Measuring Geodesic Distances via the Uniformization Theorem

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Abstract. According to the Uniformization Theorem any surface can be conformally mapped into a flat domain, that is, a domain with zero Gaussian curvature. The *conformal factor* indicates the local scaling introduced by such a mapping. This process could be used to compute geometric quantities in a simplified flat domain. For example, the computation of geodesic distances on a curved surface can be mapped into solving an eikonal equation in a plane weighted by the conformal factor. Solving an eikonal equation on the weighted plane can then be done with regular sampling of the domain using, for example, the *fast marching method*. The connection between the conformal factor on the plane and the surface geometry can be justified analytically. Still, in order to construct consistent numerical solvers that exploit this relation one needs to prove that the conformal factor is bounded.

In this paper we provide theoretical bounds over the conformal factor and introduce optimization formulations that control its behavior. It is demonstrated that without such a control the numerical results are unboundedly inaccurate. Putting all ingredients in the right order, we introduce a method for computing geodesic distances on a two dimensional manifold by using the fast marching algorithm on a weighed flat domain.

1 Introduction

Consistent and efficient distance computation on various domains is a key component in many important applications. Several papers tackle the problem of geodesic distance computation on triangulated surfaces. The celebrated *fast marching method* [7,9] enabled the solution in isotropic inhomogeneous domains that are regularly sampled. It was later generalized [3] through a geometric interpretation of the numerical update step, that enabled consistent and efficient computation of distances in anisotropic domains. So far, the fast marching method was implemented on manifolds given as either a triangulated mesh, a parametrized surface [10,8], or implicitly defined in a narrow band numerically sampled with a regular grid [5]. Traditionally, the *fast marching method* is executed on the manifold itself where some parametrization is provided. In these cases, usually there is some processing involved in order to overcome the iregularity of the numerical sampling. This is the case for the unfolding initialization

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step in [3]. Here, in order to avoid this procedure, we use a conformal mapping of a given surface and compute distances in a simplified domain. In other words, we conformally map the original curved surface into a flat plane in which we run the fast marching using the conformal factor as a local weight.

1.1 Introduction to Conformal Mapping

Let us consider a two dimensional parametrized manifold $\mathcal{X} \in \mathbb{R}^3$. It can be defined by the functions $x, y, z : \mathbb{R}^2 \to \mathbb{R}$, such that $(\alpha, \beta) \in \mathbb{R}^2$ defines a coordinate in $\mathcal{X}$ given by $\mathcal{X} = (x(\alpha, \beta), y(\alpha, \beta), z(\alpha, \beta))$. Such a parametrization induces a metric G, a scalar product $\langle u, v \rangle_G = u^T G v$, a gradient $\nabla_G \cdot = G^{-1} \nabla \cdot$ where $\nabla \cdot$ is the usual gradient with respect to α and β, and a Laplace Beltrami operator $\Delta_G \cdot = \frac{1}{\sqrt{g}} \nabla^T \left(\sqrt{g} G^{-1} \nabla \cdot \right)$ where $g = \det(G)$. We would like to map the surface $\mathcal{X}$ defined by this manifold into $D \in \mathbb{R}^2$, preserving the angles of intersections of corresponding curves. That is, given any two curves in $\mathcal{X}$, their images in D have to intersect at the same angle as in $\mathcal{X}$. A conformal mapping is a mapping function that has this property at each and every point, and can be introduced by two functions $(u(\alpha, \beta), v(\alpha, \beta))$ that map our manifold in D and obey the following condition $\nabla u = \frac{GR}{\sqrt{g}} \nabla v$, where $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This restriction over (u, v) implies four properties

1. $\Delta_G u = 0$.
2. $\Delta_G v = 0$.
3. $\langle \nabla_G u, \nabla_G v \rangle_G = 0$.
4. $\langle \nabla_G u, \nabla_G u \rangle_G = \langle \nabla_G v, \nabla_G v \rangle_G$.

This is equivalent to the Cauchy-Riemann condition if we take the metric $G = I$.

