

# Probabilistic Algorithm for Finding Roots of Linearized Polynomials

Vitaly Skachek · Ron M. Roth

**Abstract** A probabilistic algorithm is presented for finding a basis of the root space of a linearized polynomial

$$L(x) = \sum_{i=0}^t L_i x^{q^i}$$

over  $\mathbb{F}_{q^n}$ . The expected time complexity of the presented algorithm is  $O(nt)$  operations in  $\mathbb{F}_{q^n}$ .

**Keywords** Linearized polynomials · Probabilistic algorithms · Root-finding algorithms · Symbolic GCD

## 1 Introduction

Let  $q$  be a power of a prime and  $n$  be a positive integer. A *linearized polynomial over  $\mathbb{F}_{q^n}$*  (with respect to  $\mathbb{F}_q$ ) is a polynomial of the form

$$L(x) = \sum_{i=0}^t L_i x^{q^i},$$

where  $L_i \in \mathbb{F}_{q^n}$ .

Linearized polynomials were first investigated by Ore (see [12], [13]). References [9, §3.4] and [11, §4.9] contain rather extensive summaries of the properties of linearized polynomials. In particular, it is known that the mapping  $\mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$  defined by  $x \mapsto L(x)$  is a linear mapping over  $\mathbb{F}_q$ . Conversely, every

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linear mapping  $\mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$  over  $\mathbb{F}_q$  can be realized as a linearized polynomial of degree less than  $q^n$ . These properties of linearized polynomials imply that the roots of a given linearized polynomial over  $\mathbb{F}_{q^n}$  in any extension field of  $\mathbb{F}_q$  form a vector space over  $\mathbb{F}_q$ . For applications of linearized polynomials to coding theory, see [2], [5], [6], [15], or [16].

One problem that arises in some of those applications is finding a basis over  $\mathbb{F}_q$  of the root space in  $\mathbb{F}_{q^n}$  of a given linearized polynomial  $L(x)$  of degree  $q^t < q^n$  over  $\mathbb{F}_{q^n}$ . To this end, we can first find a representation of the linear mapping  $L : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$  as an  $n \times n$  matrix  $A$  over  $\mathbb{F}_q$ , according to some basis of  $\mathbb{F}_{q^n}$  over  $\mathbb{F}_q$ , and then compute a basis of the kernel of  $A$  in  $\mathbb{F}_q^n$ . The fastest algorithms currently known for finding the kernel of an  $n \times n$  matrix over  $\mathbb{F}_q$  have time complexity which grows at least as  $n^{2+\varepsilon}$ , where  $\varepsilon \approx 0.376$  (see [1] and [4]; this lower bound on the complexity does not take into account the time required to compute the matrix  $A$  from  $L(x)$ ).

Alternatively, a basis of the root space of  $L(x)$  can be found by an adaptation of Rabin's probabilistic algorithm for root finding of general polynomials over fields of even characteristic [14], taking into account the special structure and properties of linearized polynomials (see also the improved analysis of Ben-Or [3]). It can be shown that such an adapted version of Rabin's algorithm can find *one* nonzero root of a linearized polynomial in expected time complexity of  $O(nt^2)$  operations in  $\mathbb{F}_{q^n}$ .

In this note, we present a fast algorithm for finding a *whole basis* of the root space (in  $\mathbb{F}_{q^n}$ ) of a linearized polynomial over  $\mathbb{F}_{q^n}$ , in expected time complexity of  $O(nt)$  operations in  $\mathbb{F}_{q^n}$ . Hereafter, by "operations in  $\mathbb{F}_{q^n}$ " we mean any of the four arithmetic operations—addition, subtraction, multiplication, or division—in  $\mathbb{F}_{q^n}$ , as well as raising an element to the  $q$ th power (referred to here as  $q$ -exponentiation). When we represent  $\mathbb{F}_{q^n}$  as a ring of polynomials modulo an irreducible polynomial of degree  $n$  over  $\mathbb{F}_q$ , each of the four arithmetic operations in  $\mathbb{F}_{q^n}$  can be implemented using  $O(n \log^2 n \log \log n)$  arithmetic operations in  $\mathbb{F}_q$  (see [1, §8.3] and [8, §§8.3, 9.1, and 11.1]), and  $q$ -exponentiation can be implemented by  $O(\log q)$  multiplications in  $\mathbb{F}_{q^n}$ . (Moreover, for a range of values of  $q$  and  $n$ , the representation of  $\mathbb{F}_{q^n}$  according to certain normal bases over  $\mathbb{F}_q$  allows us to implement all operations in  $\mathbb{F}_{q^n}$ —including  $q$ -exponentiation—using  $O(n \log^2 n \log \log n)$  arithmetic operations in  $\mathbb{F}_q$  [7].) Hence, our algorithm has time complexity of  $O(n^2 t \log^2 n \log \log n \log q)$  operations in  $\mathbb{F}_q$ , making it faster than the aforementioned algorithms whenever  $t = o(n^\varepsilon / (\log^2 n \log \log n \log q))$ .

