Coding for New Applications in Storage Media

Ph.D. seminar

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Outline

1. Motivation (whiteboard)
2. Two models
3. Recent results
4. Not as recent results
Combinatorial model

\[ \langle s \rangle = \{0, 1, \ldots, s-1\} \]

- Binary alphabet \( \Sigma = \langle 2 \rangle \)

- Grain (of length 2) ending at location \( e \in \langle n \rangle \setminus \{0\} \) in a word \( x = (x_i)_{i \in [n]} \) overruns the value of cell \( e \) by that of cell \( e-1 \):
  \[ x_e \leftarrow x_{e-1} \]

- Grain pattern \( S \subseteq \langle n \rangle \setminus \{0\} \) contains all the grain locations and inflicts errors to \( x \) by means of operator \( \sigma_S \)

- \( S \) has overlaps if there exist \( e, e' \in S \) such that \( e' = e+1 \)
Example

\(n = 6, \ x = 102022, \ S = \{1, 3, 5\} \text{ and } S' = \{1, 2\}\)

\(x: \begin{array}{cccccc}1 & 0 & 2 & 0 & 2 & 2\end{array}\)

\(S: \begin{array}{cccccc}1 & 0 & 2 & 0 & 2 & 2\end{array}\)

\(\sigma_S(x): \begin{array}{cccccc}1 & 1 & 2 & 2 & 2 & 2\end{array}\)

\(S': \begin{array}{cccccc}1 & 0 & 2 & 0 & 2 & 2\end{array}\)

\(S': \begin{array}{cccccc}1 & 0 & 2 & 0 & 2 & 2\end{array}\)

\(\sigma_{S'}(x): \begin{array}{cccccc}1 & 1 & 0 & 0 & 2 & 2\end{array}\)
Probabilistic model: generalized Ising channel

The channel $\text{Is}(p)$ as a function of the state $S_t$ with capacity $\text{cap}(p)$:

\[
\begin{align*}
S_t &= 0 \\
0 &\xrightarrow{1} 0 \\
0 &\xrightarrow{p} Y_t \\
1 &\xrightarrow{1-p} 1 \\
X_t &\xrightarrow{1-p} 1 \\

S_t &= 1 \\
0 &\xrightarrow{1-p} 0 \\
0 &\xrightarrow{p} Y_t \\
1 &\xrightarrow{1} 1 \\
X_t &\xrightarrow{1} 1
\end{align*}
\]
Probabilistic model: generalized Ising channel with feedback

The channel $\mathcal{I}_{FB}(p)$ with capacity $\text{cap}_{FB}(p)$:
Bounds on $\text{cap}(p)$

- Capacity of the generalized Ising channel $\text{Is}(p)$

$$
\text{cap}(p) = \sup_{\text{Prob}(x)} \lim \inf_{n \to \infty} \frac{1}{n} I(X^n_1; Y^n_1) \\
= \sup_{\text{Prob}(x)} \lim \inf_{n \to \infty} \frac{1}{n} [H(Y^n_1) - H(Y^n_1 | X^n_1)]
$$

- Lower bounds

$$
\text{cap}(p) \geq \rho(\mu, \zeta, p) \triangleq \sup_{M \mu \in M, x \sim M} \left\{ H(Y_{\zeta+1} | Y_{2\zeta} X_1) - H(p) \cdot \text{Prob}(X_2 \neq X_1) \right\},
$$

- Upper bounds $\text{cap}(p) \leq \rho(p) \triangleq \\
\max(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in U \left\{ t(\beta_0, \beta_1, \beta_2) - \alpha_0 \cdot H(p) \right\}$
Bounds on $\text{cap}(p)$

- $\text{cap}(p)$
- $C_{g\varepsilon}(p)$ [ISW]
- $\rho(p)$
- $C_{M1}(p)$ [ISW]

Graph showing various curves and values for $p$ ranging from 0 to 0.5.
Bounds on $\text{cap}_{FB}(\rho)$

The main idea — recasting the problem as a dynamic program:

- System with a set of states $\mathcal{B}$ (might be finite or infinite)
- Evolves in discrete time $t$
- Takes action $a \in \mathcal{A}$ while in state $b \in \mathcal{B}$
- Suffers disturbance $d \in \mathcal{D}$ after taking action $a$ in state $b$
- Transitions to the next state $b' \in \mathcal{B}$
- Gains reward $r(b, a)$ at each step
- The goal is to maximize the average reward
  \[ \phi = \sup_{\pi} \liminf_{n \to \infty} \frac{1}{n} \mathbb{E}_{\pi} \left( \sum_{t=1}^{n} r(b_t, a_t) \right) \]
Bellman equation

\[ \phi + h(b) = \sup_{a \in A} \left( r(b, a) + \sum_{d \in D} \sigma(d \mid b, a) h(f(b, a, d)) \right) \]

