JOURNAL OF COMBINATORIAL THEORY, Series B 37, 113–141 (1984)

On the Length of Optimal TSP Circuits in Sets of Bounded Diameter

SHLOMO MORAN

Computer Science Department, Technion, Israel Institute of Technology, Haifa, Israel

Communicated by the Managing Editors

Received January 19, 1983

Let V be a set of n points in R^k. Let $d(V)$ denote the diameter of V, and $l(V)$ denote the length of the shortest circuit which passes through all the points of V. (Such a circuit is an "optimal TSP circuit".) $l^k(n)$ are the extremal values of $l(V)$ defined by $l^k(n) = \max\{l(V) \mid V \in \mathbf{V}_n^k\}$, where $\mathbf{V}_n^k = \{V \mid V \subseteq R^k, |V| = n, d(V) = 1\}$. A set $V \in \mathbf{V}_n^k$ is "longest" if $l(V) = l^k(n)$. In this paper, first some geometrical properties of longest sets in R^2 are studied which are used to obtain $l^2(n)$ for small n's, and then asymptotic bounds on $l^k(n)$ are derived. Let $\delta(V)$ denote the minimal distance between a pair of points in V, and let: $\delta^k(n) = \max\{\delta(V) \mid V \in \mathbf{V}_n^k\}$. It is easily observed that $\delta^k(n) = O(n^{-1/k})$. Hence, $c_k = \limsup_{n\to\infty} \delta^k(n)\, n^{1/k}$ exists. It is shown that for all n, $c_k n^{-1/k} \leqslant \delta^k(n)$, and hence, for all n, $l^k(n) \geqslant c_k n^{1-1/k}$. For $k = 2$, this implies that $l^2(n) \geqslant (\pi^2/12)^{1/4} n^{1/2}$, which generalizes an observation of Fejes-Toth that $\lim_{n\to\infty} l^2(n)\, n^{-1/2} \geqslant (\pi^2/12)^{1/4}$. It is also shown that $l^k(n) \leqslant [(3-\sqrt{3})k/(k-1)]\, n\delta^k(n) + o(n^{1-1/k}) \leqslant [(3-\sqrt{3})k/(k-1)]\, n^{1-1/k} + o(n^{1-1/k})$. The above upper bound is used to improve related results on longest sets in k-dimensional unit cubes obtained by Few (*Mathematika* 2 (1955), 141–144) for almost all k's. For $k = 2$, Few's technique is used to show that $l^2(n) \leqslant (\pi n/2)^{1/2} + O(1)$.

1. Introduction

Let R denote the set of the real numbers. The Euclidean traveling salesman problem (TSP) in R^k is the following: Given n points $x_1, ..., x_n$ in R^k, find the shortest circuit (i.e., closed curve) which passess through them. Such a circuit is an "optimal TSP circuit." It is easily verified that an optimal TSP circuit is a polygonal line through $x_1, ..., x_n$. In some applications it is required that the distance between any 2 points in the given set is bounded by some constant D (e.g., when the points represent nodes in a communication network. D represents the maximal distance at which 2 nodes can communicate. An optimal TSP circuit in this case corresponds to a most efficient communication protocol [6].)

0095-8956/84 $3.00

The problem addressed in this paper is the following: Given n, k, and D, what is the maximal length of an optimal TSP circuit through n points in R^k, the distance between any pair of which is at most D. Denote this length by $l^k(n, D)$. It is not hard to verify that $l^k(n, D) = D \cdot l^k(n, 1)$. Hence, we can restrict ourselves to the case where $D = 1$. For brevity, we denote $l^k(n, 1)$ by $l^k(n)$. We shall be interested in both the values $l^k(n)$ and the properties of points sets of "maximal length" which realize these values.

It had been noted (see [5]) that this problem is closely related to the following problem of "optimal packing": Allocate n points in R^k such that the distance between any pair of them $\leqslant 1$, and the minimal distance between any pair of them is maximized. Denote this "maximal minimal distance" by $\delta^k(n)$. It is easily observed that $l^k(n) \geqslant n\delta^k(n)$. Thue and others (see [4, pp. 160–166; 7]) had shown that $\delta^2(n)$ is asymptotically equal to $(\pi^2/12)^{1/4} n^{-1/2}$. The exact value of $\delta^2(n)$ for $n \leqslant 7$ (and the geometrical properties of the corresponding configuration) are given in [1]. For $k > 2$, even the asymptotic values of $\delta^k(n)$ are not known (see, e.g., [2, pp. 405–411]).

The paper has 5 sections. The rest of this section includes the necessary definitions and notations. In Section 2 some geometrical properties of sets of maximal length in R^2 are proved. These properties are then used to compute $l^2(4)$ and to give some results concerning $l^2(5)$. In Section 3 we give lower bounds on $l^k(n)$, which generalize the observation mentioned above about the connection between $\delta^k(n)$ and $l^k(n)$. In that section we also give a result on $\delta^k(n)$ which seems to be of independent interest (Theorems 3.2 and 3.2′). In Section 4 upper bounds on $l^k(n)$ are given: first we give an upper bound for arbitrary k, which improves a result on longest sets in unit cubes obtained by Few in [5], and then we use the technique of Few to give a better bound on $l^2(n)$. In Section 5 two related results are discussed.

Notations and Definitions

Let $V = \{x_1, ..., x_n\}$ be a set of n points in R^k (for some k). A *path* in V is a sequence $P = (x_{i_1} - x_{i_2} - \cdots - x_{i_m})$ of points of V. For $j = 1, ..., k-1$, $(x_{i_j} - x_{i_{j+1}})$ is an *arc* of P. An arc $(x - y)$ will be identified with the straight line segment connecting x and y. The *length* of a path P is defined by

$$l(P) = l(x_{i_1} - \cdots - x_{i_m}) = \sum_{j=1}^{m-1} \delta(x_{i_j}, x_{i_{j+1}}),$$

where $\delta(x, y)$ is the Euclidean distance between x and y.

A *Hamiltonian circuit* or a *TSP circuit* in V is a path $H = (x_1 - x_{i_2} - \cdots - x_{i_n} - x_1)$ in which $i_j \neq i_k$ for $j \neq k$. We shall identify 2 Hamiltonian circuits if one is obtained from the other by reversing the order of the points. Thus, for $n \geqslant 3$, there are $(n-1)!/2$ distinct Hamiltonian circuits on sets of n points.

DEFINITION 1.1. Let $V = \{x_1, ..., x_n\}$. Then the lenght of V, $l(V)$ is defined by

$$l(V) = \min\{l(H) \mid H \text{ is a Hamiltonian circuit in } V\}.$$

For a set V and a point x, $d(x, V) = \max\{\delta(x, y) \mid y \in V\}$. The *diameter* of V, $d(V)$, is defined by

$$d(V) = \max\{d(x, V) \mid x \in V\}$$ [1]

For positive integers n and k, let

$$\mathbf{V}_n^k = \{V \mid V \subseteq R^k, d(V) = 1, |V| = n\}.$$

DEFINITION 1.2. For a positive integer n

$$l^k(n) = \max\{l(V) \mid V \in \mathbf{V}_n^k\}$$ [2].

A set V_n^* is a "longest set" if

(i) $V_n^* \in \mathbf{V}_n^k$;
(ii) $l(V_n^*) = l^k(n)$.

DEFINITION 1.3. Let $V \subseteq R^k$ for some k. Then

$$\delta(V) = \min\{\delta(x, y) \mid x \neq y, x, y \in V\}.$$

For positive integers n and k

$$\delta^k(n) = \max\{\delta(V) \mid V \in \mathbf{V}_n^k\}.$$

The numbers $\delta^k(n)$ are sometimes called "packing constants" [3].

Most of the proofs in the paper are given for the case $k = 2$, and it will be clear from the text when they generalize to arbitrary k. $\mathbf{V}_n^2$ will be denoted by $\mathbf{V}_n$, and V_n will denote a set in $\mathbf{V}_n$. Similarly, $l(n)$ and $\delta(n)$ will denote $l^2(n)$ and $\delta^2(n)$, respectively.

2. SOME PROPERTIES OF LONGEST SETS IN R^2

In this section we prove lemmas which provide some insight on the structure of planar longest sets. We then use these lemmas to find a longest

[1] The diameter of V is sometimes denotes as "the maximal chord length of V."

[2] The use of the term "max" (and not "sup") in the definition of $l^k(n)$ is justified by the fact that $\mathbf{V}_n^k$ is homeomorphic to a compact subset of R^{kn} and that $l(V)$ is a continuous function. Similar remarks apply to a few other definitions in the paper.

