Josephson junctions

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- Superconducting quantum interference
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• Introduction – coherent tunneling of Cooper pairs

Josephson junction:
two superconductors separated by a very thin (1-2nm) dielectric


Suppose \( \psi_1 = \sqrt{n_1} e^{i \theta_1} \quad \psi_2 = \sqrt{n_2} e^{i \theta_2} \)

Define the gauge invariant phase relation:
\[
\phi = \theta_1 - \theta_2 - \frac{2\pi}{\Phi_0} \int A(r,t) \, dl
\]

Solving (on board) the coupling equations of these two wavefunctions in the tunnel barrier:

**Current- phase relation:** \( J_{sc} = J_c \sin(\phi) \quad J_c \) – the critical current

**Voltage- phase relation:** \( \frac{d\phi}{dt} = \frac{2\pi}{\Phi_0} \int E \, dl = \frac{2\pi}{\Phi_0} V \)
\[ I_s = I_c \sin(\varphi) \]
\[ \frac{d\varphi}{dt} = \frac{2\pi}{\Phi_0} V \]

\( I < I_c \) – Supercurrent
\( I > I_c \) – Supercurrent + normal current

**Josephson DC effect:**
It is possible to pass current through a Josephson junction without a voltage drop on the junction!

**The energy stored in the junction:**
Consider a current source driving the junction.

\[ t = 0 : i = 0, \varphi = 0 \]
\[ U_J = \int_0^{i_1} ivdt = (I_c \sin \varphi')(\frac{\Phi_0}{2\pi} \frac{d\varphi'}{dt})dt = \frac{\Phi_0 I_c}{2\pi} \sin \varphi' \end{split} \]
\[ \left. \int_0^{\varphi} \varphi' d\varphi' = \frac{\Phi_0 I_c}{2\pi} (1 - \cos \varphi) = E_J (1 - \cos \varphi) \right|_{\Phi_0}^{\frac{I_c}{2\pi}} \]

\[ E_J \left| \frac{\Phi_0 I_c}{2\pi} \right. \]
**JJ Driven by a DC voltage source – Josephson AC effect:**

\[ V = V_0 \]

\[ \varphi(t) = \varphi(0) + \frac{2\pi}{\Phi_0} V_0 t \]

\[ i = I_c \sin \left( \frac{2\pi}{\Phi_0} V_0 t + \varphi(0) \right) \]

\[ f_J = \frac{V_0}{\Phi_0} = \frac{2e}{h} V_0 = 483.6 \times 10^{12} V_0 \text{[Hz]} \]

\[ 10_{\mu} \rightarrow 4.83 \text{Ghz} \]

**JJ Driven by an AC voltage source:**

\[ V(t) = V_0 + V_s \cos \omega_s t \]

\[ \varphi(t) = \varphi(0) + \frac{2\pi}{\Phi_0} V_0 t + \frac{2\pi V_s}{\Phi_0 \omega_s} \sin \omega_s t \]

\[ i = I_c \sin \left( \varphi(0) + \frac{2\pi}{\Phi_0} V_0 t + \frac{2\pi V_s}{\Phi_0 \omega_s} \sin \omega_s t \right) \]

Non-linear relation for the current

For \( V_0 = n \left( \frac{\Phi_0}{2\pi} \right) \omega \), the current will have a DC component.
Real JJ – RCSJ Model:

A real JJ is consist of an ideal JJ in parallel with a resistor R and a capacitor C. This model is called the Resistively and Capacitively shunted junction model or RCSJ model.

Consider a junction biased with a dc current $I_{bias}$

The total current through the junction, the resistor and the capacitor in parallel is then given by:

$$\frac{\Phi_0}{2\pi} C \dot{\phi} + \frac{\Phi_0}{2\pi R} \phi + I_c \sin \phi = I_{bias}$$

