

Multi-Valued Model Checking Games

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Abstract. This work extends the game-based framework of μ -calculus model checking to the *multi-valued* setting. In multi-valued model checking a formula is interpreted over a Kripke structure defined over a lattice. The value of the formula is also an element of the lattice. We define a new game for this problem and derive from it a *direct* model checking algorithm that handles the multi-valued structure without any reduction. We investigate the properties of the new game, both independently, and in comparison to the automata-based approach. We show that the usual resemblance between the two approaches does *not* hold in the multi-valued setting and show how it can be regained by changing the nature of the game.

1 Introduction

Model checking [8] is a successful approach for verifying whether a system model M satisfies a specification φ , written as a temporal logic formula. In multi-valued model checking the system is defined over a lattice \mathcal{L} . Both the labelling of states and the transitions of the system are interpreted as elements from the lattice. The meaning of a formula in the model is then also given by an element of the lattice.

Multi-valued model checking has many important applications within the verification framework. For example, 3-valued model checking, where the logic is based on the lattice L_3 (see Fig. 1), has been used to reason about abstract structures or structures with partial information [2, 24, 13]. In this context the value U is used to model *uncertainty*, with the meaning that the value can either be \top or \perp . Recently, [1] has used a 6-valued logic as an extension of this approach for falsification of properties. Another useful lattice is the lattice $L_{2,2}$, with the values $\top\perp$ and $\perp\top$ representing *disagreement* (see Fig. 1). Model checking using this lattice (or its generalizations) has been used to handle inconsistent views of a system [11, 17]. Temporal logic query checking [5, 3, 15] can also be reduced to multi-valued model checking, where the elements of the lattice are sets of propositional formulas.

One way of handling the multi-valued model checking problem is the *reduction* approach, where the problem is reduced to several traditional 2-valued problems [12, 17, 18, 14, 4] or 3-valued problems [19].

As opposed to the reduction approach, the *direct* approach checks the property on the multi-valued structure directly. It thus has the advantage of a more “on-the-fly” nature. Furthermore, a direct model checker can provide auxiliary information that explains its result. Such information can help analyzing the result. For example, in [24, 13] the result of a direct model checking is used to suggest refinement of a 3-valued

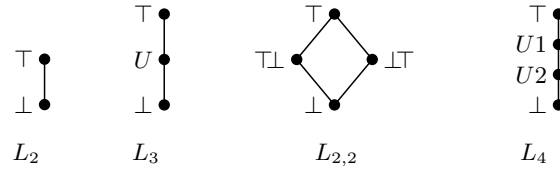


Fig. 1. Examples of Lattices

abstract structure. The same information cannot be retrieved from the model checking of two 2-valued structures [23].

Several direct model checking algorithms for various multi-valued logics have been suggested in the literature. [2, 24, 13] studied the 3-valued case of CTL ([2, 24]) and the μ -calculus ([13]). In [6] the logic LTL was considered over finite linear orders. The general multi-valued version of CTL was handled in [7]. Finally, an *almost* direct automata-based algorithm for the multi-valued μ -calculus was suggested in [4]. Their approach handled the multi-valued labelling directly, but still used a reduction to handle multi-valued transitions.

In this paper we suggest a *fully direct* model checking for the multi-valued μ -calculus, where both the multi-valued labelling and the multi-valued transitions are handled directly. The μ -calculus [20] is a powerful formalism for expressing properties of transition systems using fixpoint operators. It contains, for example, both CTL and LTL as its fragments. Our approach refers to its multi-valued semantics based on any finite distributive DeMorgan lattice.

We base our algorithm on the *game-theoretic* approach [25] and thus gain all of its advantages [24, 13]. In the traditional game-based approach to model checking two players, the verifier (called \exists loise) and the refuter (called \forall belard), try to win a game. A formula φ is true in a model M iff the verifier has a winning strategy, meaning that the verifier can win any play, no matter what the refuter does.

We adapt this approach for the multi-valued case. In particular, we now talk about the *value* of the game. It turns out that in the multi-valued case there does *not* necessarily exist a *best* strategy for \exists loise. Instead, strategies may be incomparable and the value of the game is determined by their combination.

We suggest two definitions of a multi-valued game for the μ -calculus and prove their correctness. The proof turns out to be interesting in itself, as it uses similar techniques to those used in the reduction approach of [4]. This is in spite of the fact that our approach handles the multi-valued structure directly and uses no reductions.

When comparing our definitions to the work of [4], a surprising property is revealed. The direct algorithm of [4] is based on automata [21]. Usually, the game-based and the automata-based approaches to model checking have a strong resemblance [22]. Yet, in the multi-valued setting we find that our definition of the game is different in essence from the automata-based approach of [4]. We discuss this difference and suggest an alternative multi-valued game that regains the similarity to automata. More importantly, our resulting framework in fact generalizes the work of [4], as it handles directly not only the multi-valued labelling, but also the multi-valued transitions.

The game-based approach to model checking was already generalized to the 3-valued case [24, 13]. However, it turns out that handling a general lattice, where there

is more than one intermediate value and the elements are only partially ordered, is substantially more complex (see Section 7).

The rest of the paper is organized as follows. In Section 2 we give some background on lattice theory, multi-valued μ -calculus and model checking games. In Section 3 we provide our main definition of the multi-valued model checking game and prove its correctness. A model checking algorithm, based on the game, is then described in Section 4. In Section 5 we suggest an alternative definition for the game. We then discuss the relation to the automata-theoretic approach, which yields another definition of a multi-valued game, in Section 6. Finally, we compare the general multi-valued game to the much simpler 3-valued case in Section 7.

2 Preliminaries

Lattices A *lattice* is a partially ordered set (\mathcal{L}, \leq) where for each finite subset of elements there exists a unique *greatest lower bound* (*glb*) and *least upper bound* (*lub*). The glb is also called *meet* and is denoted by $x \wedge y$ or $\bigwedge A$ (for $x, y \in \mathcal{L}, A \subseteq \mathcal{L}$). The lub is also called *join* and is denoted $x \vee y$ or $\bigvee A$ (see Fig. 1 for examples).

Throughout this paper we refer to finite distributive DeMorgan lattices. Every finite lattice is *complete*, meaning that it has a greatest element, called *top*, denoted \top , and a least element, called *bottom*, denoted \perp . In a *distributive* lattice $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all lattice elements x, y, z . In a *DeMorgan* lattice every element $x \in \mathcal{L}$ has a unique complement $\neg x \in \mathcal{L}$ such that $\neg\neg x = x$, DeMorgan's laws hold, and $x \leq y$ implies $\neg y \leq \neg x$ ¹.