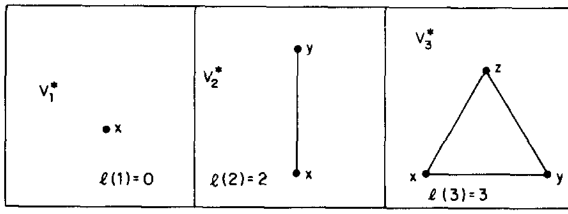

FIGURE 2.1

set V_4^* of cardinality 4, (and thus to compute $l(4)$), and to give some results concerning V_5^* and $l(5)$. Note that trivially $l(1)=0$, $l(2)=2$, and $l(3)=3$. V_n^* for $n=1,2,3$ are given in Fig. 2.1.

LEMMA 2.1. *Let $V_n=\{x_1,\ldots,x_n\}$ be a set of n points in the plane ($n \geqslant 4$), not all of them on the same line. Then an optimal TSP circuit in V_n is a simple curve (that is: a curve which does not intersect itself).*

Proof. Let $H=(x_1-x_{i_2}-\cdots-x_{i_n}-x_1)$ be a TSP circuit in V_n. We shall show that if H intersects itself, then H is not optimal. For simplicity, assume that $(i_2,\ldots,i_n)=(2,\ldots,n)$.

Suppose that for some i and j ($|i-j|>1$), the arcs (x_i-x_{i+1}) and (x_j-x_{j+1}) intersect (see Fig. 2.2).

Assume first that x_i, x_{i+1}, x_j, x_{j+1} are not collinear. Then by replacing (x_i-x_{i+1}) and (x_j-x_{j+1}) by (x_i-x_j) and $(x_{i+1}-x_{j+1})$ we obtain a TSP circuit which is shorter than H (due to the triangle inequality). The proof for the case where x_i, x_{i+1}, x_j, x_{j+1} are collinear is also not hard and is omitted. ∎

DEFINITION 2.1. Let V be a set of points. Then CON(V) is the boundary of the convex hull of V (i.e., the boundary of the smallest convex figure which contains V.)

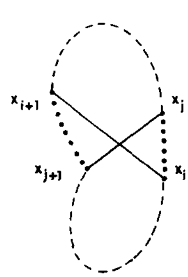

FIGURE 2.2

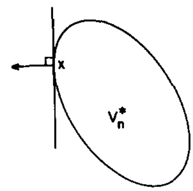

FIGURE 2.3

LEMMA 2.2. *Let $V_n^* \in \mathbf{V}_n$. If V_n^* is longest, then for each x in V_n^*, $d(x, V_n^*) = 1$ iff $x \in \mathrm{CON}(V_n^*)$.*

Proof. Since $d(V_n^*) = 1$, if $x \in V_n^*$ and $d(x, V_n^*) = 1$, then x must in $\mathrm{CON}(V_n^*)$. Hence, it suffices to show that for every x in $\mathrm{CON}(V_n^*)$, $d(x, V_n^*) = 1$. For contradiction, assume that for some $x \in \mathrm{CON}(V_n^*)$, $d(x, V_n^*) < 1$. We shall show that there is a $V_n' \in \mathbf{V}_n$ such that $l(V_n') > l(V_n^*)$: Let L be a supporting line of V_n^* through x (i.e., a line tangent to $\mathrm{CON}(V_n^*)$ at x, see Fig. 2.3). V_n' is obtained by removing x a small distance h in a direction perpendicular to L, as shown in Fig. 2.3. By doing this, $\delta(x, y)$ is increased for all $y \in V_n^*$, and hence $l(V_n') > l(V_n^*)$. On the other hand, if h is small enough, $d(x, V_n')$ is less than 1 (since $d(x, V_n^*) < 1$), and hence $d(V_n') = 1$. Thus, V_n' is in $\mathbf{V}_n$. This completes the proof of the lemma. ∎

DEFINITION 2.2 Let $x, y \in V$. Then the arc $(x - y)$ is *essential* if it participates in every optimal TSP circuit in V. $(x - y)$ is *redundant* if it participates in no optimal TSP circuit in V. $(x - y)$ is *nonessential* (*nonredundant*) if it is not essential (redundant). A point x of V is *internal in* V if its not in $\mathrm{CON}(V)$.

LEMMA 2.3. *Let V_n^* be a longest set, and let x be an internal point in* V_n^*. Then

(a) *For all $y \in V_n^*$, $(x - y)$ is nonessential.*

(b) *Let L be any line through x. Then there are $y, z \in V_n^*$ such that L separates y and z and both $(x - y)$ and $(x - z)$ are nonredundant.*

Proof. (a) Assume that for some $y \in V_n^*$, $(x - y)$ is essential. We derive a contradiction by showing that V_n^* is not longest. Suppose first that there is no z in V_n^* such that x lies on the arc $(y - z)$ as in Fig. 2.4. Then, by

FIGURE 2.4

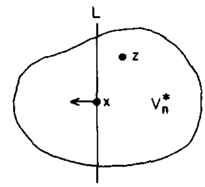

FIGURE 2.5

removing x a distance h away from y along the line containing $(x-y)$, $\delta(x, y)$ increases by h, while for any $u \in V_n^*$, if $\delta(x, u)$ decreases, it decreases by less than h, and if $x \notin \{u, v\}$, $\delta(u, v)$ remains unchanged. It follows that the length of any TSP circuit which contains $(x-y)$ (and hence of any optimal TSP circuit) increases by some positive value. By making h small enough, $d(x, V_n^*)$ remains smaller than 1 and the lengths of the nonoptimal TSP circuits remain larger than the length of the previously optimal circuits and hence $l(V_n^*)$ is increased, in contradiction with the assumption that V_n^* is longest.

The argument above does not work if there is a point z such that x lies in $(y-z)$ as in Fig. 2.4, because then removing x as before does not increase the length of the TSP circuits which use the path $(y-x-z)$. In this case, x can be removed away from y in a direction which forms a small but positive angle α with $(x-z)$, and a similar argument does apply.

(b) For contradiction, assume that there is a line L through x as in Fig. 2.5, such that for all nodes y on the left side of L, $(x-y)$ is redundant. Then, by removing x in a direction perpendicular to L as shown in Fig. 2.5, $\delta(x, z)$ increases for all z which are not on the left side of L, and hence, for all z such that $(x-z)$ is nonredundant. Hence, similarly to the proof of (a), one can increase $\delta(V_n^*)$ by removing x a small distance h in that direction. ∎

COROLLARY. *Let V_n^* be a longest set, and let x be an internal point of V_n^*. Then there are at least 3 points y_1, y_2, and y_3 in V_n^* such that $(x-y_i)$ is nonredundant $(i = 1, 2, 3)$.*

Proof. Since every TSP circuit must pass through x, there are y_1, y_2 in V_n^* such that the path $(y_1 - x - y_2)$ is in an optimal TSP circuit. Hence, $(y_1 - x)$ and $(y_2 - x)$ are nonredundant. If there is no y_3 such that $(y_3 - x)$ is nonredundant, then both $(y_1 - x)$ and $(y_2 - x)$ occur in every optimal circuit, which means that $(y_1 - x)$ and $(y_2 - x)$ are essential, in contradiction to Lemma 2.3(a).

DEFINITION 2.3. Let C be a closed curve in the plane and let D be a real number. Then C is a *curve of constant width D* if for each $x \in C$, $d(x, C) = D$.

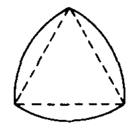

FIGURE 2.6

A "figure of constant width" is a convex figure whose boundary is a curve of constant width. Examples of curves of constant width D are a circle of diameter D and a Reuleaux triangle of side length D (see Fig. 2.6). For more about curves of constant width see [9]. We shall use the following lemma concerning these curves.

LEMMA 2.4. (a) *Every curve of constant width D has a perimeter πD.*

(b) *The area of a figure of constant with D is at most $\pi D^2/4$.*

Proof. Part (a) is Barbier's theorem; (b) follows from (a) by the isoperimetric theorem [9, pp. 51–58]. ∎

Since every set of points of diameter 1 can be embedded in a figure of constant width 1, Lemma 2.4 implies

LEMMA 2.5. *Let V_n be in $\mathbf{V}_n$. Then*

(a) *The perimeter of* $\mathrm{CON}(V_n)$ *is less than π.*

(b) *There is a convex figure whose area is less than $\pi/4$ which contains V_n.*

DEFINITION 2.4. Let k, n be given, $2 \leqslant k \leqslant n$. Then

$$\mathbf{V}_{n,k} = \{V_n \mid V_n \in \mathbf{V}_{n'} \mid V_n \cap \mathrm{CON}(V_n)| = k\};$$
$$l(n, k) = \sup\{l(V_n) \mid V_n \in \mathbf{V}_{n,k}\}.$$

LEMMA 2.6. $l(4) = 2(1 + \sqrt{3}/3) = 3.1547\ldots$.

