Online Appendix to: Discrete Derivatives of Vector Fields on Surfaces – An Operator Approach

OMRI AZENCOT
Technion – Israel Institute of Technology
MAKS OVSJANIKOV
LIX, École Polytechnique
FRÉDÉRIC CHAZAL
Geometrica, INRIA
and
MIRELA BEN-CHEN
Technion – Israel Institute of Technology

This online appendix contains challenges in the discrete setting, properties of the continuous and discrete operators associated with the Levi-Civita covariant derivative, and a periodic solution to Euler's equation.

A. CHALLENGES IN THE DISCRETE SETTING

We would like to show that the metric compatibility is impossible to achieve in the discrete setting even when functions and vector fields do not live on the same domain. In what follows we will assume that vector fields are discretized on the faces of a triangle mesh and functions are discretized on some other domain (vertices, edges, faces, etc.). However, the proof is quite general and can potentially be extended to the case where even the vector fields are also discretized on some other domain (e.g., on edges), depending on the choice of inner product.

We will use the following formulation of the metric compatibility:

$$\tilde{D}_X \mathcal{A}(\langle U, V\rangle) = \mathcal{A}(\langle \tilde{\nabla}_X U, V\rangle + \langle \tilde{\nabla}_X V, U\rangle). \tag{1}$$

Here $\tilde{D}_X$ is a covariant derivative for functions with respect to the vector field X, that is, $\tilde{D}_X$ takes a function defined on some domain (e.g., vertices or edges) and produces a function defined on the same domain. $\tilde{\nabla}_X U$ is the covariant derivative for vector fields, and the inner product is the standard inner product of vector fields in $\mathbb{R}^3$. Since the inner product $\langle U, V\rangle$ produces a function on the faces of the triangle mesh, we need an operator $\mathcal{A}$ that takes functions on faces and produces functions on vertices or edges.

We will assume that $\mathcal{A}$ has the following properties.

(1) It is linear: $\mathcal{A}(f+g) = \mathcal{A}(f) + \mathcal{A}(g)$.
(2) It maps constant functions to constant functions. In other words, if we are given a function f such that the value of f on face i is equal to its value on face j for every j, then $\mathcal{A}(f)$ is also a constant function on the target domain (e.g., vertices or edges).
(3) It is nonnegative.

Under these conditions, we have the following result.

LEMMA 1. *If $\tilde{D}_X$ is a linear operator such that $\tilde{D}_X f = 0$ if f is a constant function, and the covariant derivative for vector fields is linear: $\tilde{\nabla}_X(U_1+U_2) = \tilde{\nabla}_X U_1 + \tilde{\nabla}_X U_2$, then the metric compatibility condition (Eq. (1)), implies that $\tilde{D}_X f = 0$ for all f in the range of $\mathcal{A}$, that is, $\tilde{D}_X \mathcal{A} h = 0$ for any h.*

PROOF. We will use V_i to denote a vector field which is nonzero on face i and has unit norm, and use $e_i = \langle V_i, V_i\rangle$, as the indicator function of face i. □

(1) The metric compatibility condition implies that
$$\tilde{D}_X \mathcal{A}(e_i) = \tilde{D}_X \mathcal{A}(\langle V_i, V_i\rangle) = 2\mathcal{A}(\langle \nabla_X V_i, V_i\rangle).$$
Since $V_i = 0$ on any face other than i, we have $\langle \nabla_X V_i, V_i\rangle = a_i e_i$ for some scalar a_i. Thus,
$$\tilde{D}_X \mathcal{A}(e_i) = 2\mathcal{A}(a_i e_i) = 2a_i \mathcal{A}(e_i).$$
In other words, $\mathcal{A}(e_i)$ is an eigenvector of $\tilde{D}_X$ with eigenvalue $2a_i$. Our goal will be to show that $a_i = 0$ for all i, since in this case $\tilde{D}_X \mathcal{A}(h) = 0$ for any h.
(2) For any $i \neq j$, we have $\langle V_i, V_j\rangle = 0$. Thus,
$$0 = \tilde{D}_X \mathcal{A}(\langle V_i, V_j\rangle) = \mathcal{A}(\langle \nabla_X V_i, V_j\rangle) + \mathcal{A}(\langle \nabla_X V_j, V_i\rangle).$$
(3) Let $V = \sum_i V_i$. Note that $\langle V, V\rangle = \sum e_i = c$ a constant function on the faces. Thus $\tilde{D}_X \mathcal{A}(\langle V, V\rangle) = 0$. But
$$\begin{aligned} 0 &= \tilde{D}_X \mathcal{A}(\langle V, V\rangle) = 2\mathcal{A}(\langle \nabla_X V, V\rangle) \\ &= 2\mathcal{A}\left(\left\langle \nabla_X \sum_i V_i, \sum_i V_i\right\rangle\right) = 2\mathcal{A}\left(\left\langle \sum_i \nabla_X V_i, \sum_i V_i\right\rangle\right) \\ &= 2\mathcal{A}\left(\sum_{i,j} \langle \nabla_X V_i, V_j\rangle\right) = 2\sum_{i,j} \mathcal{A}\left(\langle \nabla_X V_i, V_j\rangle\right). \end{aligned}$$