A *join-irreducible* element x of a distributive lattice \mathcal{L} is an element $\neq \perp$ s.t. $x = y \vee z$ implies $x = y$ or $x = z$ for every $y, z \in \mathcal{L}$. We denote the set of join-irreducible elements of \mathcal{L} by $\mathcal{J}(\mathcal{L})$. For example $\perp \top \in \mathcal{J}(L_{2,2})$, but $\top \notin \mathcal{J}(L_{2,2})$ (see Fig. 1).

μ -calculus [20] Let \mathcal{P} be a finite set of atomic propositions and \mathcal{V} a set of propositional variables. We consider the logic μ -calculus in *negation normal form*, defined as follows:

$$\varphi ::= q \mid \neg q \mid Z \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \diamond \varphi \mid \square \varphi \mid \mu Z. \varphi \mid \nu Z. \varphi$$

where $q \in \mathcal{P}$ and $Z \in \mathcal{V}$. Let \mathcal{L}_μ denote the set of *closed* formulas generated by the above grammar, where the fixpoint quantifiers μ and ν are variable binders. We write η for either μ or ν . We assume that formulas are well-named, i.e. no variable is bound more than once in any formula. Thus, every variable Z *identifies* a unique subformula $fp(Z) = \eta Z. \psi$ of φ , where the set $Sub(\varphi)$ of *subformulas* of φ is defined as usual.

Semantics The *concrete semantics* of a μ -calculus formula is given with respect to a Kripke structure. A (finite) Kripke structure is a tuple $\mathcal{M} = (\mathcal{S}, \mathcal{R}, \Theta)$, where \mathcal{S} is a finite set of states, $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S}$ is a transition relation, which must be *total*, and $\Theta : \mathcal{S} \rightarrow 2^{\mathcal{P}}$ is a labelling function [8].

In this work we consider the multi-valued μ -calculus [4], where formulas are interpreted with respect to a Kripke structure defined over a lattice (also called χ Kripke structure). In a Kripke structure over a lattice \mathcal{L} , both the labelling and the transition relation have a multi-valued nature: Θ maps a state to a mapping from \mathcal{P} to elements of

¹ Since we refer to temporal logic in negation normal form, negation can be defined arbitrarily. We chose to refer to DeMorgan lattices since they are most commonly used in this context.

$\frac{s \vdash \varphi_0 \vee \varphi_1}{s \vdash \varphi_i} \exists : i \in \{0, 1\}$	$\frac{s \vdash \eta Z. \varphi}{s \vdash Z} \exists / \forall$	$\frac{s \vdash \diamond \varphi}{t \vdash \varphi} \exists : \mathcal{R}(s, t) \neq \perp$
$\frac{s \vdash \varphi_0 \wedge \varphi_1}{s \vdash \varphi_i} \forall : i \in \{0, 1\}$	$\frac{s \vdash Z}{s \vdash \varphi} \exists / \forall : \text{if } fp(Z) = \eta Z. \varphi$	$\frac{s \vdash \square \varphi}{t \vdash \varphi} \forall : \mathcal{R}(s, t) \neq \perp$

Fig. 2. The 2-valued model checking game rules for \mathcal{L}_μ .

\mathcal{L} , that is $\Theta : \mathcal{S} \rightarrow (\mathcal{P} \rightarrow \mathcal{L})$. Furthermore, \mathcal{R} maps pairs of states to lattice elements, that is $\mathcal{R} : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{L}$ (see Example 1). The totality requirement of \mathcal{R} is now given by the requirement that for each $s \in \mathcal{S}$ there exists some state $s' \in \mathcal{S}$ with $\mathcal{R}(s, s') \neq \perp$.

The *semantics* $\llbracket \varphi \rrbracket_\rho^{\mathcal{M}}$ of a \mathcal{L}_μ formula φ w.r.t. a Kripke structure $\mathcal{M} = (\mathcal{S}, \mathcal{R}, \Theta)$ over a lattice \mathcal{L} and an *environment* $\rho : \mathcal{V} \rightarrow (\mathcal{S} \rightarrow \mathcal{L})$, where ρ explains the meaning of free variables in φ , is a mapping from \mathcal{S} to \mathcal{L} .

We assume \mathcal{M} to be fixed and do not mention it explicitly anymore. With $\rho[Z \mapsto g]$ we denote the environment that maps Z to g and agrees with ρ on all other arguments. Later, when only closed formulas are considered, we will also drop the environment from the semantic brackets. In the following definition f is an element of $(\mathcal{S} \rightarrow \mathcal{L}) \rightarrow (\mathcal{S} \rightarrow \mathcal{L})$, defined by $\lambda g. \llbracket \varphi \rrbracket_{\rho[Z \mapsto g]}$ and $\nu f, \mu f$ stand for the greatest and least fixpoints of f , which exist according to [26], since the functions in $\mathcal{S} \rightarrow \mathcal{L}$ form a complete lattice under pointwise ordering and the functional f is monotone w.r.t. this ordering.

$$\begin{aligned}
\llbracket q \rrbracket_\rho &:= \lambda s. \Theta(s)(q) & \llbracket Z \rrbracket_\rho &:= \rho(Z) \\
\llbracket \neg q \rrbracket_\rho &:= \lambda s. \neg \Theta(s)(q) & \llbracket \mu Z. \varphi \rrbracket_\rho &:= \mu f \\
\llbracket \varphi_1 \vee \varphi_2 \rrbracket_\rho &:= \lambda s. \llbracket \varphi_1 \rrbracket_\rho \vee \llbracket \varphi_2 \rrbracket_\rho & \llbracket \nu Z. \varphi \rrbracket_\rho &:= \nu f \\
\llbracket \varphi_1 \wedge \varphi_2 \rrbracket_\rho &:= \lambda s. \llbracket \varphi_1 \rrbracket_\rho \wedge \llbracket \varphi_2 \rrbracket_\rho \\
\llbracket \diamond \varphi \rrbracket_\rho &:= \lambda s. \bigvee \{ \mathcal{R}(s, s') \wedge \llbracket \varphi \rrbracket_\rho(s') \mid \mathcal{R}(s, s') \neq \perp \} \\
\llbracket \square \varphi \rrbracket_\rho &:= \lambda s. \bigwedge \{ \neg \mathcal{R}(s, s') \vee \llbracket \varphi \rrbracket_\rho(s') \mid \mathcal{R}(s, s') \neq \perp \}
\end{aligned}$$

Given φ , (\mathcal{M}, s) and \mathcal{L} , computing the value of $\llbracket \varphi \rrbracket^{\mathcal{M}}(s)$ is called the *multi-valued model checking problem*. A regular Kripke structure \mathcal{M} can be viewed as a Kripke structure over lattice L_2 (see Fig. 1), by referring to the set of transitions and the set of atomic propositions that label a state by their characteristic functions. In this case we write $(M, s) \models \varphi$ for $\llbracket \varphi \rrbracket^{\mathcal{M}}(s) = \top$ and $(M, s) \not\models \varphi$ for $\llbracket \varphi \rrbracket^{\mathcal{M}}(s) = \perp$.

