SUPPLEMENTARY MATERIALS: Recovering Hidden Components in Multimodal Data with Composite Diffusion Operators*

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Appendices

A. Proof of Proposition 3.1 for the operator Q_ϵ. In this appendix we show that the asymptotic expansion of the operator Q_ϵ, presented in Subsection 3.1, is given by

$$Q_\epsilon f\,(x) = \int k_\epsilon(x,x')\frac{f(x')\mu(x')}{\hat{d}_\epsilon(x')}dV(x') = f(x) - \frac{m_2}{m_0}\epsilon^2\left(\Delta f(x) - \frac{f\Delta\mu}{\mu}(x)\right) + O(\epsilon^4), \tag{A.1}$$

where $\hat{d}_\epsilon(x') = \int k_\epsilon(x,x')\mu(x)dV(x)$ and m_0 and m_2 are manifold related constants.

Proof. As shown in [SM1] (Appendix B, Lemma 8), the asymptotic expansion of an appropriately scaled kernel $k_\epsilon(x,x')$, defined similarly to (3.1), applied to any smooth function $g(x)$ on $\mathcal{M}$, is given by

$$K_\epsilon g(x) = \int k_\epsilon(x,x')g(x')dV(x') = m_0 g(x) - m_2\epsilon^2\left(\Delta g(x) - \omega(x)g(x)\right) + O(\epsilon^4), \tag{A.2}$$

where $\omega(x)$ is a function that depends on the curvature.

Therefore, for Q_ϵ, consider $g(x) = f(x)\mu(x)/\hat{d}_\epsilon(x)$, and its asymptotic expansion is given by

$$Q_\epsilon f\,(x) = m_0\frac{f(x)\mu(x)}{\hat{d}_\epsilon(x)} - m_2\epsilon^2\left(\Delta\left(\frac{f\mu}{\hat{d}_\epsilon}\right)(x) - \omega(x)\frac{f(x)\mu(x)}{\hat{d}_\epsilon(x)}\right) + O(\epsilon^4). \tag{A.3}$$

In addition, for $\hat{d}_\epsilon(x)$, consider $g(x) = \mu(x)$ and then $\hat{d}_\epsilon(x) = m_0\mu(x) - m_2\epsilon^2\left(\Delta\mu(x) - \omega(x)\mu(x)\right) + O(\epsilon^4)$. When ϵ is sufficiently small, we have,

$$\left(\hat{d}_\epsilon\right)^{-1} = (m_0\mu)^{-1}\left(1 + \frac{m_2}{m_0}\epsilon^2\left(\frac{\Delta\mu}{\mu} - \omega\right)\right) + O(\epsilon^4). \tag{A.4}$$

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By substituting $\hat{d}_\epsilon$ in (A.3) with (A.4), when ϵ is sufficiently small, we obtain the following asymptotic expansion

$$\begin{aligned} Q_\epsilon f\,(x) &= f(x) - \frac{m_2}{m_0}\epsilon^2\left(\Delta f(x) - \omega(x)f(x) + \omega(x)f(x) - f\frac{\Delta\mu}{\mu}(x)\right) + O(\epsilon^4) \\ &= f(x) - \frac{m_2}{m_0}\epsilon^2\left(\Delta f(x) - f\frac{\Delta\mu}{\mu}(x)\right) + O(\epsilon^4). \end{aligned} \tag{A.5}$$

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B. Proof of Proposition 3.2.

For simplicity, we present the proof of Proposition 3.2 for $\epsilon_2 = \epsilon_1 = \epsilon$. For $\epsilon_2 \neq \epsilon_1$, the proof is similar up to some notation changes. The asymptotic expansion of the operators G_ϵ and H_ϵ, defined in Subsection 3.2, is given by

$$G_\epsilon f(x) = f(x) - \epsilon^2\left(\Delta^{(1)}f(x) + \phi^*\Delta^{(2)}(\phi^*)^{-1}f(x)\right) \tag{B.1}$$

$$- \epsilon^2\left(\phi^*\frac{2\nabla^{(2)}(\phi^*)^{-1}f\cdot\nabla^{(2)}\mu^{(2)}}{\mu^{(2)}}(x) - \frac{f\Delta^{(1)}\mu^{(1)}}{\mu^{(1)}}(x)\right) + O(\epsilon^4) \tag{B.2}$$

$$H_\epsilon f(x) = f(x) - \epsilon^2\left(\phi^*\Delta^{(2)}(\phi^*)^{-1}f(x) + \Delta^{(1)}f(x)\right) \tag{B.3}$$

$$- \epsilon^2\left(\frac{2\nabla^{(1)}f\cdot\nabla^{(1)}\mu^{(1)}}{\mu^{(1)}}(x) - f\phi^*\frac{\Delta^{(2)}\mu^{(2)}}{\mu^{(2)}}(x)\right) + O(\epsilon^4) \tag{B.4}$$

Proof. From Proposition 3.1, for $x \in \mathcal{M}^{(\ell)}$, we have

$$P_\epsilon^{(\ell)}f\,(x) = f(x) - \epsilon^2\left(\Delta^{(\ell)}f + \frac{2\nabla^{(\ell)}f\cdot\nabla^{(\ell)}\mu^{(\ell)}}{\mu^{(\ell)}}\right)(x) + O(\epsilon^4) \tag{B.5}$$

$$Q_\epsilon^{(\ell)}f\,(x) = f(x) - \epsilon^2\left(\Delta^{(\ell)}f - \frac{f\Delta^{(\ell)}\mu^{(\ell)}}{\mu^{(\ell)}}\right)(x) + O(\epsilon^4). \tag{B.6}$$

For better readability, we assume without loss of generality that the kernel functions $k_\epsilon^{(1)}$ and $k_\epsilon^{(2)}$ are scaled such that the constants $m_0^{(1)}$, $m_2^{(1)}$, $m_0^{(2)}$ and $m_2^{(2)}$ are equal to 1, similarly to [SM1, Appendix B] and [SM3, Appendix A]. We omit these constants in the following appendices as well.

For the operator $G_\epsilon f(x) = \phi^*P_\epsilon^{(2)}(\phi^*)^{-1}Q_\epsilon^{(1)}f(x)$, where $x \in \mathcal{M}^{(1)}$, consider $g(y) = \left((\phi^*)^{-1}Q_\epsilon^{(1)}f\right)(y)$, where $y = \phi(x)$, and place the expansion of $\left((\phi^*)^{-1}Q_\epsilon^{(1)}f\right)(y)$ into

$\left(\phi^* P_\epsilon^{(2)} g\right)(x)$:

$$G_\epsilon f(x) = \left(\phi^* P_\epsilon^{(2)} g\right)(x) \tag{B.7}$$

$$= \phi^* \left[g - \epsilon^2 \left(\Delta^{(2)} g + \frac{2\nabla^{(2)} g \cdot \nabla^{(2)} \mu^{(2)}}{\mu^{(2)}} \right) \right](x) + O(\epsilon^4) \tag{B.8}$$

$$= f(x) - \epsilon^2 \left(\Delta^{(1)} f - \frac{f \Delta^{(1)} \mu^{(1)}}{\mu^{(1)}} \right)(x) \tag{B.9}$$

$$- \epsilon^2 \left(\phi^* \Delta^{(2)} (\phi^*)^{-1} f + \phi^* \frac{2\nabla^{(2)} (\phi^*)^{-1} f \cdot \nabla^{(2)} \mu^{(2)}}{\mu^{(2)}} \right)(x) + O(\epsilon^4). \tag{B.10}$$

Similarly, for H_ϵ we get

$$H_\epsilon f(x) = f(x) - \epsilon^2 \left(\Delta^{(1)} f(x) + \phi^* \Delta^{(2)} (\phi^*)^{-1} f(x) \right) \tag{B.11}$$

$$- \epsilon^2 \left(\frac{2\nabla^{(1)} f \cdot \nabla^{(1)} \mu^{(1)}}{\mu^{(1)}}(x) - f \phi^* \frac{\Delta^{(2)} \mu^{(2)}}{\mu^{(2)}}(x) \right) + O(\epsilon^4). \tag{B.12}$$

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Remark B.1. The difference between the asymptotic expansions of the operators G_ϵ and H_ϵ and the alternating diffusion operator shown in Appendix D, is in the term $f \frac{\Delta^{(\ell)} \mu^{(\ell)}}{\mu^{(\ell)}}$, which appears in G_ϵ and H_ϵ. In the alternating diffusion operator the expressions representing the two manifolds are similar and given by $\frac{2\nabla^{(\ell)} f \cdot \nabla^{(\ell)} \mu^{(\ell)}}{\mu^{(\ell)}}$.