## 2 Symbolic GCD

In this section, we summarize several definitions and properties that will be used in the sequel. Most properties can be found in Ore [12, Ch. 1].

Let  $L(x)$  and  $M(x)$  be linearized polynomials over  $\mathbb{F}_{q^n}$ . The *symbolic product* of  $L(x)$  with  $M(x)$  is defined by

$$L(x) \otimes M(x) = L(M(x)).$$

Symbolic product satisfies associativity and distributivity (with respect to ordinary polynomial addition), but in general it does not satisfy commutativity; i.e.  $L(x) \otimes M(x)$  and  $M(x) \otimes L(x)$  are typically not equal.

Let  $L(x)$  and  $M(x)$  be linearized polynomials over  $\mathbb{F}_{q^n}$  where  $M(x) \neq 0$ . Using an algorithm akin to ordinary “long division,” one can find unique linearized polynomials  $Q(x)$  and  $R(x)$  over  $\mathbb{F}_{q^n}$  such that

$$L(x) = Q(x) \otimes M(x) + R(x) \quad \text{and} \quad \deg R(x) < \deg M(x). \quad (1)$$

When  $R(x) = 0$ , we say that  $M(x)$  is a *right symbolic divisor* of  $L(x)$ . The polynomial  $M(x)$  is a right symbolic divisor of  $L(x)$ , if and only if  $M(x)$  divides  $L(x)$  in the ordinary sense (see [12, p. 561] for the “only if” part; the “if” part is easy to prove).

Let  $L(x)$  and  $M(x)$  be linearized polynomials over  $\mathbb{F}_{q^n}$ , not both zero. A *right symbolic greatest common divisor* of  $L(x)$  and  $M(x)$  is a monic linearized polynomial  $G(x)$  over  $\mathbb{F}_{q^n}$  of a largest degree such that  $G(x)$  is a right symbolic divisor of both  $L(x)$  and  $M(x)$ .

**Proposition 1** [12, Theorem 1] *Let  $L(x)$  and  $M(x)$  be linearized polynomials over  $\mathbb{F}_{q^n}$ , not both zero. The right symbolic greatest common divisor of  $L(x)$  and  $M(x)$  is unique and equals the return value of the algorithm in Figure 1.*

The unique right symbolic greatest common divisor of  $L(x)$  and  $M(x)$  will be denoted by  $\text{rgcd}(L(x), M(x))$ .

**Proposition 2** [12, Theorem 2] *Let  $L(x)$  and  $M(x)$  be linearized polynomials over  $\mathbb{F}_{q^n}$ , not both zero. Then*

$$\text{rgcd}(L(x), M(x)) = \text{gcd}(L(x), M(x)),$$

where  $\text{gcd}(L(x), M(x))$  is the monic (ordinary) greatest common divisor of  $L(x)$  and  $M(x)$ .

Similarly to (1), for any two linearized polynomials  $L(x)$  and  $M(x) \neq 0$  over  $\mathbb{F}_{q^n}$  there exist unique linearized polynomials  $Q(x)$  and  $R(x)$  over  $\mathbb{F}_{q^n}$  such that

$$L(x) = M(x) \otimes Q(x) + R(x) \quad \text{and} \quad \deg R(x) < \deg M(x).$$

When  $R(x) = 0$ , we say that  $M(x)$  is a *left symbolic divisor* of  $L(x)$ . In general, the set of left symbolic divisors of a given linearized polynomial may differ from its set of right symbolic divisors. However, we do have the following result.