Denoting by J the Jacobian of the mapping $(\alpha, \beta) \to (u, v)$, the previous conditions can be written as

$$\begin{aligned}
\begin{pmatrix} \|\nabla_G u\|_G^2 & \langle \nabla_G u, \nabla_G v \rangle_G \\ \langle \nabla_G u, \nabla_G v \rangle_G & \|\nabla_G v\|_G^2 \end{pmatrix} = \|\nabla_G u\|_G^2 I &\Leftrightarrow (\nabla_G u, \nabla_G v)^T G (\nabla_G u, \nabla_G v) = \|\nabla_G u\|_G^2 I \\
&\Leftrightarrow (\nabla u, \nabla v)^T G^{-1} (\nabla u, \nabla v) = \|\nabla_G u\|_G^2 I \\
&\Leftrightarrow J G^{-1} J^T = \|\nabla_G u\|_G^2 I \\
&\Leftrightarrow G^{-1} = \|\nabla_G u\|_G^2 J^{-1} J^{-T} \\
&\Leftrightarrow J^T J = G \|\nabla_G u\|_G^2 .
\end{aligned}$$

Hence, any mapping is conformal with respect to a metric G if and only if there exists a scalar function μ, refered to as the *conformal factor*, such that its jacobian J satisfies $J^T J = \mu^2 G$. We also note that

$$\left\| \begin{pmatrix} du \\ dv \end{pmatrix} \right\|^2 = \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix}^T J^T J \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} = \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix}^T \mu^2 G \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} = \mu^2 \left\| \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} \right\|_G^2 .$$

It follows that $\left\| \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} \right\|_G = \frac{1}{\mu} \left\| \begin{pmatrix} du \\ dv \end{pmatrix} \right\|$.

Such a mapping would allow us to compute distances on any metric space with a generalized metric G using the computation of distance in an inhomogeneous isotropic flat manifold.

2 Construction of a Discrete Harmonic Map

We start with a theorem that would be useful for our conformal map construction.

Theorem 1. *Given a metric G defined on a regular domain D, and a function f defined on ∂D, the solution f of the following problem*

$$\underset{\substack{f \in C^2(D) \\ f(x)=f_0(x)\ \forall x \in \partial D}}{\operatorname{argmin}} \left\{ \int_D \|\nabla_G f\|_G^2 \right\}$$

satisfies $\Delta_G f = 0$ and $f(x) = f_0(x)\ \forall x \in \partial D$.

The main idea when constructing a discrete conformal map according to Polthier [6] is to find a triangulation $\mathfrak{T} = \{T_1, \ldots, T_{N_T}\}$ (where T_i is a triangle, and N_T is the number of triangles) of our map with N_V vertices, and search for a continuous function u minimizing the Dirichlet energy. For example, we could find u given by $u(\gamma) = u_0(\gamma) + \sum_{i=1}^{N_V} u_i \phi_i(\gamma)$, where u_i are some coefficients, and ϕ_i are functions satisfying

1. $\phi_i \in C^0(M)$
2. $\phi_i(V_j) = \delta_{ij} \quad \forall i, j \in \{1, \ldots, N_V\}$
3. ϕ_i is linear in each triangle.

V_j designating the jth vertex of $\mathfrak{T}$. After introducing these prerequisites, one can construct the function u, denoted as the discrete harmonic map, using the minimization problem expression of the harmonic function. It can be shown [6] that the discrete Laplace Beltrami operator applied to u at a vertex V_i can be expressed as

$$\Delta u(V_i) = \sum_{\text{edges } (i,j) \text{ at } i} (\cot(\theta_{ij}) + \cot(\psi_{ij}))(u_i - u_j),$$

where $u_j = u(V_j)$ and θ_{ij} and ψ_{ij} represent the angles supporting the edge V_iV_j, where V_j is a neighbor of V_i , and $u_i = u(V_i)$. We then have to solve the following system of equations to find an harmonic function u

$$\sum_{\text{edges } (i,j) \text{ at } i} (\cot(\theta_{ij}) + \cot(\psi_{ij}))(u_i - u_j) = 0, \quad \forall i. \tag{1}$$