C. Proof of Proposition 3.3. For simplicity, we present the proof of Proposition 3.3 for $\epsilon_2 = \epsilon_1 = \epsilon$. For $\epsilon_2 \neq \epsilon_1$, the proof is similar up to some notation changes. For the operators S_ϵ and A_ϵ, defined in Subsection 3.3, we present the derivation of the asymptotic expansion and prove Proposition 3.3.

Proof. For $S_\epsilon f(x)$, place the asymptotic expansions of G_ϵ and H_ϵ, shown in Proposition

3.2, into $S_\epsilon f(x) = (G_\epsilon f(x) + H_\epsilon f(x))/2$ to obtain:

$$
\begin{aligned}
S_\epsilon f(x) =& \frac{1}{2} f(x) - \frac{\epsilon^2}{2}\left(\Delta^{(1)} f(x) + \phi^* \Delta^{(2)} (\phi^*)^{-1} f(x)\right) && \text{(C.1)}\\
&- \frac{\epsilon^2}{2}\left(\phi^* \frac{2\nabla^{(2)}(\phi^*)^{-1} f \cdot \nabla^{(2)} \mu^{(2)}}{\mu^{(2)}}(x) - \frac{f \Delta^{(1)} \mu^{(1)}}{\mu^{(1)}}(x)\right) && \text{(C.2)}\\
&+ \frac{1}{2} f(x) - \frac{\epsilon^2}{2}\left(\phi^* \Delta^{(2)} (\phi^*)^{-1} f(x) + \Delta^{(1)} f(x)\right) && \text{(C.3)}\\
&- \frac{\epsilon^2}{2}\left(\frac{2\nabla^{(1)} f \cdot \nabla^{(1)} \mu^{(1)}}{\mu^{(1)}}(x) - f \phi^* \frac{\Delta^{(2)} \mu^{(2)}}{\mu^{(2)}}(x)\right) + O(\epsilon^4) && \text{(C.4)}\\
=& f(x) - \epsilon^2 \left(\Delta^{(1)} f(x) + \phi^* \Delta^{(2)} (\phi^*)^{-1} f(x)\right) && \text{(C.5)}\\
&- \frac{\epsilon^2}{2}\left(\phi^* \frac{2\nabla^{(2)}(\phi^*)^{-1} f \cdot \nabla^{(2)} \mu^{(2)}}{\mu^{(2)}}(x) - f \phi^* \frac{\Delta^{(2)} \mu^{(2)}}{\mu^{(2)}}(x)\right) && \text{(C.6)}\\
&- \frac{\epsilon^2}{2}\left(\frac{2\nabla^{(1)} f \cdot \nabla^{(1)} \mu^{(1)}}{\mu^{(1)}}(x) - \frac{f \Delta^{(1)} \mu^{(1)}}{\mu^{(1)}}(x)\right) + O(\epsilon^4). && \text{(C.7)}
\end{aligned}
$$

For $A_\epsilon f(x)$, place the asymptotic expansions of G_ϵ and H_ϵ, shown in Proposition 3.2, into $A_\epsilon f(x) = (G_\epsilon f(x) - H_\epsilon f(x))/2$ to obtain:

$$
\begin{aligned}
A_\epsilon f(x) =& \frac{1}{2} f(x) - \frac{\epsilon^2}{2}\left(\Delta^{(1)} f(x) + \phi^* \Delta^{(2)} (\phi^*)^{-1} f(x)\right) && \text{(C.8)}\\
&- \frac{\epsilon^2}{2}\left(\phi^* \frac{2\nabla^{(2)}(\phi^*)^{-1} f \cdot \nabla^{(2)} \mu^{(2)}}{\mu^{(2)}}(x) - \frac{f \Delta^{(1)} \mu^{(1)}}{\mu^{(1)}}(x)\right) && \text{(C.9)}\\
&- \frac{1}{2} f(x) + \frac{\epsilon^2}{2}\left(\phi^* \Delta^{(2)} (\phi^*)^{-1} f(x) + \Delta^{(1)} f(x)\right) && \text{(C.10)}\\
&+ \frac{\epsilon^2}{2}\left(\frac{2\nabla^{(1)} f \cdot \nabla^{(1)} \mu^{(1)}}{\mu^{(1)}}(x) - f \phi^* \frac{\Delta^{(2)} \mu^{(2)}}{\mu^{(2)}}(x)\right) + O(\epsilon^4) && \text{(C.11)}\\
=& \frac{\epsilon^2}{2}\left(\frac{2\nabla^{(1)} f \cdot \nabla^{(1)} \mu^{(1)}}{\mu^{(1)}}(x) + \frac{f \Delta^{(1)} \mu^{(1)}}{\mu^{(1)}}(x)\right) && \text{(C.12)}\\
&- \frac{\epsilon^2}{2}\left(\phi^* \frac{2\nabla^{(2)}(\phi^*)^{-1} f \cdot \nabla^{(2)} \mu^{(2)}}{\mu^{(2)}}(x) + f \phi^* \frac{\Delta^{(2)} \mu^{(2)}}{\mu^{(2)}}(x)\right) + O(\epsilon^4). && \text{(C.13)}
\end{aligned}
$$

■

D. Comparison to Alternating diffusion. In this appendix, we review the asymptotic expansion of the alternating diffusion operator from [SM5, SM2] and show that it is not self-adjoint. For simplicity, we assume that $\epsilon_2 = \epsilon_1 = \epsilon$. For $\epsilon_2 \neq \epsilon_1$, the derivations are similar up to some notation changes.

The asymptotic expansion of the alternating diffusion operator can be derived similarly to Appendix B and Appendix C. This operator is defined by $P_\epsilon^{AD} f(x) = \phi^* P_\epsilon^{(2)} (\phi^*)^{-1} P_\epsilon^{(1)} f(x)$.

By placing the asymptotic expansion of $P_\epsilon^{(\ell)}$ from Proposition 3.1 in this definition we get

$$P_\epsilon^{AD} f(x) = f(x) - \epsilon^2 \left(\Delta^{(1)} f + \frac{2\nabla^{(1)} f \cdot \nabla^{(1)} \mu^{(1)}}{\mu^{(1)}} \right)(x) \tag{D.1}$$

$$- \epsilon^2 \left(\phi^* \Delta^{(2)} (\phi^*)^{-1} f + \phi^* \frac{2\nabla^{(2)} (\phi^*)^{-1} f \cdot \nabla^{(2)} \mu^{(2)}}{\mu^{(2)}} \right)(x) + O(\epsilon^4) \tag{D.2}$$

$$= f(x) - \epsilon^2 \left(\Delta^{(1)} f + \phi^* \Delta^{(2)} (\phi^*)^{-1} f \right)(x) \tag{D.3}$$

$$- \epsilon^2 \left(\frac{2\nabla^{(1)} f \cdot \nabla^{(1)} \mu^{(1)}}{\mu^{(1)}} + \phi^* \frac{2\nabla^{(2)} (\phi^*)^{-1} f \cdot \nabla^{(2)} \mu^{(2)}}{\mu^{(2)}} \right)(x) + O(\epsilon^4). \tag{D.4}$$

We now show that the limit operator of alternating diffusion, $P^{AD} = \lim_{\epsilon\to 0} \left(I - P_\epsilon^{AD} \right) / \epsilon^2$, where I denotes the identity operator, is not self-adjoint. We separate P^{AD} into two additive terms, the first, denoted by $P^{AD(1)}$, which contains elements related to the first manifold, i.e. elements from (D.1), and the second, denoted by $P^{AD(2)}$, which contains elements related to the second manifold, i.e. elements from (D.2). We will show that each of these operators is not self-adjoint, and therefore, P^{AD} is not self-adjoint, from the linearity of the inner product and from the additivity of these operators.

