VECTOR REPRESENTATION OF GRAPH DOMINATION

NOGA ZEWI

ABSTRACT. We study a function on graphs, denoted by “Gamma”, representing vectorially the domination number of a graph, in a way similar to that in which the Lovász Theta function represents the independence number of a graph. This function is a lower bound on the homological connectivity of the independence complex of the graph, and hence is of value in studying matching problems by topological methods. Not much is known at present about the $\Gamma$ function, in particular there is no known procedure for its computation for general graphs. In this paper we compute the precise value of Gamma for trees and cycles, and to achieve this we prove new lower and upper bounds on Gamma, formulated in terms of known domination and algebraic parameters of the graph. We also use the Gamma function to prove a fractional version of a strengthening of Ryser’s conjecture.

1. Introduction

Let $G = (V, E)$ be a graph. The Gamma function $\Gamma(G)$ of $G$ is defined as follows: A vector representation of a graph $G$ is an assignment $P$ of a vector $P(v) \in \mathbb{R}^d$ for some fixed $d$ to every vertex $v$ of the graph, such that $P(u) \cdot P(v) \geq 1$ whenever $u, v$ are adjacent in $G$ and $P(u) \cdot P(v) \geq 0$ for all vertices $u$ and $v$. The value $|P|$ of $P$ is the minimum of $\sum_{v \in V} \alpha(v)$, over all non-negative vectors $\alpha \in \mathbb{R}^V$ which satisfy $\sum_{v \in V} \alpha(v)P(v) \cdot P(u) \geq 1$ for every vertex $u \in V$. $\Gamma(G)$ is defined to be the supremum of $|P|$ over all vector representations of $G$.

The Gamma function was introduced by Aharoni, Berger and Meshulam in [3, 5]. It is similar in spirit to the Theta function defined by Lovász in [11], using as it does vector representation to mimic domination, just as the Theta function uses vector representations to mimic independence of vertices.

The main result in [3] links $\Gamma(G)$ and the homological connectivity of the independence complex of $G$. The independence complex $I(G)$ of $G$ is the simplicial complex on the vertex set $V$ whose simplices are all independent sets of $G$. The homological connectivity $\eta(C)$ of a simplicial complex $C$ is $\min \left\{ i \mid \tilde{H}^i(C, \mathbb{R}) \neq 0 \right\} + 1$, where $\tilde{H}^i(C, \mathbb{R})$ denotes the i-th reduced homology group of $C$ with real coefficients.

One reason for the interest in $\eta(I(G))$ is that there exists a combinatorial theorem using it. Suppose that $V_1, V_2, \ldots, V_m$ is a system of sets, and that $G$ is a graph on $V = \bigcup_{i=1}^m V_i$. An independent system
of representatives (ISR) of \( V_1, V_2, \ldots, V_m \) with respect to \( G \) is a choice of elements \( x_1 \in V_1, x_2 \in V_2, \ldots, x_m \in V_m \) such that \( x_i, x_j \) are distinct and not adjacent in \( G \) for \( i \neq j \). The following was proved in [4]:

**Theorem 1.1** ([4]). Suppose \( G \) is a graph on \( V = \bigcup_{i=1}^{m} V_i \), where \( V_1, V_2, \ldots, V_m \) are disjoint sets that form a partition of \( V \). If \( \eta(I(G[\bigcup_{i \in I} V_i])) \geq |I| \) for all \( I \subseteq [m] \), then the partition \( \{V_i \mid i \in [m]\} \) has an ISR with respect to \( G \).

To be applied combinatorially, this theorem should be combined with lower bounds on \( \eta(I(G)) \) of combinatorial nature. To this day, all known lower bounds on \( \eta(I(G)) \) are related to domination parameters of \( G \), except for one which is related to the largest eigenvalue of the Laplacian matrix of \( G \). For a subset \( S \subseteq V \) of vertices in a graph \( G = (V, E) \) let \( \hat{N}(S) \) denote the set of vertices that are adjacent to at least one vertex of \( S \), and let \( N(S) = S \cup \hat{N}(S) \). \( S \) is a dominating set in \( G \) if \( N(S) = V \). \( S \) is a totally dominating set in \( G \) if \( V = \hat{N}(S) \) (i.e., a vertex cannot dominate itself). A few known lower bounds on \( \eta(I(G)) \) are:

- [13] \( \hat{\gamma}(G)/2 \), where \( \hat{\gamma}(G) \) is the total domination number of \( G \), which is the minimal size of a totally dominating set in the graph \( G \).
- [14] \( i\gamma(G) \), the independent domination number, which is the maximum, over all independent sets \( I \) in \( G \), of the minimal size of a set \( S \), such that \( I \subseteq N(S) \).
- [3] \( \Gamma(G) \) - the Gamma function defined above.
- [3] \( \frac{|V|}{\lambda_{\max}(G)} \), where \( \lambda_{\max}(G) \) is the largest eigenvalue of the Laplacian matrix of \( G \) (see Subsection 3.4 for the definition of the Laplacian matrix of a graph).

The heart of the proof of the \( \Gamma(G) \leq \eta(I(G)) \) theorem from [3] was the study of the relations between \( \eta(I(G)), \Gamma(G), \) and the largest eigenvalue of the Laplacian matrix of the graph. The combination of the \( \Gamma(G) \leq \eta(I(G)) \) theorem and Theorem 1.1 was used in [3] in order to derive a new Hall type result for hypergraphs.

A hypergraph \( H \) is a set of subsets, called edges, of the ground set \( V \), whose elements are called vertices. A hypergraph is called \( r \)-uniform if all its edges are of the same size \( r \). A matching in \( H \) is a subhypergraph \( M \subseteq H \), such that \( e \cap e' = \emptyset \) for all pairs of distinct edges \( e, e' \in M \). The matching number \( \nu(H) \) of \( H \) is the cardinality \( |M| \) of a largest matching \( M \) in \( H \). The fractional matching number \( \nu^*(H) \) of \( H \) is the maximum of \( \sum_{e \in H} f(e) \) over all non-negative functions \( f : H \to \mathbb{R} \) such
that $\sum_{v \in e} f(v) \leq 1$ for all $v \in V$. Let $\{H_i\}_{i=1}^m$ be a family of hypergraphs. A **system of disjoint representatives** (SDR) of $\{H_i\}_{i=1}^m$ is a matching $e_1, e_2, \ldots, e_m$, such that $e_i \in H_i$ for $1 \leq i \leq m$.

**Theorem 1.2.** [3] Suppose that $\{H_i\}_{i=1}^m$ is a family of $r$-uniform hypergraphs. If $\nu^*(\bigcup_{i \in I} H_i) > r(|I| - 1)$ for all $I \subseteq [m]$, then $\{H_i\}_{i=1}^m$ has an SDR.

### 2. Summary of Results

There is no known procedure for computing the Gamma function for general graphs. We shall prove some inequalities on $\Gamma$ that will enable us to calculate $\Gamma$ for trees and for cycles. Finding a procedure (however inefficient) for calculating $\Gamma$ remains an open problem.

In another direction, we shall show an application of the $\Gamma$ function - a proof of a fractional version of a strengthened version of Ryser’s conjecture.