Proof. Let V_4^* be a longest set of 4 points. Then V_4^* is either in $\mathbf{V}_{4,3}$ or in $\mathbf{V}_{4,4}$. Hence, $l(4) = \max\{l(4,3), l(4,4)\}$. We shall prove first that $l(4,4) \leqslant \pi$, and then that $l(4,3) = 2(1 + \sqrt{3}/3) > \pi$.

Let V_4 be in $\mathbf{V}_{4,4}$. Then $l(V_4)$ is the perimeter of V_4, which by Lemma 2.5 is less than π. Hence, $l(4,4) \leqslant \pi$.[3]

Let $V_4' = \{x, y, z, u\}$, where x, y, and z are the vertices of an equilateral triangle of side length 1, and u is the center of this triangle (see Fig. 2.7).

[3] In fact, one can show that $l(4,4) = 2 + 1/\cos 15° = 3.0353^{(-)}\ldots$.

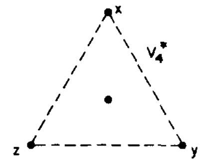

FIGURE 2.7

Then, as one can easily verify

$$l(V_4') = l(u - y - x - z - u) = 2(1 + \sqrt{3}/3) > \pi.$$

Hence, $l(4) \geqslant l(V_4') > l(4,4)$. Therefore, a longest set in $\mathbf{V}_4$ must be in $\mathbf{V}_{4,3}$. It remains to show that V_4' is, in fact, a longest set: Let $V_4^* = \{x', y', z', u'\}$ be a longest set, and let u' be the internal point of V_4^*. Using the same technique that was used in the proof of Lemma 2.2, one can show that $\{x', y', z'\}$ are the vertices of an equilateral triangle of side length 1. Also, by Lemma 2.3 and its corollary, each of the arcs $(x' - y')$, $(x' - z')$, $(x' - u')$ is nonredundant. Hence, all of the possible 3 TSP circuits have the same length, which implies that $\delta(u', y') = \delta(u', z') = \delta(u', x')$ and hence that u' is the center of the triangle. The lemma follows. ∎

An interesting corollary to the last two lemmas is the following: Let $n \geqslant 4$, and let V_n^* be a longest set in $\mathbf{V}_n$. Then V_n^* is not in $\mathbf{V}_{n,n}$.

Deriving $l(n)$ and V_n^* for $n \geqslant 5$ seems to be hard. Using Lemmas 2.1–2.5, we have been able to prove some results concerning $l(5)$ and V_5^*. These results are stated below.

LEMMA 2.7. $l(5,3) \approx 3.2175^{(-)}\ldots$ *Moreover,* $l(5,3) = l(V_{5,3}^*)$, *where* $V_{5,3}^*$ *is defined by (see Fig.* 2.8*):*

(1) $\{x, y, z\}$ *are the vertices of an equilateral triangle of side length* 1.

(2) u *is the center of the triangle.*

(3) v *lies on the height from* x *to* $(y - z)$ *and* $\delta(v, z) - \delta(v, u) = 1 - \sqrt{3}/3 = 0.4226^{(+)}\ldots$.

The proof that the set $V_{5,3}^*$ defined in Lemma 2.7 above is indeed longest

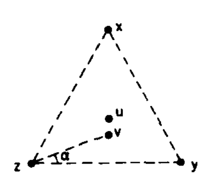

FIGURE 2.8

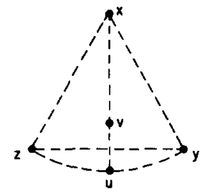

FIGURE 2.9

(in $\mathbf{V}_{5,3}$) is rather involved, and is postponed to the Appendix. Let $\alpha = \measuredangle vzy$. Then one can check that, by (3):

$$\delta(v, z) - \delta(v, u) = 1/(2 \cos \alpha) - [\sqrt{3}/6 - (\operatorname{tg} \alpha)/2]$$
$$= (1/\cos \alpha + \operatorname{tg} \alpha - \sqrt{3}/3)/2 = 1 - \sqrt{3}/3.$$

This implies that a longest set in this case is obtained when $\alpha \approx 19.79°$. The value for $l(V_{5,3}^*)$ follows by computing the length of one of the optimal routes— $(x - u - v - y - z - x)$, say:

$$l(x - u - v - y - z - x) \approx \sqrt{3}/2 - (\operatorname{tg} 19.79°)/2 + 1/(2 \cos 10.79°) + 2$$
$$\approx 3.2175.$$

LEMMA 2.8. *Let V_5^* be a longest set in $\mathbf{V}_5$. Then $V_5^* \in \mathbf{V}_{5,4}$.*

Proof. In view of Lemma 2.7, we have the show that there is a $V_5' \in \mathbf{V}_{5,4}$ such that $l(V_5') > l(5, 3) \approx 3.2175$. Such a V_5' is given in Fig. 2.9: x, y, and z form an equilateral triangle of side length 1, u and v lie on the bisector of $\measuredangle zxy$, $\delta(x, u) = 1$, and

$$\delta(u, v) = \delta(y, u)/2 = 1/(4 \cos 15°) = 1/(2(3^{1/2} + 2)^{1/2}) = (2 - \sqrt{3})^{1/2}/2.$$

In V', all the 4 circuits which do not intersect themselves, are optimal. (Note that, by Lemma 2.1, these are the only candidates for optimal circuits.)

$$l(V_5') = \delta(x, y) + \delta(y, u) + \delta(u, v) + \delta(v, z) + \delta(z, x)$$
$$= 1 + 1/(2 \cos 15°) + 1/(4 \cos 15°) + d(v, z) + 1$$
$$\approx 2 + 3/(4 \cos 15°) + 0.5153505$$
$$\approx 3.291807 > l(5, 3). \quad \blacksquare$$

We conjecture that V_5' is a longest set, though we do not yet have a formal proof for this.

3. A Lower Bound on $l^k(n)$

In this section we derive a lower bound on $l^k(n)$. The results are stated and proved for $k = 2$, but are easily generalized to arbitrary k.

Let $V_n \in \mathbf{V}_n$ be such that for all x, y in V_n, $\delta(x, y) \geqslant r$. Then clearly, $l(V_n) \geqslant nr$. Taking r to be $\delta(n)$, we have that $l(n) \geqslant n\delta(n)$. Let

$$c_2 = \limsup_{n \to \infty} \delta(n) \sqrt{n}.$$

Then, by the discussion above

$$c_2 \leqslant \limsup_{n \to \infty} l(n)/\sqrt{n}.$$

We shall generalize this observation to the following stronger result:

Theorem 3.1. *For all n, $l(n) \geqslant c_2\sqrt{n}$.*

Theorem 3.1 follows easily from

Theorem 3.2. *For all n, $\delta(n)\sqrt{n} \geqslant c_2$.*

Theorem 3.2 seem to be of independent interest, since it implies not only that $c_2 = \lim_{n \to \infty} \delta(n) \sqrt{n}$, but also that c_2 is a lower bound of $\delta(n) \sqrt{n}$. The k-dimensional version of Theorem 3.2 is

Theorem 3.2′. *Let $c_k = \limsup \delta^k(n)\, n^{1/k}$. Then for all n, $\delta^k(n)\, n^{1/k} \geqslant c_k$.*

The key lemma for the above theorems is Lemma 3.2, which uses a relation between the "packing constants" and densities of "sparse sets," as described below:

Let R be a planar figure of area A, and let S be a finite set of points contained in R. Then the density of S in R is $|S|/A$. The set S is "sparse" if for each pair of points x, y in S, $\delta(x, y) \geqslant 1$. The packing constants $\delta(n)$ correspond to the density of sparse sets of points in certain planar figures in the following way: Suppose that for some n and c, $\delta(n) \geqslant c/\sqrt{n}$. Then, by using appropriate scaling, one can obtain a planar figure R of diameter $\sqrt{n}/c$ which contains a sparse set of n points. Using the fact that, by Barbier's theorem, the area of R cannot exceed $\pi n/(4c^2)$, we have that the density of S in R is at least $4c^2/\pi$. On the other hand, if we can embed a sparse set S in a circle C of diameter $\sqrt{n}/c$ such that the density of S in $C \geqslant 4c^2/\pi$, then, since the area of C is $n\pi/(4c^2)$, S contains at least n points. This implies (again by scaling), that $\delta(n) \geqslant c/\sqrt{n}$. The next lemma summarizes the above.

LEMMA 3.1. (a) *If $\delta(n) \geqslant c/\sqrt{n}$, then there is a sparse set S of n points contained in a figure R of diameter $\leqslant \sqrt{n}/c$, and (hence) the density of S in R is at least $4c^2/\pi$.*

(b) *If a sparse set S is contained in a circle C of diameter $\sqrt{n}/c$ such that the density of S in C is at least $4c^2/\pi$, then $\delta(n) \geqslant c/\sqrt{n}$.*

Due to the fact that for all k, the k-dimensional set of given diameter and maximal volume is a k-dimensional sphere[4], Lemma 3.1 has a simple generalization to the k-dimensional case. In this generalization, $\sqrt{n}/c$ is replaced by $n^{1/k}/c$, and $4c^2/\pi$ is replaced by c^k/w_k, where w_k is the volume of the k-dimensional sphere of diameter 1.

