\mathsf{e})=1$, $\chi(\mathsf{e})=2$	$0 \; \overline{1}$	$\overline{2} \; 0$
$\nu(\mathsf{e})=2$, $\chi(\mathsf{e})=2$	$0 \; 0$	$\overline{1} \; \overline{2}$

Matrix function $\mathsf{A}_{\mathcal{G}}^{(\mathcal{N})}$

- $\left[\mathsf{A}_{\mathcal{G}}^{(\mathcal{N})}(z,h)\right]_{v,v'\in V} = \begin{cases} z^{\nu(\mathsf{e})}h^{\chi(\mathsf{e})} & \mathsf{e}=(v,v')\in E^{(\mathcal{N})} \\ 0 & \text{otherwise} \end{cases}$

- For $q=2$,

$$\mathsf{A}_{\mathcal{G}}^{(\mathcal{N})}(z,h) = \begin{array}{c|cccccc} & 00 & \overline{0}1 & 0\overline{1} & \overline{1}0 & 1\overline{0} & 11 \\ \hline 00 & 1 & 0 & hz & hz & 0 & h^2 \\ \overline{0}1 & h & 0 & 0 & 0 & h^2z & h \\ 0\overline{1} & h & 0 & 0 & h^2z & 0 & h \\ \overline{1}0 & h & 0 & h^2z & 0 & 0 & h \\ 1\overline{0} & h & h^2z & 0 & 0 & 0 & h \\ 11 & h^2 & hz & 0 & 0 & hz & 1 \end{array}$$

Main theorem

- Applying special cases of lemmas from [MR92]: optimizing convex functions subject to linear equality and inequality constraints
- Asymptotic upper bound on the number of paths with average number of crossovers $\sim 2p$ and number of confusing grains $\leq 2\tau$

$$K^{(\mathcal{N})} = \inf_{z\in(0,1],\ h\in(0,\infty)} \{\log_q \lambda(\mathsf{A}_{\mathcal{G}}^{(\mathcal{N})}(z,h)) - 2\tau \log_q z - 2p \log_q h\}$$

Theorem. *Let* $\tau \in (0,1)$, *then*[a]

$$R_q(\tau) \geq \varrho_q^{(\mathcal{N})}(\tau) = \sup_{p\in[0,1]} \left\{2\mathsf{H}_q(p) - K^{(\mathcal{N})}\right\}$$

[a] Asymptotic version of $M_q(n,t) \geq \frac{|\mathcal{X}|^2}{4\mathsf{W}_t(\mathcal{X})}$

Merging states of $\mathcal{G}^{(\mathcal{N})}$

- Similar to the standard procedure for reducing the number of states in a presentation of a constrained system while preserving its spectral radius
- The states of V_0 can be merged into superstate 0, V_1 — into superstate 1, V_2 — into superstate 2
- Reduced matrix $\mathcal{A}_{\mathcal{G}}^{(\mathcal{N})}$:

$$\mathcal{A}_{\mathcal{G}}^{(\mathcal{N})} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1+(q-1)h^2 & 2(q-1)hz & (q-1)(q-2)h^2z^2 \\ 1 & 2h+(q-2)h^2 & (q-1)h^2z & 0 \\ 2 & 2h+(q-2)h^2 & 0 & 0 \end{array}$$

Overlaps allowed

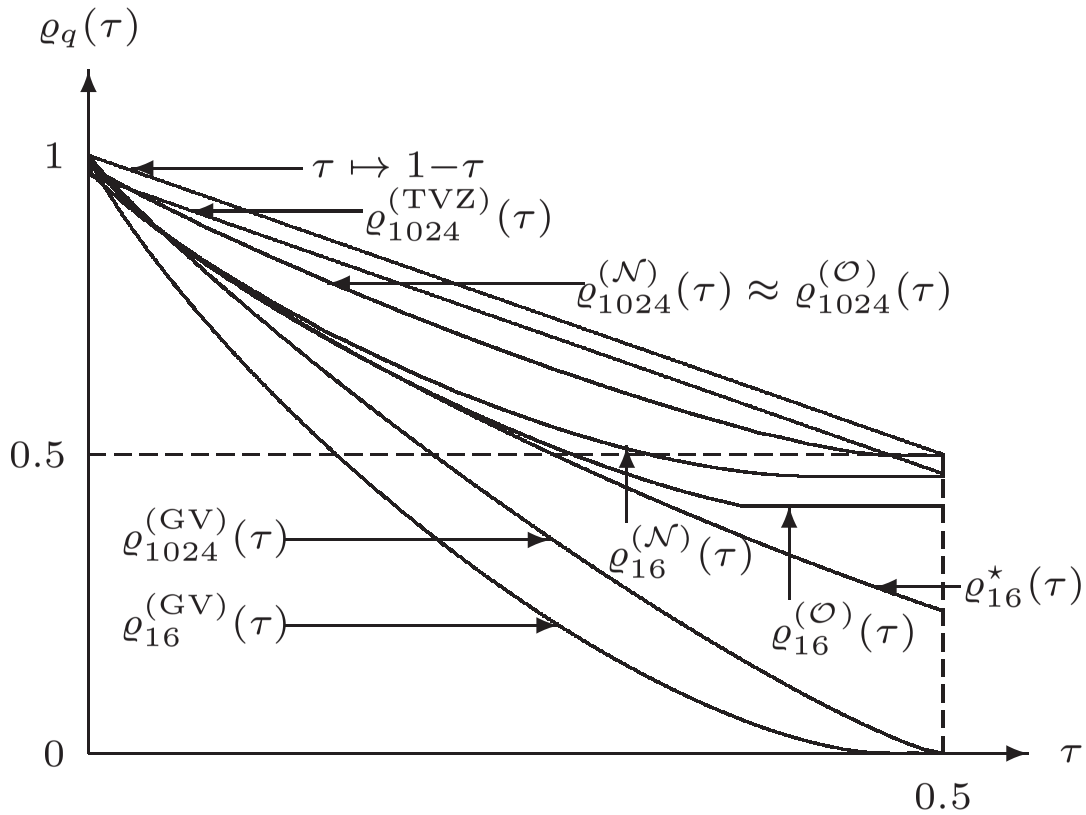
$$\left[\mathsf{A}_{\mathcal{G}}^{(\mathcal{O})}(z,h,m)\right]_{v,v'\in V} = \begin{cases} z^{\omega(\mathsf{e})}h^{\chi(\mathsf{e})}m^{\mu(\mathsf{e})} & \mathsf{e}=(v,v')\in E^{(\mathcal{O})} \\ 0 & \text{otherwise} \end{cases}$$