- \( \phi \) — maximum average reward (capacity)
- \( a \) — action
- \( b \) — state
- \( d \) — disturbance
- \( r \) — the reward function
- \( \sigma \) — describes the channel
- \( f \) — describes the system evolution in time
Bellman equation (specialized)

\[ \phi + h(b) = \sup_{(\gamma, \delta) \in [0,1-b] \times [0,b]} u(\gamma, \delta, b; h), \]

where

\[ u(\gamma, \delta, b; h) \triangleq H(\sigma) + H(p)(\delta + \gamma - 1) + (1 - \sigma) \cdot h \left( \frac{p}{1-\sigma} (1-b-\gamma) \right) \]

\[ + \sigma \cdot h \left( \frac{1}{\sigma} (\delta + (1-p)(1-b-\gamma)) \right), \]

and

\[ \sigma \triangleq (1-p)(1+\delta-\gamma) + (2p-1) \cdot b \]
Solution of Bellman equation

Let $p_0 \approx 0.398324$ be the unique solution of the equality

$$\frac{p}{1-p} = \frac{2^{H(p)}}{2^{H(p)}+1}$$
on $[0,1]$ and let $b^* = \frac{1}{2^{H(p)}+1}$. For $p \in [0, p_0]$,

$$\phi^* = H(b^*) - b^* \cdot H(p),$$

$$h^*(b) = \begin{cases} 
H(p)b + \phi^* & b \in [0, b^*] \\
H(b) & b \in [b^*, 1-b^*] \\
H(p)(1-b) + \phi^* & b \in [1-b^*, 1]
\end{cases}.$$
Iterative method

- Start with $h_0 \triangleq 0 \leq h^*$
- $h_i(b) = \sup_{(\gamma, \delta) \in [0,1-b] \times [0,b]} u(\gamma, \delta, b; h_{i-1}) \leq h^*(b) + i \cdot \phi^*$
- Normalize $h^*(0) = 0$
- $\phi^* \geq \frac{h_i(0)}{i}$
- Discretize the range $b \in [0, 1]$
- Best lower bound on $p \in [0.525, 1]$; on $p \in [p_0 \approx 0.398324, 0.525]$ the best lower bound is given by the Elishco–Permuter encoder
Bounds on $\text{cap}_{FB}(p)$

$$p \mapsto H\left(\frac{1}{2^{H(p)}+1}\right) - \frac{H(p)}{2^{H(p)}+1}$$
Combinatorial model (cont.)

- Words \( \mathbf{x}, \mathbf{x}' \in \Sigma^n \) are \( t \)-confusable if there exist \( S, S' \) of size \( t \) at most for which \( \sigma_S(\mathbf{x}) = \sigma_{S'}(\mathbf{x}') \)

- Code \( C \subseteq \Sigma^n \) is \( t \)-grain-correcting if no two distinct codewords are \( t \)-confusable

- Largest size \( M(n, t) \), rate \( R(\tau) = \lim_{n \to \infty} \frac{1}{n} \log_2 M(n, \lfloor \tau n \rfloor) \)

- Code \( C \subseteq \Sigma^n \) is \( \infty \)-grain-detecting if \( \sigma_S(\mathbf{x}) \notin C \) for any \( \mathbf{x} \in C \) and any grain pattern \( S \neq \emptyset \)
### Best known bounds on $M(n, t)$

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<th>$t$</th>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>4</td>
<td>6</td>
<td>8</td>
<td>16</td>
<td>26</td>
<td>44</td>
<td>88(72)</td>
<td>176(112)</td>
<td>352(210$^\dagger$)</td>
<td>682*(372)</td>
<td>1260*(702$^\dagger$)</td>
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<tr>
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<td>22</td>
<td>32</td>
<td>64 (44)</td>
<td>128 (68$^\diamond$)</td>
<td>256 (88)</td>
<td>512 (136$^\diamond$)</td>
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<td>32</td>
<td>64 (38)</td>
<td>128 (64)</td>
<td>128 (76)</td>
<td>256 (128)</td>
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<thead>
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<th>16</th>
<th>17</th>
<th>18</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>2304 (1272)</td>
<td>4368*$^\ast$(2400$^\dagger$)</td>
<td>8190*$^\ast$(4522)</td>
<td>15420*$^\ast$(8428)</td>
<td>29126*$^\ast$(15348)</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>512 (176)</td>
<td>1024 (312$^\diamond$)</td>
<td>1792 (418$^\diamond$)</td>
<td>3584 (836$^\diamond$)</td>
<td>6144 (1318$^\diamond$)</td>
</tr>
<tr>
<td>3</td>
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<td>256 (152)</td>
<td>512 (260$^\diamond$)</td>
<td>1024 (304)</td>
<td>2048 (520$^\diamond$)</td>
<td>2048 (608)</td>
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* $^\star$ — [KZ13], $^\dagger$ and $^\diamond$ — [GYD13]
Bounds on $R(\tau)$
Additional results

- Minimum redundancy of $\infty$-grain-detecting codes between $0.5 \log_2 n$ and $1.5 \log_2 n$
- Explicit constructions of grain-correcting codes correcting large number of grain errors