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Using Parts 1 and 2 previously (which states that the cross-terms cancel out), this further simplifies to

$$0 = \tilde{D}_X \mathcal{A}(\langle V, V\rangle) = 2\sum_i \mathcal{A}(\langle \nabla_X V_i, V_i\rangle) = 2\sum_i a_i \mathcal{A}(e_i).$$

(4) Since $\mathcal{A}(e_i)$ is an eigenvector of $\tilde{D}_X$ with eigenvalue $2a_i$, the previous part can be rewritten as (by summing eigenvectors with the same eigenvalue) $\sum_j \lambda_j \phi_j = 0$, where λ_j are all distinct and nonzero, and $\phi_j = \sum_i \mathcal{A}(e_i)$ such that $2a_i = \lambda_j$.

We claim that ϕ_j are all linearly independent. To see this, suppose that $\phi_k = \sum_{j\neq k} b_j \phi_j$, for some k, such that $\{\phi_j\}$ are linearly independent and $b_j \neq 0$. Then, since $\tilde{D}_X \phi_i = \lambda_i \phi_i$, we have

$$\sum_{j\neq k} b_j \lambda_j \phi_j = \sum_{j\neq k} b_j \lambda_k \phi_j,$$

which implies (since b_j is nonzero) that $\lambda_k = \lambda_j$ for all j, which is a contradiction.

(5) Since ϕ_j are all linearly independent, $\sum_j \lambda_j \phi_j = 0$, implies that $\lambda_j \phi_j = 0$ for all j. Thus either $\lambda_j = 0$ or $\phi_j = 0$. But $\phi_j = \sum_i \mathcal{A}(e_i)$ for some index i, and A is assumed nonnegative, $\sum_i \mathcal{A}(e_i) = 0$ only if $\mathcal{A}(e_i) = 0$ for every i. But this means that $\lambda_j = a_i = 0$. Therefore $a_i = 0$ for all i, which implies that $\tilde{D}_X \mathcal{A}(h) = 0$ for all h. □

B. PROPERTIES OF THE CONTINUOUS OPERATORS ASSOCIATED WITH THE LEVI-CIVITA COVARIANT DERIVATIVE

The following lemmas all deal with smooth manifolds. We will assume each manifold is compact and without boundary. Moreover, we will assume all vector fields are smooth.

LEMMA 2. *For a closed oriented surface M without boundary, $\nabla_U V = 0\ \forall\ U$ if and only if $V = 0$ or M is a flat torus.*

PROOF. First, note that if M is not a genus-1 surface then, according to the Hopf index theorem [Morita 2001 page 256], there must be some point p such that $V(p) = 0$. But then pick another point p' and construct a vector field Z such that the flow lines of Z connect p and p'. Since $\nabla_Z V = 0$, we have that V is parallel transported along the flow lines of Z. As parallel transport is an isometry, this implies that $V(p') = V(p) = 0$ and thus $V = 0$ everywhere, since p' was arbitrary. Let us assume now that M is a torus. Since $\nabla_U V = 0\ \forall\ U$, parallel transport along any two paths must commute, so there is no curvature and thus M must be a flat torus. □

LEMMA 3. *Two vector fields U and V are equal if and only if $\nabla_U W = \nabla_V W$ for all vector fields W.*

PROOF. Recall from the definition of parallel transport that, if $\nabla_X V = 0$, then V is preserved by parallel transport along the trajectories of X. Suppose $X \neq 0$, so that there is some point p, such that $\Phi_t(p) \neq p$ for some t. Then, for any vector field V, $V(\Phi_t(p))$ is the parallel transport of $V(p)$ along the trajectory of X from p to $\Phi_t(p)$. As the parallel-transported image of $V(p)$ is uniquely defined, it is easy to build two vector fields V_1 and V_2 such that $V_1(p) = V_2(p)$ but $V_1(\Phi_t(p)) \neq V_2(\Phi_t(p))$, a contradiction. □