#### 2.1. Computing Gamma for trees and cycles

Let $G = (V, E)$ be a graph. Recall that a dominating set in $G$ is a subset $S \subseteq V$ which satisfies $N(S) = V$. The **domination number** $\gamma(G)$ of $G$ is the size of a dominating set of smallest cardinality. Its fractional relaxation, denoted by $\gamma^*(G)$, is the minimum of $\sum_{v \in V} g(v)$ over all non-negative functions $g : V \to \mathbb{R}$ which satisfy $\sum_{u \in N(v)} g(u) \geq 1$ for all $v \in V$. From linear programming duality it follows that $\gamma^*(G)$ can be defined equivalently as the maximum of $\sum_{v \in V} f(v)$ over all non-negative functions $f : V \to \mathbb{R}$ which satisfy $\sum_{u \in N(v)} f(u) \leq 1$ for all $v \in V$. Thus $\gamma^*(G)$ is also the fractional relaxation of $\rho(G)$, the **Neighborhood packing number**, which is the maximal cardinality of a set of vertices, such that the distance between any two vertices in this set is at least three. Clearly, we have $\rho \leq \gamma^* \leq \gamma$. Our first main result relates these graph parameters to the Gamma function:

**Theorem 2.1.** For every graph $G$:

$$\rho(G) \leq \Gamma(G) \leq \gamma^*(G).$$

The proof of the above theorem is given in Section 4. It is known that for a tree $T$, $\rho(T) = \gamma^*(T) = \gamma(T)$ (see for example [12]), and hence the above theorem implies the following corollary:

**Corollary 2.2.** For every tree $T$:

$$\Gamma(T) = \rho(T) = \gamma(T).$$

Our second main result is the calculation of the value of the Gamma function for cycles. The following lower bound on the value of Gamma for $C_n$, the cycle on $n$ vertices, was given in [3]:
Theorem 2.3 ([3]).

\[ \Gamma(C_n) \geq \begin{cases} 
\left\lceil \frac{n+1}{3} \right\rceil & n \neq 3k + 2 \\
\frac{n+1}{3} - \frac{1}{2} & n = 3k + 2 
\end{cases} \]

Also, in [14] it was proved that \( \eta(I(C_n)) = \left\lceil \frac{n+1}{3} \right\rceil \) for all \( n \). This, together with the above theorem, and the fact that \( \Gamma \) serves as a lower bound on \( \eta \) implies that \( \Gamma(C_n) = \eta(I(C_n)) = \left\lceil \frac{n+1}{3} \right\rceil \) for \( n \neq 3k + 2 \).

We show that the lower bound from [3] is also tight in the case that \( n = 3k + 2 \):

Theorem 2.4. \( \Gamma(C_n) \leq \frac{n+1}{3} - \frac{1}{2} \) for \( n = 3k + 2 \).

The proof of the above theorem appears in Section 5. The proof relies mainly on the non-singularity of optimal vector representations and the study of the relations between the Gamma function and the largest eigenvalue of the Laplacian matrix of the graph.

2.2. A fractional version of a strengthening of Ryser’s conjecture. Recall the definition of an \( r \)-uniform hypergraph given in Section 1. An \( r \)-uniform hypergraph is called \( r \)-partite if its vertex set \( V \) can be partitioned into sets \( V_1, V_2, \ldots, V_r \) (called the “sides” of the hypergraph) in such a way that each edge meets each \( V_i \) in precisely one vertex.

Recall also the definitions of the matching number \( \nu(H) \) and the fractional matching number \( \nu^*(H) \) of a hypergraph \( H \) given in Section 1. A cover of a hypergraph \( H \) is a subset of \( V \) meeting all edges of \( H \). The covering number \( \tau(H) \) of \( H \) is the minimal size of a cover of \( H \). From linear programming duality it follows that \( \nu^*(H) \) can be defined alternatively as the fractional covering number \( \tau^*(H) \) of \( H \), which is the minimum of \( \sum_{v \in V} g(v) \) over all non-negative functions \( g : V \to \mathbb{R} \) such that \( \sum_{v : e \ni v} g(v) \geq 1 \) for all \( e \in H \).

Obviously we have that \( \tau \geq \nu \) for all hypergraphs. As to the converse inequality, it is not difficult to see that \( \tau \leq r\nu \) for \( r \)-uniform hypergraphs. The following conjecture was made in the early 1970’s (the conjecture appeared in the Ph.D thesis of Henderson, a student of Ryser, and it was independently conjectured in a stronger form by Lovász [10]):

Conjecture 2.5 (Ryser’s conjecture). For an \( r \)-partite \( r \)-uniform hypergraph \( \tau(H) \leq (r - 1)\nu(H) \).

For \( r = 2 \) the conjecture is just König’s theorem [9], which states that in a bipartite graph \( \tau = \nu \).

The case \( r = 3 \) of the above conjecture was proved by Aharoni in [1] using topological methods. For \( r > 3 \) the conjecture remains wide open, but if the fractional version of this conjecture is considered, then Füredi proved in [6] that \( \tau^* \leq (r - 1)\nu \) for \( r \)-partite \( r \)-uniform hypergraphs.
Let $H$ be an $r$-partite $r$-uniform hypergraph with sides $V_1, V_2, \ldots, V_r$. Also, let $w : V \to \mathbb{R}$ be the weight function which satisfies $w(v) = r - 1$ for all $v \in V_1$, and $w(v) = 1$ otherwise. The \textit{unbalanced covering number} of the hypergraph $H$, denoted by $\tau_{ub}(H)$, is the minimal total sum of weights of the vertices in a cover of $H$. Obviously we have that $\tau_{ub} \geq \tau$ for all hypergraphs. The following stronger conjecture was made by Aharoni [2]:

**Conjecture 2.6 ([2]).** Suppose that $H$ is an $r$-partite $r$-uniform hypergraph with sides $V_1, V_2, \ldots, V_r$. Then $\tau_{ub}(H) \leq (r - 1)\nu(H)$.

Obviously, for $r = 2$ the above conjecture is equivalent to Ryser’s conjecture. The $r = 3$ case of the above conjecture is actually implied by the proof of the $r = 3$ case of Ryser’s conjecture in [1]. In this paper we use the Gamma function in order to prove a fractional version of the above conjecture. Suppose that $H$ is an $r$-partite $r$-uniform hypergraph with sides $V_1, V_2, \ldots, V_r$. The \textit{fractional unbalanced covering number} $\tau_{ub}^*(H)$ of $H$ is the minimum of $\sum_{v \in V_1} (r - 1)g(v) + \sum_{v \notin V_1} g(v)$ over all non-negative functions $g : V \to \mathbb{R}$ which satisfy $\sum_{v : v \in e} g(v) \geq 1$ for all $e \in H$.

**Theorem 2.7.** Suppose that $H$ is an $r$-partite $r$-uniform hypergraph with sides $V_1, V_2, \ldots, V_r$. Then $\tau_{ub}^*(H) \leq (r - 1)\nu(H)$.

The proof of the above theorem appears in Section 6. The proof uses similar ideas to those used in [1], but it makes use of the Gamma function instead of the matching width that was used in [1]. We note that the above theorem is not implied by the proof of $\tau^* \leq (r - 1)\nu$ from [6].

2.3. \textbf{Paper Organization.} The rest of the paper is organized as follows. In Section 3 we give the necessary preliminaries and definitions needed later on. In section 4 we compute the value of the Gamma function for trees by proving Theorem 2.1. Then in Section 5 we compute the value of the Gamma function for cycles by proving Theorem 2.4. Finally, in Section 6 we prove the fractional version of the strengthening of Ryser’s theorem (Theorem 2.7). We end this paper by highlighting some interesting open problems and conjectures raised by this work (Section 7).

3. \textbf{Preliminaries and Definitions}

3.1. \textbf{Vector Representations.} Recall that a \textit{vector (domination) representation} of a graph $G$ is an assignment $P$ of a vector $P(v) \in \mathbb{R}^d$ for some fixed $d$ to every vertex $v$ of the graph, such that $P(u) \cdot P(v) \geq 1$ whenever $u, v$ are adjacent in $G$ and $P(u) \cdot P(v) \geq 0$ for all vertices $u$ and $v$. 
With every vector representation $P$ we associate the $|V| \times |V|$ matrix $M_P$, which is the positive semidefinite matrix defined by $(M_P)_{u,v} = P(u) \cdot P(v)$ for every $u,v \in V$. The following lemma is a consequence of the properties of a positive semidefinite matrix:

**Lemma 3.1.** Suppose that $M$ is a $|V| \times |V|$ matrix. Then there exists a vector representation $P$ of $G$ such that $M = M_P$ if and only if $M$ is a positive semidefinite matrix with non-negative entries, such that $M_{u,v} \geq 1$ whenever $u,v$ are adjacent in $G$.

