

Completeness and Universality Properties of Graph Invariants and Graph Polynomials

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Completeness and Universality Properties of Graph Invariants and Graph Polynomials

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Abstract

Graph polynomials are powerful and well-developed tools to express graph parameters. Usually graph polynomials are compared to each other by ad-hoc means allowing to decide whether a newly defined graph polynomial generalizes (or is generalized) by another one. We study their distinctive power and introduce the notions of *dp-completeness* and **universality** of graph polynomials in order to formalize dependencies between them.

Many known graph polynomials satisfy linear recurrence relations with respect to some set of edge- or vertex-elimination operations. Inspired by the work of Brylawski and Oxley on the Tutte polynomial, we define several classes of graph polynomials according to their recurrence relations, and prove *dp-completeness* and universality results for those classes. We also extend the results above to classes of *labeled* graph polynomials.

We provide several observations regarding computational complexity of the discussed *dp-complete* graph polynomials. Notably, we exploit definability properties of these polynomials to show that they can be computed efficiently when input graphs are limited by certain parameter, and present some explicit algorithms.

Abbreviations and Notations

FOL	—	First Order Logic
$MSOL$	—	Monadic Second Order Logic
$PNGI$	—	Parameterized Numeric Graph Invariant
$a \prec_{\omega} b$	—	a precedes b in order ω
$TCI(R)$	—	The transitive closure of the relation R
$G_1 \simeq G_2$	—	Graphs G_1 and G_2 are isomorphic
$G_1 \sqcup G_2$	—	Disjoint Union of two graphs
$G(A)$	—	Spanning subgraph of G with edge set A
$G[U]$	—	Induced subgraph of G with vertex set U
$k(G)$	—	Number of connected components of the graph G
$k_{cov}(G)$	—	Number of connected components of the graph G covered by edges, i.e. not singletons
$r(G)$	—	Rank of the graph G , defined as $ V - k(G)$
$s(G)$	—	Nullity of the graph G , defined as $ E - r(G)$
<i>bridge</i>	—	An edge $e = \{u, v\}$ that connects between two vertices such that there is no path between them if the edge e is removed
<i>loop</i>	—	An edge that connects a vertex to itself.

Chapter 1

Introduction

Let \mathcal{G} be the class of all graphs. A graph invariant is a function $f : \mathcal{G} \rightarrow \mathcal{R}$ which maps graphs into some range \mathcal{R} such that for isomorphic graphs $G_1 \simeq G_2$, f gets the same value: $f(G_1) = f(G_2)$. Graph invariants vary in their range \mathcal{R} . For example, when \mathcal{R} is a two-elements set $\{0, 1\}$, we speak of *graph properties*, e.g. being connected, planar, Eulerian, Hamiltonian, etc. When \mathcal{R} is either \mathbb{Z} or \mathbb{R} , such graph invariant is called a *graph parameter*, or a *numeric graph invariant*. Examples of numeric graph invariants are number of vertices, number of edges, number of connected components, number of proper 3-colorings etc. When \mathcal{R} is a ring of polynomials, $\mathbb{Z}[\bar{X}]$ or $\mathbb{R}[\bar{X}]$, over some set of indeterminates \bar{X} , such graph invariants are called *graph polynomials*. Graph polynomials encode an infinite number of graph invariants in their evaluations, coefficients, degrees and zeros. Among other graph invariants encoded by a graph polynomial, there are graph polynomials too; they can be obtained, for example, by substitution of variables, or by some transformation over set of coefficients. In such cases we speak about one graph polynomial *generalizing* another.

Historically, graph polynomials emerged one by one, and they were compared to each other by ad hoc means, mainly to make more or less precise whether a new graph polynomial was or was not a generalization of some graph polynomial previously defined in the literature.

To show that certain graph polynomial p *does not generalize* some graph invariant q , authors usually show two graphs G_1 and G_2 , such that $p(G_1) = p(G_2)$, but $q(G_1) \neq q(G_2)$. To show that p *does generalize* q , a different technique is usually applied: authors show a supposedly simple transformation

which uniformly produces q from p . These transformations vary in different papers, using variable substitutions only, or allowing simple algebraic operations in the polynomial ring, or simple graph transformations.