LEMMA 4. *A vector field U is divergence free if and only if ∇_U is anti-symmetric with respect to the inner product on the surface, that is, if and only if $\int_M \langle \nabla_U X, Y\rangle dx = -\int_M \langle \nabla_U Y, X\rangle dx$ for all vector fields X and Y.*

PROOF. Suppose U is divergence free. Then, using the metric compatibility of the covariant derivative, we have

$$\int_M (\langle \nabla_U X, Y\rangle + \langle \nabla_U Y, X\rangle)\, dx = \int_M \nabla_U \langle X, Y\rangle dx = \int_M \operatorname{div}(U)\langle X, Y\rangle dx = 0,$$

where the second-to-last equality uses Stokes' theorem and integration by parts. Now, suppose that ∇_U is anti-symmetric. Then by the same argument we get $\int_M \operatorname{div}(U)\langle X, Y\rangle dx = 0$ for any X and Y. Suppose $f = div(U)$ is not zero. Then there exists a point p such that $f(p) = \epsilon > 0$. Let Ω be a small neighborhood of p such that $f(p)$ does not change sign and is strictly greater than 0. By constructing a vector field X that vanishes outside of Ω, and considering $\int_M \operatorname{div}(U)\langle X, X\rangle dx$, it is easy to see that this integral must be positive. But this contradicts the assumption of anti-symmetry. □

C. PROPERTIES OF THE DISCRETE OPERATORS ASSOCIATED WITH THE LEVI-CIVITA COVARIANT DERIVATIVE

LEMMA 5. *Let $\tilde{\Gamma}_{U,t} = \exp(t\tilde{\nabla}_U)$, where $\tilde{\nabla}_U$ is the matrix representation of the discrete covariant derivative operator defined in the main article, and* exp *is matrix exponentiation. Then*

$$\tilde{\nabla}_U(V)(p) = \frac{d}{dt}(\tilde{\Gamma}_{U,t}(V)(p))|_{t=0}. \tag{2}$$

PROOF. We have $\frac{d}{dt}\tilde{\Gamma}_{U,t} = \frac{d}{dt}exp(t\tilde{\nabla}_U) = \tilde{\nabla}_U exp(t\tilde{\nabla}_U)$, where we can use standard matrix derivative rules, as $\tilde{\nabla}_U$ does not depend on t. Hence for $t = 0$, we get $\frac{d}{dt}\tilde{\Gamma}_{U,t}|_{t=0} = \tilde{\nabla}_U$, as required. □

LEMMA 6. *If $\tilde{D}_U$ uses a full basis, then the operator $\tilde{\nabla}$ is invariant to rigid transformations. Namely, let $M = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a mesh embedded with coordinates $X \in \mathbb{R}^3$, and let U, V be two tangent vector fields on M. In addition, let $\mathcal{T}$ be a global rigid transformation $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^3$. Then*

$$(\tilde{\nabla}^{\mathcal{T}(X)})_{\mathcal{T}(U)}\mathcal{T}(V) = \mathcal{T}((\tilde{\nabla}^X)_U V), \tag{3}$$

where $\tilde{\nabla}^X$ is the discrete covariant derivative operator on a mesh embedded with coordinates X.

PROOF. Since we only deal with vector quantities (e.g., the input vectors and the edge vectors of the mesh) and intrinsic scalar quantities (e.g., triangle areas), it is clear that the definition is invariant to global translation.

Let $\mathcal{T}$ be given by a global rotation matrix R^T. Consider the gradient of the three components of V in the face $i \in \mathcal{F}$, namely the 3×3 matrix $G^X_{V,i}$ whose columns are $[(\nabla^X \mathcal{A} v_x)(i), (\nabla^X \mathcal{A} v_y)(i), (\nabla^X \mathcal{A} v_z)(i)]$, where $\mathcal{A}$ is an intrinsic averaging operator and ∇^X is the gradient of nonconforming elements on a mesh embedded with coordinates X.