Model Checking Games The *2-valued model checking game* $\Gamma_{\mathcal{M}}(s_0, \varphi_0)$ on a (regular) Kripke structure $\mathcal{M} = (\mathcal{S}, \mathcal{R}, \Theta)$ with $s_0 \in \mathcal{S}$ and a formula $\varphi_0 \in \mathcal{L}_\mu$ is played by players \exists loise (the prover) and \forall belard (the refuter) in order to determine the truth value of φ_0 in s_0 , cf. [25]. Configurations are elements of $\mathcal{C} \subseteq \mathcal{S} \times \text{Sub}(\varphi_0)$, and written $t \vdash \psi$. Each *play* of $\Gamma_{\mathcal{M}}(s_0, \varphi_0)$ is a maximal sequence of configurations that starts with $s_0 \vdash \varphi_0$. The game rules are presented in Fig. 2. Each rule is marked by \exists / \forall to indicate which player makes the move. A rule is applied when the player is in configuration C_i , which is of the form of the upper part of the rule. C_{i+1} is then the configuration in the lower part of the rule. The rules shown in the first and third columns present a choice which the player can make. Since no choice is possible when applying the rules in the second column, both players can apply them. If no rule can be applied the play terminates. This happens in *terminal* configurations of the form $t \vdash p$ or $t \vdash \neg p$.

Winning Criteria: Player \exists wins a play C_0, C_1, \dots iff

1. the play terminates in $t \vdash q$ with $\Theta(t)(q) = \top$ or $t \vdash \neg q$ with $\Theta(t)(q) = \perp$, or
2. the outermost variable that occurs infinitely often is of type ν .

Player \forall wins a play $C_0, C_1 \dots$ iff

3. the play terminates in $t \vdash q$ with $\Theta(t)(q) = \perp$ or $t \vdash \neg q$ with $\Theta(t)(q) = \top$, or
4. the outermost variable that occurs infinitely often is of type μ .

A (memoryless) *strategy* for player Q is a partial function $\sigma : \mathcal{C} \rightarrow \mathcal{C}$, such that its domain is the set of configurations where player Q moves. Player Q plays a game according to a strategy σ if all his choices agree with σ . A strategy for player Q is called a *winning strategy* if player Q wins every play where he plays according to this strategy.

We have the following relation between the game and the semantics.

Theorem 1. [25] *For a regular Kripke structure $\mathcal{M} = (\mathcal{S}, \mathcal{R}, \Theta)$, $s \in \mathcal{S}$, and $\varphi \in \mathcal{L}_\mu$:*

- (a) $\llbracket \varphi \rrbracket^{\mathcal{M}}(s) = \top$ iff Player \exists has a winning strategy for $\Gamma_{\mathcal{M}}(s, \varphi)$,
- (b) $\llbracket \varphi \rrbracket^{\mathcal{M}}(s) = \perp$ iff Player \forall has a winning strategy for $\Gamma_{\mathcal{M}}(s, \varphi)$

3 A Multi-Valued Game for the μ -Calculus

In this section we investigate the multi-valued model checking problem from the game-theoretic point of view. For the rest of the section let \mathcal{M} be a Kripke structure over lattice \mathcal{L} , s_0 a state in \mathcal{M} and φ_0 a μ -calculus formula. We suggest a *multi-valued* model checking game, $\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)$, for evaluating φ_0 in state s_0 of \mathcal{M} .

The new game is still played by two players, \exists loise and \forall belard, and the moves of the players are defined as in the 2-valued game (see Fig. 2). In particular, in the rules of the third column the players can make a move along any transition whose value is not \perp . However, the concept of winning needs to be adapted. In fact, to capture the multi-valued nature of the problem, we no longer talk about *winning* a play versus *losing* it. Instead, we now associate with each play a *value* which is an element from the lattice.

In our definitions we take the point of view of \exists loise (we could dually describe the game from the point of view of \forall belard)). Intuitively, we think of the value of a play as a measure for how close \exists loise is to winning; Winning of \exists loise in the 2-valued case now corresponds to the *top* value. Winning of \forall belard corresponds to the *bottom* value, but more values are possible. In these terms, the goal of the players is no longer to *win* the play. Instead, the goal of \exists loise is to maximize the resulting value, whereas the goal of \forall belard is to minimize this value.

Notation We refer to the configurations of $\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)$ as *nodes* in a *game graph*, divided to \vee -nodes, where \exists loise plays, versus \wedge -nodes, where \forall belard plays. Moves between configurations are *edges* in the graph. Each edge (move) has a value from the lattice: moves that use a transition of the model get its value. The rest get the \top value. We abuse the notation of the transition relation and denote the value of an edge from n to n' by $\mathcal{R}(n, n')$. We refer to edges with values $\neq \top, \perp$ as *indefinite* edges.

Example 1. Consider the Kripke structure \mathcal{M} of Fig. 3 over lattice \mathcal{L} , s.t. $x, y, z, w \in \mathcal{L}$. The labels of the transitions define their values. Unlabelled transitions have value \top . The states labelling denotes that $\Theta(s_0)(r) = z$, $\Theta(s_0)(h) = w$ and $\Theta(s_1)(q) = \Theta(s_2)(q) = \top$, where q, r, h are atomic propositions. Fig. 3 also shows the game-graph of $\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)$, where $\varphi_0 = \diamond q \wedge (r \vee h)$. Again, the edges are labelled by their values.

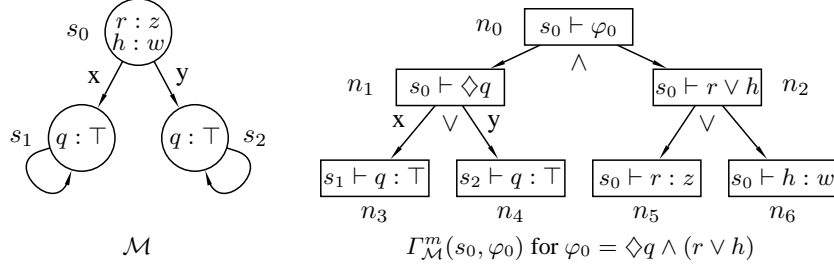


Fig. 3. Example of a Multi-Valued Game

3.1 Plays and their Values

A play in $\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)$ is defined as before. To understand how we determine the value of a multi-valued play, consider again a 2-valued play. As explained above, if the winner is \exists loise, then in the multi-valued context we view its value as \top . Similarly, if the winner is \forall belard, then we view the value as \perp . However, in the multi-valued case we have two extensions, which introduce more values. First, the terminal nodes $(t \vdash q, t \vdash \neg q)$ are no longer classified as winning or losing, but they have a value which results from the value of q in the state t . This affects the values of finite plays.