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**Input:** linearized polynomials  $L(x) \neq 0$  and  $M(x)$  over  $\mathbb{F}_{q^n}$ ;  
 $\overline{R}_{-1}(x) \leftarrow M(x)$ ;  $R_0(x) \leftarrow L(x)$ ;  
 for ( $i \leftarrow 1$ ;  $R_{i-1}(x) \neq 0$ ;  $i++$ )  
 $R_i(x) \leftarrow R_{i-2} - Q_i(x) \otimes R_{i-1}(x)$ , where  $\deg R_i(x) < \deg R_{i-1}(x)$ ;  
 normalize  $R_{i-2}(x)$  to be monic;  
**Output:**  $R_{i-2}(x)$ .

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**Fig. 1** Algorithm for computing  $\text{rgcd}(L(x), M(x))$ .

**Lemma 3** *A linearized polynomial  $M(x)$  over  $\mathbb{F}_{q^n}$  is a right symbolic divisor (or an ordinary divisor) of the polynomial  $x^{q^n} - x$ , if and only if  $M(x)$  is a left symbolic divisor of that polynomial.*

*Proof* Starting with the “only if” part, suppose that

$$x^{q^n} - x = P(x) \otimes M(x)$$

for some linearized polynomial  $P(x)$  over  $\mathbb{F}_{q^n}$ . Next write

$$x^{q^n} - x = M(x) \otimes Q(x) + R(x) \quad (2)$$

for two linearized polynomials  $Q(x)$  and  $R(x)$  such that  $\deg R(x) < \deg M(x)$ . Computing the symbolic product of  $P(x)$  with both sides of (2) we obtain

$$\begin{aligned} P(x) \otimes (x^{q^n} - x) &= P(x) \otimes M(x) \otimes Q(x) + P(x) \otimes R(x) \\ &= (x^{q^n} - x) \otimes Q(x) + P(x) \otimes R(x). \end{aligned}$$

Now, the two polynomials  $P(x) \otimes (x^{q^n} - x)$  and  $(x^{q^n} - x) \otimes Q(x)$  vanish at each element of  $\mathbb{F}_{q^n}$ ; therefore, so must  $P(x) \otimes R(x)$ . On the other hand, since  $\deg R(x) < \deg M(x)$ , we have

$$\deg(P(x) \otimes R(x)) < \deg(P(x) \otimes M(x)) = \deg(x^{q^n} - x) = q^n.$$

Hence,  $P(x) \otimes R(x) = 0$  and, so,  $R(x) = 0$ .

The proof of the “if” part is similar.  $\square$

### 3 Finding roots of linearized polynomials

Figure 2 presents a probabilistic algorithm for finding a basis over  $\mathbb{F}_q$  of the roots in  $\mathbb{F}_{q^n}$  of a given linearized polynomial  $L(x)$  over  $\mathbb{F}_{q^n}$ . Assuming that  $L(x) \neq 0$ , we let  $t$  denote  $\log_q \deg L(x)$ .

By Proposition 2 we have that the computed linearized polynomial  $G(x)$  in Figure 2 equals  $\gcd(L(x), x^{q^n} - x)$ . So, the roots of  $G(x)$  in  $\mathbb{F}_{q^n}$  are the

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**Input:** linearized polynomial  $L(x)$  over  $\mathbb{F}_{q^n}$ ;  
 $G(x) \leftarrow \text{rgcd}(L(x), x^{q^n} - x)$ ; /\* use the algorithm in Figure 1 \*/  
denote  $r = \log_q \deg G(x)$ ;  
compute a linearized polynomial  $H(x)$  such that  $x^{q^n} - x = G(x) \otimes H(x)$ ;  
for ( $j \leftarrow 0$ ;  $j < r$ ;  $j++$ ) {  
  do {  
    select at random an element  $z_j \in \mathbb{F}_{q^n}$ ;  
  }  
  while  $H(z_j)$  is in the linear span of  $\{H(z_\ell)\}_{\ell=0}^{j-1}$  over  $\mathbb{F}_q$ ;  
}

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**Output:** basis elements  $H(z_0), H(z_1), \dots, H(z_{r-1})$ .

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**Fig. 2** Algorithm for finding a basis of the root space of  $L(x)$  in  $\mathbb{F}_{q^n}$ .

roots of  $L(x)$  in that field, and the dimension of the linear space of these roots is  $r = \log_q \deg G(x)$ . Lemma 3 implies that  $G(x)$  is also a left symbolic divisor of  $x^{q^n} - x$  and, thus, the polynomial  $H(x)$  in Figure 2 is well defined.