From the definition of the gradient, it is easy to check that

$$G^X_{V,i} = -\left(E^X_i\right)^T \mathcal{C}_i \mathcal{A} V / \triangle_i, \tag{4}$$

where E^X_i is a 3×3 matrix whose rows are the rotated vector edges of the face i, $\mathcal{C}_i$ is a $3 \times |\mathcal{E}|$ matrix which chooses the edges in the face i, V is a $|\mathcal{F}| \times 3$ matrix where the i-th row represents the vector in face i, and $\triangle_i$ is the area of face i. Similarly, for the rotated mesh we have

$$G^{XR}_{VR,i} = -\left(E^{XR}_i\right)^T \mathcal{C}_i \mathcal{A} V R / \triangle_i, \tag{5}$$

since $\mathcal{C}_i$ is combinatorial, $\mathcal{A}$ and $\triangle_i$ are intrinsic, and rotating the vector field can be expressed as post-multiplying by R. Similarly, rotating the coordinates (and thus the edge vectors) of X can also be expressed as post-multiplying by R, hence we have $E_i^{XR} = E_i^X R$. Combined with (4) and (5) we get

$$G_{VR,i}^{XR} = R^T G_{V,i}^X R. \tag{6}$$

By definition, we have that $(\tilde{D}_U^X V)(i) = (\tilde{D}_U^X v_x, \tilde{D}_U^X v_y, \tilde{D}_U^X v_z)(i) = U(i)G_{V,i}^X$, where $U(i)$ is the vector U in the face i. Hence, plugging in (6) we get

$$\begin{aligned}\big(\tilde{D}_{UR}^{XR}(VR)\big)(i) &= (UR)(i)G_{VR,i}^{XR} = U(i)RR^T G_{V,i}^X R \\ &= U(i)G_{V,i}^X R, \end{aligned} \tag{7}$$

hence $(\tilde{D}_{UR}^{XR}(VR))(i) = (\tilde{D}_U^X V)(i)R$.

It is straightforward to check that by projecting out the normal component we get $(\tilde{\nabla}_{UR}^{XR}(VR))(i) = (\tilde{\nabla}_U^X(V))(i)R$, as required. □

LEMMA 7. *Suppose $\mathcal{A}$ is a positive local averaging operator $\mathcal{A} : (\mathcal{F} \to \mathbb{R}) \to (\mathcal{E} \to \mathbb{R})$. Namely, $\mathcal{A}$ is strictly positive and averages only those values of faces neighboring the edge. Then, $\mathcal{A}$ has an empty kernel for any mesh with at least one odd-degree vertex.*

PROOF. Given a mesh $\mathcal{M} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ with vertices, edges, and faces, respectively, we assume to have an odd-degree vertex $v_i \in \mathcal{V}$. We denote by $t_{ij} \in \mathcal{N}_1(v_i)$ the ordered adjacent faces (1-ring) of v_i, such that t_{ij} and $t_{i(j+1 \mod |\mathcal{N}_1(v_i)|)}$ are neighboring faces. Let $f : \mathcal{F} \to \mathbb{R}$ be a function in the kernel of $\mathcal{A}$, that is, $\mathcal{A}(f) = 0$. Notice that this equation holds pointwise, that is, for every edge of $\mathcal{M}$. Suppose $x_1 = f(t_{i1})$ and assume without loss of generality that x_1 is positive, then $x_2 = f(t_{i2})$ is negative, since $\mathcal{A}$ is positive and averages only those values of faces neighboring the edge. If we continue, we get $x_3 = f(t_{i3}) > 0$, and so forth. However, since the degree of v_i is odd, we obtain that x_1 should be negative. Thus $x_1 = 0$. It follows that $f = 0$ everywhere by applying the same argument on the edges of the dual graph. □

D. PERIODIC SOLUTION TO EULER'S EQUATION

In this section we consider the evolution of an incompressible inviscous fluid on a two-dimensional sphere. Our goal is to show that, if the velocity field at time 0 equals $V(0) = U_0 + J\nabla\phi_j$, where U_0 is a killing vector field, J an operator that rotates a given vector field by $\pi/2$ in each tangent plane, and ϕ_j an eigenfunction of the Laplace-Beltrami operator corresponding to the j-th eigenvalue, then the solution to the Euler equation at time t will have the form

$$V(t) = U_0 + \sum_i a_i(t) J\nabla\phi_i.$$

Here, $a_i(t)$ are scalar-valued functions and ϕ_i are eigenfunctions of the Laplace-Beltrami operator corresponding to the same j-th eigenvalue. Thus, $V(t)$ is a linear combination of a KVF and a rotated gradient of an eigenfunction corresponding to the j-th eigenvalue for all times t. Moreover, we would also like to show that the vorticity $\omega(t) = \mathrm{curl}(V(t))$ is advected isometrically by the flow.