In view of Lemma 3.1(b) above, Theorem 3.2 will follow if we show that for each r, there is a circle C of radius r which contains a sparse set S, such that the density of S in C is at least $4c_2^2/\pi$. (Recall that $c_2 = \limsup_{n\to\infty} \delta(b)\sqrt{n}$.) This will follow from Lemma 3.2, for which we need the following definitions:

DEFINITION 3.1. For each positive real number t, let R_t be a set of diameter t and area A_t, and let S_t be a finite set of points contained in R_t. For a given $r \geqslant 0$, $R_{t,r}$ is the set obtained by deleting from R_t all the points whose distance from the boundary of R_t is less than r. Let $A_{t,r}$ be the area of $R_{t,r}$, and let $S_{t,r} = S_t \cap R_{t,r}$. We say that the family $\{(R_t, S_t)\}$ is *blanced* if for each fixed r, the following hold:

$$\lim_{t\to\infty} \frac{A_{t,r}}{A_t} = 1; \tag{3.1.1}$$

$$\lim_{t\to\infty} \frac{|S_{t,r}|}{|S_t|} = 1. \tag{3.1.2}$$

DEFINITION 3.2. A family $\{(R_t, S_t)\}$ as above has density e if it is balanced and

$$\limsup_{t\to\infty} \frac{|S_t|}{A_t} = e. \tag{3.2}$$

EXAMPLE 1. Let m be a positive integer, and let G_m be the lattice

$$G_m = \left\{ \left(\frac{p}{m}, \frac{q}{m}\right) \;\middle|\; p, q \text{ are integers} \right\}.$$

For each t, let R_t be a set of constant width t, and let $G_{m,t} = G_m \cap R_t$. Then $\{R_t, G_{m,t})\}$ has density m^2.

[4] The author is indebted to M. Perles for bringing this fact to his attention.

EXAMPLE 2. For each t, let R_t be as in Example 1, and let S_t be a sparse set of maximum possible cardinality contained in R_t. Then $\{(R_t, S_t)\}$ has a density $\geqslant 4c_2^2/\pi$ (this follows from Lemma 3.1(a) and the definition of c_2).

DEFINITION 3.3. *Let G_m be as in Example 1, and let B be a bounded region of the plane. Then*

$$N(B, m) = |G_m \cap B|.$$

The proofs of the following propositions are easy and omitted.

PROPOSITION 3.1. *Let B be a convex figure of area $F > 0$. Then*

$$\lim_{m \to \infty} \frac{N(B, m)}{m^2} = F,$$

and the convergence is uniform (i.e., it does not depend on the specific location of B in the plane).

PROPOSITION 3.2. *Let $\{(R_t, S_t)\}$ be a family of density e and let r be a fixed number. Then*

$$\limsup_{t \to \infty} \frac{|S_{t;r}|}{A_t} = e.$$

LEMMA 3.2. *Let $\{(R_t, S_t)\}$ be a family of density e. Then for each $F > 0$, there is a circle C of area F and a number t such that $|C \cap S_t| \geqslant \lceil eF \rceil$*[5].

Proof. For simplicity, let $e = 1$. Assume for contradiction that for all C of area F, and for all t, $|C \cap S_t| < F$. Let $F = I + h$, where I is an integer and $0 < h \leqslant 1$, and let $\delta = h/F > 0$. Then for each t, each circle of area F contains at most $I = F(1 - \delta)$ points of S_t. Let $\varepsilon > 0$ be such that $(1 - \varepsilon)^2 > (1 - \delta)(1 + \varepsilon)$, and let $r = (F/\pi)^{1/2}$ (i.e., $\pi r^2 = F$). By Propositions 3.1 and 3.2 there exist t and m (m depends on t) such that,

(i) $|N(C, m)/m^2 - F| < \varepsilon F$ for every circle C of area F;

(ii) $||S_{t;r}|/A_t - 1| < \varepsilon$;

(iii) $|N(R_t, m)/m^2 - A_t| < \varepsilon A_t$.

Let $G_{m,t} = G_m \cap R_t$ and let $C_{m,t}$ be the set of all circles of radius r (and area F) whose centers belong to $G_{m,t}$. For each $x \in S_t$ and $C \in \mathbf{C}_{m,t}$ let

$$n_x = |\{C \mid C \in \mathbf{C}_{m,t}, x \in C\}|,$$
$$n_c = |\{x \mid x \in S_t \cap C\}|.$$

[5] $\lceil X \rceil$ denotes the smallest integer not smaller than X.

Note that if x is in $R_{t,r}$, then

$$n_x = |\{u \mid u \in G_m, \delta(u, x) \leqslant r\}|,$$

and that, under the assumption that the lemma is false, $n_c \leqslant F(1-\delta)$ for all $C \in \mathbf{C}_{m,t}$. Finally, let

$$P = \{(x, C) \mid C \in \mathbf{C}_{m,t}, x \in C \cap S_t\} \qquad \text{and let} \quad p = |P|.$$

We derive a contradiction by computing p by two different methods:

Method 1:

$$p = \sum_{C \in \mathbf{C}_{m,t}} n_C \leqslant F(1-\delta)\,|\mathbf{C}_{m,t}| < F(1-\delta)\, m^2(1+\varepsilon)\, A_t.$$

The last inequality follows from (iii), since $|\mathbf{C}_{m,t}| = N(R_t, m)$.

Method 2:

$$p = \sum_{x \in S_t} n_x \geqslant \sum_{x \in S_{t,r}} n_x \geqslant |S_{t,r}|\, m^2 F(1-\varepsilon) > A_t(1-\varepsilon)\, m^2 F(1-\varepsilon).$$

The second inequality follows from (i), and the last inequality from (ii).

By combining the above result and cancelling equal terms, we get $(1-\varepsilon)^2 < (1-\delta)(1+\varepsilon)$, which contradicts the assumption on ε. ∎

Proof of Theorem 3.2. By Lemma 3.1(b), it is enough to show that for every circle C there is a sparse set S whose density in $C \geqslant 4c_2^2/\pi$. For each positive real number t, let S_t be a sparse set of width t and of maximum possible cardinality, and let R_t be a set of constant width containing S_t. Then, by Lemma 3.1(a) and the definitions, the family $\{(R_t, S_t)\}$ has density $\geqslant 4c_2^2/\pi$ (see Example 2). Let C be a given circle of area F. Then by Lemma 3.2, there is a replica C' of C and a number t such that $|C' \cap S_t| \geqslant \lceil F \cdot 4c_2^2/\pi \rceil \geqslant F \cdot 4c_2^2/\pi$. This implies that the density of $C' \cap S_t$ in C' is at least $4c_2^2/\pi$. Since S_t is a sparse set. so is $C' \cap S_t$. The theorem follows. ∎

COROLLARY. *For each n,*

$$\delta(n) \geqslant (\pi^2/12)^{1/4} m^{-1/2} \qquad \text{and} \qquad l(n) \geqslant (\pi^2/12)^{1/4} n^{1/2}.$$

Proof. By Theorems 3.1 and 3.2, using the result (due to Thue and others, see [2]) that $\lim_{n\to\infty} \delta(n)\sqrt{n} = (\pi^2/12)^{1/4}$. ∎

4. An Upper Bound on $l^k(n)$

In this section first we derive an upper bound on $l^k(n)$ expressed in terms of $\delta^k(n)$, and then derive from it an upper bound expressed in terms of k and n only. We show that our result improves a result of Few on longest sets in unit k-dimensional cubes [5] for almost all k's, and then use the technique of Few to improve our result for $k = 2$.

Lemma 4.1. *For each k and n $(k, n \geqslant 2)$,*

$$l^k(n) \leqslant l^k(n-1) + (3-\sqrt{3})\,\delta^k(n).$$

Proof. As before, we shall prove the lemma for $k = 2$, since the generalization to arbitrary k will be obvious. Let $V_n \in \mathbf{V}_n$. We shall prove that

$$l(V_n) \leqslant l(n-1) + (3-\sqrt{3})\,\delta(n).$$

Let $x, y \in V_n$ be such that $\delta(x, y)$ is minimized. Then $\delta(x, y) \leqslant \delta(n)$. Let z be the median of the interval (x, y) and let $V_{n-1} = [V_n - \{x, y\}] \cup \{z\}$. Then $l(V_{n-1}) \leqslant l(n-1)$. Hence, it suffices to prove that $l(V_n) \leqslant l(V_{n-1}) + (3-\sqrt{3})\,\delta(x, y)$. Let H be an optimal TSP circuit in V_{n-1}, and let $u, v \in V_{n-1}$ be such that the path $(u - z - v)$ is included in H (see Fig. 4.1). A TSP circuit H' for V_n is obtained by replacing $(u - z - v)$ in H by either $(u - x - y - v)$ or $(u - y - x - v)$, whichever is shorter.