The value $|P|$ of a vector representation can be defined in the following way:

**Definition 3.2.** A non-negative vector $\alpha \in \mathbb{R}^V$ is said to be dominating for $P$ if $\sum_{v \in V} \alpha(v)P(v) \cdot P(u) \geq 1$ for every vertex $u$ (or equivalently: $M_P \cdot \alpha \geq 1_{|V|}$). The value $|P|$ of $P$ is the minimum of $\sum_{v \in V} \alpha(v)$, over all vectors $\alpha$ that are dominating for $P$.

Using linear programming duality one can obtain the following equivalent dual definition of $|P|$:

**Definition 3.3.** A non-negative vector $\beta \in \mathbb{R}^V$ is said to be dually dominating for $P$ if $\sum_{v \in V} \beta(v)P(v) \cdot P(u) \leq 1$ for every vertex $u$ (or equivalently: $M_P \cdot \beta \leq 1_{|V|}$). The value $|P|$ of $P$ is the maximum of $\sum_{v \in V} \beta(v)$, over all vectors $\beta$ that are dually dominating for $P$.

### 3.2. The Gamma function and the weak Gamma function.

Recall that $\Gamma(G)$ is defined to be the supremum of $|P|$ over all vector representations $P$ of $G$. For our purposes it will often be useful to consider a variant of $\Gamma$, called the weak Gamma function, and denoted by $\hat{\Gamma}$. A weak vector representation is a vector representation in which all the vectors are of length at least one, that is $||P(v)||^2 = \sum_{1 \leq i \leq d}(P^2(v))_i \geq 1$ for every $v \in V$. The weak Gamma function $\hat{\Gamma}$ is defined to be the supremum of $|P|$ over all weak vector representations $P$ of $G$. The relation between $\Gamma(G)$ and $\hat{\Gamma}(G)$ is the same as that between the total domination number $\tilde{\gamma}(G)$ and the regular (weak) domination number $\gamma(G)$: the fact that all the vectors in a weak vector representation are of length at least one means that a vector can 'dominate' itself. Thus, while $\Gamma$ uses vectors to mimic total domination, $\hat{\Gamma}$ uses vectors to mimic regular (weak) domination.

**Remark 3.4.** Obviously, we have that $\hat{\Gamma}(G) \leq \Gamma(G)$ for every graph $G$ (since in $\hat{\Gamma}$ we are taking the supremum over a partial subset of the vector representations - only the weak ones). As to the other direction, in [16] (Page 25, Proposition 12) it is proved that $\Gamma(G) \leq 2\hat{\Gamma}(G)$, and we conjecture that equality holds in general.
The following is an analogue of Lemma 3.1 for the case of weak vector representations:

**Lemma 3.5.** Suppose that $M$ is a $|V| \times |V|$ matrix. Then there exists a weak vector representation $P$ of $G$ such that $M = MP$ if and only if $M$ is a positive semidefinite matrix with non-negative entries, such that $M_{u,v} \geq 1$ whenever $u, v$ are adjacent in $G$ and $M_{v,v} \geq 1$ for all $v \in V$.

### 3.3. Duplication of vertices

In our proofs it will often be of value for us to study the behavior of the Gamma function when some of the vertices of the graph are duplicated. For this end we introduce the following definitions. Let $G$ be a graph, and $a \in \mathbb{Z}_+^V$ a non-negative vector. We define the *independent duplication of $G$ with respect to $a$* to be the graph obtained from $G$ by replacing every vertex $v$ in $G$ with an independent set of vertices of size $a_v$. Similarly, we define the *clique duplication of $G$ with respect to $a$* to be the graph obtained from $G$ by replacing every vertex $v$ in $G$ with a clique of size $a_v$. Formally:

**Definition 3.6.** Let $G$ be a graph, and $a \in \mathbb{Z}_+^V$ a non-negative vector. The *independent duplication of $G$ with respect to $a$*, denoted by $G_a$, is the graph on the vertex set $\bigcup_{v \in V} \{v\} \times [a_v]$, where $(u, i)$ and $(v, j)$ are adjacent in $G_a$ if and only if $u$ and $v$ are adjacent in $G$. Similarly, the *clique duplication of $G$ with respect to $a$*, denoted by $G_a$, is the graph on the vertex set $\bigcup_{v \in V} \{v\} \times [a_v]$, where $(u, i)$ and $(v, j)$ are adjacent in $G_a$ if and only if $u$ and $v$ are adjacent in $G$ or $u = v$.

### 3.4. The Laplacian matrix

The computation of the value of the Gamma function for cycles depends heavily on studying the relation between the largest eigenvalue of the Laplacian matrix of the graph and the Gamma function. In this subsection we restate some known facts, needed later on, regarding the eigenvalues of the Laplacian matrix.

For a graph $G$ we denote by $L_G$ the Laplacian matrix of $G$, which is the $|V| \times |V|$ positive semidefinite matrix given by:

$$L_G(u,v) = \begin{cases} 
\text{deg}(u), & u = v \\
-1, & (u,v) \in E \\
0, & \text{otherwise}
\end{cases}$$

We denote by $\lambda_{\text{max}}(G)$ the largest eigenvalue of $L_G$. The second smallest eigenvalue of the Laplacian matrix, called the *spectral gap*, is a parameter of central importance in a variety of problems. In particular, it controls the expansion properties of $G$, and the convergence rate of a random walk on $G$ (for a detailed survey, see [8]). Some known facts (for example, see [7]) regarding the Laplacian matrix and its eigenvalues are:
Fact 3.7. Let $G$ be a graph on $n$ vertices. Denote the complement of $G$ by $\overline{G}$. Then:

1. If $G$ has $c$ connected components, then $\text{rank}(L_G) = n - c$.
2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n = \lambda_{\text{max}}$ denote the eigenvalues of the Laplacian, where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.
   Then $\lambda_i(\overline{G}) = n - \lambda_{n-i+2}(G)$ for every $2 \leq i \leq n$.
3. $\lambda_{\text{max}}(G) \leq n$. If $G$ has $\tau$ connected components, then the multiplicity of $n$ as an eigenvalue of $L_G$ is $\tau - 1$.

From the fact above, we have that $\lambda_{\text{max}}(G)$ is equal to the spectral gap of the complement of $G$.

Another well known fact is the following:

Fact 3.8. $\lambda_{\text{max}}(G) = \max_{\|x\|=1} x^T L_G x = \max_{\|x\|=1} \sum_{(u,v) \in E(G)} (x_u - x_v)^2$.

And as a corollary we have:

Corollary 3.9. If $H$ is a subgraph of $G$ (not necessarily induced), then $\lambda_{\text{max}}(H) \leq \lambda_{\text{max}}(G)$.

4. Computation of Gamma for Trees - Proof of Theorem 2.1

The proof of Theorem 2.1 breaks into two theorems. The first one says that the neighborhood packing number $\rho(G)$ of $G$ serves as a lower bound on the Gamma function, while the second one says that the fractional domination number $\gamma^*(G)$ of $G$ serves as an upper bound on the Gamma function.

We start with the first theorem. Recall the definition of the independent domination number given in Section 1. We define a fractional version of this parameter as follows:

Definition 4.1 (Independent fractional domination number). For a subset $U \subseteq V$ of vertices, we denote by $\gamma^*(G, U)$ the minimum of $\sum_{v \in V} g(v)$ over all non-negative functions $g : V \rightarrow \mathbb{R}$ which satisfy $\sum_{v : v \in N(u)} g(v) \geq 1$ for all $u \in U$. The independent fractional domination number, denoted by $i\gamma^*(G)$, is the maximum of $\gamma^*(G, I)$ over all independent sets $I$ in $G$.