It was shown by Stewart-McCumber that the nonlinear dynamic of this system is governed by the dimensionless parameter:

$$\beta_c = \frac{2\pi I_c R}{\Phi_0 RC}$$

- $\beta_c < 1$  I(V) - Non hysteretic
- $\beta_c > 1$  I(V) - Hysteretic
We will now try to compare this dynamics to an equivalent mechanical problem of a point mass in an external potential. We can rewrite this equation as:

\[ C\left(\frac{\Phi_0}{2\pi}\right)^2 \ddot{\varphi} + \frac{1}{R} \left(\frac{\Phi_0}{2\pi}\right)^2 \dot{\varphi} = \frac{\Phi_0}{2\pi} (I_{bias} - I_c \sin \varphi) \]

\( \varphi \) can be identified as the displacement coordinate of a point mass \( m = C\left(\frac{\Phi_0}{2\pi}\right)^2 \)

And the above equation as the equation of motion of the mass \( m \) in a potential energy:

\[ U(\varphi) = \frac{\Phi_0}{2\pi} (-I_{bias} \varphi - I_c \cos \varphi) \]

U is called “tilted washboard potential”

For \( I_{bias} = I_c \) - the minima in the potential vanish and the mass starts rolling down the washboard, causing a finite voltage across the junction.
Superconducting Quantum Interference

A superconducting loop with one or more JJ’s is called SQUID - Superconducting Quantum Interference device. One junction – “RF SQUID”
Two junctions – “DC SQUID” – used as a sensitive detector of magnetic flux.

\[ I_s = I_c \sin(\varphi) \]
\[ \frac{d\varphi}{dt} = \frac{2\pi}{\Phi_0} V \]

\[ i = i_1 + i_2 = I_c \sin \varphi_1 + I_c \sin \varphi_2 = 2I_c \cos\left(\frac{\varphi_1 - \varphi_2}{2}\right)\sin\left(\frac{\varphi_1 + \varphi_2}{2}\right) \]

\[ \varphi_1 - \varphi_2 = 2\pi n + \frac{2\pi \Phi}{\Phi_0} \]

\[ i = 2I_c \cos\left(\frac{\pi \Phi}{\Phi_0}\right)\sin\left(\varphi_1 + \frac{\pi \Phi}{\Phi_0}\right) \]

Suppose that the inductance of the loop is \( L \).

\[ i_1 = \bar{I} + I_{cir} ; \quad i_2 = \bar{I} - I_{cir} \]

\[ \bar{I} = I/2 = (i_1 + i_2)/2 ; \quad I_{cir} = (i_1 - i_2)/2 \]

\( I_{cir} \) is the circulating current and generates flux \( LI_{cir} \).

The total flux is then

\[ \Phi = \Phi_{ext} + LI_{cir} = \Phi_{ext} + \frac{LI_c}{2} (\sin \varphi_1 - \sin \varphi_2) \]

\[ = \Phi_{ext} + LI_c \sin\left(\frac{\varphi_1 - \varphi_2}{2}\right)\cos\left(\frac{\varphi_1 + \varphi_2}{2}\right) \]

\[ = \Phi_{ext} - LI_c \sin\left(\frac{\pi \Phi}{\Phi_0}\right)\cos\left(\varphi_1 + \frac{\pi \Phi}{\Phi_0}\right) \]
Magnetic flux modulates the maximal superconducting current:

The special case $LI_{\text{cir}} \ll \Phi_{\text{ext}}$ - negligible inductance

$$I_{\text{SC max}} = 2I_c \cos\left(\frac{\pi \Phi}{\Phi_0}\right)$$

$$\Phi_0 = \frac{h}{2e} = 2 \times 10^{-15} \text{ Wb} - \text{the flux quantum}$$
The dc SQUID:

- DC Bias current
- Magnetic flux to voltage transformer

Max sensitivity

$$\Phi_0$$

$$\Phi$$

$$\Phi_0$$

$$\Phi$$

Superconducting Quantum Interference Device (SQUID) Applications:

- Resonant gravitation Wave detectors
- Biomagnetism

Applications:

T. solo3.abac.com/gwicnstitute/

www.sfn.org

Supercouding Quantum Interference Device

Max sensitivity
The I(v) curve will be:

\[ \Phi = n\Phi_0 \]

\[ \Phi = (n + \frac{1}{2})\Phi_0 \]

The dc SQUID equivalent electronic scheme

Remark: additional notes will be given on board in the class