For $P^{AD(1)}$, given $f, g \in C^\infty\left(\mathcal{M}^{(1)}\right)$,

$$\begin{aligned}
\left\langle P^{AD(1)} f, g \right\rangle_{\mathcal{M}^{(1)}} &= \int_{\mathcal{M}^{(1)}} \left(\Delta^{(1)} f + \frac{2\nabla^{(1)} f \cdot \nabla^{(1)} \mu^{(1)}}{\mu^{(1)}} \right)(x) g(x) \mu^{(1)}(x) dV^{(1)}(x) \\
&= \int_{\mathcal{M}^{(1)}} \left(\Delta^{(1)} f(x) \right) g(x) \mu^{(1)}(x) dV^{(1)}(x) \\
&+ \int_{\mathcal{M}^{(1)}} \left(2\nabla^{(1)} f \cdot \nabla^{(1)} \mu^{(1)} \right)(x) g(x) dV^{(1)}(x) \qquad \text{(D.5)} \\
&= \int_{\mathcal{M}^{(1)}} \left(\Delta^{(1)} g + \frac{2\nabla^{(1)} g \cdot \nabla^{(1)} \mu^{(1)}}{\mu^{(1)}} \right)(x) \mu^{(1)}(x) f(x) dV^{(1)}(x) \\
&+ \int_{\mathcal{M}^{(1)}} \left(g \frac{\Delta^{(1)} \mu^{(1)}}{\mu^{(1)}} \right)(x) \mu^{(1)}(x) f(x) dV^{(1)}(x) \\
&- \int_{\mathcal{M}^{(1)}} \left(\frac{2\nabla^{(1)} g \cdot \nabla^{(1)} \mu^{(1)}}{\mu^{(1)}} + 2g \frac{\Delta^{(1)} \mu^{(1)}}{\mu^{(1)}} \right)(x) \mu^{(1)}(x) f(x) dV^{(1)}(x) \qquad \text{(D.6)} \\
&= \int_{\mathcal{M}^{(1)}} \left(\Delta^{(1)} g - g \frac{\Delta^{(1)} \mu^{(1)}}{\mu^{(1)}} \right)(x) \mu^{(1)}(x) f(x) dV^{(1)}(x) \qquad \text{(D.7)} \\
&\neq \left\langle f, P^{AD(1)} g \right\rangle_{\mathcal{M}^{(1)}}, \qquad \text{(D.8)}
\end{aligned}$$

where the transition between (D.5) and (D.6), is based on Green's first identity (for manifolds without a boundary).

Similarly, for $P^{AD(2)}$, given $f,g\in C^\infty\left(\mathcal{M}^{(1)}\right)$,

$$
\begin{aligned}
\left\langle P^{AD(2)}f,g\right\rangle_{\mathcal{M}^{(1)}} &= \int_{\mathcal{M}^{(1)}}\left(\phi^*\Delta^{(2)}(\phi^*)^{-1}f\right)(x)g(x)\mu^{(1)}(x)dV^{(1)}(x) \\
&\quad+\int_{\mathcal{M}^{(1)}}\phi^*\frac{2\nabla^{(2)}(\phi^*)^{-1}f\cdot\nabla^{(2)}\mu^{(2)}}{\mu^{(2)}}(x)g(x)\mu^{(1)}(x)dV^{(1)}(x) &\text{(D.9)}\\
&=\int_{\mathcal{M}^{(2)}}\left((\phi^*)^{-1}g\mu^{(2)}\Delta^{(2)}(\phi^*)^{-1}\right)(y)f(y)dV^{(2)}(y)\\
&\quad+\int_{\mathcal{M}^{(2)}}\left(2(\phi^*)^{-1}g\nabla^{(2)}(\phi^*)^{-1}f\cdot\nabla^{(2)}\mu^{(2)}\right)(y)dV^{(2)}(y) &\text{(D.10)}\\
&=\int_{\mathcal{M}^{(2)}}\left(\mu^{(2)}(\phi^*)^{-1}f\Delta^{(2)}(\phi^*)^{-1}g\right)(y)dV^{(2)}(y)\\
&\quad+\int_{\mathcal{M}^{(2)}}\left(2(\phi^*)^{-1}f\nabla^{(2)}(\phi^*)^{-1}g\cdot\nabla^{(2)}\mu^{(2)}\right)(y)dV^{(2)}(y)\\
&\quad+\int_{\mathcal{M}^{(2)}}\left((\phi^*)^{-1}f(\phi^*)^{-1}g\Delta^{(2)}\mu^{(2)}\right)(y)dV^{(2)}(y)\\
&\quad-\int_{\mathcal{M}^{(2)}}\left(2(\phi^*)^{-1}f\nabla^{(2)}(\phi^*)^{-1}g\cdot\nabla^{(2)}\mu^{(2)}\right)(y)dV^{(2)}(y)\\
&\quad+\int_{\mathcal{M}^{(2)}}\left(2(\phi^*)^{-1}f(\phi^*)^{-1}g\Delta^{(2)}\mu^{(2)}\right)(y)dV^{(2)}(y) &\text{(D.11)}\\
&=\int_{\mathcal{M}^{(2)}}\left((\phi^*)^{-1}f\Delta^{(2)}(\phi^*)^{-1}g\right)(y)\mu^{(2)}(y)dV^{(2)}(y)\\
&\quad-\int_{\mathcal{M}^{(2)}}\left((\phi^*)^{-1}f(\phi^*)^{-1}g\frac{\Delta^{(2)}\mu^{(2)}}{\mu^{(2)}}\right)(y)\mu^{(2)}(y)dV^{(2)}(y) &\text{(D.12)}\\
&=\int_{\mathcal{M}^{(1)}}\left(\phi^*\Delta^{(2)}(\phi^*)^{-1}g-g\phi^*\frac{\Delta^{(2)}\mu^{(2)}}{\mu^{(2)}}\right)(x)f(x)\mu^{(1)}(x)dV^{(1)}(x) &\text{(D.13)}\\
&\neq\left\langle f,P^{AD(2)}g\right\rangle_{\mathcal{M}^{(1)}}, &\text{(D.14)}
\end{aligned}
$$

where the transitions from (D.9) to (D.10) and from (D.12) to (D.13) are based on $\mu^{(1)}(x)dV^{(1)}(x)=\mu^{(2)}(y)dV^{(2)}(y)$ and $y=\phi(x)$. In addition, the transition between (D.10) and (D.11) is based on Green's first identity.

Finally, due to linearity, we can combine both operators and conclude that P^{AD} is not self-adjoint (nor anti-self-adjoint).

Remark D.1. Note that based on a similar derivation, it can be shown that the limit operators of G_ϵ and H_ϵ, i.e. $G=\lim_{\epsilon\to 0}\left(G_\epsilon-I\right)/\epsilon^2$ and $H=\lim_{\epsilon\to 0}\left(H_\epsilon-I\right)/\epsilon^2$, are not self-adjoint as well.