Furthermore, the moves are also multi-valued, due to the multi-valued nature of the model's transitions. The value that a player gains in the play also depends on the values of the transitions that were used. Intuitively, one can think of the moves of \exists loise as attempts at proving the formula and the moves of \forall belard as attempts at refuting it. In this context, the use of indefinite edges in the multi-valued case is interpreted as a weak attempt at proving or refuting (depending on the player).

Recall that we think of the value of the play as a measure for how close \exists loise is to winning. Therefore, when evaluating a play we take the point of view of \exists loise. Conceptually, we first give the play a *base* value, while ignoring the values of edges used. We then update the resulting value based on the edges.

Definition 1. For a terminal node $n = t \vdash q$, we define $val(n)$ to be $\Theta(t)(q)$. For $n = t \vdash \neg q$ we define $val(n)$ to be $\neg\Theta(t)(q)$.

Definition 2. For a play p in the game, we define its base value, $base(p)$, as follows. If p is finite, $base(p) = val(n)$, where n is the terminal node in which p ends. If p is infinite, then $base(p) = \top$ if p is won by \exists loise in the 2-valued game. Otherwise $base(p) = \perp$.

We update the base value by taking into consideration the values of the edges used by *both* players in the play. Intuitively, when \exists loise plays, she tries to show an evidence for truth. For her evidence to be “convincing”, she needs to both continue to a position which is good for her, *and* also use an edge with a high value (which corresponds in a way to high certainty). Consequently, the value of the play is given by the *glb* of the value of the edge and the value of the rest of the play. On the other hand, when \forall belard plays, he tries to refute. When looking at the situation from the point of view of \exists loise, she succeeds in her goal better if \forall belard either reaches a position that is good for her, *or* if he uses an edge of low value (alternatively: high negated value), in which case the certainty of his refutation is low. Therefore, the value of the play in this case is given by the *lub* of the *negation* of the value of the edge and the value of the rest of the play. This intuition leads to a bottom-up computation of the value of a play.

Definition 3. Let $p = n_0, n_1, \dots, n_k$ be a finite prefix of a play, and let $x \in \mathcal{L}$ be a base value. We define $\text{update}(p, x)$ by reverse induction. Initially, $\text{val}_k = x$. Given val_i , we define val_{i-1} depending on the player that made the move from n_{i-1} to n_i . If it is \exists loise, then $\text{val}_{i-1} = \mathcal{R}(n_{i-1}, n_i) \wedge \text{val}_i$. If the player is \forall belard, then $\text{val}_{i-1} = \neg \mathcal{R}(n_{i-1}, n_i) \vee \text{val}_i$. Finally, we let $\text{update}(p, x) = \text{val}_0$.

Note that edges with value \top do not change the base value since $\top \wedge x = x$ and $\neg \top \vee x = x$ for all $x \in \mathcal{L}$ (since in a DeMorgan lattice $\neg \top = \perp$).

Definition 3 is directly applicable to defining the value of a finite play by taking x to be the base of p . Unfortunately, it is not suitable for infinite plays.

To handle infinite plays, we use the following key observations. We say that a prefix p_i of a play p is *total* if for each player, the set of values of edges used by the player in p_i is equal to the set of values used by the same player in p . Since the underlying lattice is finite, the set of values of edges used in the play by each player is finite. Thus, there always exists a *finite total* prefix of the (possibly infinite) play. Furthermore, it turns out that computing the value of the play by considering only such a (finite) prefix is sufficient, in the following sense. We define the value $\text{val}(p_i)$ of a prefix p_i of a play p similarly to the definition of the value of a finite play, except that the base value is set to the base value of the entire play p . That is, $\text{val}(p_i) = \text{update}(p_i, \text{base}(p))$. We now have the property that the value of *any* total prefix of p is the same.

Lemma 1. Let p_i, p_j be two finite total prefixes of a play p . Then $\text{val}(p_i) = \text{val}(p_j)$.

In other words, the play has a *limit* value. This property is surprising since the sequence of values of increasingly longer prefixes is not necessarily monotonic. Lemma 1 also implies that any finite total prefix of the play is a good representative for computing this value. We therefore define the value of a play as follows.

Definition 4. For a play p , $\text{val}(p) = \text{update}(p_i, \text{base}(p))$, where p_i is the minimal total prefix of p .

Example 2. Consider again the game described in Fig. 3. Terminal nodes in the game-graph are labelled by their values. One possible play in the game is $\langle n_0, n_1, n_3 \rangle$. Its value is $\neg \top \vee (x \wedge \top) = x$. Another example is the play $\langle n_0, n_2, n_5 \rangle$ whose value is $\neg \top \vee (\top \wedge z) = z$. More plays exist.

3.2 Strategies and their Values

As always, to relate the game to model checking, we need to talk about strategies, rather than a single play. In the 2-valued game we talked about *winning* strategies and we were guaranteed that exactly one player had one. In the multi-valued case, we no longer talk about winning. Instead, we talk about the *gain* of each player in the game. We therefore need to replace the notion of winning strategies by strategies for gaining a value.

Consider again the 2-valued game. A winning strategy for \exists loise in the 2-valued game guarantees that every play, where \exists loise plays by the strategy is winning for \exists loise (or has value \top). On the other hand, a non-winning strategy for \exists loise is such that there exists a play where \exists loise plays by the strategy, but the play is winning for \forall belard (has value \perp). Thus, we can say that a winning strategy for \exists loise ensures the

value \top , whereas a non-winning strategy ensures only \perp (as it ensures a value $\geq \perp$, but not better than that). Furthermore, each strategy is either winning for \exists loise or non-winning. Thus, strategies are comparable, and there always exists a *best* strategy. The best strategy is a winning strategy if one exists, or a non-winning one otherwise.

When we move to the general multi-valued case, a strategy for \exists loise is defined as usual. However, unlike the usual case, here plays can have many values, which may be *incomparable* to one another. Given a strategy σ_{\exists} for \exists loise, the value that will be achieved in practice depends on the choices of \forall belard. We want the value of σ_{\exists} to be a *lower bound* on the set of all possible values that can be achieved in plays where \exists loise plays by σ_{\exists} , with the meaning that the strategy ensures a value which is greater or equal than its value. We choose the *greatest* possible lower bound, which characterizes the strategy as precisely as possible.

Definition 5. For a strategy σ_{\exists} for \exists loise, $val(\sigma_{\exists}) = \bigwedge \{val(p) \mid p \text{ is a play by } \sigma_{\exists}\}$.

This definition implies that \exists loise can always achieve a value $\geq val(\sigma_{\exists})$ in any play where she plays by the strategy σ_{\exists} . Note that since $val(\sigma_{\exists})$ is given by the glb of possibly incomparable values, it is possible that there does *not* exist a play with value $val(\sigma_{\exists})$ by this strategy. Still, the strategy cannot ensure a strictly better (higher) value.