After u has been computed, we have to find another conjugate discrete harmonic function v, such that $\nabla v = \frac{GR}{\sqrt{g}} (\nabla u)$. Next, we have to compute the gradient of u and perform a rotation by $\frac{\pi}{2}$. For that goal, Polthier [6] proposed to define a mid-edge grid. For each edge (V_i, V_j), define a vertex at the mid-edge as $V_s^* = \frac{V_i + V_j}{2}$. This way, each triangle (V_1, V_2, V_3) is associated with a new triangle (V_1^*, V_2^*, V_3^*). If we define Ψ_r, the function associated to the vertex V_r^* in the mid-edge grid (or, equivalently to the edge (V_i, V_j) in the regular grid) we can show that

$$\begin{pmatrix} v_3 - v_1 \\ v_3 - v_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (u_2 - u_1)\cot(\theta_{21}) + (u_2 - u_3)\cot(\theta_{23}) \\ (u_2 - u_1)\cot(\theta_{21}) + (u_3 - u_1)\cot(\theta_{31}) \end{pmatrix},$$

where v_r, v_s are the values of v on the mid-edge vertices V_r^*, V_s^* located along the edges $(V_i, V_j), (V_j, V_k)$ (respectively), and θ_{jk} is the oriented angle supporting the edge (j, k).

We end up with an algorithm, summarized for example in [4,6], that computes the mid-edge conformal flattening.

Algorithm 1. Mid-Edge discrete conformal map

Require: $\mathfrak{T}$ triangulation of the space Ω

Choose a face to cut, $C = \{V_{i_c}, V_{j_c}, V_{k_c}\} \in \mathfrak{T}$, and solve:

$$\sum_{j \in \mathcal{N}(i)} (u_i - u_j)\left(\cot(\theta_{ij}) + \cot(\psi_{ij})\right) = 0 \quad \forall i \notin \{i_c, j_c, k_c\}$$

Set arbitrary value for u on C and solve :

$$\begin{pmatrix} v_j - v_k \\ v_j - v_l \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (u_l - u_k)\cot(\theta_{lk}) + (u_l - u_j)\cot(\theta_{lj}) \\ (u_l - u_k)\cot(\theta_{lk}) + (u_j - u_k)\cot(\theta_{jk}) \end{pmatrix}$$

For the mid-edge vertex $V_r^* = \frac{V_p + V_q}{2}$, set the value of the conformal map on the midedge grid

$$u_r^* = \frac{u_p + u_q}{2}, \quad v_r^* = v_r$$

We also have the value of the conformal factor for each triangle $T_k = (V_p, V_q, V_r)$

$$\begin{aligned} \mu(T_k) &= \|\nabla u(x_q)\| \\ &= \left(\frac{1}{2 \text{ area } T_q} \left((u_r - u_q)^2 \cot(\theta_p) + (u_p - u_q)^2 \cot(\theta_r) + (u_r - u_p)^2 \cot(\theta_q) \right) \right)^{\frac{1}{2}}. \end{aligned}$$

3 Fast Marching on the Conformal Map

In the following experiments, we conformally mapped several functions into $\mathbb{R}^2$ and run the fast marching algorithm on the conformal map using the conformal factor as a local scaling of a uniform isotropic metric tensor. That is, we numerically solve the eikonal equation $||\nabla f(x,y)\| = \mu(x,y)$. When mapping a surface, we have to take care of the boundary conditions. The way we define the boundary of our target map is important, and can help us control the conformal factor and thereby the numerical accuracy of our scheme. Without controlling the boundary, all the points of the surface boundary could be mapped to a line. While uniforming the metric and solving one problem, we encounter a new one, that is, a non-uniform conformal factor. The conformal factor observes the curvature of the surface on one hand, but, yields a challenging highly non-uniformly sampled numerical domain to operate on the other.

In our first example, Figure 1, we map the surface $z = f(x,y) = \exp(-0.2x^2 - 0.5y^2)$ without controlling the boundary.