To show that $V(t) = U_0 + \sum_i a_i(t)J\nabla\phi_i$, for all t, recall the vorticity formulation of the Euler equation.

(1) $V(t) = J\nabla\psi(t)$;
(2) $\omega(t) = -L\psi(t)$; and
(3) $\frac{d}{dt}\omega(t) = -D_{V(t)}\omega(t)$;

where $V(t)$ is the velocity field, ω the vorticity, ψ is called the *stream function*, L is the Laplace-Beltrami operator, and $D_{V(t)}$ the covariant derivative (of functions) in the direction of $V(t)$ (see, e.g., Taylor [1996, page 536, Eq. (1.27.)]).

Suppose $\psi(0) = \phi_1 + \phi_j$ where ϕ_1 corresponds to the first nonzero eigenfunction of the Laplace-Beltrami (note that $J\nabla\phi_1$ is a killing vector field). Thus we have $w(0) = -L\psi(0) = -(\lambda_1\phi_1 + \lambda_j\phi_j)$ and $V(0) = -J\nabla\phi_1 - J\nabla\phi_j$. Now

$$\begin{aligned} L\frac{d}{dt}\psi(0) = D_{V(0)}L\psi(0) &= \langle V(0), \nabla L\psi(0)\rangle \\ &= \langle J\nabla\phi_1 + J\nabla\phi_j, \lambda_1\nabla\phi_1 + \lambda_j\nabla\phi_j\rangle \\ &= \langle J\nabla\phi_1, \lambda_j\nabla\phi_j\rangle + \langle J\nabla\phi_j, \lambda_1\nabla\phi_1\rangle \\ &= (\lambda_j - \lambda_1)\langle J\nabla\phi_1, \nabla\phi_j\rangle. \end{aligned}$$

Now, since $U_0 = J\nabla\phi_1$ is a killing vector field, $L\langle U_0, \nabla f\rangle = \langle U_0, \nabla Lf\rangle$ for any f, which implies in particular that $L\langle U_0, \nabla\phi_j\rangle = \lambda_j\langle U_0, \nabla\phi_j\rangle$ and therefore $\langle U_0, \nabla\phi_j\rangle$ is an eigenfunction of L corresponding to the j-th eigenvalue. Note this implies that $\frac{d}{dt}\psi(0)$ is contained in the span of those eigenfunctions corresponding to the j-th eigenvalue. Moreover, using the same argument as before, the same is true for any t. Thus we have $\psi(t) = \phi_1 + \sum_i a_i(t)\phi_i$ and $V(t) = U_0 + \sum_i a_i(t)J\nabla\phi_i$, for all t, where $a_i(t)$ are scalar-valued functions of time and ϕ_i are eigenfunctions corresponding to the j-th eigenvalue of L.

Note that $V(t)$ is not a killing vector field for any time t. However, as we will show, the vorticity function ω is advected isometrically by $V(t)$.

For this, note that $w(t) = -L\psi(t) = -(\lambda_1\phi_1 + \lambda_j \sum_i a_i(t)\phi_i)$ for all t, and

$$\frac{d}{dt}\omega(t) = -D_{V(t)}\omega(t) = (\lambda_j - \lambda_1)\langle J\nabla\phi_1, \sum_i a_i(t)\nabla\phi_i\rangle.$$

Now consider another PDE for the evolution of ω (which would a priori give a different flow).

$$\frac{d}{dt}\omega(t) = -D_{J\nabla\phi_1}\omega(t) = \lambda_j\left\langle J\nabla\phi_1, \sum_i a_i(t)\nabla\phi_i\right\rangle.$$

Note that, when $\omega(t)$ has the form as previously given these two equations only differ by a scalar, that is, the speed of evolution. Moreover, note that when $w(0) = -(\lambda_1\phi_1 + \lambda_j\phi_j)$ then $w(t)$ will have this form for all t for both PDEs. Thus regardless of, w is advected by $V(t)$ or by a constant $U_0 = J\nabla\phi_1$, the trajectory will be the same. Since we know that $J\nabla\phi_1$ is a killing vector field, this means that $\omega(t)$ is advected isometrically by $V(t)$.