Similarly to the phenomenon of several values achieved by a single strategy, it may be the case that \exists loise has several different strategies, with incomparable values. \exists loise chooses which strategy to use. We therefore define the value that she achieves in the *game* to be the *least upper bound* on the values of all her strategies. It implies that \forall belard cannot enforce any value which is strictly lower than the value of the game.

Definition 6. Let $\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)$ be a multi-valued game. Then

$$val(\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)) = \bigvee \{\alpha \mid \exists \text{loise has a strategy } \sigma_{\exists} \text{ with value } val(\sigma_{\exists}) = \alpha\}.$$

Note that in the general case, \exists loise does *not* necessarily have a *best* strategy that achieves the lub. However, if the lattice has a total order then such a best strategy exists.

Example 3. In the game of Fig. 3 \exists loise has two possible moves from n_1 and n_2 (the nodes where she moves). She thus has four possible (memoryless) strategies – one for each combination. Consider for example the strategy σ_1 in which \exists loise always proceeds to the left successor. The choice in n_0 is of \forall belard, therefore there are two possible plays by this strategy: $\langle n_0, n_1, n_3 \rangle$ (when \forall belard chooses the left successor of n_0) and $\langle n_0, n_2, n_5 \rangle$ (when \forall belard chooses the right successor of n_0) whose values are x and z respectively (see Example 2). Since the choice between the plays is of \forall belard, the value of the strategy is the glb of their values. That is, $val(\sigma_1) = x \wedge z$. This means that by σ_1 , \exists loise can only ensure a value which is $\geq x \wedge z$, where possibly $x \wedge z$ is strictly smaller than both x and z (see for example $\perp\top$ and $\top\perp$ in $L_{2,2}$).

Similarly, we get $val(\sigma_2) = x \wedge w$, $val(\sigma_3) = y \wedge z$ and $val(\sigma_4) = y \wedge w$. Since \exists loise chooses which strategy to use, the value of the game is then $val(\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)) = val(\sigma_1) \vee val(\sigma_2) \vee val(\sigma_3) \vee val(\sigma_4) = (x \wedge z) \vee (x \wedge w) \vee (y \wedge z) \vee (y \wedge w)$. If all the latter values are incomparable, then \exists loise does *not* have a unique *best* strategy. By distributivity, $val(\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)) = (x \wedge (z \vee w)) \vee (y \wedge (z \vee w)) = (x \vee y) \wedge (z \vee w)$. An inspection of the model shows that this is the value of $\llbracket \varphi_0 \rrbracket^{\mathcal{M}}(s_0)$, which demonstrates the correctness of the game (see Theorem 2 in the following section).

Remark 1. One can think of the value of the game in the regular 2-valued case (from the point of view of \exists loise), as defined by the following formula

$$\exists\sigma_{\exists}\forall\sigma_{\forall} : val(outcome(\sigma_{\exists}, \sigma_{\forall})) = \top$$

where σ_{\forall} denotes a strategy for \forall belard and $outcome(\sigma_{\exists}, \sigma_{\forall})$ is the unique play defined by the combination of σ_{\exists} and σ_{\forall} . This formula describes the condition for a game to be won by \exists loise: it requires that \exists loise has a winning strategy σ_{\exists} , meaning that for each possible strategy σ_{\forall} of \forall belard, the resulting play is winning for \exists loise (has value \top).

Similarly, in the multi-valued case, the definition of $val(\sigma_{\exists})$ can be rephrased as $val(\sigma_{\exists}) = \bigwedge_{\sigma_{\forall}} \{val(outcome(\sigma_{\exists}, \sigma_{\forall}))\}$. This makes

$$val(\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)) = \bigvee_{\sigma_{\exists}} \{ \bigwedge_{\sigma_{\forall}} \{val(outcome(\sigma_{\exists}, \sigma_{\forall}))\} \}$$

That is, we replace the \exists and \forall quantifiers by the lub and glb operators respectively, since there are no longer *best* strategies for \exists loise and \forall belard. A similar phenomenon happens when considering probabilistic games [10], where it is possible that the limit probability in which \exists loise wins is 1, but there is no strategy that achieves probability 1. Instead, for every probability, as close to 1 as we want, there is a strategy that achieves it. There also, the \exists and \forall quantifiers are replaced by *supremum* and *infimum* respectively.

3.3 Correctness

Theorem 2. *Let \mathcal{M} be a Kripke structure over lattice \mathcal{L} , s_0 a state in \mathcal{M} and φ_0 a μ -calculus formula. Then $val(\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)) = \llbracket \varphi_0 \rrbracket^{\mathcal{M}}(s_0)$.*

To prove Theorem 2 we first give an alternative definition for $val(\Gamma_{\mathcal{M}}^m(s_0, \varphi_0))$, which mainly results from Birkhoff representation theorem for finite distributive lattices.

Lemma 2. $val(\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)) = \bigvee \{ \alpha \mid \exists \text{loise has a strategy } \sigma_{\exists} \text{ with } val(\sigma_{\exists}) \geq \alpha \}$
 $= \bigvee \{ \alpha \in \mathcal{J}(\mathcal{L}) \mid \exists \text{loise has a strategy } \sigma_{\exists} \text{ with } val(\sigma_{\exists}) \geq \alpha \}$

We now use similar techniques to those used in the reduction approach of [4]. There, the multi-valued model checking problem is reduced to several 2-valued model checking problems. First, to avoid a technical problem with negated atomic propositions, the formula is transformed to a formula with no negation symbols, by replacing each negated proposition $\neg q$ by a new atomic proposition q' . The labelling function Θ of \mathcal{M} is extended to Θ' by setting $\Theta'(s)(q') = \neg\Theta(s)(q)$. Then, the Kripke structure \mathcal{M} over \mathcal{L} is reduced to several Kripke Modal Transition Systems (KMTSs).

Definition 7. [16] A Kripke Modal Transition System (KMTS) is a tuple $\tilde{\mathcal{M}} = (\tilde{\mathcal{S}}, R^+, R^-, \tilde{\Theta})$ with a must transition relation $R^+ \subseteq \mathcal{S} \times \mathcal{S}$ and a may transition relation $R^- \subseteq \mathcal{S} \times \mathcal{S}$. The labelling is given by $\tilde{\Theta} : \tilde{\mathcal{S}} \rightarrow (\mathcal{P} \rightarrow L_3)$.