Without loss of generality assume that $(u - x - y - v)$ is the shorter one, that is,

$$l(u - x - y - v) \leqslant l(u - y - x - v). \tag{1}$$

Then

$$l(V_n) \leqslant l(H') = l(H) + l(u - x - y - v) - l(u - z - v).$$

Hence, it suffices to show that

$$l(u - x - y - v) - l(y - z - v) \leqslant (3-\sqrt{3})\,\delta(x, y).$$

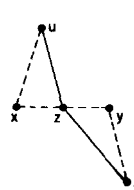

Figure 4.1

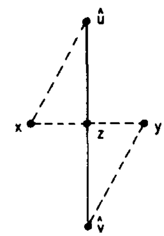

FIGURE 4.2

Note also that x and y were chosen so that

$$\delta(x, y) = \min\{\delta(s, t) \mid s \neq t, \{s,t\} \subseteq \{u, v, x, y\}\}. \tag{2}$$

For a given pair of points (u, v), let $f(u, v) = l(u - x - y - v) - l(u - z - v)$, and let $M = \max\{f(u, v) \mid u, v$ satisfy (1) and (2). (Note that even if u, v, x, y are not restricted to be coplanar, $f(u, v)$ is maximized when u, v, x, y are coplanar.) To prove the lemma, it suffices to show that $M \leqslant (3 - \sqrt{3})\, \delta(x, y)$.

Let $\bar{u}$, $\bar{v}$ be such that $\bar{u}$, x, $\bar{v}$, y form a parallelogram, in which $\delta(\bar{u}, x) = \delta(\bar{u}, y) = d(\bar{v}, x) = d(\bar{v}, y) = \delta(x, y)$ (see Fig. 4.2). Then

$$f(\bar{u}, \bar{v}) = 3\delta(x, y) - \sqrt{3}\, \delta(x, y) = (3 - \sqrt{3})\, \delta(x, y).$$

Hence, the lemma will follow if we can show that $M = f(\bar{u}, \bar{v})$. To prove this, we prove the following claim:

Claim. $f(\bar{u}, \bar{v}) \geqslant f(u, v)$ for all u, v which satisfy (1) and (2).

Proof of the Claim. Note that $f(u, v)$ can be written as

$$f(u, v) = f_1(u) + f_2(v) + \delta(x, y),$$

where

$$f_1(t) = \delta(t, x) - \delta(t, z); \qquad f_2(t) = \delta(t, y) - \delta(t, z).$$

The claim now follows by the following observations:

Observation (a). Let t be any point, and let t' be in the arc $(t - z)$, (see Fig. 4.3). Then, by the triangle inequality, $f_1(t') \geqslant f_1(t)$ and $f_2(t') \geqslant f_2(t)$.

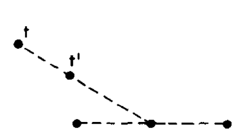

FIGURE 4.3

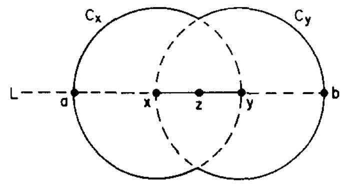

FIGURE 4.4

Observation (b). Let $r=\delta(x, y)$, and let C_x (C_y) be circles of radius r and centers x (y), respectively. Let D_x (D_y) denote the discs bounded by C_x (C_y). Then, by (2), u and v cannot lie in the interior of D_x (D_y) (see Fig. 4.4).

Let $D=D_x \cup D_y$, and let C be the boundary of D. Observations (a) and (b) imply that $f_1(v)$ and $f_2(u)$ (and hence also $f(u, v)$), are maximized for u and v which satisfy (2) when both u and v are in C.

Observation (c). Let L be the line containing $(x-y)$, and let a be the unique point in $C_x \cap C \cap L$, and b be the unique point in $C_y \cap C \cap L$ (see Fig. 4.4). Then, when u moves from a to b along C, $f_1(u)$ increases monotonically and $f_2(u)$ decreases monotonically.

Observation (d): If both u and v are in C, then $l(u-x-y-v) \leqslant l(u-y-x-v)$ is equivalent to $\delta(u, x) \leqslant \delta(v, x)$. This, together with Observations (b) and (c), implies that under constraints (1) and (2), $f(u, v)$ is maximized when $\delta(u, x)=\delta(v, x)$, that is, when u is the reflection of v in L. Note that in this case $f_1(u)=f_1(v)$ and $f_2(u)=f_2(v)$. Moreover, by symmetry, $f(u, v)$ attains its maximum for some u, v in $C \cap C_x$. Thus, the problem of computing M reduces to the following maximization problem:

$$\text{maximize } f_1(u)+f_2(u)(=f_1(u)+f_2(v)),$$
$$\text{subject to } u \in C \cap C_x.$$

Using polar coordinates by substituting $u=(r, \alpha)$ (see Fig. 4.5), we get

$$\text{maximize } r(1+2\cos(\alpha/2)-(5+4\cos\alpha)^{1/2})=f(\alpha),$$
$$\text{subject to } 0 \leqslant \alpha \leqslant 2\pi/3, \ (r=\delta(x, y)).$$

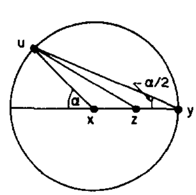

FIGURE 4.5

(In polar coordinates, $\delta(u,x)=r$, $\delta(u,y)=2r\cos(\alpha/2)$, and $\delta(u,z)=r(5+4\cos\alpha)^{1/2}/2$). Taking the derivative with respect to α, we get

$$f'(\alpha)=r(-\sin(\alpha/2)+2\sin\alpha/(5+4\cos\alpha)^{1/2}).$$

Since $f'(\alpha)>0$ for $\alpha\in(0,2\pi/3]$, f attains its maximum when $\alpha=2\pi/3$, i.e., when $u=\bar{u}$, $v=\bar{v}$. This completes the proof of the claim, and hence of the lemma. ∎

THEOREM 4.1. *For all k and n*

$$l^k(n)/(n\delta^k(n))\leqslant(3-\sqrt{3})\,k/(k-1)+o(1).$$

Proof. Let $c_k=\lim_{n\to\infty}\delta^k(n)\,n^{1/k}$ (the existence of c_k follows from Theorem 3.2′). Then

$$\delta^k(n)=c_k n^{-1/k}+o(n^{-1/k}).$$

The theorem will follow if we can show that

$$l^k(n)\leqslant[(3-\sqrt{3})k/(k-1)]\,c_k n^{1-1/k}+o(n^{1-1/k}).$$

By Lemma 4.1, $l^k(n)\leqslant l^k(n-1)+(3-\sqrt{3})\,\delta^k(n)$. Hence,

$$l^k(n)\leqslant(3-\sqrt{3})[\delta^k(2)+\cdots+\delta^k(n)]=(*).$$

Let $\delta^k(x)$ be a continuous, nonincreasing real extension of $\delta^k(n)$. Then $\delta^k(x)=c_k x^{-1/k}+o(x^{-1/k})$. We get that

$$\begin{aligned} l^k(n)\leqslant(*)&<(3-\sqrt{3})\int_2^n\delta^k(x)\,dx\\ &=(3-\sqrt{3})\int_2^n[c_k x^{-1/k}+o(x^{-1/k})]\,dx\\ &=[(3-\sqrt{3})k/(k-1)]\,c_k n^{1-1/k}+o(n^{1-1/k}). \end{aligned}$$ ∎

THEOREM 4.2. *For all k,* $l^k(n)<[(3-\sqrt{3})k/(k-1)]\,n^{1-1/k}+o(n^{1-1/k})$.

Proof. In view of Theorem 4.1, it is enough to show that for large enough n, $\delta^k(n)<n^{-1/k}$.