It is not difficult to verify that $\rho(G) \leq i\gamma^*(G) \leq i\gamma(G)$. Hence, in order to prove the first part of Theorem 2.1 it suffices to prove the following slightly stronger theorem:

Theorem 4.2. For every graph $G$:

$$i\gamma^*(G) \leq \Gamma(G).$$

Proof. Let $I$ be an independent set of vertices in the graph such that $i\gamma^*(G) = \gamma^*(G, I)$. Our goal will be to construct a vector representation $P$ of $G$ such that $|P| \geq \gamma^*(G, I)$. Since $\Gamma$ is defined to be the supremum over the values of all vector representations, this will imply that $i\gamma^*(G) = \gamma^*(G, I) \leq \Gamma(G)$.
Without loss of generality, suppose that $V = \{v_1, v_2, \ldots, v_n\}$, and $I = \{v_1, v_2, \ldots, v_d\}$. For every $v \in V$ let $P(v)$ be the zero-one $d+1$-dimensional vector, whose $i$-th entry is the indicator function of the set $N(v_i)$ for $1 \leq i \leq d$, and whose $d+1$ entry is the indicator function of the set $V \setminus I$. Formally:

$$(P(v))_i = \begin{cases} 
1, & (1 \leq i \leq d \text{ and } v \in N(v_i)) \text{ or } (i = d+1 \text{ and } v \notin I) \\
0, & \text{otherwise}
\end{cases}$$

We claim that $P$ is a valid vector representation of $G$. Obviously, $P(u) \cdot P(v) \geq 0$ for all $u, v \in V$.

Suppose that $u, v$ are neighbors in $G$. If one of the vertices, say $u$, is in $I$, then $u = v_i$ for some $1 \leq i \leq d$, and $v \in N(v_i)$. In this case $(P(u))_i = (P(v))_i = 1$, which implies (since all the coordinates of $P(u)$ and $P(v)$ are non-negative) that $P(u) \cdot P(v) \geq 1$. Otherwise, both vertices are not in $I$, which implies that $P(u)_{d+1} = P(v)_{d+1} = 1$, and this in turn implies that $P(u) \cdot P(v) \geq 1$ also in this case.

It remains to show that $|P| \geq \gamma^*(G,I)$. Suppose that $\alpha \in \mathbb{R}^V$ is dominating for $P$, such that $\sum_{v \in V} \alpha(v) = |P|$. Then for all $v \in I$:

$$1 \leq \sum_{u \in V} \alpha(u)P(u) \cdot P(v) = \sum_{u \in N(v)} \alpha(u)\overbrace{P(u) \cdot P(v)}^{=1} + \sum_{u \notin N(v)} \alpha(u)\overbrace{P(u) \cdot P(v)}^{=0} = \sum_{u \in N(v)} \alpha(u)$$

Thus, $\alpha : V \to \mathbb{R}$ is a non-negative function which satisfies $\sum_{u \in N(v)} \alpha(u) \geq 1$ for every vertex $v \in I$. From Definition 4.1 this implies that $\gamma^*(G,I) \leq \sum_{v \in V} \alpha(v) = |P|$, concluding the proof of the theorem. $\square$

We move on to the proof of the second part of Theorem 2.1:

**Theorem 4.3.** For every graph $G$:

$$\Gamma(G) \leq \gamma^*(G).$$

**Proof.** Suppose that $P$ is a vector representation of $G$, and $g : V \to \mathbb{R}$ is a non-negative function which satisfies $\sum_{u \in N(v)} g(u) \geq 1$ for all $v \in V$ and $\sum_{v \in V} g(v) = \gamma^*$. Our goal will be to show that $|P| \leq \sum_{v \in V} g(v) = \gamma^*$ by constructing a vector $\alpha \in \mathbb{R}^V$ which is dominating for $P$, and satisfies $\sum_{v \in V} \alpha(v) \leq \sum_{v \in V} g(v) = \gamma^*$.

Our main observation is that $\Gamma(G[N(v)]) \leq 1$ for every $v \in V$, where $G[N(v)]$ is the subgraph of $G$ induced by the vertices in $N(v)$. This is true since the simplicial complex $I(G[N(v)])$ is disconnected which implies $\eta(I(G[N(v)])) = 1$. This in turn implies that $\Gamma(G[N(v)]) \leq 1$ due to the fact that $\Gamma$ serves as a lower bound on $\eta$. 

Next we show how to construct the vector $\alpha$. Fix $v \in V$. Note that the restriction $P|_{N(v)}$ of $P$ to the set $N(v)$ is a valid vector representation of $G[N(v)]$, and therefore $|P|_{N(v)}| \leq 1$. This guarantees the existence of a vector $\alpha^v \in \mathbb{R}^{N(v)}$, such that $\sum_{u \in N(v)} \alpha^v(u) \leq 1$, and for every $w \in N(v)$:

$$\sum_{u \in N(v)} \alpha^v(u) P(u) \cdot P(w) \geq 1$$

Let $\alpha \in \mathbb{R}^{V}$ be the vector which satisfies $\alpha(u) = \sum_{v: u \in N(v)} \alpha^v(u) \cdot g(v)$ for every $u \in V$. It remains to show that $\alpha$ is dominating for $P$ and that $\sum_{u \in V} \alpha(u) \leq \sum_{v \in V} g(v)$.

But $\alpha$ is dominating for $P$ since for every $w \in V$:

$$\sum_{u \in V} \alpha(u) P(u) \cdot P(w) = \sum_{u \in V} \left( \sum_{v: u \in N(v)} \alpha^v(u) \cdot g(v) \right) P(u) \cdot P(w) \geq \sum_{v \in V} g(v) \left( \sum_{u \in N(v)} \alpha^v(u) \cdot P(u) \cdot P(w) \right) \geq 0$$

Also, $\sum_{u \in V} \alpha(u) \leq \sum_{v \in V} g(v)$ since:

$$\sum_{u \in V} \alpha(u) = \sum_{v \in V} \left( \sum_{u \in N(v)} \alpha^v(u) \cdot g(v) \right) = \sum_{v \in V} g(v) \left( \sum_{u \in N(v)} \alpha^v(u) \right) \leq \sum_{v \in V} g(v)$$

which concludes the proof of the theorem.

\[\square\]

5. Computation of Gamma for Cycles - Proof of Theorem 2.4

The proof of Theorem 2.4 consists of three main steps. The first main step is the following lemma, which says that optimal vector representations are non-singular. For a vector representation $P$ let $\text{dim}(P)$ denote the dimension of the vector space $\text{span}\{P(v)|v \in V\}$ (or equivalently: $\text{dim}(P) = \text{rank}(M_P)$). A vector representation is said to be non-singular if $\text{dim}(P) < |V|$.

**Lemma 5.1** (Optimal vector representations are non-singular). For every weak vector representation $P$ there exists a non-singular weak vector representation $P'$ such that $|P'| \geq |P|$. 
The proof of the above lemma is given in Subsection 5.1. This lemma alone is enough to prove that \( \hat{\Gamma}(C_n) \leq \frac{n+1}{3} - \frac{1}{2} \) for \( n = 3k + 2 \), and our second main step is proving the following stronger statement that will help us later in obtaining an upper bound on \( \Gamma(C_n) \):

**Lemma 5.2** (\( \hat{\Gamma} \) of duplicated cycles). Recall the definition of the clique duplication \( G_\pi \) of a graph \( G \) given in Definition 3.6. Suppose that \( C \) is a cycle on \( n \) vertices, \( n = 3k + 2 \), and \( a \in \mathbb{Z}_+^n \) is a non-negative vector. Then \( \hat{\Gamma}(C_a) \leq \frac{n+1}{3} - \frac{1}{2} \).

The proof of the above lemma appears in Subsection 5.2. Choosing \( a \) to be the all ones vector in this lemma gives the following corollary:

**Corollary 5.3.** \( \hat{\Gamma}(C_n) \leq \frac{n+1}{3} - \frac{1}{2} \) for \( n = 3k + 2 \).

We do not know whether \( \hat{\Gamma} \) is always larger than \( \Gamma \) (see Remark 3.4), and therefore we cannot apply the above corollary directly in order to prove Theorem 2.4. Hence, the last main step in the proof is proving that the following weaker relation between \( \Gamma \) and \( \hat{\Gamma} \) indeed holds, and this together with the stronger Lemma 5.2 imply Theorem 2.4:

**Lemma 5.4.** For every graph \( G \): \( \Gamma(G) \leq \sup_{a \in \mathbb{Z}_+^n} \hat{\Gamma}(G_\pi) \).

The proof of the above lemma appears in Subsection 5.3 and it relies mainly on the study of the relation between the Gamma function and the largest eigenvalue of the Laplacian matrix of the graph.