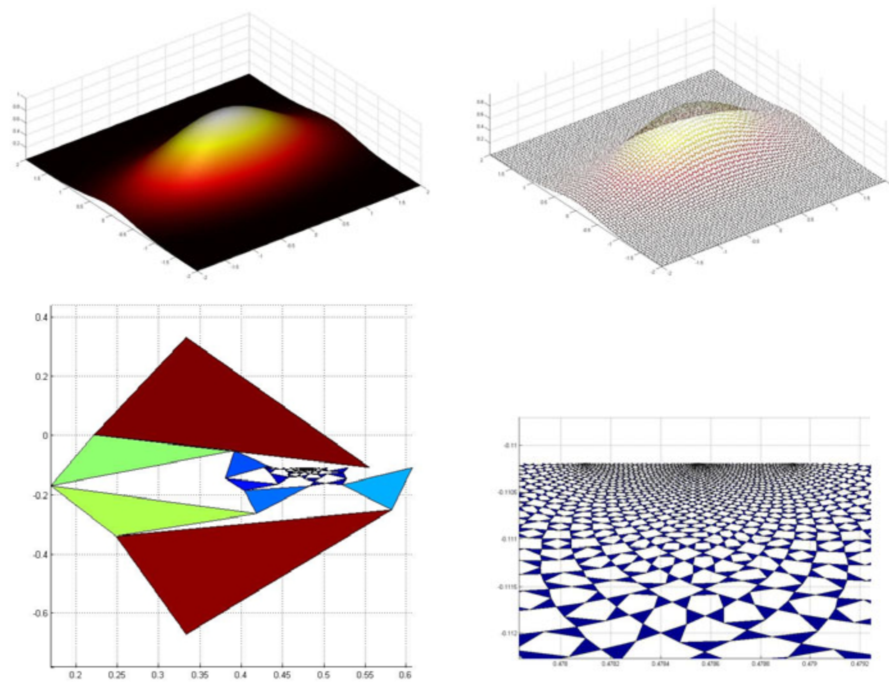

Fig. 1. Left to right, top to bottom: Original surface, midedges surface, conformal map, and zoom in

If we zoom in the area with the smallest triangles we observe that there are three points around which small triangles are concentrated. These points correspond to the corners of the original surface. When we compute the geodesic distances from the corner point $(-2,-2)$ to the rest of the surface points, the result presented in Figure 2 demonstrates numerical inaccuracies caused by the lack of control over the conformal factor.

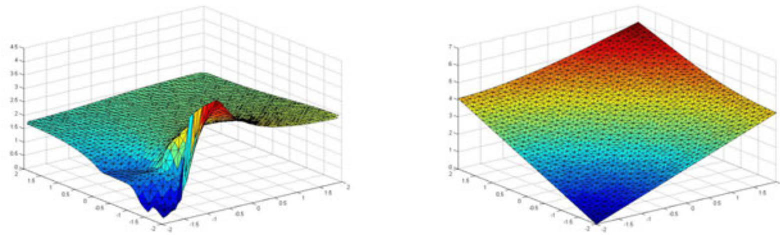

Fig. 2. Geodesic distance from the point (-2,-2) computed with FMM on the conformal map (Left) and with the FMM on the triangulated domain (Rigth)

Our next challenge would be to bound the ratio between the smallest conformal factor and the largest one on the map. Actually, in the above example, the areas ratio is in the order of 10^{-13} and the conformal factor ratio is 10^{-7}. Therefore, it is not trivial to numerically approximate geodesic distances using the FMM on the uniform grid obtained by sampling an arbitrary conformal map. Next, we try to overcome this problem by manipulating the boundary points of the conformal map.

3.1 Controlling the Conformal Factor

We would like to bound the minimal conformal factor. For that goal, we start by studying the computational aspect of the problem. We could try to manipulate the boundary conditions. In Polthier's algorithm, the scheme involves in finding u and v. We find u by solving the system of equations (1). More precisely, this system of equations is defined for each vertex i that does not belong to the boundary of our domain. Define A to be the matrix of cotangent weights, such that the previous equations can be written as $Au = 0$. Let us define $\tilde{A}$ to be the matrix obtained by removing from A the rows and columns that correspond to boundary points. As an example, if the point n belongs to the boundary of our domain, we remove from A the n^{th} row and the n^{th} column. We introduce also P the matrix whose rows are the rows of A corresponding to the removed points from A, and $\tilde{u}$ a vector representing the values of u along the boundary in a lexicographic order. $\tilde{u}$ is filled with the u_i where $i \in \mathcal{B}$, $\mathcal{B}$ being the set of indices of the points along the boundary.