Specifically, given an element $\alpha \in \mathcal{J}(\mathcal{L})$, a reduced KMTS \mathcal{M}_{α} is defined by setting

$$\begin{aligned} \Theta_{\alpha}(s)(q) &= \Theta(s)(q) \geq \alpha \\ R_{\alpha}^+(s, s') &= R(s, s') \geq \alpha \\ R_{\alpha}^-(s, s') &= (\neg R(s, s')) \not\geq \alpha \end{aligned}$$

Note that in this definition of the KMTS \mathcal{M}_{α} the value of $\Theta_{\alpha}(s)(q)$ is in fact defined over L_2 . The formula is then interpreted over the KMTS \mathcal{M}_{α} w.r.t. a 2-valued semantics, with the main difference being that

$$\begin{aligned} \llbracket \diamond \varphi \rrbracket_{\rho}^{\mathcal{M}_{\alpha}} &:= \lambda s. \bigvee \{ R_{\alpha}^{+}(s, s') \wedge \llbracket \varphi \rrbracket_{\rho}^{\mathcal{M}_{\alpha}}(s') \mid \text{all } s' \} \\ \llbracket \square \varphi \rrbracket_{\rho}^{\mathcal{M}_{\alpha}} &:= \lambda s. \bigwedge \{ \neg R_{\alpha}^{-}(s, s') \vee \llbracket \varphi \rrbracket_{\rho}^{\mathcal{M}_{\alpha}}(s') \mid \text{all } s' \} \end{aligned}$$

It then holds that $(\mathcal{M}_{\alpha}, s_0) \models \varphi_0 \Leftrightarrow \alpha \leq \llbracket \varphi_0 \rrbracket^{\mathcal{M}}(s_0)$ [4], and the following is implied.

Lemma 3. [4] $\llbracket \varphi_0 \rrbracket^{\mathcal{M}}(s_0) = \bigvee \{ \alpha \in \mathcal{J}(\mathcal{L}) \mid (\mathcal{M}_{\alpha}, s_0) \models \varphi_0 \}$.

Now, to prove the correctness of our multi-valued game we combine Lemmas 2 and 3 with the following lemma. Together they imply $\text{val}(F_{\mathcal{M}}^m(s_0, \varphi_0)) = \llbracket \varphi_0 \rrbracket^{\mathcal{M}}(s_0)$.

Lemma 4. $\bigvee \{ \alpha \in \mathcal{J}(\mathcal{L}) \mid \exists \text{loise has a strategy } \sigma_{\exists} \text{ with } \text{val}(\sigma_{\exists}) \geq \alpha \} = \bigvee \{ \alpha \in \mathcal{J}(\mathcal{L}) \mid (\mathcal{M}_{\alpha}, s_0) \models \varphi_0 \}$.

Proof. We refer to the 2-valued game for KMTSs, defined in [23]. This game is similar to the 2-valued game for Kripke structures. The difference is that \exists loise uses only must-transitions, whereas \forall belard uses may-transitions. The winning conditions are as before, with the exception that a player can get stuck (if R^{+} or R^{-} is not total), in which case he loses. Theorem 1 holds for this case as well. In our case this means that \exists loise has a winning strategy in the 2-valued game over \mathcal{M}_{α} iff $(\mathcal{M}_{\alpha}, s_0) \models \varphi_0$.

To prove Lemma 4 we show a 1-1 correspondence between strategies of \exists loise with value $\geq \alpha$ in the multi-valued game over \mathcal{M} and winning strategies for \exists loise in the 2-valued game over \mathcal{M}_{α} , for $\alpha \in \mathcal{J}(\mathcal{L})$. We use the following property of a distributive lattice \mathcal{L} . If $\alpha \in \mathcal{J}(\mathcal{L})$ and $y, z \in \mathcal{L}$, then $\alpha \leq y \vee z$ iff $\alpha \leq y$ or $\alpha \leq z$ [9].

Let σ be a strategy for \exists loise with $\text{val}(\sigma) \geq \alpha$. We show that the same strategy is winning in the 2-valued game. Consider a play played by σ in the 2-valued case. We show that \exists loise wins it. We know that in the multi-valued game its value is $\geq \alpha$.

First, if the play is infinite (in the 2-valued case) then the value y of each edge used by \forall belard is such that $\neg y \not\geq \alpha$ (otherwise it does not exist as a may-edge). Thus for the value of the play to be $\geq \alpha$, its base value has to be \top . This is because only edges of \forall belard can increase the value and by the previous property they cannot increase a base value of \perp to be $\geq \alpha$, since α is join-irreducible. Since the base value is \top , we conclude that the play fulfills the winning criteria of \exists loise in the 2-valued game.

If the play is finite (in the 2-valued case), we first rule out the possibility that \exists loise is stuck. If \exists loise is stuck it means that the strategy defines for her to use an edge with value $y \not\geq \alpha$ (that does not exist as a must-edge in the 2-valued case). The same reasoning as before shows that for the value of the play to be $\geq \alpha$, there had to be an earlier edge of \forall belard with value y s.t. $\neg y \geq \alpha$, but such an edge does not exist as a may-edge in the 2-valued play, which leads to contradiction. Thus, either \forall belard gets stuck, in which case \exists loise wins, or the play ends in a terminal node of the form $n = s \vdash q$. In the latter case, we again conclude by the same reasons that $\text{val}(n) \geq \alpha$, thus $\Theta(s)(q) \geq \alpha$ and in the KMTS \mathcal{M}_{α} this implies $\Theta_{\alpha}(s)(q) = \top$ and \exists loise wins.

For the other direction, let σ be a winning strategy for \exists loise in the 2-valued game. Once again, we show that the same strategy has a value $\geq \alpha$ in the multi-valued game, with the exception that if σ does not define a move from some configuration, we extend it arbitrarily. To prove that $\text{val}(\sigma) \geq \alpha$ we show that the value of every play where \exists loise plays by (the extended) σ in the multi-valued game has value $\geq \alpha$.

Consider such a play. First, if the same play exists in the 2-valued game, then it is winning for \exists loise, making its base value \top . Furthermore, all the edges used by \exists loise

are must-edges, with values $\geq \alpha$. Since only edges of \exists loise can decrease the value of the play, this ensures that the value of the play is $\geq \alpha$.

If the play does not exist in the 2-valued game, it means that one of two possibilities occurred. The first is that \forall belard used an edge that does not exist as a may-edge in the 2-valued game, meaning that its value y fulfills $\neg y \geq \alpha$. But this immediately increases the value of the suffix of the play from that point to be $\geq \alpha$. By the same reasons as before the prefix of the play does not decrease the value below α , and thus it remains $\geq \alpha$. The second possibility is that \exists loise used an edge that does not exist in the 2-valued game. This could only happen if the play reached a configuration where σ was extended. This means that originally, in the 2-valued game, this configuration was not reachable by σ . But this implies that in order to reach it \forall belard made a move that was not possible in the 2-valued game, and we return to the first possibility. \square

4 Solving the Multi-Valued Game

In this section we discuss how to solve the multi-valued model checking game. Given a game $\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)$ our purpose is to compute its value. By Theorem 2 this gives us the result of the multi-valued model checking problem for \mathcal{M} , s_0 and φ_0 . Since the game is defined directly on the multi-valued Kripke structure, we get a *direct* model checking algorithm for the multi-valued problem, that has all the advantages of the game-theoretic approach [24, 13].