5.1. **Optimal vector representations are non-singular - Proof of Lemma 5.1.**

*Proof of Lemma 5.1.* Suppose that \( \dim(P) = |V| \) (otherwise there is nothing to prove). For every vector \( a \in \mathbb{R}^{|V|} \), let \( M_P(a) \) denote the matrix obtained from \( M_P \) by replacing the \((v, v)\) entry of \( M_P \) with \( a_v \) for every \( v \in V \). Let \( K := \{ a \in \mathbb{R}^{|V|} \mid 1 \leq a_v \leq (M_P)_{v,v} \quad \forall v \in V \} \). Our goal will be to show that there exists a vector \( a \in K \) such that \( \mu_{\min}(M_P(a)) = 0 \).

This will imply that \( M_P(a) \) is a positive semidefinite matrix with non-negative entries, such that \((M_P(a))_{u,v} \geq 1\) whenever \( u, v \) are adjacent in \( G \), and \((M_P(a))_{v,v} \geq 1\) for every \( v \in V \), and hence there exists a weak vector representation \( P' \) such that \( M_{P'} = M_P(a) \). Moreover, \( M_P(a) = M_{P'} \) is non-singular, and hence \( \dim(P') < |V| \). Finally, all the values of the entries of \( M_{P'} \) are greater or equal than the values of the corresponding entries in \( M_P \), and this will imply that \( |P'| \geq |P| \) which concludes the proof of the lemma.
Let $M' := M_P(1_{|V|})$. We argue that $M'$ is not positive definite. Indeed, take $u, v \in V$ such that $u$ is adjacent to $v$. Then \[
abla \begin{pmatrix} M_{u,v} & M_{v,u} \\ M_{u,v} & M_{u,u} \end{pmatrix} = \begin{pmatrix} 1 & \geq 1 \\ \geq 1 & 1 \end{pmatrix}
abla\] is a principal minor of $M'$ with a non-positive determinant. Hence $M'$ cannot be positive definite, and in particular, $\mu_{\min}(M') \leq 0$, where $\mu_{\min}(M')$ is the smallest eigenvalue of $M'$. On the other hand our assumption that $\dim(P) = |V|$ implies that $M_P$ is (strictly) positive definite, and hence $\mu_{\min}(M_P)$ is strictly positive. Noting that $\mu_{\min}(M_P(a))$ is continuous on the compact set $K$ we conclude that there exists $a \in K$ such that $\mu_{\min}(M_P(a)) = 0$ which concludes the proof of the lemma. \qed

5.2. $\hat{\Gamma}$ of duplicated cycles - Proof of Lemma 5.2.

Proof of Lemma 5.2. For convenience, let $V = \{v_1, v_2, \ldots, v_n\}$ and $V_a := \bigcup_{i=1}^n \{v_i\} \times \{a_i\}$ denote the vertex sets of $G$ and $G_{\pi}$ respectively. Suppose that $\{P_t\}_{t \in \mathbb{N}}$ is a sequence of weak vector representations of $C_{\pi}$, such that $\lim_{t \to \infty} |P_t| = \hat{\Gamma}(C_{\pi})$. From Lemma 5.1 without loss of generality we may assume that $P_t$ are non-singular vector representations, that is $\dim(P_t) < |V_a|$ for all $t \in \mathbb{N}$.

It is well-known (see for example [15], Lemmas 2.4. and 2.5.) that every linear program of the form $\max \{c \cdot x | A \cdot x \leq b, A \in \mathbb{R}^{n \times n}\}$ has an optimal solution with at least $n - \text{rank}(A)$ zero entries. This implies for every $t \in \mathbb{N}$ the existence of a vector $\beta_t \in \mathbb{R}^{V_a}$ which is dually dominating for $P_t$ and satisfies $\sum_{v \in V_a} \beta_t(v) = |P_t|$ with at least $|V_a| - \dim(P_t) \geq 1$ zero entries. Since $V_a$ is finite, there exists $v' \in V_a$ such that $\beta_t(v') = 0$ for infinitely many values of $t$, without loss of generality suppose that $v'$ correspond to the vertex $v_1$ in the original graph $G$. Our goal will be to show that $\sum_{v \in V_a} \beta_t(v) \leq \frac{n+1}{2} - \frac{1}{2}$ for every such $t$. This will conclude the proof of the lemma due to our assumptions that $\sum_{v \in V_a} \beta_t(v) = |P_t|$ and $\lim_{t \to \infty} |P_t| = \hat{\Gamma}(C_{\pi})$.

We observe that it suffices to prove the lemma only for the case in which $a_{v_1} = 1$. We claim that if this is not the case, then $\hat{\Gamma}(C_{\pi}) \leq \hat{\Gamma}(C_{\pi})$, where $a' \in \mathbb{Z}^n_\pi$ is the vector defined by:

$$a'(v) = \begin{cases} a(v), & v \neq v' \\ a(v) - 1, & v = v' \end{cases}$$

and hence it suffices to prove the lemma for the vector $a'$ instead of $a$. In order to show that $\hat{\Gamma}(C_{\pi}) \leq \hat{\Gamma}(C_{\pi})$, it is enough to show an infinite sequence of weak vector representation $P'_t$ of $C_{\pi}$ each with value at least $|P_t|$. But one can check that letting $P'_t$ be the vector representation $P_t$ restricted to $V_a \setminus \{v'\}$ for every $t$ such that $\beta_t(v') = 0$ satisfies the latter requirement.
Fix $t \in \mathbb{N}$ such that $\beta_t(v') = 0$. Since $\beta_t$ is dually dominating for $P_t$, for every $i \in [n]$ the following inequality holds:

$$\sum_{k=1}^{n} \sum_{j=1}^{a_{v_i}} \beta_t(v_k,j) P_t(v_k,j) \cdot P_t(v_t,1) \leq 1$$  \hspace{1cm} (5.1)

For every $i \in [n]$ let $b_i$ be the sum of the values of $\beta_t$ on the duplicated vertices of $v_i$. Formally:

$$b_i = \sum_{j=1}^{a_{v_i}} \beta_t(v_i,j)$$

Since $P_t$ is a weak vector representation and from Equation (5.1) for every $i \in [n]$ we have that:

$$b_{(i-1)(\mod n)} + b_i + b_{(i+1)(\mod n)} \leq 1$$

Let $S$ be the subset of the above inequalities which include only the inequalities that correspond to $i \neq 2(\mod 3)$. First we note that since $n = 3k + 2$ there are $\frac{2(n+1)}{3} - 1$ $i$'s of this form, and hence $|S| = \frac{2(n+1)}{3} - 1$. Also, for every $i > 1$ $b_i$ appears in exactly two different inequalities in $S$, while $b_1$ appears in exactly one inequality in $S$. Therefore summing up all the inequalities of $|S|$ we obtain:

$$b_1 + 2 \sum_{i=2}^{n} b_i \leq \frac{2(n + 1)}{3} - 1$$

But our assumptions that $\beta_t(v_1,1) = 0$ and $a_{v_1} = 1$ imply that $b_1 = 0$, and this in turn implies that

$$\sum_{v \in V_a} \beta_t(v) = \sum_{i=1}^{n} b_i \leq \frac{n+1}{3} - \frac{1}{2}$$

which concludes the proof of the lemma.

5.3. $\Gamma(G) \leq \sup_{a \in \mathbb{Z}_+} \hat{\Gamma}(G_a) \cdot \Gamma(G)$ - Proof of Lemma 5.4. The proof of Lemma 5.4 relies on two main lemmas. The first lemma presents a new relation between the maximal eigenvalue of the Laplacian matrix of the graph and the Gamma function. In [3] it was proved that $\sup_{a \in \mathbb{Z}_+} \frac{|V(G_a)|}{\lambda_{\max}(G_a)}$ is an upper bound on $\Gamma$ for every graph $G$. We complement this upper bound by showing that $\frac{|V(G)|}{\lambda_{\max}(G)}$ is a lower bound on $\Gamma$ (and also on $\hat{\Gamma}$) for every graph $G$.