$$K^{(\mathcal{O})} = \sup_{0\le\eta\le 2\tau}\ \inf_{z\in(0,1],\ h,m\in(0,\infty)} \{\log_q \lambda(\mathsf{A}_{\mathcal{G}}^{(\mathcal{O})}(z,h,m)) - (2\tau-\eta)\log_q z - \eta\log_q m - 2p\log_q h\}$$

Lower bound on the rate $\varrho_q^{(\mathcal{O})}(\tau) = \sup_{p\in[0,1]}\left\{2\mathsf{H}_q(p) - K^{(\mathcal{O})}\right\}$

$$\mathcal{A}_{\mathcal{G}}^{(\mathcal{O})} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1+(q-1)h^2 & 2(q-1)hz+(q-1)(q-2)h^2z^2 \\ 1 & 2h+(q-2)h^2 & h^2m+2(q-2)h^2z \end{array}$$

Lower bounds $\varrho_q^{(j)}(\tau)$



Bounds $\varrho_{1024}^{(\mathrm{TVZ})}(\tau)$ and $\varrho_{16}^{\star}(\tau)$ rely on the code

$\mathcal{C}^\star = \left\{\mathbf{c} = (c_i)_{i\in[n]} \in \Sigma^n : c_i \neq c_{i+1} \text{ for any } i \in [n-1]\right\}$

Upper bound

- $\mathcal{C}$ is a binary t-grain-correcting code of length n
- Set $\mathcal{B}_t(\mathbf{x})$ of all words $\mathbf{w} \in [2]^n$ for which there is $\mathcal{S}$ of size t at most such that $\sigma_{\mathcal{S}}(\mathbf{x}) = \mathbf{w}$
- Lower bound $\psi_t(r)$ on $|\mathcal{B}_t(\mathbf{c})|$ that depends only on the number of runs $r = \mathsf{r}(\mathbf{c})$ in $\mathbf{c}$; $\mathsf{N}(r)$ is the number of words with r runs
- By sphere-packing arguments,

$$\Psi = \sum_{\mathbf{c} \in \mathcal{C}} \psi_t(\mathsf{r}(\mathbf{c})) \leq \sum_{\mathbf{c} \in \mathcal{C}} |\mathcal{B}_t(\mathbf{c})| \leq 2^n$$

- $|\mathcal{C}| \leq \sum_{r=1}^{\rho} \mathsf{N}(r) + \left\lfloor \frac{\Psi - \sum_{r=1}^{\rho} \mathsf{N}(r)\psi_t(r)}{\psi_t(\rho+1)} \right\rfloor$ where ρ is the largest such that $\sum_{r=1}^{\rho} \mathsf{N}(r)\psi_t(r) \leq 2^n$

Upper bound (cont.)

- Lower bounds on $|\mathcal{B}_t(\mathbf{c})|$ without overlaps

$$\psi_t^{(\mathcal{N})}(r) = 1 + \sum_{s=1}^{\min\{t,\lfloor (r-1)/3 \rfloor\}} \frac{1}{s!} \prod_{s'=0}^{s-1} (r-1-3s');$$