Remark D.2. When reversing the kernel order, i.e. $\tilde{P}^{AD}_\epsilon f(x)=P^{(1)}_\epsilon\phi^*P^{(2)}_\epsilon(\phi^*)^{-1}f(x)$, the asymptotic expansion of the resulting alternating diffusion operator is given by a similar expression, up to the forth order terms, $O(\epsilon^4)$. Therefore, constructing the difference operator, A_ϵ from Subsection 3.3, using two alternating diffusion operators with reversed order, i.e. $A_\epsilon^{AD} f(x) = \frac{1}{2}(P_\epsilon^{AD} - \tilde{P}_\epsilon^{AD}) f(x)$, will result in cancellation of all second order terms, $A_\epsilon^{AD} f(x) = O(\epsilon^4)$.

E. Proof of Proposition 3.4. Define the limit operator of $A_{\epsilon_1,\epsilon_2}$, where $\epsilon_2 = \alpha\epsilon$ and $\epsilon_1 = \epsilon$, $\alpha > 0$, by $A_\alpha = \lim_{\epsilon\to 0} A_{\epsilon_1,\epsilon_2}/\epsilon^2$. We show in this appendix that jA_α is self-adjoint, by equivalently showing that A_α is anti-self-adjoint.

The asymptotic expansion of $A_\alpha : C^\infty(\mathcal{M}^{(1)}) \to C^\infty(\mathcal{M}^{(1)})$ is given by:

$$A_\alpha f(x) = \frac{1}{2}\left(\frac{2\nabla^{(1)} f \cdot \nabla^{(1)}\mu^{(1)}}{\mu^{(1)}}(x) + \frac{f\Delta^{(1)}\mu^{(1)}}{\mu^{(1)}}(x)\right) \tag{E.1}$$

$$-\frac{\alpha^2}{2}\left(\phi^* \frac{2\nabla^{(2)}(\phi^*)^{-1} f \cdot \nabla^{(2)}\mu^{(2)}}{\mu^{(2)}}(x) + f\phi^* \frac{\Delta^{(2)}\mu^{(2)}}{\mu^{(2)}}(x)\right). \tag{E.2}$$

This is obtained from Proposition 3.3, for $A_{\epsilon_1,\epsilon_2}/\epsilon^2$ when $\epsilon \to 0$ and $\epsilon_2 = \alpha\epsilon_1 = \alpha\epsilon$.

Proof. Denote by $A_\alpha^{(1)}$ the terms in the asymptotic expansion of A_α which are related to the first manifold, i.e. (E.1). Similarly, denote by $A_\alpha^{(2)}$ the terms which are related to the second manifold, i.e. (E.2). In order to show that A_α is anti-self-adjoint we will first show that each of these partial operators are anti-self-adjoint and then, from the linearity of the inner product and the additivity of these terms, this result naturally extends to A_α.

For $A_\alpha^{(1)}$, given $f, g \in C^\infty(\mathcal{M}^{(1)})$,

$$\left\langle A_\alpha^{(1)} f, g\right\rangle_{\mathcal{M}^{(1)}} = \int_{\mathcal{M}^{(1)}} \left(\frac{f\Delta^{(1)}\mu^{(1)}}{2\mu^{(1)}} + \frac{\nabla^{(1)} f \cdot \nabla^{(1)}\mu^{(1)}}{\mu^{(1)}}\right)(x) g(x) \mu^{(1)}(x) dV^{(1)}(x) \tag{E.3}$$

$$= \int_{\mathcal{M}^{(1)}} \left(\frac{1}{2} f\Delta^{(1)}\mu^{(1)}\right)(x) g(x) dV^{(1)}(x) \tag{E.4}$$

$$+ \int_{\mathcal{M}^{(1)}} \left(\nabla^{(1)} f \cdot \nabla^{(1)}\mu^{(1)}\right)(x) g(x) dV^{(1)}(x) \tag{E.5}$$

$$= \int_{\mathcal{M}^{(1)}} \left(\frac{1}{2} fg\Delta^{(1)}\mu^{(1)}\right)(x) dV^{(1)}(x) \tag{E.6}$$

$$- \int_{\mathcal{M}^{(1)}} \left(\nabla^{(1)} \cdot \left(g\nabla^{(1)}\mu^{(1)}\right)\right)(x) f(x) dV^{(1)}(x) \tag{E.7}$$

$$= -\int_{\mathcal{M}^{(1)}} \left(\frac{1}{2} g\Delta^{(1)}\mu^{(1)} + \nabla^{(1)} g\nabla^{(1)}\mu^{(1)}\right)(x) f(x) dV^{(1)}(x) \tag{E.8}$$

$$= \int_{\mathcal{M}^{(1)}} \left(\frac{g\Delta^{(1)}\mu^{(1)}}{2\mu^{(1)}} + \frac{\nabla^{(1)} g \cdot \nabla^{(1)}\mu^{(1)}}{\mu^{(1)}}\right)(x) f(x) \mu^{(1)}(x) dV^{(1)}(x) \tag{E.9}$$

$$= -\left\langle f, A_\alpha^{(1)} g\right\rangle_{\mathcal{M}^{(1)}}, \tag{E.10}$$

where the transition between (E.5) and (E.7) is based on Green's first identity (for manifolds without a boundary).

Similarly, for $A_\alpha^{(2)}$, given $f,g\in C^\infty(\mathcal{M}^{(1)})$,

$$\begin{aligned}
\left\langle A_\alpha^{(2)}f,g\right\rangle_{\mathcal{M}^{(1)}} &= -\int_{\mathcal{M}^{(1)}}\alpha^2\left(f\phi^*\frac{\Delta^{(2)}\mu^{(2)}}{2\mu^{(2)}}\right)(x)g(x)\mu^{(1)}(x)dV^{(1)}(x)\\
&\quad -\int_{\mathcal{M}^{(1)}}\alpha^2\left(\phi^*\frac{\nabla^{(2)}(\phi^*)^{-1}f\cdot\nabla^{(2)}\mu^{(2)}}{\mu^{(2)}}\right)(x)g(x)\mu^{(1)}(x)dV^{(1)}(x) \quad \text{(E.11)}\\
&= -\int_{\mathcal{M}^{(2)}}\alpha^2\left((\phi^*)^{-1}f\frac{\Delta^{(2)}\mu^{(2)}}{2\mu^{(2)}}(\phi^*)^{-1}g\right)(y)\mu^{(2)}(y)dV^{(2)}(y)\\
&\quad -\int_{\mathcal{M}^{(2)}}\alpha^2\left(\frac{\nabla^{(2)}(\phi^*)^{-1}f\cdot\nabla^{(2)}\mu^{(2)}}{\mu^{(2)}}(\phi^*)^{-1}g\right)(y)\mu^{(2)}(y)dV^{(2)}(y) \quad \text{(E.12)}\\
&= -\int_{\mathcal{M}^{(2)}}\alpha^2\left(\frac{1}{2}(\phi^*)^{-1}g(\phi^*)^{-1}f\Delta^{(2)}\mu^{(2)}\right)(y)dV^{(2)}(y)\\
&\quad +\int_{\mathcal{M}^{(2)}}\alpha^2\left((\phi^*)^{-1}g\Delta^{(2)}\mu^{(2)}(\phi^*)^{-1}f\right)(y)dV^{(2)}(y)\\
&\quad +\int_{\mathcal{M}^{(2)}}\alpha^2\left(\nabla^{(2)}(\phi^*)^{-1}g\cdot\nabla^{(2)}\mu^{(2)}(\phi^*)^{-1}f\right)(y)dV^{(2)}(y) \quad \text{(E.13)}\\
&= \int_{\mathcal{M}^{(2)}}\alpha^2\left((\phi^*)^{-1}g\frac{\Delta^{(2)}\mu^{(2)}}{2\mu^{(2)}}(\phi^*)^{-1}f\right)(y)\mu^{(2)}(y)dV^{(2)}(y)\\
&\quad +\int_{\mathcal{M}^{(2)}}\alpha^2\left(\frac{\nabla^{(2)}(\phi^*)^{-1}g\cdot\nabla^{(2)}\mu^{(2)}}{\mu^{(2)}}(\phi^*)^{-1}f\right)(y)\mu^{(2)}(y)dV^{(2)}(y) \quad \text{(E.14)}\\
&= \int_{\mathcal{M}^{(1)}}\alpha^2\left(g\phi^*\frac{\Delta^{(2)}\mu^{(2)}}{2\mu^{(2)}}\right)(x)f(x)\mu^{(1)}(x)dV^{(1)}(x)\\
&\quad +\int_{\mathcal{M}^{(1)}}\alpha^2\left(\phi^*\frac{\nabla^{(2)}(\phi^*)^{-1}g\cdot\nabla^{(2)}\mu^{(2)}}{\mu^{(2)}}\right)(x)f(x)\mu^{(1)}(x)dV^{(1)}(x) \quad \text{(E.15)}\\
&= -\left\langle f,A_\alpha^{(2)}g\right\rangle_{\mathcal{M}^{(1)}}, \quad \text{(E.16)}
\end{aligned}$$