Lemma 5.5. For every graph $G$:

$$\frac{|V(G)|}{\lambda_{\max}(G)} \leq \hat{\Gamma}(G) \leq \Gamma(G) \leq \sup_{a \in \mathbb{Z}_+} \frac{|V(G_a)|}{\lambda_{\max}(G_a)}.$$
The second lemma says that the maximal eigenvalue of the Laplacian matrix of the independent duplication of $G$ with respect to some vector $a$ is equal to that of the clique duplication of $G$ with respect to the same vector $a$.

**Lemma 5.6.** For every connected graph $G$ and a vector $a \in \mathbb{Z}^V_+$:

$$\lambda_{\text{max}}(G_a) = \lambda_{\text{max}}(G_\pi).$$

Before proving the above lemmas we show how they imply Lemma 5.4:

**Proof of Lemma 5.4.**

$$\Gamma(G) \leq \sup_{a \in \mathbb{Z}^V_+} \frac{|V(G_a)|}{\lambda_{\text{max}}(G_a)} \quad \text{(Due to the righthand inequality of Lemma 5.5)}$$

$$= \sup_{a \in \mathbb{Z}^V_+} \frac{|V(G_\pi)|}{\lambda_{\text{max}}(G_\pi)} \quad \text{(Due to Lemma 5.6)}$$

$$\leq \sup_{a \in \mathbb{Z}^V_+} \Gamma(G_\pi) \quad \text{(Due to the lefthand inequality of Lemma 5.5)}$$

□

We proceed to the proofs of Lemmas 5.5 and 5.6. In order to prove these lemmas we need the following lemma which shows the following relation between the maximal eigenvalue of the Laplacian matrix of a graph and the maximal degree of the graph:

**Lemma 5.7.** Let $G$ be a graph, denote by $\Delta(G)$ the maximum degree of a vertex in $G$. Then $\lambda_{\text{max}}(G) \geq \Delta(G) + 1$. Moreover, if $G$ is connected, and $|V(G)| > \Delta(G) + 1$, then $\lambda_{\text{max}}(G) > \Delta(G) + 1$.

**Proof.** Denote by $S_\Delta$ the graph which is a star with $\Delta + 1$ vertices and $\Delta$ edges. Since the complement of $S_\Delta$ has two connected components, Fact 3.7 implies that the multiplicity of $\Delta + 1$ as an eigenvalue of $L_{S_\Delta}$ is one, and hence $\lambda_{\text{max}}(S_\Delta) = \Delta + 1$. Since $G$ contains a vertex of degree $\Delta$, $S_\Delta$ is a subgraph of $G$. Corollary 3.9 then implies that $\lambda_{\text{max}}(G) \geq \lambda_{\text{max}}(S_\Delta) = \Delta + 1$.

For the proof of the second part of the lemma, suppose that $G$ is connected and that it has more than $\Delta + 1$ vertices. Denote by $S'_\Delta$ the graph obtained from $S_\Delta$ by adding a new vertex $v$ and connecting it to one of the vertices of degree one in $S_\Delta$. Since $G$ is connected, and has more than $\Delta + 1$ vertices, $S'_\Delta$ is a subgraph of $G$. Therefore it suffices to show that $\lambda_{\text{max}}(S'_\Delta) > \Delta + 1$, and then Corollary 3.9 will imply that $\lambda_{\text{max}}(G) \geq \lambda_{\text{max}}(S'_\Delta) > \Delta + 1$. 

\[ \]
Observe that the matrix \( B := (\Delta + 1)I - L_{S_{\Delta}'} \) is of the form:

\[
B = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 0 \\
1 & \Delta & 0 & \cdots & 0 & 0 \\
1 & 0 & \ddots & 0 & \vdots & \vdots \\
\vdots & \vdots & 0 & \Delta & 0 & 0 \\
1 & 0 & \cdots & 0 & \Delta - 1 & 1 \\
0 & 0 & \cdots & 0 & 1 & \Delta \\
\end{pmatrix}
\]

Computation of the determinant of \( B \) by permutation expansion gives that \( \det(B) = -(1 + \Delta) \cdot \Delta^{\Delta - 2} < 0 \). Hence \( B \) has a negative eigenvalue, and this implies that \( L_{S_{\Delta}'} \) has an eigenvalue larger than \( \Delta + 1 \). So we have that \( \lambda_{\text{max}}(S_{\Delta}') > \Delta + 1 \), concluding the proof of the second part of the lemma. \( \square \)

**Proof of Lemma 5.5.** We only need to show that \( \frac{|V(G)|}{\lambda_{\text{max}}(G)} \leq \hat{\Gamma}(G) \). Take the \( |V| \times |V| \) matrix \( M := \lambda_{\text{max}}(G)I - L_G \). Since all the eigenvalues of \( M \) are non-negative, \( M \) is a positive semidefinite matrix. Also, \( M \) has non-negative entries, and whenever \( u, v \) are adjacent in \( G \) we have that \( (M)_{u,v} = 1 \).

Hence, from Lemma 3.1 we have that \( M = M_P \) for some vector representation \( P \) of \( G \).

Let \( \beta \in \mathbb{R}^V \) be the vector which satisfies \( \beta(v) = \frac{1}{\lambda_{\text{max}}(G)} \) for all \( v \in V \). We argue that \( \beta \) is dually dominating for \( P \) since for every \( v \in V \):

\[
(M_P \cdot \beta)_v = \frac{1}{\lambda_{\text{max}}} \left( \lambda_{\text{max}} - \deg(v) + \sum_{u: (u,v) \in E} 1 \right) = 1
\]

We conclude that \( |P| \geq \sum_{v \in V} \beta(v) = \frac{|V|}{\lambda_{\text{max}}} \) (Actually \( \beta \) is also dominating for \( P \), and hence \( |P| \leq \frac{|V|}{\lambda_{\text{max}}} \)), which implies that \( |P| = \frac{|V|}{\lambda_{\text{max}}} \). Since \( \Gamma \) is defined to be the supremum over the values of all vector representations, this implies that \( \frac{|V|}{\lambda_{\text{max}}} \leq \Gamma(G) \).

In order to show the stronger statement that \( \frac{|V|}{\lambda_{\text{max}}} \leq \hat{\Gamma}(G) \) it suffices to show that \( P \) is a weak vector representation, or equivalently that every diagonal entry of \( M_P \) is of value at least 1. But this is true since Lemma 5.7 implies that \( \lambda_{\text{max}}(G) \geq \deg(v) + 1 \) for every vertex \( v \). \( \square \)

**Proof of Lemma 5.6.** If \( G \) has \( \Delta + 1 \) vertices, then \( \overline{G_{\sigma}} \) and \( \overline{G_{\pi}} \) are both disconnected, and hence Fact 3.7 implies that the multiplicity of \( \sum_{v \in V} a_v = |V(G_a)| = |V(G_{\pi})| \) as an eigenvalue of \( L_{G_a} \), and also as an eigenvalue of \( L_{G_{\pi}} \), is non-zero. This implies that \( \lambda_{\text{max}}(G_a) = \lambda_{\text{max}}(G_{\pi}) = \sum_{v \in V} a_v \). Hence we may assume that \( G \) has more than \( \Delta + 1 \) vertices.
First we note that $G_a$ is a subgraph of $G_{\pi}$, and hence from Corollary 3.9 we have that $\lambda_{\text{max}}(G_a) \leq \lambda_{\text{max}}(G_{\pi})$. So it remains to prove that $\lambda_{\text{max}}(G_a) \geq \lambda_{\text{max}}(G_{\pi})$.