Then, it can be shown that there exists a matrix M whose columns are taken from the identity matrix and from the matrix $\tilde{A}^{-1}P$ such that $u = M\tilde{u}$. It can be also shown that there exist matrices K_i such that $\mu(x_i)^2 = u^T K_i u = \tilde{u}'\left(M' K_i M\right)\tilde{u}$.

We would like to control the ratio between the smallest conformal factor and the largest one. We do so by maximizing the following expression

$$\begin{aligned} &\max_{u_j} \frac{\min_i \mu(x_i)^2}{\max_i \mu(x_i)^2}, \\ &\text{s.t.} \\ &u_j \in [0,1], \forall j \in \mathcal{B}. \end{aligned}$$

Actually, the conformal map we get contains some irregularities as some regions of our map are associated with high conformal factor, that are numerically realized as large triangles while some other regions to small conformal factors that correspond to small triangles. Then, when using the conformal factor, we should work with fine grid determined by the smallest triangle to preserve the numerical accuracy captured by the triangulated mesh.

The above problem can be reformulated as

$$\begin{aligned}&\max_{\tilde{u}_j}\left[\frac{\min_i \tilde{u}'\left(M'K_iM\right)\tilde{u}}{\max_i \tilde{u}'\left(M'K_iM\right)\tilde{u}}\right],\\&\text{s.t.}\\&\tilde{u}_j\in[0,1],\forall j.\end{aligned}$$

Since $\tilde{u}$ represents the first coordinate of the boundary points, to avoid foldovers, we have to make sure that its coordinates are increasing and decreasing at most once. The coordinates of $\tilde{u}$ have to grow up to an index from which they decrease. This constraint can be written as

$$Au \leqslant 0,\ A=\begin{pmatrix}1 & -1 & 0 & \dots & \dots & \dots\\ 0 & 1 & -1 & 0 & \dots & \dots\\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots\\ 0 & \dots & -1 & 1 & 0 & \dots\\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots\\ 0 & \dots & \dots & \dots & -1 & 1\end{pmatrix}.$$

Actually, without the previous constraint, we could get a conformal map with foldovers as shown in Fig. 3.

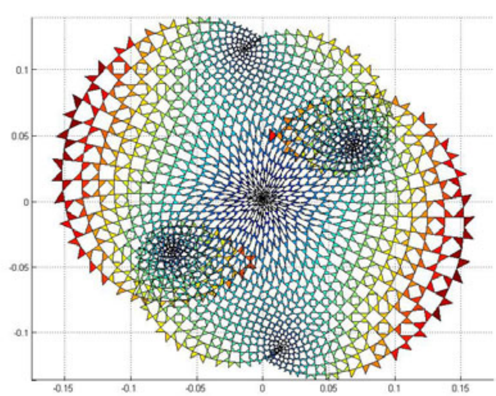

Fig. 3. Unconstrained optimal conformal map

Since the conformal factor can be normalized by restricting $\tilde{u}_j \in [0,1], \forall j$, we can rewrite our problem and its dual.

$$\begin{cases} \max\limits_{\tilde{u}} \left[\min\limits_{i} \tilde{u}' \left(M' K_i M \right) \tilde{u} \right], \\ \text{s.t.} \\ \tilde{u}_j \in [0,1], \forall j \\ A\tilde{u} \leqslant 0. \end{cases} \Rightarrow \begin{cases} \min\limits_{i} \left[\max\limits_{\tilde{u}} \tilde{u}' \left(M' K_i M \right) \tilde{u} \right]. \\ \text{s.t.} \\ \tilde{u}_j \in [0,1], \forall j \\ A\tilde{u} \leqslant 0. \end{cases}$$

This leads us to the solution of the non-convex optimization problem

$$\begin{aligned} &\max_{\tilde{u}} \tilde{u}' K \tilde{u} \\ &\text{s.t.} \\ &B\tilde{u} \leqslant b. \end{aligned} \tag{2}$$

Solving Problem (2) by manipulating the values of u along the boundary, the areas ratio in our example can be increased to 0.34 and the conformal factor ratio becomes 0.59. We can then obtain accurate results, see Fig. 4 and can compare the error between consistent geodesic distances (computed with the Tosca toolbox[1]), and the geodesic distances computed with FMM on a flat regularly sampled domain. We notice that in this case, the error is of the same order as that of the FMM.