where the transitions from (E.11) to (E.12) and from (E.14) to (E.15) are based on $\mu^{(1)}(x)dV^{(1)}(x)=\mu^{(2)}(y)dV^{(2)}(y)$ and $y=\phi(x)$. In addition, the transition between (E.12) and (E.13) is based on Green's first identity.

Finally, combining these results for $A_\alpha^{(1)}$ and $A_\alpha^{(2)}$ we get:

$$\begin{aligned}
\langle jA_\alpha f,g\rangle_{\mathcal{M}^{(1)}} &= \left\langle j\left(A_\alpha^{(1)}+A_\alpha^{(2)}\right)f,g\right\rangle_{\mathcal{M}^{(1)}} \quad \text{(E.17)}\\
&= j\left\langle A_\alpha^{(1)}f,g\right\rangle_{\mathcal{M}^{(1)}}+j\left\langle A_\alpha^{(2)}f,g\right\rangle_{\mathcal{M}^{(1)}} \quad \text{(E.18)}\\
&= -j\left\langle f,-A_\alpha^{(1)}g\right\rangle_{\mathcal{M}^{(1)}}-j\left\langle f,A_\alpha^{(2)}g\right\rangle_{\mathcal{M}^{(1)}} \quad \text{(E.19)}\\
&= -j\left\langle f,\left(A_\alpha^{(1)}+A_\alpha^{(2)}\right)g\right\rangle_{\mathcal{M}^{(1)}} = \langle f,jA_\alpha g\rangle_{\mathcal{M}^{(1)}}. \quad \text{(E.20)}
\end{aligned}$$

■

Remark E.1. By performing a similar derivation for the operator S_ϵ, it can be shown to be self-adjoint as well.

F. Proof of Proposition 3.5. We prove here that $\forall f \in C^\infty\left(\mathcal{M}^{(1)}\right)$, if $\mathrm{supp} f \subset \mathring{\Omega}_\alpha$, then $A_\alpha f(x) = 0$, where, as defined in Section 2, $\Omega_\alpha = \left\{x \in \mathcal{M}^{(1)} :\ \nabla\phi|_x = \alpha \mathrm{I}\right\}$, $\alpha > 0$.

Proof. As presented in Proposition 3.3 and in Appendix E, the asymptotic expansion of the operator A_α is given by

$$A_\alpha f(x) = \frac{1}{2}\left(\frac{2\nabla^{(1)} f \cdot \nabla^{(1)}\mu^{(1)}}{\mu^{(1)}}(x) + \frac{f\Delta^{(1)}\mu^{(1)}}{\mu^{(1)}}(x)\right) \tag{F.1}$$

$$-\frac{\alpha^2}{2}\left(\phi^*\frac{2\nabla^{(2)}(\phi^*)^{-1} f \cdot \nabla^{(2)}\mu^{(2)}}{\mu^{(2)}}(x) + f\phi^*\frac{\Delta^{(2)}\mu^{(2)}}{\mu^{(2)}}(x)\right). \tag{F.2}$$

Consider $x \in \mathcal{M}^{(1)}$, $y = \phi(x) \in \mathcal{M}^{(2)}$ and $f \in C^\infty\left(\mathcal{M}^{(1)}\right)$. With the chosen coordinates around x and y, we calculate the following gradient of f:

$$\left(\nabla^{(2)}(\phi^*)^{-1} f\right)\Big|_y = \left(\nabla^{(2)} f \circ \phi^{-1}\right)\Big|_y = \nabla^{(1)} f|_x \nabla^{(2)}\phi^{-1}|_y. \tag{F.3}$$

In addition, calculating the gradient of the density function of the manifold $\mathcal{M}^{(2)}$, given by $\mu^{(2)}(y) = J(y)\mu^{(1)}\left(\phi^{-1}(y)\right)$, where $J(y) = \left|det\left(\nabla^{(2)}\phi^{-1}(y)\right)\right|$, leads to:

$$\nabla^{(2)}\mu^{(2)}|_y = \nabla^{(2)}\left(J\mu^{(1)} \circ \phi^{-1}\right)\Big|_y \tag{F.4}$$

$$= \nabla^{(2)} J|_y \left(\mu^{(1)} \circ \phi^{-1}\right)\Big|_y + J|_y \nabla^{(1)}\mu^{(1)}|_x \nabla^{(2)}\phi^{-1}|_y. \tag{F.5}$$

By substituting these derivations in expression (F.2), we get:

$$A_\alpha f(x) = \frac{1}{2}\left(\frac{2\nabla^{(1)} f \cdot \nabla^{(1)}\mu^{(1)}}{\mu^{(1)}}(x) + \frac{f\Delta^{(1)}\mu^{(1)}}{\mu^{(1)}}(x)\right) \tag{F.6}$$

$$-\frac{\alpha^2}{2}\frac{2\nabla^{(1)} f|_x \nabla^{(2)}\phi^{-1}|_{\phi(x)} \cdot \nabla^{(2)} J|_{\phi(x)}\mu^{(1)}}{J|_{\phi(x)}\mu^{(1)}|_x} \tag{F.7}$$

$$-\frac{\alpha^2}{2}\frac{2\nabla^{(1)} f|_x \nabla^{(2)}\phi^{-1}|_{\phi(x)} \cdot \nabla^{(1)}\mu^{(1)}|_x \nabla^{(2)}\phi^{-1}|_{\phi(x)}}{\mu^{(1)}|_x} \tag{F.8}$$

$$-\frac{\alpha^2}{2} f\frac{\Delta^{(2)}\mu^{(2)}|_{\phi(x)}}{\mu^{(2)}|_{\phi(x)}}. \tag{F.9}$$

Then, if $\mathrm{supp} f \subset \mathring{\Omega}_\alpha$, for $x \in \mathring{\Omega}_\alpha$ we have $\nabla^{(2)}\phi^{-1}|_{\phi(x)} = \frac{1}{\alpha}\mathrm{I}$, where I denotes the $d \times d$ identity matrix, and $J|_{\phi(x)} = \alpha^{-d}$. In addition, for such x, we have $\mu^{(2)}(\phi(x)) = \alpha^{-d}\mu^{(1)}(x)$.