The main step of the proof is showing the existence of an eigenvector $x \neq 0$ of $L_{G_{\pi}}$, which corresponds to the eigenvalue $\lambda_{\text{max}}(G_{\pi})$, such that $x_{(v,i)} = x_{(v,j)}$ for all $v \in V$, and $i, j \in [a_v]$. This will conclude the proof of the lemma since this will imply that

$$\lambda_{\text{max}}(G_a) = \max_{\|z\| = 1} z^T L_{G_a} z \geq x^T L_{G_a} x = \sum_{(u,v) \in E} \sum_{i=1}^{a_u} \sum_{j=1}^{a_v} (x_{(u,i)} - x_{(v,j)})^2 =$$

$$\sum_{(u,v) \in E} \sum_{i=1}^{a_u} \sum_{j=1}^{a_v} (x_{(u,i)} - x_{(v,j)})^2 + \sum_{v \in V} \sum_{1 \leq i < j \leq a_v} (x_{(v,i)} - x_{(v,j)})^2 = x^T L_{G_a} x = \lambda_{\text{max}}(G_{\pi})$$

Suppose that $y \neq 0$ is an eigenvector of $L_{G_a}$ that corresponds to the eigenvalue $\lambda_{\text{max}}(G_{\pi})$. Let $x$ be the vector which satisfies $x_{(v,i)} = \frac{1}{a_v} \sum_{j=1}^{a_v} y_{(v,j)}$ for every $v \in V$, and $i \in [a_v]$. Since for every $v \in V$ and $i, j \in [a_v]$ the vertices $(v, i)$ and $(v, j)$ are isomorphic in $G_{\pi}$, we have that $x$ is an eigenvector of $L_{G_{\pi}}$, which corresponds to the eigenvalue $\lambda_{\text{max}}(G_{\pi})$. It remains to prove that $x \neq 0$.

It suffices to show that there exists $v \in V$ such that $\sum_{j=1}^{a_v} y_{(v,j)} \neq 0$. Suppose in contradiction that $\sum_{j=1}^{a_v} y_{(v,j)} = 0$ for every $v \in V$. Since $y \neq 0$ there exists $v_0 \in V$ and $i_0 \in [a_{v_0}]$, such that $y_{(v_0, i_0)} \neq 0$.

From the definition of the Laplacian matrix we have that:

$$(L_{G_{\pi}} y)_{(v_0, i_0)} = \left( a_{v_0} - \sum_{u: (u, v_0) \in E} a_u \right) y_{(v_0, i_0)} + \sum_{u: (u, v_0) \in E} (-1) \cdot \left( \sum_{j=1}^{a_u} y_{(v,j)} \right) + \left( -\sum_{j \in \{a_{v_0}\} \setminus \{i_0\}} y_{(v_0, j)} \right) =$$

$$\left( a_{v_0} + \sum_{u: (u, v_0) \in E} a_u \right) y_{(v_0, i_0)}$$

This implies that $\lambda_{\text{max}}(G_{\pi}) = a_{v_0} + \sum_{u: (u, v_0) \in E} a_u$. But since $G$ is connected and $|V(G)| > \Delta(G) + 1$ we also have that $V(G_{\pi})$ is connected and $|V(G_{\pi})| > \Delta(G_{\pi}) + 1$ and therefore Lemma 5.5 implies that

$$\lambda_{\text{max}}(G_{\pi}) > \Delta(G_{\pi}) + 1 \geq a_{v_0} + \sum_{u: (u, v_0) \in E} a_u$$

- a contradiction (the last inequality is due to the fact that the degree of the vertex $(v_0, i_0)$ in $G_{\pi}$ equals $a_{v_0} + \sum_{u: (u, v_0) \in E} a_u$).

\[\square\]
Proof of Theorem 2.7

The main tool used for the proof of Theorem 2.7 is the following theorem from [3] (this theorem follows from the combination of Theorem 1.1 and the $\Gamma(G) \leq \eta(I(G))$ theorem proved in [3]):

**Theorem 6.1 ([3]).** Suppose that $G$ is a graph on $V = \bigcup_{i=1}^{m} V_i$, where $V_1, V_2, \ldots, V_m$ are disjoint sets that form a partition of $V$. If $\Gamma(G[\bigcup_{i \in I} V_i]) \geq |I|$ for all $I \subseteq [m]$, then the system $\{V_i | i \in [m]\}$ has an ISR with respect to $G$.

For our purposes we need a variant of the above theorem that applies also when the sets $V_1, V_2, \ldots, V_m$ are not disjoint. Indeed, we were able to prove such a variant exists when $\Gamma$ is replaced by the weak Gamma function $\hat{\Gamma}$:

**Theorem 6.2.** Suppose that $G$ is a graph on $V = \bigcup_{i=1}^{m} V_i$, where $V_1, V_2, \ldots, V_m$ are subsets of $V$ (not necessarily disjoint). If $\hat{\Gamma}(G[\bigcup_{i \in I} V_i]) \geq |I|$ for all $I \subseteq [m]$, then the system $\{V_i | i \in [m]\}$ has an ISR with respect to $G$.

**Proof.** Recall the definition of the clique duplication given in Subsection 3.3. Let $G' = (V', E')$ be the clique duplication of $G$ with respect to the vector $a$, where $a_v = |\{i \in [m] | v \in V_i\}|$ for all $v \in V$. Also, for every $i \in [m]$ let $V'_i := \bigcup_{v \in V_i} \{v\} \times \{a_v\}$. It is easy to check that $V'_1, V'_2, \ldots, V'_m$ are disjoint sets that form a partition of $V'$, and that the system $\{V_i | i \in [m]\}$ has an ISR with respect to $G$ if and only if the system $\{V'_i | i \in [m]\}$ has an ISR with respect to $G'$. Hence in order to prove the theorem it suffices to show that the system $\{V'_i | i \in [m]\}$ has an ISR with respect to $G'$.

Our main observation is that $\hat{\Gamma} \left( G' \mid \bigcup_{j \in J} V'_j \right) \geq \hat{\Gamma} \left( G \mid \bigcup_{j \in J} V_j \right)$ for every $J \subseteq [m]$. This is true since for every weak vector representation $P$ of $G \mid \bigcup_{j \in J} V_j$ the vector representation $Q$ of $G' \mid \bigcup_{j \in J} V'_j$ given by $Q(v, i) = P(v)$ for all $v \in V, i \in a_v$ is a valid weak vector representation of $G' \mid \bigcup_{j \in J} V'_j$ with value at least $|P|$ (note that we are using here the fact that $P$ is a weak vector representation, and this is where the fact that $\Gamma$ is replaced by $\hat{\Gamma}$ is crucial for us).

Concluding, for all $J \subseteq [m]$ we have that:

$$\Gamma \left( G' \mid \bigcup_{j \in J} V'_j \right) \geq \hat{\Gamma} \left( G' \mid \bigcup_{j \in J} V'_j \right) \geq \hat{\Gamma} \left( G \mid \bigcup_{j \in J} V_j \right) \geq |J|$$

Theorem 6.1 then implies that the system $\{V'_i | i \in [m]\}$ has an ISR with respect to $G'$ concluding the proof of the theorem. \qed
Another observation made in [3] that will be of value for us is the following. Let \( H \) be a hypergraph. The line graph \( L(H) \) of \( H \) is the graph on the vertex set \( H \), where \( \{e, e'\} \in L(H) \) if and only if \( e' \cap e \neq \emptyset \).

It was observed in [3] that \( \Gamma(L(H)) \geq \frac{\nu^r(H)}{r} \) for an \( r \)-uniform hypergraph \( H \). This can be seen by taking the vector representation \( P : H \to \mathbb{R}^V \) which assigns to every \( e \in H \) the incidence vector of \( e \), and noticing that \( |P| \geq \frac{\nu^r(H)}{r} \). We note that the inequality \( \Gamma(L(H)) \geq \frac{\nu^r(H)}{r} \) also holds when \( \Gamma \) is replaced by \( \hat{\Gamma} \) since the above vector representation \( P \) is actually a weak vector representation.