Given that the algorithm in Figure 2 halts, it is rather straightforward to see that it returns a basis of the root space of  $G(x)$  and, hence, of  $L(x)$ . The rest of this section is devoted to analyzing the time complexity of the algorithm.

**Lemma 4** *The polynomial  $G(x)$  in Figure 2 can be computed using less than  $3(n+1)(t+1)$  operations in  $\mathbb{F}_{q^n}$ .*

*Proof* When  $t \leq n$  (the typical case), we apply the algorithm in Figure 1 to  $R_{-1}(x) = x^{q^n} - x$  and  $R_0(x) = L(x)$ . Otherwise, we switch the roles of  $R_{-1}(x)$  and  $R_0(x)$ .

Denote by  $\nu$  the largest value of  $i$  in Figure 1 for which  $R_i(x) \neq 0$  and, for  $i = -1, 0, 1, \dots, \nu$ , let  $\tau_i$  stand for  $(\log_q \deg R_i) + 1$ . Using symbolic long division to implement each iteration in the main loop in Figure 1, iteration  $i$  requires less than

$$3(\tau_{i-2} - \tau_{i-1} + 1)\tau_{i-1}$$

operations in  $\mathbb{F}_{q^n}$ . Hence, the overall number of operations in  $\mathbb{F}_{q^n}$  that are required to compute  $G(x)$  (without applying the last normalization step in Figure 1) is less than three times the value of

$$\begin{aligned} & \sum_{i=1}^{\nu+1} (\tau_{i-2} - \tau_{i-1} + 1)\tau_{i-1} \\ & \leq (\tau_{-1} - \tau_0 + 1)\tau_0 + \sum_{i=2}^{\nu+1} (\tau_{i-2}(\tau_{i-2} - 1) - \tau_{i-1}(\tau_{i-1} - 1)) \\ & \leq \tau_{-1}\tau_0 \\ & = (n+1)(t+1). \end{aligned}$$

The result follows.  $\square$

**Lemma 5** *The polynomial  $H(x)$  in Figure 2 can be computed using less than  $3(n-r+1)(r+1)$  operations in  $\mathbb{F}_{q^n}$  (where  $r = \log_q \deg G(x)$ ).*

*Proof* Compute  $H(x)$  by symbolic long division.  $\square$

**Lemma 6** *Given the polynomial  $H(x)$ , the expected number of operations in  $\mathbb{F}_{q^n}$  needed to compute the basis elements  $H(z_0), H(z_1), \dots, H(z_{r-1})$  in Figure 2 is less than  $3n(r+2)$ .*

*Proof* In iteration  $j$  (which selects  $z_j$ ), the values

$$H(z_0), H(z_1), \dots, H(z_{j-1})$$

are linearly independent over  $\mathbb{F}_q$ . Since  $\deg H(x) = q^{n-r}$  and  $H(x)$  (being a right symbolic divisor of  $x^{q^n} - x$ ) divides  $x^{q^n} - x$  in the ordinary sense, it

follows that the size of the kernel of the linear mapping  $x \mapsto H(x)$  is  $q^{n-r}$ . Therefore, when  $z_j$  is randomly selected from  $\mathbb{F}_{q^n}$ , the probability that  $H(z_j)$  is not in the linear span of  $\{H(z_\ell)\}_{\ell=0}^{j-1}$  equals

$$\frac{q^n - q^{n-r+j}}{q^n} = 1 - q^{j-r},$$

and the expected number of random selections until  $H(z_j)$  satisfies this property is

$$\frac{1}{1 - q^{j-r}} = 1 + \frac{1}{q^{r-j} - 1} \leq 2.$$

Summing over  $j$ , the expected overall number of elements that are randomly selected in Figure 2 is