Many graph polynomials described in the literature satisfy linear recurrence relations with respect to certain edge- or vertex-elimination operations. A catalogue of graph polynomials discussed in the thesis and the survey of the literature is given in Chapter 2.

Let S be a class of graph polynomials that satisfies certain linear recurrence relation. Is there a graph polynomial in S that generalizes all the others graph polynomials of S ? In order to answer this, we use in this thesis relations \preceq_{dp} and \preceq_{subst} between graph polynomials, which express respectively the ability of graph polynomials to distinguish between graphs and the power to encode other graph invariants by a variable substitution. Based on those relations we define notions of *universality* and *dp-completeness* of graph polynomials with respect to certain class of graph invariants. We introduce several classes of graph invariants with respect to their satisfied recurrence relations, and provide universality and *dp-completeness* results for these classes.

The main part of this thesis (Chapters 2 - 4) is dedicated to this structural theory. We classify several known graph polynomials and state their *dp-completeness* and universality features where applicable. Then, in Chapter 3 we introduce new graph polynomials and show their *dp-completeness* and, if possible, universality, with respect to their classes. Finally, in Chapter 4 we extend our framework to labeled graphs and labeled graph invariants, providing results similar to those obtained in the Chapter 3 for unlabeled graphs.

Our results in complexity theory are presented in Chapter 5. We use definability features of the new graph polynomials we introduce, in order to show that the general parameterized complexity results available in [19, 12, 41, 16, 45] are applicable. Then in Appendix A we show explicit efficient algorithms for computing of new graph polynomials on graph classes of bounded tree-width that significantly improve theoretical run time upper bounds provided by the general theorems. Moreover, we show an explicit algorithm for computing of bivariate matching polynomial on graph classes of bounded clique-width. This result is completely new: it does not follow from the general theorems above.

The rest of this chapter is organized as follows: we first provide the necessary definitions and define notation (Section 1.1). Then we define our tools for comparing graph invariants and introduce the classes of graph invariants we want to study (Section 1.2). Finally, in Section 1.3 we give an overview of obtained results.

1.1 Preliminaries

1.1.1 Used notation of graphs and graph-like structures

Undirected graphs

A *simple* graph G is a pair $\langle V, E \rangle$, where V is a (final) set of vertices, and E is a binary relation $E \subseteq V \times V$ representing the edges of graph. In *multigraph* instead, multiple edges are allowed, so, for every pair in $V \times V$, a multiplicity m_e of the edge is given. The standard presentation of a graph is its *adjacency matrix*, defined as

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix},$$

where $a_{i,j} \in \mathbb{N}$ represents the multiplicity m_e of the edge $e = \{v_i, v_j\}$ between vertices v_i and v_j in graph $G = \langle V, E \rangle$ with vertex set $V = \{v_1, v_2, \dots, v_n\}$.

Trivially, both the adjacency matrix and the edge relation E of an undirected graph are symmetric.

Alternatively, an undirected graph can be represented by a *two-sorted structure* $G = \langle V, E, R \rangle$, where V is a vertex set, E is an edge set, and R is a binary relation $R \subseteq V \times E$ that defines the incidence of vertices and edges. Here, the relation R is limited such that any edge is incident to exactly two vertices (or just one vertex in case of a loop). On the other hand, the multiplicity of edges does not require any special representation.

The standard presentation of a graph $G = \langle V, E, R \rangle$ with vertex set is $V = \{v_1, v_2, \dots, v_n\}$ and edge set is $E = \{e_1, e_2, \dots, e_m\}$ is its *incidence*

matrix, defined as

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{pmatrix},$$

where

$$b_{i,j} = \begin{cases} 1 & \text{if } (v_i, e_j) \in R \\ 0 & \text{otherwise} \end{cases}.$$

Directed graphs

A directed graph (or a *digraph*) is D is a pair $\langle V, E \rangle$, where V is a (final) set of vertices, and E is a binary relation $E \subseteq V \times V$ representing the directed arcs of the graph, not necessarily symmetric. The entry $a_{i,j}$ of the adjacency matrix of a digraph represents the number of arcs from v_i to v_j .