**Lemma 6.3.** For an \( r \)-uniform hypergraph \( H \), \( \hat{\Gamma}(L(H)) \geq \frac{\nu^r(H)}{r} \).

As was the case in [1], our main theorem (Theorem 2.7) follows from the following deficiency version of Theorem 6.2:

**Lemma 6.4 (Deficiency version of Theorem 6.2).** Suppose that \( G \) is a graph on \( V = \bigcup_{i=1}^{m} V_i \), where \( V_1, V_2, \ldots, V_m \) are subsets of \( V \) (not necessarily disjoint). For all \( J \subseteq [m] \) let:

\[
\text{def}(J) := |J| - \hat{\Gamma} \left( G \mid \bigcup_{j \in J} V_j \right)
\]

and let \( d := \max_{J \subseteq [m]} \text{def}(J) \). Then the system \( \{V_i \mid i \in [m]\} \) has a partial ISR of size at least \( m - d \) with respect to \( G \).

**Proof.** First, we note that \( d \geq 0 \), since \( \text{def}(\emptyset) = 0 \). We split into two cases.

The first case is when \( d = 0 \). In this case \( \hat{\Gamma} \left( G \mid \bigcup_{j \in J} V_j \right) \geq |J| \) for all \( J \subseteq [m] \), and hence from Theorem 6.2 we have that the system \( \{V_i \mid i \in [m]\} \) has an ISR of size \( m \) with respect to \( G \).

The second case is when \( d > 0 \). Let \( \hat{G} \) be the graph obtained from \( G \) by adding \( d \) isolated vertices \( v_1, v_2, \ldots, v_d \), and for every \( j \in [m] \) let \( \hat{V}_j := V_j \cup \{v_1, v_2, \ldots, v_d\} \). Our main observation is that

\[
(6.1) \quad \hat{\Gamma} \left( \hat{G} \mid \bigcup_{j \in J} \hat{V}_j \right) \leq \hat{\Gamma} \left( G \mid \bigcup_{j \in J} V_j \right) + d
\]

for all \( J \subseteq [m] \). To see this suppose that \( \hat{P} \) is a weak vector representation of \( \hat{G} \mid \bigcup_{j \in J} \hat{V}_j = \left( G \mid \bigcup_{j \in J} V_j \right) \cup \{v_1, v_2, \ldots, v_d\} \) and \( \hat{\beta} \) is dually dominating for \( \hat{P} \) such that \( \sum_{v \in \cup_{j \in J} \hat{V}_j} \hat{\beta}(v) = |P| \).

Let \( P \) and \( \beta \) be the restrictions of \( \hat{P} \) and \( \hat{\beta} \) to \( \cup_{j \in J} V_j \) respectively. Then it is easy to check that \( P \) is a weak vector representation of \( G \mid \bigcup_{j \in J} V_j \), and \( \beta \) is dually dominating for \( P \). Also, since all the
vectors in $\tilde{P}$ are of length at least one we have that $\beta(v_i) \leq 1$ for all $1 \leq i \leq d$. But this means that

$$|P| \geq \sum_{v \in \bigcup_{j \in J} V_j} \beta(v) \geq \sum_{v \in \bigcup_{j \in J} \tilde{V}_j} \beta(v) - d = |\tilde{P}| - d$$

Concluding, for every weak vector representation $\tilde{P}$ of $\tilde{G} | \bigcup_{j \in J} \tilde{V}_j$ we have found a weak vector representation $P$ of $G | \bigcup_{j \in J} V_j$ of size at least $|\tilde{P}| - d$ which implies (6.1). From (6.1), for all $J \subseteq [m]$ we have that

$$\hat{\Gamma} \left( \tilde{G} | \bigcup_{j \in J} \tilde{V}_j \right) \leq \hat{\Gamma} \left( G | \bigcup_{j \in J} V_j \right) + d = (|J| - \text{def}(J)) + d \geq |J|$$

Similarly to the previous case, Theorem 6.2 implies that the system $\{\tilde{V}_i \mid i \in [m]\}$ has an ISR of size $m$ with respect to $\tilde{G}$. This implies in turn that the system $\{V_i \mid i \in [m]\}$ has a partial ISR of size at least $m - d$ with respect to $G$. □

We are now ready to prove our main theorem:

**Proof of Theorem 2.7.** Let $V_1 := \{v_1, v_2, \ldots, v_m\}$, and for every $j \in [m]$ let:

$$H'_j = \{e \setminus \{v_j\} \mid e \in E, e \cap V_1 = \{v_j\}\}$$

Also, let:

$$H' = \bigcup_{j \in [m]} H'_j = \{e \setminus \{v\} \mid e \in H and e \cap V_1 = \{v\}\}$$

Finally, let $d := \max_{J \subseteq [m]} [\text{def}(J)]$ be the deficiency of the system of subsets $H'_1, H'_2, \ldots, H'_m$ in the line graph $L(H')$, and let $J \subseteq [m]$ be the subset of $[m]$ for which the maximum in the definition of $d$ is attained.

Our main observation is that $\bigcup_{j \in J} H'_j$ is an $r - 1$-uniform hypergraph, and hence Lemma 6.3 implies that

$$\tau^* \left( \bigcup_{j \in J} H'_j \right) = \nu^* \left( \bigcup_{j \in J} H'_j \right) \leq (r - 1)\hat{\Gamma} \left( L \left( \bigcup_{j \in J} H'_j \right) \right) = (r - 1)(|J| - d)$$

This implies the existence of a non-negative function $g : V \setminus V_1 \to \mathbb{R}$ such that $\sum_{v \in V \setminus V_1 : e \in e} g(v) \geq 1$ for all $e \in \bigcup_{j \in J} H'_j$, and $\sum_{v \in V \setminus V_1 : e} g(v) \leq (r - 1)(|J| - d)$.

Let $h : V \to \mathbb{R}$ be the function which assigns the value 1 to every $v \in V_1 \setminus \{v_j \mid j \in J\}$ and the value $g(v)$ to every $v \in V \setminus V_1$. Then it can be verified $\sum_{v \in e} h(v) \geq 1$ for every $e \in H$, and hence from the
The definition of $\tau^*_u$ we have that:

$$\tau^*_u(H) \leq (r - 1) \sum_{v \in V_1} h(v) + \sum_{v \in V \setminus V_1} h(v) \leq (r - 1)(m - |J|) + (r - 1)(|J| - d) = (r - 1)(m - d)$$

The proof of the theorem is completed by noticing that Lemma 6.4 implies that the system \( \{H'_i \mid i \in [m]\} \) has a partial ISR of size at least \( m - d \) with respect to \( L(H') \), which implies in turn that \( \nu(H) \geq m - d \).

\[\square\]

### 7. Discussion and Open Problems

In this paper we made a first attempt to calculate the value of the Gamma function for specific graphs, by calculating its value for trees and cycles. Obviously, the most important open problem raised by this work is to find a general procedure for computing Gamma for all graphs.

**Open problem 7.1.** Find a general procedure for computing Gamma for all graphs.

In our proofs we extensively used a function closely related to the Gamma function $\Gamma$ called the weak Gamma function and denoted by $\hat{\Gamma}$ (see Subsection 3.2 for the precise definition of the weak Gamma function). Our proofs show that $\Gamma = \hat{\Gamma}$ for cycles and trees and we conjecture that equality holds in general.

**Conjecture 7.2.** $\Gamma(G) = \hat{\Gamma}(G)$ for every graph $G$.

Recall the definitions of the independent duplication $G_a$ and the clique duplication $G_\pi$ of a graph $G$ given in Subsection 3.3. Another closely related conjecture is the following:

**Conjecture 7.3.** $\Gamma(G) = \sup_{a \in \mathbb{Z}_+^V} \Gamma(G_a) = \sup_{a \in \mathbb{Z}_+^V} \Gamma(G_\pi)$ for every graph $G$.

The discussion in the previous sections prove the conjecture for cycles and trees.

**Acknowledgment.** I thank Ron Aharoni for introducing me to the subject of the paper, for help and support while writing the paper, and for helpful comments on earlier drafts of this paper. I also thank Yohay Kaplan for comments on earlier drafts of this paper.

**References**


Department of Computer Science, Technion, Haifa, Israel 32000

E-mail address, Noga Zewi: nogaz@cs.technion.ac.il