$$\sum_{j=0}^{r-1} \left(1 + \frac{1}{q^{r-j} - 1}\right) < r + 2.$$

Now, for each selected element  $z_j$ , we compute  $H(z_j)$  using at most  $3(n-r) + 1$  operations in  $\mathbb{F}_{q^n}$ . Then, we check whether  $H(z_j)$  is in the linear span of the set  $\{H(z_\ell)\}_{\ell=0}^{j-1}$ . To this end, we assume that the  $j$  elements of this set have been written as row vectors in  $\mathbb{F}_q^n$  thereby forming a  $j \times n$  matrix over  $\mathbb{F}_q$ , and that this matrix has been brought to an upper-echelon form; we then append  $H(z_j)$  as a  $(j+1)$ st row to that matrix. Checking whether that row is linearly dependent of the previous rows can be done by Gaussian elimination, which, in turn, requires no more than  $2j + 1$  operations in  $\mathbb{F}_{q^n}$  (specifically, each addition of rows in the matrix amounts to one addition in  $\mathbb{F}_{q^n}$ , and each multiplication by a scalar from  $\mathbb{F}_q$  is over-counted as one multiplication in  $\mathbb{F}_{q^n}$ ). Hence, the overall expected number of operations in  $\mathbb{F}_{q^n}$  needed to find  $H(z_0), H(z_1), \dots, H(z_{r-1})$  is at most

$$\begin{aligned} \sum_{j=0}^{r-1} \left(3(n-r) + 1 + (2j+1)\right) \left(1 + \frac{1}{q^{r-j} - 1}\right) \\ < 3(n-r)(r+2) + 4 \sum_{j=0}^{r-1} (j+1) \\ < 3n(r+2). \end{aligned}$$

□

Summing up the results of Lemmas 4, 5, and 6, we conclude that the overall number of operations in  $\mathbb{F}_{q^n}$  of the algorithm in Figure 2 is less than  $9(n+1)(t+2)$ .

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## References

1. A.V. AHO, J.E. HOPCROFT, J.D. ULLMAN, *The Design and Analysis of Computer Algorithms*. Reading, Massachusetts: Addison-Wesley, 1974.
2. D. AUGOT, P. CHARPIN, N. SENDRIER, *Studying the locator polynomials of minimum weight codewords of BCH codes*, *IEEE Trans. Inform. Theory*, 38 (1992), 960–973.
3. M. BEN-OR, *Probabilistic algorithms in finite fields*, *Proc. 22nd Annual IEEE Symp. Foundations of Computer Science (FOCS'1981)*, Nashville, Tennessee (1981), 394–398.
4. D. COPPERSMITH, S. WINOGRAD, *Matrix multiplication via arithmetic progressions*, *J. Symb. Comput.*, 9 (1990), 251–280.
5. P. DELSARTE, *Bilinear forms over a finite field, with applications to coding theory*, *J. Comb. Theory A*, 25 (1978), 226–241.
6. E.M. GABIDULIN, *Theory of codes with maximum rank distance*, *Probl. Inform. Transm.*, 21 (1985), 1–12.
7. S. GAO, J. VON ZUR GATHEN, D. PANARIO, V. SHOUP, *Algorithms for exponentiation in finite fields*, *J. Symb. Comput.*, 29 (2000), 879–889.
8. J. VON ZUR GATHEN, J. GERHARD, *Modern Computer Algebra*. Cambridge, UK: Cambridge University Press, 1999.
9. R. LIDL, H. NIEDERREITER, *Finite Fields*, Second Edition. Cambridge, UK: Cambridge University Press, 1997.
10. P. LOIDREAU, *A Welch–Berlekamp like algorithm for decoding Gabidulin codes*, *Proc. 4th International Workshop on Coding and Cryptography (WCC'2005)*, Bergen, Norway (2005), 36–45.
11. F.J. MACWILLIAMS, N.J.A. SLOANE, *The Theory of Error-Correcting Codes*. Amsterdam, The Netherlands: North-Holland, 1977.
12. O. ORE, *On a special class of polynomials*, *Trans. Amer. Math. Soc.*, 35 (1933), 559–584.
13. O. ORE, *Contributions to the theory of finite fields*, *Trans. Amer. Math. Soc.*, 36 (1934), 243–274.
14. M.O. RABIN, *Probabilistic algorithms in finite fields*, *SIAM J. Comput.*, 9 (1980), 273–280.
15. R.M. ROTH, *Maximum-rank array codes and their application to crisscross error correction*, *IEEE Trans. Inform. Theory*, 37 (1991), 328–336.
16. R.M. ROTH, *Probabilistic crisscross error correction*, *IEEE Trans. Inform. Theory*, 43 (1997), 1425–1436.