Let $\delta^k(n)=d$. Then it is possible to pack n disjoint k-dimensional spheres of diameter d in a k-dimensional sphere of diameter $1+d/2$. It is known that the ratio between the sum of the volumes of the small spheres and the volume of the large sphere cannot exceed some constant $E_k<1$. The ratio between the volume of one small sphere and the volume of the large sphere is $[d/(1+d/2)]]^k$. Hence

$$n[d/(1+d/2)]^k \leqslant E_k,$$

or, equivalently,

$$d^k \leqslant (1+d/2)^k E_k/n.$$

For large enough n, $(1+d/2)^k E_k < 1$ (recall that $d=\delta^k(n)$) which implies that, for large enough n, $\delta^k(n) < n^{-1/k}$. ∎

A problem similar to the one discussed in this section was discussed in [5, 8], where a bound on the length of the shortest road through n points in the k-dimensional unit cube was investigated. In [5] it was shown that this bound cannot exceed

$$[k(2(k-1))^{(1-k)/2k} + o(1)]n^{1-1/k}. \tag{4.1}$$

Using an argument similar to the one in Theorem 4.2, but replacing the k-dimensional sphere of diameter $1+d/2$ by a k-dimensional cube of side length $1+d/2$, and using the fact that the volume of the k-dimensional sphere of radius 1 is $\pi^{k/2}/\Gamma(k/2+1)$, one can show that this bound cannot exceed

$$\{[2(3-\sqrt{3})k/(k-1)](\Gamma(k/2+1))^{1/k}(\pi^{-1/2}) + o(1)\}\, n^{1-1/k}. \tag{4.2}$$

For large k we have

$$(4.1) \approx [(k/2)^{1/2} + o(1)]\, n^{1-1/k} \approx 0.7071\sqrt{k}\, n^{1-1/k}$$

and

$$(4.2) \approx [(3-\sqrt{3})(2k/\pi e)^{1/2} + o(1)]\, n^{1-1/k} \approx 0.6136\sqrt{k}\, n^{1-1/k}.$$

In fact, (4.2) gives a better bound than (4.1) already for $k=7$ (a constant of $2.370^{(-)}$ vs. a constant of $2.413^{(+)}$). However, for $k<7$ the technique used in [5] provides a better bound on $l^k(n)$. In particular, one can use that technique to prove

THEOREM 4.3. $l(n) = (\pi n/2)^{1/2} + O(1)$.

Proof (sketch). Let $V_n \in \mathbf{V}_n$ be given. Then V_n can be embedded in a figure C of diameter 1 which, by Barbier's theorem, is of area $\leqslant \pi/4$.

Let $t = (\pi/2n)^{1/2}$, and let $\mathbf{L}_0$, $\mathbf{L}_1$ be the sets of lines defined by

$$\mathbf{L}_0 = \{(x, y) \mid y = nt \text{ for some integer } n\};$$

$$\mathbf{L}_1 = \{(x, y) \mid y = (n + \tfrac{1}{2})t \text{ for some integer } n\}.$$

For a point v in R^2, let $\delta(v, \mathbf{L}_i)$ be the shortest distance from v to a line in $\mathbf{L}_i$. Then for each v, $\delta(v, \mathbf{L}_0) + \delta(v, \mathbf{L}_1) = t/2$. Hence,

$$\sum_{v \in V_n} \delta(v, \mathbf{L}_0) + \sum_{v \in V_n} \delta(v, \mathbf{L}_1) = nt/2.$$

Hence, for $i = 0$ or $i = 1$, it holds that

$$\sum_{v \in V_n} \delta(v, \mathbf{L}_i) \leqslant nt/4. \tag{4.3.1}$$

Without loss of generality, assume that (4.3.1) holds for $i = 0$. Consider the TSP circuit composed of:

(a) The line segments in $\mathbf{L}_0 \cap C$.

(b) Portions of the boundary of C connecting these line segments to a path.

(c) For each point v in V_n, the shortest line segment connecting v to the path described above, each such segment counted twice.

(d) A segment connecting the first and last points of V_n traversed along the described path.

The sum of the lengths of the line segments described in (a) is equal approximately to the area of C divided by t, and hence it is at most $\pi/4t + O(1)$.

The sum of the lengths of the segments in (b) is $O(1)$. The sum of the lengths of the segments in (c) (each taken twice) is $nt/2$, and the segment (d) is of length $\leqslant 1$. Altogether, the total length of the described circuit is $\pi/4t + nt/2 + O(1)$. The theorem follows. ∎

5. Two Related Results

Two problems which are related to the problem discussed in this paper are:

(1) Minimal tree: Given a set V of n points in R^k, find a tree (i.e., a connected graph without circuits) on V such that the length of the tree, defined as the sum of the lengths of its arcs, is minimal. Denote this length by $l_T^k(V)$.

(2) Steiner tree: Given a set V as above, find a set $V' \supseteq V$ such that $l_T^k(V')$ is minimal. Formally, for a given V, the length of the Steiner tree of V is defined by

$$l_S^k(V) = \min_{V' \supseteq V} \{l_T^k(V')\}.$$

Note. The existence of a set $V' \supseteq V$ such that $l_T^k(V')$ is minimal follows from the observation that for every set V' which contains V there exists a set V'' which contains V such that

$$l_T^k(V'') \leqslant l_T^k(V') \qquad \text{and} \qquad |V''| \leqslant 2|V| - 2.$$

(Thus, in the definition of a Steiner tree we can add the restriction: $|V'| \leqslant 2|V| - 2$, which implies that the minimum is attained.) V'' is constructed from V' in the following manner: Let T be a tree of minimal length on V'. Delete from V' all the points which are not in V and have degree at most 2 in T. In the resulting tree every point not in V has a degree at least 3. The observations follows.

The corresponding problems for graphs of bounded diameter are: For each n find:

(1) $l_T^k(n) = \max_{V \in \mathbf{V}_n^k} \{l_T^k)\}$.

(2) $l_S^k(n) = \max_{V \in \mathbf{V}_n^k} \{l_S^k(V)\} = \max_{V \in \mathbf{V}_n^k} \{\min_{V' \supseteq V} \{l_T^k(V')\}\}$

Theorem 5.1. *For each k and n,*

$$1 - 1/n \leqslant \frac{l_T^k(n)}{n\delta^k(n)} \leqslant (3 - \sqrt{3})k/(k-1) + o(1).$$

Proof. The upper bound follows immediately from the upper bound on $l^k(n)$ (Theorem 4.1). The lower bound follows from the fact that a tree on n points has $n - 1$ edges and from the definition of $\delta^k(n)$. ∎

Theorem 5.2. *For each k and n,*

$$\frac{1}{2} \leqslant \frac{l_S^k(n)}{n\delta^k(n)} \leqslant k/(k-1) + o(1)$$

Proof. The upper bound follows from the observation that $l_S^k(n) \leqslant l_S^k(n-1) + \delta^k(n)$. (In the proof of Lemma 4.1, simply add $(z - x)$ and $(z - y)$ to the Steiner tree for V_{n-1}, see Fig. 5.1.)

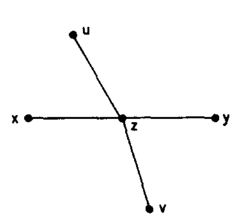

Figure 5.1

The lower bound follows from the lower bound on $l^k(n)$, by the fact that the existence of a Steiner tree of length l implies the existence of a Hamiltonian circuit of length $\leqslant 2l$. ▮

APPENDIX

Proof of Lemma 2.7. Let $V = V^*_{5,3} = \{x, y, z, u, v\}$ be a longest set in $\mathbf{V}_{5,3}$. We shall prove that V satisfies conditions (1)–(3) of Lemma 2.7. As in Lemma 2.6, we may assume that $\{x, y, z\}$ are the vertices of an equilateral triangle, T. There are 12 Hamiltonian circuits in V, each uses either one or two sides of T. We denote as $(x-y)$-*circuit* a circuit that intersects T with the edge $(x-y)$, as $(x-y-z)$-*circuit* a circuit that intersects T with the edges $(x-y)$ and $(y-z)$, etc. The 12 circuits are listed below:

$$\begin{aligned}
(x-y-z)\text{-circuits: } C_1 &= (x-y-z-u-v-x)\\
C_2 &= (x-y-z-v-u-x)\\
(y-z-x)\text{-circuits: } C_3 &= (x-u-v-y-z-x)\\
C_4 &= (x-v-u-y-z-x)\\
(z-x-y)\text{-circuits: } C_5 &= (x-y-u-v-z-x)\\
C_6 &= (x-y-v-u-z-x)\\
(x-y)\text{-circuits: } C_7 &= (x-y-u-z-v-x)\\
C_8 &= (x-y-v-z-u-x)\\
(y-z)\text{-circuits: } C_9 &= (x-u-y-z-v-x)\\
C_{10} &= (x-v-y-z-u-x)\\
(z-x)\text{-circuits: } C_{11} &= (x-u-y-v-z-x)\\
C_{12} &= (x-v-y-u-z-x)
\end{aligned}$$

Conditions (1)–(3) of Lemma 27 are equivalent to the following

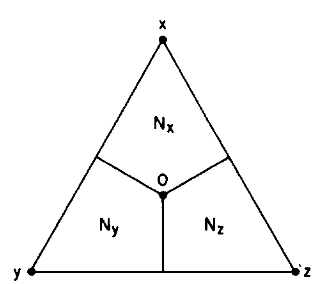

FIGURE A.1

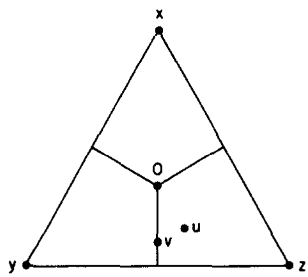

FIGURE A.2

statement: In the longest set V, u and v lie on one of T's heights, such that the lengths of all the (Hamiltonian) circuits that do not intersect themselves are equal to $l(V)$. (If that height is $\mathbf{h}_x$ and u lies above v, then these circuits are C_2, C_3, C_5, C_6, C_8, and C_{11}.) The proof of the lemma proceeds by the following two claims:

Claim 1. Both u and v lie on one of T's heights.