We are then left with:

$$A_\alpha f(x) = \frac{1}{2}\left(\frac{2\nabla^{(1)} f \cdot \nabla^{(1)}\mu^{(1)}}{\mu^{(1)}}(x) + \frac{f\Delta^{(1)}\mu^{(1)}}{\mu^{(1)}}(x)\right) \tag{F.10}$$

$$-\frac{\alpha^2}{2}\left(\frac{2\nabla^{(1)} f\alpha^{-1} \cdot \nabla^{(1)}\mu^{(1)}\alpha^{-1}}{\mu^{(1)}}(x) + f\frac{\Delta^{(2)}\mu^{(2)}}{\mu^{(2)}}(\phi(x))\right) \tag{F.11}$$

$$= \frac{1}{2}\left(\frac{f\Delta^{(1)}\mu^{(1)}}{\mu^{(1)}}(x) - \alpha^2 f\frac{\Delta^{(2)}\mu^{(2)}}{\mu^{(2)}}(\phi(x))\right) \tag{F.12}$$

$$= \frac{1}{2}\left(\frac{f\Delta^{(1)}\mu^{(1)}}{\mu^{(1)}}(x) - \alpha^2 f\frac{\alpha^{-d-2}\Delta^{(1)}\mu^{(1)}}{\alpha^{-d}\mu^{(1)}}(x)\right) \tag{F.13}$$

$$= 0, \tag{F.14}$$

where we use the fact that for $x \in \mathring{\Omega}_\alpha$, $\Delta^{(2)}\mu^{(2)}\left(\phi(x)\right) = \alpha^{-d-2}\Delta^{(1)}\mu^{(1)}(x)$.

Therefore, we showed that if $\text{supp} f \subset \mathring{\Omega}_\alpha$, then $A_\alpha f(x) = 0$. ■

G. Interpretation of the operators and diffeomorphism in the discrete setting. Note that in the current definition of the discrete operators $\mathbf{S}$ and $\mathbf{A}$, we apply operators defined on $\mathcal{M}^{(1)}$ and operators defined on $\mathcal{M}^{(2)}$ to the same functions. Specifically, applying $\mathbf{H}$ to $\mathbf{v}^{(1)}$, a discretization of $f \in C^\infty\left(\mathcal{M}^{(1)}\right)$, implies that the function f is first pushed forward to $\mathcal{M}^{(2)}$ and then discretized. Namely, the discrete operators, $\mathbf{G}$ and $\mathbf{H}$, embody both the continuous operators, $P^{(\ell)}_{\epsilon_\ell}$ and $Q^{(\ell)}_{\epsilon_\ell}$, and the diffeomorphism, ϕ, i.e. $\mathbf{G}$ is the discrete counterpart of $\phi^* P^{(2)}_{\epsilon_2}(\phi^*)^{-1}Q^{(1)}_{\epsilon_1}$ and $\mathbf{H}$ is the discrete counterpart of $P^{(1)}_{\epsilon_1}\phi^* Q^{(2)}_{\epsilon_2}(\phi^*)^{-1}$. When the two datasets significantly differ in their densities or metrics, the discrete operators do not necessarily embody the diffeomorphism. In this case, when the operator $\mathbf{A}$ is applied to the vector $\mathbf{v}^{(1)}$, explicitly given by $\mathbf{A}\mathbf{v}^{(1)} = \mathbf{P}^{(2)}\mathbf{Q}^{(1)}\mathbf{v}^{(1)} - \mathbf{P}^{(1)}\mathbf{Q}^{(2)}\mathbf{v}^{(1)}$, the subtracted expressions may be in different domains, i.e. $\mathbf{P}^{(2)}\mathbf{Q}^{(1)}\mathbf{v}^{(1)} \in \mathcal{M}^{(2)}$ and $\mathbf{P}^{(1)}\mathbf{Q}^{(2)}\mathbf{v}^{(1)} \in \mathcal{M}^{(1)}$. Moreover, the application of $\mathbf{Q}^{(2)}$ to $\mathbf{v}^{(1)}$ may be erroneous as well. One option to resolve this is by defining the following operators

$$\tilde{\mathbf{S}} = \mathbf{Q}^{(1)}\mathbf{S}\mathbf{P}^{(1)} \tag{G.1}$$

$$\tilde{\mathbf{A}} = \mathbf{Q}^{(1)}\mathbf{A}\mathbf{P}^{(1)}. \tag{G.2}$$

Using these definitions, by applying the operator $\tilde{\mathbf{A}}$ to $\mathbf{v}^{(1)}$, for example, we obtain $\tilde{\mathbf{A}}\mathbf{v}^{(1)} = \mathbf{Q}^{(1)}\mathbf{P}^{(2)}\mathbf{Q}^{(1)}\mathbf{P}^{(1)}\mathbf{v}^{(1)} - \mathbf{Q}^{(1)}\mathbf{P}^{(1)}\mathbf{Q}^{(2)}\mathbf{P}^{(1)}\mathbf{v}^{(1)}$. Therefore, the two subtracted terms now begin and end with kernels representing the density and metric properties of $\mathcal{M}^{(1)}$.

The operators $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{A}}$ are symmetric and anti-symmetric, respectively, and preserve the same asymptotic behavior. A second option is to use concepts from [SM4], which presents a method for recovering a functional map between two shapes, and include such a functional map, between the two manifolds, in the construction of the operators $\mathbf{S}$ and $\mathbf{A}$. We note that in the experimental results, presented in Section 5 and Section 6, both operator forms $\tilde{\mathbf{S}}$, $\tilde{\mathbf{A}}$, and $\mathbf{S}$, $\mathbf{A}$, led to comparable results. This is due to the similarity of the two manifolds in these applications.

H. Proof of Corollary 6.1. In this appendix we prove Corollary 6.1. For simplicity, we assume here that $\epsilon_2 = \epsilon_1 = \epsilon$ ($\alpha = 1$). For $\epsilon_2 \neq \epsilon_1$, the derivations are similar up to some notation changes, as in Appendix F.

Consider $\mathcal{E}^{(1)} \subset \mathbb{R}^p$ and $\mathcal{E}^{(2)} \subset \mathbb{R}^p$ such that $\mathcal{E}^{(\ell)} = \mathcal{M}^{(\ell)} \oplus \mathcal{F}^{(\ell)}$, where $\mathcal{M}^{(\ell)} \subset \mathbb{R}^{p_1}$, $\mathcal{F}^{(\ell)} \subset \mathbb{R}^{p_2}$, $p = p_1 + p_2$, $\ell = 1, 2$, and $\phi : \mathcal{E}^{(1)} \to \mathcal{E}^{(2)}$ satisfies $\phi(\mathcal{M}^{(1)} \oplus \mathcal{F}^{(1)}) = \mathcal{M}^{(1)} \oplus \tilde{\phi}(\mathcal{F}^{(1)})$, where $\tilde{\phi} : \mathcal{F}^{(1)} \to \mathcal{F}^{(2)}$ is a smooth diffeomorphism. In addition, assume that $\mu^{(\ell)}(\mathbf{s}^{(\ell)}) = \mu_m^{(\ell)}(\mathbf{m}^{(\ell)})\mu_f^{(\ell)}(\mathbf{f}^{(\ell)})$, where $\mu^{(\ell)}$ is the probability density on $\mathcal{E}^{(\ell)}$, $\mu_m^{(\ell)}$ is the marginal density of $\mu^{(\ell)}$ on $\mathcal{M}^{(\ell)}$, $\mu_f^{(\ell)}$ is the marginal density of $\mu^{(\ell)}$ on $\mathcal{F}^{(\ell)}$ and $\mathbf{s}^{(\ell)}(t) = \mathbf{m}^{(\ell)}(t) + \mathbf{f}^{(\ell)}(t)$, where $\mathbf{s}^{(\ell)} \in \mathcal{E}^{(\ell)}$, $\mathbf{m}^{(\ell)} \in \mathcal{M}^{(\ell)}$ and $\mathbf{f}^{(\ell)} \in \mathcal{F}^{(\ell)}$.