As usual, we solve the game by processing the game-graph and evaluating each node in it. The difference as opposed to the 2-valued case is that we need to propagate values from the lattice. We demonstrate this change for the *alternation-free* fragment of the μ -calculus, where no nesting of fixpoints is allowed.

We partition the game graph to *Maximal Strongly Connected Components* (MSCCs) and determine a (total) order on them, reflected by their numbers: Q_1, \dots, Q_k . The order fulfills the rule that if $i < j$ then there are no edges from Q_i to Q_j . Such an order exists because the MSCCs of the game-graph form a *directed acyclic graph* (DAG).

The components are handled bottom-up. Consider a single Q_i . We label each node $n \in Q_i$ with a value, denoted $res(n)$, as follows. For a terminal node n , $res(n) = val(n)$. For an \vee -node n we set $res(n)$ to be $\bigvee\{\mathcal{R}(n, n') \wedge res(n') \mid \mathcal{R}(n, n') \neq \perp\}$. Similarly, if n is an \wedge -node then $res(n) = \bigwedge\{\neg\mathcal{R}(n, n') \vee res(n') \mid \mathcal{R}(n, n') \neq \perp\}$.

To handle Q_i 's that form a non-trivial MSCC, we use the following observation: when dealing with the alternation-free μ -calculus, an infinite play has exactly one variable that occurs infinitely often [25]. Therefore, if Q_i is a non-trivial MSCC then it contains exactly one fixpoint variable Z . In this case we first label the nodes in Q_i with temporary values, $temp(n)$, that are updated iteratively. For nodes of the form $n_w = t \vdash Z$ we initialize $temp(n_w)$ to \top if Z is of type ν , or to \perp if Z is of type μ (the rest remain uninitialized). We then apply the previous rules until the temporary values do not change anymore. Finally, we set $res(n) = temp(n)$ for every node n in Q_i . Intuitively, this algorithm imitates the iterative computation of the fixpoint.

Several optimizations can be made on this computation. For example, consider an \vee -node n with a successor n' for which $res(n')$ is already computed. Furthermore, suppose that the values of edges leading to the rest of the successors of n have values

$\frac{s \vdash \diamond\varphi}{(s, t) \vdash \diamond\varphi} \exists : \mathcal{R}(s, t) \neq \perp$	$\frac{(s, t) \vdash \diamond\varphi}{(s, t) \vdash \top} \forall$	$\frac{(s, t) \vdash \diamond\varphi}{t \vdash \varphi} \forall$
$\frac{s \vdash \square\varphi}{(s, t) \vdash \square\varphi} \forall : \mathcal{R}(s, t) \neq \perp$	$\frac{(s, t) \vdash \square\varphi}{(s, t) \vdash \perp} \exists$	$\frac{(s, t) \vdash \square\varphi}{t \vdash \varphi} \exists$

Fig. 4. New rules for $\Gamma_{\mathcal{M}}^d(s_0, \varphi_0)$.

$\leq \mathcal{R}(n, n') \wedge \text{res}(n')$. This means that the rest of the successors cannot increase the result of the lub over all successors of n and we can immediately set $\text{res}(n)$ to be $\mathcal{R}(n, n') \wedge \text{res}(n')$, regardless of whether or not the rest of its successors were handled. Such optimizations can spare us the need to process big subgraphs.

Theorem 3. *Let \mathcal{M} be a Kripke structure over \mathcal{L} . Then for every state s_0 in \mathcal{M} and every closed μ -calculus formula φ_0 , we have that $\text{val}(\Gamma_{\mathcal{M}}^m(s_0, \varphi_0)) = \text{res}(s_0 \vdash \varphi_0)$. We conclude that $\llbracket \varphi_0 \rrbracket^{\mathcal{M}_0}(s_0) = \text{res}(s_0 \vdash \varphi_0)$.*

5 Avoiding Multi-Valued Edges in the Game

Recall that the multi-valued edges used in the game posed a problem when we wanted to define the value of an infinite play. Our treatment of such plays relied on the finite nature of the lattice. In this section we suggest a different way of overcoming the problem. The new definition makes the value of a play much simpler to define.

The idea is to split each move along a multi-valued transition (of the model) into two moves: first the player who is supposed to play chooses a transition. Then, the opponent chooses whether he wants to examine the value of the transition or to continue in the play. If he chooses the value of the transition, the play ends with this value. This means that there are no longer multi-valued edges in the game. We only have multi-valued terminal nodes. That is, we reduce the multi-valued edges into more multi-valued terminal nodes. We emphasize that the reduction is performed in the game level, rather than the model level. The underlying Kripke structure still has multi-valued transitions.

Formally, given a Kripke structure \mathcal{M} over lattice \mathcal{L} , a state s_0 and a μ -calculus formula φ_0 , we define $\Gamma_{\mathcal{M}}^d(s_0, \varphi_0)$ (where d stands for *definite* edges) as follows. The configurations of the game are as before, with additional configurations of the form $(s, t) \vdash \diamond\varphi$, $(s, t) \vdash \square\varphi$, $(s, t) \vdash \top$ and $(s, t) \vdash \perp$ that act as intermediate configurations for the new rules. The rules are given by Fig. 2, where the rules in the third column are replaced by the rules in Fig. 4.

For example, in a configuration of the form $s \vdash \diamond\varphi$, \exists loise chooses, as usual, a transition that is supposed to show evidence for $\diamond\varphi$. Since it is a move of \exists loise, we have the meaning of the lub of all possibilities. However, the next move is a move of \forall belard, with the meaning of the glb between the two options. This means that for each possibility of \exists loise we examine the glb of both the value of the transition and the value of the rest of the play. Configurations of the form $s \vdash \square\varphi$ are handled dually.

Configurations of the form $(s, t) \vdash \top$ and $(s, t) \vdash \perp$ are (new) terminal configurations. A configuration of the form $(s, t) \vdash \top$ is reached when \forall belard challenges the transition that \exists loise chose from $s \vdash \diamond\varphi$. It expresses the fact that we are interested in the value of $\mathcal{R}(s, t)$ that determines the certainty in which \exists loise tries to prove the existential property. Dually, a configuration of the form $(s, t) \vdash \perp$ is reached when \exists loise

challenges the transition that \forall belard chose from $s \vdash \Box\varphi$. In this case, we are interested in the value of $\neg\mathcal{R}(s, t)$, since from the point of view of \exists loise, her chances of proving are better as the value of $\mathcal{R}(s, t)$ used by \forall belard for refutation is lower (alternatively: $\neg\mathcal{R}(s, t)$ is higher). Following this intuition, we add the following definition.