Proof. We shall assume that Claim 1 is false, and derive a contradiction. Denote by N_x the set of points t in T s.t. $\delta(t,x) \leqslant \min\{\delta(t,y), \delta(t,z)\}$. N_y and N_z are defined similarly (see Fig. A.1). We consider three cases:

Case (i). For some $t \in \{x, y, z\}$, exactly one out of $\{u, v\}$ is an interior point of N_t (see Fig. A.2). Without loss of generality we may assume that u is an interior point of N_z, and that v is in N_y (not necessarily as an interior point). Then we have the following inequalities:

$$\delta(u,z) < \min\{\delta(u,y), \delta(u,x)\}$$
$$\delta(v,y) \leqslant \min\{\delta(v,x), \delta(v,z)\}.$$

The above inequalities imply that C_2, C_3, C_4, and C_5 are longer than C_6, and hence are not optimal. They also imply that C_9 and C_{11} are longer than C_8, hence C_9 and C_{11} are not optimal either. But all the remaining circuits use the edge $(u-z)$, which must therefore be an essential edge, in contradiction with Lemma 2.3(a)[6].

Case (ii). Both u and v are interior points of some N_t for some t. Without loss of generality $t = z$. Then

$$\delta(u,z) < \min\{\delta(u,y), \delta(u,x)\}$$
$$\delta(v,z) < \min\{\delta(v,y), \delta(v,x)\}.$$

[6] Actually, with the straightforwards generalization of Lemma 2.3(a) to longest sets in $\mathbf{V}_{n,k}$.

We may also assume that

$$\delta(u, y) + \delta(v, x) \leqslant \delta(v, y) + \delta(u, x).$$

Then C_7 is shorter than C_9, C_{10}, C_{11}, C_{12} and is not longer than C_8. C_3 and C_4 are longer than C_2 and C_1, respectively. Thus, the only candidates for optimal circuits are C_1, C_2, C_5, C_6, C_7, and C_8. To show that this yields a contradiction we use an extension of the idea in Lemma 2.3(a): Let $S = \{(z-u), (z-v)\}$ and $L = \{(x-u), (x-v), (y-u), (y-v)\}$. Each of the candidate circuits above uses one or two edges of S, and the same number of edges of L. By removing u and v a small distance h away from z along $(z-u)$ and $(z-v)$, respectively, the lengths of the edges in S increase by h, the lengths of the edges in L decrease by less than h, and the lengths of the rest of the edges do not decrease. Hence, the lengths of all the candidate circuits above increase. By taking h small enough, a set V^1 which is longer than V is obtained, a contradiction.

Case (iii). u and v are not interior points of any N_t; that is, u and v lie on different heights of T, as shown in Fig. A.3, with u on $\mathbf{h}_y$ and v on $\mathbf{h}_x$. In this case we have:

$$l(C_2) = l(C_3) = l(C_6) < \min\{l(C_1), l(C_4), l(C_5)\}$$

and

$$l(C_8) < l(C_i) \quad \text{for} \quad i = 7, 9, 10, 11, 12.$$

Subcase (iii.1) $l(C_8) < l(C_2)$. In this case C_8 is a unique optimal circuit, in contradiction to Lemma 2.3(a), (b).

Subcase (iii.2). $l(C_2) < l(C_8)$. In this case the optimal circuits are C_2, C_3, C_6. This means that $(u-v)$ is essential, in contradiction the Lemma 2.3(a).

Subcase (iii.3). $l(C_2) = l(C_8)$. The proof of this case is a little more

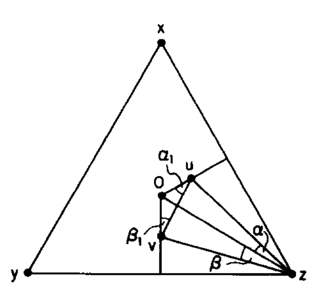

FIGURE A.3

involved, and is sketched below. The equality above is easily seen to be equivalent to

$$\delta(z, u) + \delta(z, v) = 1 + \delta(u, v) \tag{$*$}$$

It also implies that C_8 is optimal and hence

$$l(V) = 1 + 2(\delta(u, z) + \delta(v, z)). \tag{$**$}$$

Hence, if V is longest then $(**)$ is maximized under the constraint $(*)$. We shall show that this cannot happen if both u and v are different from the center O.

Let $\alpha = \measuredangle uzO$, $\beta = \measuredangle vzO$, $\alpha_1 = \measuredangle vuO$, $\beta_1 = \measuredangle uvO$ (see Fig. A.3). Assume first that $\alpha = \beta$ (and hence $\alpha_1 = \beta_1$). Then[7]

$$\delta(z, u) = \delta(z, v) = \frac{1}{2\cos(30 - \alpha)}$$

$$\delta(u, v) = 2\delta(z, u)\sin\alpha = \frac{\sin\alpha}{\cos(30 - \alpha)}$$

Thus, by $(*)$

$$\frac{1}{\cos(30 - \alpha)} = 1 + \frac{\sin\alpha}{\cos(30 - \alpha)}$$

and hence $\alpha \approx 5.2644$, which implies that $l(V) = l(C_8) \approx 3.2020$, which is smaller than $l(V^*)$ (≈ 3.2175), where V^* is the set defined in Lemma 2.7.

Second, assume that $\alpha > \beta$, and hence $\alpha_1 < \beta_1$. For a small h, let $u(h)$ and $v(h)$ be the points on $\mathbf{h}_y$ and $\mathbf{h}_x$ for which

$$\delta(z, u(h)) = \delta(z, u) + h$$

$$\delta(z, v(h)) = \delta(z, v) - h$$

For infinitesimal h we have

$$\delta(u(h), u) \approx \frac{h}{\sin(30 - \alpha)},$$

$$\delta(v(h), v) \approx \frac{h}{\sin(30 - \beta)}$$

and

$$\delta(u(h), v(h)) - \delta(u, v) \approx \delta(v(h), v)\cos\beta_1 - \delta(u(h), u)\cos\alpha_1.$$

[7] All angles are measured in degrees.

Thus, if we define

$$\frac{d(u,v)}{d(h)} = \frac{\delta(u(h),v(h)) - \delta(u,v)}{h}$$

then we get

$$\lim_{h\to 0}\left(\frac{d(u,v)}{d(h)}\right)$$

$$= \frac{\cos\beta_1}{\sin(30-\beta)} - \frac{\cos\alpha_1}{\sin(30-\alpha)} < 0. \quad (\text{since } \alpha > \beta \text{ and } \alpha_1 < \beta_1).$$

Let $V(h) = \{x, y, z, u(h), v(h)\}$, and let $C_i(h)$ be C_i in $V(h)$. For small negative h, $C_8(h)$ and $C_2(h)$ are shorter than all other circuits in $V(h)$. By the definition of $u(h)$ and $v(h)$, $l(C_8(h)) = l(C_8)$ and by the inequality above,

$$l(C_2(h)) = l(C_2) + \delta(u(h), v(h)) - \delta(u,v) > l(C_2).$$

Since $l(C_2) = l(C_8)$, this implies that $l(C_2(h)) > l(C_8(h))$. Thus

$$l(V(h)) = l(C_8(h)) = l(C_8) = l(V)$$

and hence, by the assumption on V, $V(h)$ is longest. But $V(h)$ has a unique optimal circuit—$C_8(h)$—in contradiction with Lemma 2.3. This completes the proof of Claim 1.

Note. The proof above implies not only that u and v lie on one of the heights, but also that they are not interior points of N_t for $t \in \{x, y, z\}$ (see Fig. A.4).

Assume that u and v lie on $\mathbf{h}_x$, and that u lies above v (as in Fig. A.4). It is easily observed that the $(x-y-z)$-circuits are reflections through $\mathbf{h}_x$ of the $(y-z-x)$ circuits, that the $(x-y)$-circuits are (similar) reflections of the $(z-x)$-circuits, and that C_5 is a reflection of C_6. Using this, one can verify that the only candidates for optimal circuits are C_2, C_5, C_8, and their

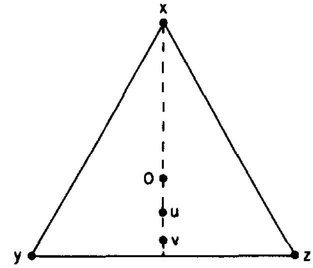

FIGURE A.4

reflections. If all of them are optimal, then (2) and (3) of Lemma 2.7 must be satisfied, and hence the set V is the one described in Lemma 2.7 (and shown in Fig. 2.8), as claimed. Thus, it remains to show that the circuits C_2, C_5, and C_8 are optimal.