Denote $\Omega_f = \left\{\mathbf{f}^{(1)}(t) \in \mathcal{F}^{(1)} : \nabla\tilde{\phi}|_{\mathbf{f}^{(1)}} = \mathrm{I}\right\} \subset \mathcal{F}^{(1)}$, where I denotes a $p_2 \times p_2$ identity matrix, and define $A = \lim_{\epsilon\to 0} A_\epsilon/\epsilon^2$.

Corollary 6.1 states that for all $g \in C^\infty\left(\mathcal{E}^{(1)}\right)$, if $\mathrm{supp} g \subset \mathcal{M}^{(1)} \oplus \mathring{\Omega}_f$, then $Ag = 0$. Hence, if $Ag = \lambda g$, $g \neq 0$, then, $\mathrm{supp} g \subset \mathcal{M}^{(1)} \oplus \Omega_f^c$.

Proof. We first note that since $\mathcal{E}^{(1)} = \mathcal{M}^{(1)} \oplus \mathcal{F}^{(1)}$, the eigenfunctions of $A|_{\mathcal{F}^{(1)}}$, i.e. the restriction of A to $\mathcal{F}^{(1)}$, multiplied by a non-zero function defined on $\mathcal{M}^{(1)}$, are eigenfunctions of A. Second, note that $\nabla^{(1)}\phi \neq \mathrm{I}$ when $\nabla^{(1)}\tilde{\phi} \neq \mathrm{I}$, since

$$\nabla^{(1)}\phi_{p\times p} = \begin{bmatrix} \mathrm{I}_{p_1\times p_1} & \mathbf{0}_{p_1\times p_2} \\ \mathbf{0}_{p_2\times p_1} & \nabla^{(1)}\tilde{\phi}_{p_2\times p_2} \end{bmatrix} \tag{H.1}$$

where $\mathbf{0}_{d_1\times d_2}$ denotes a zero matrix of size $d_1 \times d_2$. Third, from the relation between the probability density functions on the two manifolds, we have $\mu_m^{(2)}(\mathbf{m}^{(2)}) = \mu_m^{(1)}(\mathbf{m}^{(1)})$ and $\mu_f^{(2)}(\mathbf{f}^{(2)}) = J_{\tilde{\phi}}\Big|_{\mathbf{f}^{(2)}} \mu_f^{(1)}(\mathbf{f}^{(1)})$, where $J_{\tilde{\phi}}\Big|_{\mathbf{f}^{(2)}} = \left|\det\left(\nabla^{(2)}\tilde{\phi}^{-1}(\mathbf{f}^{(2)})\right)\right|$, since $J_\phi|_{\mathbf{s}^{(2)}} = J_{\tilde{\phi}}\Big|_{\mathbf{f}^{(2)}}$ and $\mu^{(2)}(\mathbf{s}^{(2)}) = J_\phi|_{\mathbf{s}^{(2)}}\, \mu^{(1)}(\mathbf{s}^{(1)})$, $\mathbf{s}^{(2)} = \phi(\mathbf{s}^{(1)})$.

Therefore, we can derive the following expressions for $g \in C^\infty\left(\mathcal{E}^{(1)}\right)$, ϕ^{-1} and $\mu^{(\ell)}$:

$$\nabla^{(1)} g|_{\mathbf{s}^{(1)}} = \begin{bmatrix} \nabla_m^{(1)} g\Big|_{\mathbf{m}^{(1)}} \\ \nabla_f^{(1)} g\Big|_{\mathbf{f}^{(1)}} \end{bmatrix} \quad \nabla^{(2)}\phi^{-1}\Big|_{\phi(\mathbf{s}^{(1)})} = \begin{bmatrix} \mathrm{I}_{p_1\times p_1} & \mathbf{0}_{p_1\times p_2} \\ \mathbf{0}_{p_2\times p_1} & \nabla^{(2)}\tilde{\phi}^{-1}\Big|_{\phi(\mathbf{f}^{(1)})} \end{bmatrix} \tag{H.2}$$

$$\nabla^{(\ell)}\mu^{(\ell)}\Big|_{\mathbf{s}^{(1)}} = \begin{bmatrix} \mu_f^{(\ell)}(\mathbf{f}^{(\ell)})\, \nabla_m^{(\ell)}\mu_m^{(\ell)}\Big|_{\mathbf{m}^{(\ell)}} \\ \nabla_f^{(\ell)}\mu_f^{(\ell)}\Big|_{\mathbf{f}^{(\ell)}} \mu_m^{(\ell)}(\mathbf{m}^{(\ell)}) \end{bmatrix} \tag{H.3}$$

$$\Delta^{(\ell)}\mu^{(\ell)}\Big|_{\mathbf{s}^{(\ell)}} = \mu_f^{(\ell)}(\mathbf{f}^{(\ell)})\, \Delta_m^{(\ell)}\mu_m^{(\ell)}\Big|_{\mathbf{m}^{(\ell)}} + \Delta_f^{(\ell)}\mu_f^{(\ell)}\Big|_{\mathbf{f}^{(\ell)}} \mu_m^{(\ell)}(\mathbf{m}^{(\ell)}) \tag{H.4}$$

$$\nabla^{(1)} g|_{\mathbf{s}^{(1)}} \nabla^{(2)}\phi^{-1}|_{\phi(\mathbf{s}^{(1)})} = \begin{bmatrix} \nabla_m^{(1)} g|_{\mathbf{m}^{(1)}} \\ \nabla_f^{(1)} g|_{\mathbf{f}^{(1)}}\, \nabla^{(2)}\tilde{\phi}^{-1}\Big|_{\tilde{\phi}(\mathbf{f}^{(1)})} \end{bmatrix} \tag{H.5}$$

$$\nabla^{(1)}\mu^{(1)}|_{\mathbf{s}^{(1)}} \nabla^{(2)}\phi^{-1}|_{\phi(\mathbf{s}^{(1)})} = \begin{bmatrix} \mu_f^{(1)}(\mathbf{f}^{(1)})\, \nabla_m^{(1)}\mu_m^{(1)}\Big|_{\mathbf{m}^{(1)}} \\ \nabla_f^{(1)}\mu_f^{(1)}\Big|_{\mathbf{f}^{(1)}}\, \nabla^{(2)}\tilde{\phi}^{-1}\Big|_{\tilde{\phi}(\mathbf{f}^{(1)})} \mu_m^{(1)}(\mathbf{m}^{(1)}) \end{bmatrix}. \tag{H.6}$$

According to Appendix F the operator $A = \lim_{\epsilon\to 0} A_\epsilon/\epsilon^2$ is given by

$$\begin{aligned}
Ag(x) =& \frac{1}{2}\left(\frac{2\nabla^{(1)}g\cdot\nabla^{(1)}\mu^{(1)}}{\mu^{(1)}}(x) + \frac{g\Delta^{(1)}\mu^{(1)}}{\mu^{(1)}}(x)\right)\\
&-\frac{1}{2}\frac{2\nabla^{(1)}g|_x\nabla^{(2)}\phi^{-1}|_{\phi(x)}\cdot\nabla^{(2)}J|_{\phi(x)}\mu^{(1)}}{J|_{\phi(x)}\mu^{(1)}|_x}\\
&-\frac{1}{2}\frac{2\nabla^{(1)}g|_x\nabla^{(2)}\phi^{-1}|_{\phi(x)}\cdot\nabla^{(1)}\mu^{(1)}|_x\nabla^{(2)}\phi^{-1}|_{\phi(x)}}{\mu^{(1)}|_x}\\
&-\frac{1}{2}g\frac{\Delta^{(2)}\mu^{(2)}|_{\phi(x)}}{\mu^{(2)}|_{\phi(x)}}.
\end{aligned} \tag{H.7}$$