with overlaps

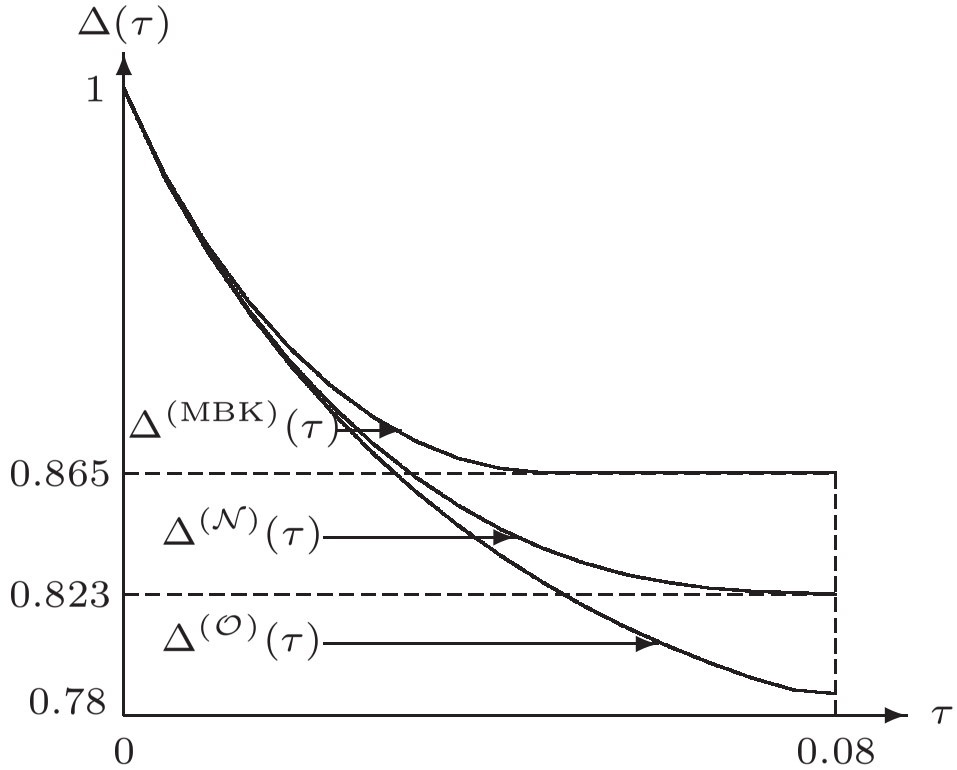
$$\psi_t^{(\mathcal{O})}(r) = \sum_{s=0}^{\min\{t,r-1\}} \binom{r-1}{s}$$

Theorem. *Let $\mathcal{C}$ be a binary t-grain-correcting code of length n, then $|\mathcal{C}| \leq \Delta^{(j)}(n,t)$ for $j \in \{\mathcal{N}, \mathcal{O}\}$ where*

$$\Delta^{(j)}(n,t) = 2\sum_{r=1}^{\rho} \binom{n-1}{r-1} + \left\lfloor \frac{2^n - 2\sum_{r=1}^{\rho} \binom{n-1}{r-1} \psi_t^{(j)}(r)}{\psi_t^{(j)}(\rho+1)} \right\rfloor$$

Upper bounds $\Delta^{(j)}(\tau)$

$n = 200$, $t \in \{1, 2, \ldots, 16\}$, $\tau = t/n$, $\Delta^{(j)}(\tau) = \Delta^{(j)}(n, \tau n)$



Asymptotically, $\Delta^{(\mathcal{N})}(\tau)$ and $\Delta^{(\mathrm{MBK})}(\tau)$ are the same; $\Delta^{(\mathcal{O})}(\tau)$ grows like $\mathsf{H}_2(\mathsf{x})$ where x is the smallest positive solution of $\mathsf{H}_2(\mathsf{x}) + \mathsf{x} \cdot \mathsf{H}_2(\tau/\mathsf{x}) = 1$

Constructions of grain-correcting codes

Theorem (due to Mazumdar *et al.*)**.** *For any* $n \in \mathbb{Z}^+$,

$$M_2(n, \lfloor n/2 \rfloor) = 2^{\lceil n/2 \rceil}$$

For even n,

$$\mathcal{C}_n = \left\{ \mathbf{c} = (c_i)_{i \in [n]} : c_{2s} = c_{2s+1} \text{ for any } s \in [n/2] \right\}$$

For odd n, $\mathcal{C}_n = (0 \mid \mathcal{C}_{n-1}) \cup (1 \mid \mathcal{C}_{n-1})$

Theorem. *For odd* n*, the above code is the only binary* $\lfloor n/2 \rfloor$*-grain-correcting code of length* n *and size* $2^{\lceil n/2 \rceil}$.

Constructions of grain-correcting codes (cont.)

Theorem. *Let $n \geq 5$ be an odd integer. Then*

$$M_2(n, \lfloor n/2 \rfloor - 1) = 2^{\lceil n/2 \rceil}.$$

Theorem. *Let $n \geq 4$ be an even integer. Then*

$$M_2(n, n/2 - 1) = 2^{n/2} + 2.$$

Theorem. *Let $m \geq 2$ be an integer and let $n = 2^m - 1$. Then*

$$M_2(n, 1) \geq 2^{n-m} + 2^{(n-1)/2}.$$

Sizes $M_2(n,t)$ of largest t-grain-correcting codes of length n

t \ n	2	3	4	5	6	7	8	9
1	**2**	***4***	**6**	**8**	16	26*	44	
2			**4**	***8***	**10**	**16**	22	
3					**8**	***16***	**18**	**32**