Claim 2. C_2,C_5, and C_8 are optimal in V.

Proof. Assume that the claim is false. Then the longest set in $\mathbf{V}_{5,3}$, V, differs from the set V^* in Fig. 2.8. Let $V^* = \{x, y, z, u^*, v^*\}$. By the note above u cannot lie above u^*. We consider three cases, according to the locations of u and v.

Case (i): $u = u^*$ and v lies above v^*. Then, by the triangle inequality

$$\delta(u^*, v^*) + \delta(v^*, z) > \delta(u, v) + \delta(v, z),$$

hence

$$l(V) \leqslant l(C_2) < l(C_2^*) = l(V^*)$$

which contradicts the assumption on V.

Case (ii). $u = u^*$ and v lies below v^*. Then

$$\delta(y, v) < \delta(y, v^*),$$

hence

$$l(V) \leqslant l(C_8) < l(C_8^*) = l(V^*),$$

which again contradicts the assumption on V.

Case (iii). u lies below u^*. In this case $\delta(u, x) > \delta(u, y)$, which implies that C_2 is longer than C_5, and hence is not optimal. Hence, the only candidates for optimality are C_5, C_8 and their reflections through $\mathbf{h}_x$. If any of them is not optimal then Lemma 2.3 is violated. It follows that both C_5 and C_8 must be optimal. Thus

$$\begin{aligned} l(C_5) &= 2 + \delta(y, v) + \delta(u, v) + \delta(u, z) = l(C_8) \\ &= 1 + 2\delta(y, v) + \delta(u, z) + \delta(u, z) + \delta(u, x). \end{aligned}$$

In order to use the Lagrange multipliers theorem we restate the above equality, after rearranging and cancelling equal terms, in terms of the scalar variables u' and v' whose range of definition is $\mathbf{h}_x$ as follows: The *constraint*

$$\delta(v', z) + \delta(u', x) = 1 + \delta(u', v') \tag{\&}$$

is satisfied at $u' = u$ and $v' = v$. The constraint (&) is also satisfied at

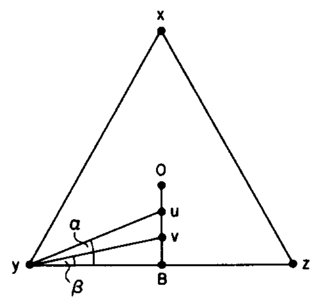

FIGURE A.5

$u' = u_1$ and $v' = v_1$, where v_1 is at the bottom B of $\mathbf{h}_x$ (see Fig. A.5), and u_1 satisfies $\text{tg}(\sphericalangle u_1 yz) = (\sqrt{3} - 1)/2$ (we leave the verification of this fact to the reader). Let $V_1 = \{x, y, z, u_1, v_1\}$. Then since $l(V_1) < l(V^*)$, V_1 is not optimal; hence $V_1 \neq V$, which implies that $v_1 \neq v$. It follows that both u and v are internal points of the segments (O, B). Hence, by the assumption that V is longest, the function

$$l(V) = l(C_5) = 2 + \delta(y, v') + \delta(u', v') + \delta(u', z) = f(u', v') \qquad (\&\&)$$

attains a *local* maximum, under the constraint (&), at $u' = u$ and $v' = v$. Let $\alpha' = \sphericalangle u'yz$, $\beta' = \sphericalangle v'yz$, $\alpha = \sphericalangle uyz$ and $\beta = \sphericalangle vyz$. Then multiplying (&) and (&&) above by 2 and expressing them in terms of α' and β', we get (after rearranging terms) that under the constraint

$$\frac{1}{\cos\beta'} - 2\,\text{tg}\,\alpha' + \text{tg}\,\beta' = \text{Const}_1$$

the function

$$\frac{1}{\cos\beta'} + \frac{1}{\cos\alpha'} + \text{tg}\,\alpha' - \text{tg}\,\beta' + \text{Const}_2$$

attains a *local* maximum at $\alpha' = \alpha$ and $\beta' = \beta$. Thus, by the Lagrange multipliers theorem there exists a constant λ for which the partial derivatives of the function $G(\alpha', \beta')$ defined below with respect to α' and β' vanish at $\alpha' = \alpha$ and $\beta' = \beta$.

$$G(\alpha', \beta') = \frac{1}{\cos\beta'} + \frac{1}{\cos\alpha'} + \text{tg}\,\alpha' - \text{tg}\,\beta' + \lambda\left(\frac{1}{\cos\beta'} - 2\,\text{tg}\,\alpha' + \text{tg}\,\beta'\right).$$

Taking derivatives, we get that at $\alpha' = \alpha$ and $\beta' = \beta$:

$$\frac{dG}{d\alpha'} = \frac{\sin\alpha + 1 - 2\lambda}{\cos^2\alpha} = 0$$

$$\frac{dG}{d\beta'} = \frac{(1+\lambda)\sin\beta + \lambda - 1}{\cos^2\beta} = 0.$$

Thus, we get

$$\sin\alpha = 2\lambda - 1,$$

$$\sin\beta = \frac{1-\lambda}{1+\lambda},$$

hence

$$\cos\alpha = 2\sqrt{\lambda - \lambda^2}$$

$$\operatorname{tg}\alpha = \frac{2\lambda - 1}{2\sqrt{\lambda - \lambda^2}}$$

$$\cos\beta = \frac{2\sqrt{\lambda}}{1+\lambda}$$

$$\operatorname{tg}\beta = \frac{1-\lambda}{2\sqrt{\lambda}}.$$

Given α and β, the length of C_8 as a function of λ is

$$\begin{aligned} 1 &+ \frac{1}{\cos\beta} + \frac{1}{2\cos\alpha} + \frac{\sqrt{3} - \operatorname{tg}\alpha}{2} \\ &= 1 + \frac{1+\lambda}{2\sqrt{\lambda}} + \frac{1}{4\sqrt{\lambda-\lambda^2}} + \frac{\sqrt{3}}{2} - \frac{2\lambda - 1}{4\sqrt{\lambda - \lambda^2}} \\ &= \frac{2+\sqrt{3}}{2} + \frac{1+\lambda}{2\sqrt{\lambda}} + \frac{1-\lambda}{2\sqrt{\lambda-\lambda^2}}. \end{aligned}$$

In a similar way, the length of C_5 is shown to be

$$2 + \frac{\lambda}{2\sqrt{\lambda - \lambda^2}} + \frac{\sqrt{\lambda}}{2}.$$

Since $l(C_5) = l(C_8)$, we obtain

$$\frac{1}{\sqrt{\lambda}} + \frac{1-2\lambda}{\sqrt{\lambda-\lambda^2}} + \sqrt{3} - 2 = 0.$$

Solving the above for λ, we get

$$\lambda \approx 0.7089055$$

which implies that $\alpha \approx 24.70$, $\beta \approx 9.81$, and that

$$l(V) = l(C_5) = l(C_8) \approx 3.201257 < l(V^*) \approx 3.2175$$

and hence V is not longest. This contradiction completes the proof of Claim 2, and hence of Lemma 2.7. ∎

Acknowledgments

I would like to thank Yaron Gold for stimulating discussions which led to this research. I would also like to thank an anonymous referee for correcting some errors in the original manuscript and for numerous suggestions which improved the presentation of the paper.

References

1. P. Bateman and F. Erdos, Geometrical extrema suggested by a lemma of Besicovitch, *Amer. Math. Monthly* **58** (1951), 306–314.
2. H. S. M. Coxeter, "Introduction to Geometry," 2nd ed. Wiley, New York, 1969.
3. P. Erdos, A. Meir, V. Y. Sos, and P. Turan, On some applications of graph theory I, *Discrete Math.* **2** (1972), 207–228.
4. L. Fejes-Toth, "Regular Figures," Pergamon Press, Oxford, 1964.
5. L. Few, The shortest path and shortest road through n points, *Mathematika* **2** (1955), 141–144.
6. Y. I. Gold and W. R. Franta, An efficient collision-free protocol for prioritized access control of cable or radio channels with (fixed) arbitrary spatial node configuration, *Comput. Networks* **7** (2) (1983), 83–98.
7. A. Thue, On nogle geometriks taltheoretiske theoremer, *Fordht. Skand. Naturforsk* **14** (1982), 352–353.
8. S. Verblunski, On the shortest path through a number of points, *Proc. Amer. Math. Soc.* **2** (1951), 904–913.
9. I. M. Yaglom and V. G. Boltyanski, "Convex Figures," Holt, Rinehart & Winston, New York, 1961.