By substituting expressions (H.2) - (H.6) and $\mu^{(\ell)}(\mathbf{s}^{(1)}) = \mu_m^{(\ell)}(\mathbf{m}^{(1)})\mu_f^{(\ell)}(\mathbf{f}^{(1)})$ into (H.7), we get:

$$
\begin{aligned}
Ag(\mathbf{s}^{(1)}) =& \frac{1}{2}\left(\frac{2\nabla_m^{(1)} g|_{\mathbf{m}^{(1)}} \cdot \nabla_m^{(1)} \mu_m^{(1)}|_{\mathbf{m}^{(1)}}}{\mu_m^{(1)}|_{\mathbf{m}^{(1)}}} + \frac{g\Delta_m^{(1)} \mu_m^{(1)}|_{\mathbf{m}^{(1)}}}{\mu_m^{(1)}|_{\mathbf{m}^{(1)}}}\right) \\
&+\frac{1}{2}\left(\frac{2\nabla_f^{(1)} g|_{\mathbf{f}^{(1)}} \cdot \nabla_f^{(1)} \mu_f^{(1)}|_{\mathbf{f}^{(1)}}}{\mu_f^{(1)}|_{\mathbf{f}^{(1)}}} + \frac{g\Delta_f^{(1)} \mu_f^{(1)}|_{\mathbf{f}^{(1)}}}{\mu_f^{(1)}|_{\mathbf{f}^{(1)}}}\right) \\
&-\frac{1}{2}\frac{2\nabla_f^{(1)} g|_{\mathbf{f}^{(1)}} \nabla^{(2)}\tilde{\phi}^{-1}|_{\tilde{\phi}(\mathbf{f}^{(1)})} \cdot \nabla^{(2)} J_{\tilde{\phi}}|_{\tilde{\phi}(\mathbf{f}^{(1)})}\mu_f^{(1)}|_{\mathbf{f}^{(1)}}}{J_{\tilde{\phi}}|_{\tilde{\phi}(\mathbf{f}^{(1)})}\mu_f^{(1)}|_{\mathbf{f}^{(1)}}} \\
&-\frac{1}{2}\frac{2\nabla_m^{(1)} g|_{\mathbf{m}^{(1)}} \cdot \nabla_m^{(1)} \mu_m^{(1)}|_{\mathbf{m}^{(1)}}}{\mu_m^{(1)}|_{\mathbf{m}^{(1)}}} \\
&-\frac{1}{2}\frac{2\nabla_f^{(1)} g|_{\mathbf{f}^{(1)}} \nabla^{(2)}\tilde{\phi}^{-1}|_{\tilde{\phi}(\mathbf{f}^{(1)})} \cdot \nabla_f^{(1)} \mu_f^{(1)}|_{\mathbf{f}^{(1)}} \nabla^{(2)}\tilde{\phi}^{-1}|_{\tilde{\phi}(\mathbf{f}^{(1)})}}{\mu_f^{(1)}|_{\mathbf{f}^{(1)}}} \\
&-\frac{1}{2}\frac{g\Delta_f^{(2)} \mu_f^{(2)}|_{\tilde{\phi}(\mathbf{f}^{(1)})}}{\mu_f^{(2)}|_{\tilde{\phi}(\mathbf{f}^{(1)})}} - \frac{1}{2}\frac{g\Delta_m^{(1)} \mu_m^{(1)}|_{\mathbf{m}^{(1)}}}{\mu_m^{(1)}|_{\mathbf{m}^{(1)}}} \qquad \text{(H.8)} \\
=& \frac{1}{2}\left(\frac{2\nabla_f^{(1)} g|_{\mathbf{f}^{(1)}} \cdot \nabla_f^{(1)} \mu_f^{(1)}|_{\mathbf{f}^{(1)}}}{\mu_f^{(1)}|_{\mathbf{f}^{(1)}}} + \frac{g\Delta_f^{(1)} \mu_f^{(1)}|_{\mathbf{f}^{(1)}}}{\mu_f^{(1)}|_{\mathbf{f}^{(1)}}}\right) \\
&-\frac{1}{2}\frac{2\nabla_f^{(1)} g|_{\mathbf{f}^{(1)}} \nabla^{(2)}\tilde{\phi}^{-1}|_{\tilde{\phi}(\mathbf{f}^{(1)})} \cdot \nabla^{(2)} J_{\tilde{\phi}}|_{\tilde{\phi}(\mathbf{f}^{(1)})}\mu_f^{(1)}|_{\mathbf{f}^{(1)}}}{J_{\tilde{\phi}}|_{\tilde{\phi}(\mathbf{f}^{(1)})}\mu_f^{(1)}|_{\mathbf{f}^{(1)}}} \\
&-\frac{1}{2}\frac{2\nabla_f^{(1)} g|_{\mathbf{f}^{(1)}} \nabla^{(2)}\tilde{\phi}^{-1}|_{\tilde{\phi}(\mathbf{f}^{(1)})} \cdot \nabla_f^{(1)} \mu_f^{(1)}|_{\mathbf{f}^{(1)}} \nabla^{(2)}\tilde{\phi}^{-1}|_{\tilde{\phi}(\mathbf{f}^{(1)})}}{\mu_f^{(1)}|_{\mathbf{f}^{(1)}}} \\
&-\frac{1}{2}\frac{g\Delta_f^{(2)} \mu_f^{(2)}|_{\tilde{\phi}(\mathbf{f}^{(1)})}}{\mu_f^{(2)}|_{\tilde{\phi}(\mathbf{f}^{(1)})}} \qquad \text{(H.9)} \\
=& A|_{\mathcal{F}^{(1)}} g(\mathbf{f}^{(1)}), \qquad \text{(H.10)}
\end{aligned}
$$

where we used $\mu_m^{(2)}|_{\phi(\mathbf{m}^{(1)})} = \mu_m^{(1)}|_{\mathbf{m}^{(1)}}$ and $\Delta_m^{(2)}\mu_m^{(2)}|_{\phi(\mathbf{m}^{(1)})} = \Delta_m^{(1)}\mu_m^{(1)}|_{\mathbf{m}^{(1)}}$ to obtain the last term in (H.8).

This derivation states that $Ag(\mathbf{s}^{(1)}) = A|_{\mathcal{F}^{(1)}} g(\mathbf{f}^{(1)})$. Therefore, under the assumptions stated in the beginning of this appendix, the considered setting is equivalent to the setting in Proposition 3.5, with the manifolds $\mathcal{F}^{(\ell)}$, $\ell = 1, 2$, the smooth diffeomorphism $\tilde{\phi} : \mathcal{F}^{(1)} \to \mathcal{F}^{(2)}$ and $g \in C^\infty\left(\mathcal{F}^{(1)}\right)$. We can now apply Proposition 3.5 to (H.10) and obtain that for all $g \in C^\infty\left(\mathcal{F}^{(1)}\right)$, if $\mathrm{supp} g \subset \mathring{\Omega}_f$, then $A|_{\mathcal{F}^{(1)}} g(\mathbf{f}^{(1)}) = 0$. Due to the definition of $\mathcal{E}^{(\ell)}$ as a direct sum of $\mathcal{M}^{(\ell)}$ and $\mathcal{F}^{(\ell)}$, we can define $g \in C^\infty\left(\mathcal{E}^{(1)}\right)$ and obtain that for all $g \in C^\infty\left(\mathcal{E}^{(1)}\right)$, if $\mathrm{supp} g \subset \mathcal{M}^{(1)} \oplus \mathring{\Omega}_f$, then $Ag(\mathbf{s}^{(1)}) = 0$, which concludes the proof. ∎

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