We repeat the experiment for another surface given by the peaks function of Matlab with the same boundary condition, see Fig. 4.

So far, we demonstrated the difficulties of working with conformal mapping and showed that manipulating the boundary conditions can lead to a consistent scheme. Next provide more motivation for maximizing the conformal factor.

4 Bounding the Conformal Factor

Let us consider S, a smooth surface embedded in $\mathbb{R}^3$, and G its induced metric. If $u : S \to \mathbb{R}$ is a function defined on the surface, we can define another metric $\bar{G} = e^{2u}G$, that is *conformal* to the original metric, since the two metrics are proportional. The Gaussian curvature $\bar{k}$ of the new metric changes by [2]

$$\bar{k} = e^{-2u}(k - \Delta_G u)$$

where k is the original Gaussian curvature, and Δ_G the Laplace-Beltrami operator. In the case of a conformal mapping to the plane, the target curvature of the new metric is zero. Then, the above relation becomes

$$\Delta_G u = k.$$

Let us introduce a fundamental property of the Laplace-Beltrami operator:

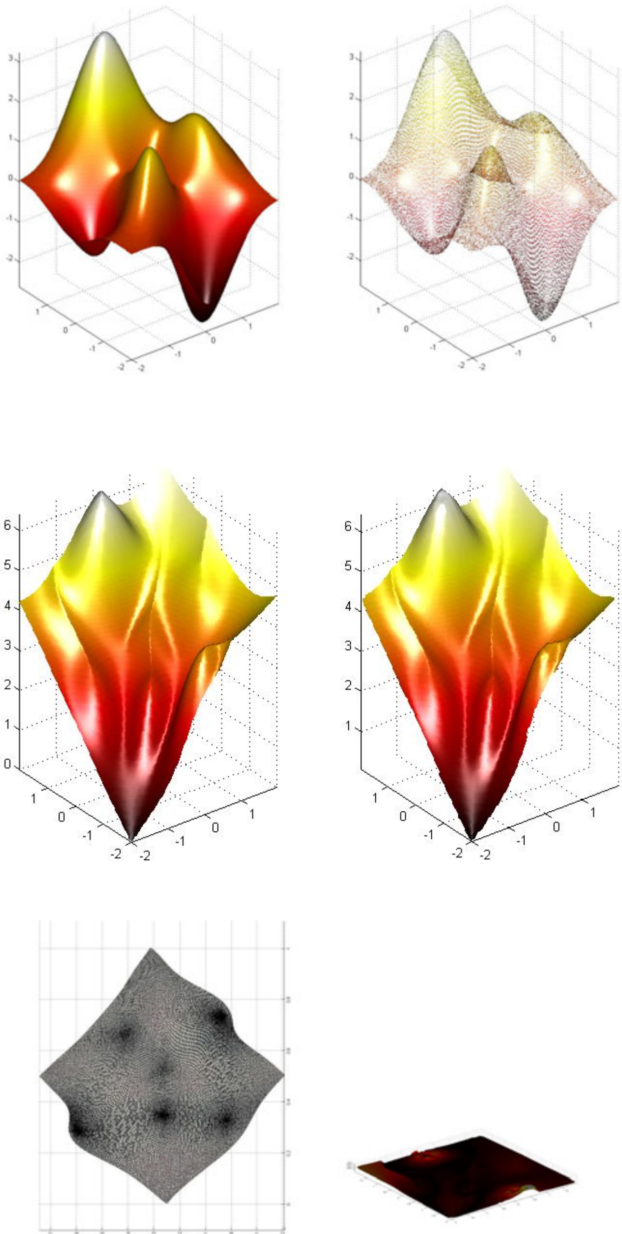

Fig. 4. Left to right, top to bottom: Original surface, midedges surface, geodesic distance with FMM on the surface, geodesic distance with FMM on the conformal map,conformal map optimized for max $\frac{\mu_{\min}}{\mu_{\max}}$, the difference between the geodesic distances