Definition 8. For a terminal node $n = (s, t) \vdash \top$, we define $\text{val}(n)$ to be $\mathcal{R}(s, t)$. For $n = (s, t) \vdash \perp$ we define $\text{val}(n)$ to be $\neg\mathcal{R}(s, t)$.

Since there are no longer multi-valued edges in the game, the value of a play is now determined to be the base value, as defined earlier (see Definition 2) – no update is needed. The rest of the definitions of strategies, their values and the value of the game remain unchanged. Theorem 4 ensures that the correctness of the game is maintained.

Theorem 4. Let \mathcal{M} be a Kripke structure over lattice \mathcal{L} , with a state s_0 and let φ_0 be a μ -calculus formula. Then $\text{val}(\Gamma_{\mathcal{M}}^d(s_0, \varphi_0)) = \text{val}(\Gamma_{\mathcal{M}}^m(s_0, \varphi_0))$.

6 Discussion: Games versus Automata

In this paper we have investigated the multi-valued model checking problem from the game-theoretic point of view. In [4] the same problem was considered from the automata-theoretic point of view. There, model checking is performed by checking the nonemptiness of an automaton that represents the product of the model and the checked formula. In this section we discuss the essential difference between the two approaches in the multi-valued case.

It is well-known that the game-based and the automata-based approaches are closely related in the 2-valued setting: an accepting run corresponds to a winning strategy for \exists loise and vice versa [22]. Surprisingly, the same relation does *not* hold anymore in the multi-valued case. More specifically, in [4] *extended alternating automata (EAAs)* were used as the basis for model checking. To capture the multi-valued nature they referred to the *value* of an accepting run. They showed that there always exists an accepting run of the EAA with a *maximal* value. This maximal value defines the value of the emptiness of the automaton. In the multi-valued game, on the other hand, it is *not* necessarily the case that there exists a strategy of \exists loise with a maximal value. This clearly demonstrates the discrepancy between automata and games in the multi-valued setting.

It is possible to regain the relation to the automata-theoretic approach by defining the game differently. The alternative game is still played over the same game-graph, but the moves are different. Initially, \exists loise makes a statement with respect to the value of the initial node n_0 , denoted $\text{bet}(n_0)$. In each node she proceeds by associating (possibly a subset) of its successors with a value in a consistent way based on the type of the node: in an \vee -node n the values have to fulfill the rule $\text{bet}(n) = \bigvee\{\mathcal{R}(n, n') \wedge \text{bet}(n') \mid \mathcal{R}(n, n') \neq \perp\}$. In an \wedge -node the values have to fulfill the rule $\text{bet}(n) = \bigwedge\{\neg\mathcal{R}(n, n') \vee \text{bet}(n') \mid \mathcal{R}(n, n') \neq \perp\}$. Once a bet is made on the value of a node it cannot be changed. The role of \forall belard is then to choose one successor n' for which \exists loise needs to continue and prove the value $\text{bet}(n')$. Intuitively, \forall belard will try to choose a successor for which the value is incorrect.

In this definition we return to talking about *winning* versus losing. Intuitively, \exists loise wins if she manages to proceed without contradictions. Formally, if \exists loise is stuck

(meaning she cannot associate the successors of a node with values by the above rules) then \forall belard wins. If the play ends in a terminal node of the form $s \vdash q$ or $s \vdash \neg q$, then \exists loise wins iff the value she gave the node matches its real value ($\Theta(s)(q)$ or $\neg\Theta(s)(q)$ resp.). In an infinite play the winner is determined by the 2-valued winning conditions.

Note that here \exists loise moves in *both* types of nodes, which changes the basic nature of the game. However, we now have the desired property that the game is equivalent to the definition used in the context of EAA. It now holds that an accepting run of the automaton with value α corresponds to a winning strategy for \exists loise with an initial bet of value α , and vice versa. Thus, there exists a *maximum* value for which \exists loise has a winning strategy and this value is the multi-valued model checking result.

Our definition of the game is in fact more general than the automaton used in [4] as it handles the multi-valued transitions of the Kripke structure directly.

7 Comparison to the 3-Valued Game

One of the most useful applications of multi-valued model checking is the 3-valued case. In [24, 13] the regular model checking game has been generalized to a 3-valued game over a KMTS (see Definition 7). A KMTS \mathcal{M} can be viewed as a Kripke structure over lattice L_3 by giving the must transitions in R^+ value \top , the may transitions in $R^- \setminus R^+$ value U and the rest value \perp . In this section we compare the game of [13] to our general multi-valued game $\Gamma_{\mathcal{M}}^m(s, \varphi)$ and point out the main differences that make the 3-valued game much simpler.

When considering the 3-valued case, it is possible to give the indefinite value U an intuitive meaning of a *tie*. We can thus still talk about the notion of *winning* in a way that corresponds to the three possible values $\{\top, U, \perp\}$ in the logic (see L_3 in Fig. 1) [13]. This is unlike the multi-valued case where we need to talk about the general notion of a *value* of a play or a game. The correspondence between the value of the game and the formula is then given by a variant of Theorem 1, with an additional possibility [13]:

(c) $\llbracket \varphi \rrbracket^{\mathcal{M}}(s) = U$ iff no player has a winning strategy for $\Gamma_{\mathcal{M}}(s, \varphi)$

Another major difference arises from the fact that the lattice L_3 has a *total* order, meaning that all values are comparable. As a result, in the 3-valued case a strategy has a *precise* value (rather than a lower bound) and the same holds for the game. That is, strategies are comparable and there always exists a *best* strategy (either winning or non-winning) that determines the value of the game.

The combination of these differences results in another interesting property of the 3-valued game. As in the general multi-valued case, the result of the play in the 3-valued case also depends on the values of the edges that were used. However, in [13] this effect is captured by a *consistency* requirement that says that in order to win, the winner has to use only must edges (with value \top). The surprising part is that the opponent can use either type. Recall that in the general multi-valued case, on the other hand, we need to consider not only the edges that one player uses, but also those used by the opponent.

This results from the fact that in the 3-valued case only one intermediate result is possible. Furthermore, because of the total order on the elements of the lattice, a value cannot be achieved by a combination of values that are all different from it. Thus the values of the edges that the opponent uses in the 3-valued game cannot improve the

result for the other player beyond a tie (or U). They are therefore irrelevant when we determine a *winner* in the play – recall that in the 3-valued case we are interested in the *winner* of the play. This is no longer the case in the multi-valued case, where we are interested in the (more general notion of a) *value* that each player achieves and this value can be achieved by a combination of several values, possibly incomparable ones.

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