Definition 1. *A linear differential operator L of order n on a domain Ω in $\mathbb{R}^d$ given by*

$$Lf = \sum_{\|\alpha\|\leqslant n} a_\alpha(x)\partial^\alpha f$$

is called elliptic if for every x in Ω and every non-zero ξ in $\mathbb{R}^d$,

$$\sum_{\|\alpha\|=n} a_\alpha(x)\xi^\alpha \neq 0.$$

Lemma 1. *The Laplace-Beltrami operator is an elliptic operator.*

Proof. *We have* $\Delta_G f = \operatorname{trace}\left(G^{-1}\nabla^2 f\right) + v^t\nabla f$ *where* $v_j = \frac{1}{\sqrt{g}}\sum_i \partial_i\left(\sqrt{g}g^{ij}\right)$. *Then, the Δ_G highest order derivative terms are given by* $\operatorname{trace}\left(G^{-1}\nabla^2 f\right)$. *Taking a vector $\xi \neq 0 \in \mathbb{R}^2$, we have, with the notation of Lemma 1,* $\sum_{\|\alpha\|=2} a_\alpha(x)\xi^\alpha = \operatorname{trace}\left(\xi^T G^{-1}\xi\right) \neq 0$ *since G^{-1} is a positive definite matrix. This proves that the Laplace-Beltrami operator is elliptic.*

The following lemma gives us an upper bound over the conformal factor when the target domain is bounded.

Lemma 2. *Given a C^∞ domain $C \in \mathbb{R}^2$, with a metric G, there exists a function b such that for any function $f : C :\to \mathbb{R}$ s.t. $\forall p \in \partial C : f(p) = 0$, and a positive real number k such that $\|\Delta_G f\| \leqslant k$, we have*

$$\sup_{x\in C}\{\|f(x)\|\} \leqslant b(k).$$

Proof. According to the elliptic regularity theorem, for any $q \in]1,\infty[$, if C is regular, if Δ_G is an elliptic operator, and if $\Delta_G f \in L^q(C)$, then $f \in W^{2,q}(C)$ where $W^{2,q}(C)$ is the $(2,q)$-Sobolev space of C, and there exists a function $g_C^G(q)$ that depends only on C, G and q such that

$$\|f\|_{W^{2,q}} \leqslant g_C^G(q)\|\Delta_G f\|_{L^q}.$$

Moreover, the Sobolev injection theorem states that if $q > 2$, then there exists a function $h_C^G(q)$ that depends only on C and q such that

$$\|f\|_{C^2(C)} \leqslant h_C^G(q)\|f\|_{W^{2,q}}$$

where $\|f\|_{C^1(C)} = \sup_{x\in C}\{\|f(x)\|\}$. *We can then conclude that*

$$\sup_{x\in C}\{\|f(x)\|\} \leqslant h_C^G(q)g_C^G(q)\mu(C)k = b(k).$$

Using the relation $u = \log \mu$, we can choose the conformal factor such that $\mu = 1$ on ∂C. The previous lemma states that $\log \mu$ is upper bounded, which proves that μ is lower and upper bounded, and that

$$\frac{\sup |\mu|}{\inf |\mu|} \leqslant e^{2b(k)}.$$

This bound justifies using the conformal map for numerically computing geometric measures like geodesic distances. We can then conclude that since it is possible to find a boundary condition for the conformal factor that leads to a global upper bound over the ratio, our optimization programming on the conformal factor is justified. The computation of geometric quantities in the conformal mapping in this case is thereby consistent.

5 Conclusions

Conformal mapping a surface to a plain is a powerful as analysis procedure. Still, in order to justify its usage as a computational tool one needs to control the numerical behavior of this mapping. We proved that a lower bound over the ratio between the minimal and the maximal conformal factor exists. We demonstrated that this theoretical bound does not help much in practice. Next, we formulized optimization problems that maximize this ratio. It allowed us to efficiently and accurately compute geodesic distances using regular sampling of the plain.

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