The incidence matrix of a digraph is defined by

$$b_{i,j} = \begin{cases} -1 & \text{if } e_j \text{ leaves the vertex } v_i \\ 1 & \text{if } e_j \text{ enters the vertex } v_i \\ 0 & \text{otherwise} \end{cases},$$

A digraph D is an orientation of an undirected graph G if $b_{i,j}^G = |b_{i,j}^D|$ for every i, j .

An *oriented incidence matrix* of an undirected graph G is the incidence matrix, in the sense of directed graphs, of any orientation of G . That is, in the column of edge e , there is a $+1$ in the row corresponding to one vertex of e and a -1 in the row corresponding to the other vertex of e , and all other rows have 0. All oriented incidence matrices of G differ only by negating some set of columns.

Labeled graphs

A graph can be *vertex-labeled* and/or *edge-labeled*. This means that additionally to V and E , there are sets of labels Λ_V and Λ_E , and respective vertex and edge labeling functions $lab_V : V \mapsto \Lambda_V$ and $lab_E : E \mapsto \Lambda_E$. We intentionally use the term *labeling* instead of coloring, to prevent possible

confusion with colorings in context of chromatic and generalized chromatic polynomials.

Special graphs

We use in the text the following special graphs:

- Null graph: \emptyset – a graph with no vertices and no edges;
- Empty graph: E_n – a graph with n vertices and no edges;
- Path: P_n – a path graph with n vertices;
- Star: S_n – a tree with one root and n leaves adjacent to the root;
- Cycle: C_n – a cycle graph with n vertices;
- Clique: K_n – a *complete* graph with n vertices such that every two vertices are connected by an edge;

Definition 1 Rank and Nullity

Let $G = \langle V, E \rangle$ be an undirected graph. The **rank**¹ of a graph $r(G)$ is defined based on the number of the connected components in the graph $k(G)$ (including the isolated vertices):

$$r(G) = |V| - k(G)$$

and the **nullity** of the graph $s(G)$ is defined by $s(G) = |E| - r(G)$.

Definition 2 Graph Isomorphism

Graphs $G_1 = \langle V_1, E_1 \rangle$ and $G_2 = \langle V_2, E_2 \rangle$ are isomorphic (denoted as $G_1 \simeq G_2$) if there is a bijection $f : V_1 \cup E_1 \rightarrow V_2 \cup E_2$ such that

$$e = \{u, v\} \in E_1 \longleftrightarrow f(e) = \{f(u), f(v)\} \in E_2$$

Definition 3 Subgraphs, Induced Subgraphs and Spanning Subgraphs

A graph $G' = \langle U, F \rangle$ is a **subgraph** of a graph $G = \langle V, E \rangle$ if

$$U \subseteq V \text{ and } F \subseteq (U^2 \cap E)$$

¹Historically, the rank of a graph denoted the rank of its **oriented incidence matrix**, cf. for example [2].

A graph $G' = \langle U, F \rangle$ is an **induced subgraph** of a graph $G = \langle V, E \rangle$ if

$$U \subseteq V \text{ and } F = (U^2 \cap E)$$

A graph $G' = \langle U, F \rangle$ is an **spanning subgraph** of a graph $G = \langle V, E \rangle$ if

$$U = V \text{ and } F \subseteq (U^2 \cap E)$$

Definition 4 Disjoint Union

Given two graphs $G_1 = \langle V_1, E_1 \rangle$ and $G_2 = \langle V_2, E_2 \rangle$ with disjoint vertex sets, the disjoint union $G_1 \sqcup G_2$ is defined by

$$G_1 \sqcup G_2 = \langle V_1 \cup V_2, E_1 \cup E_2 \rangle$$

Spanning trees and spanning forests

Definition 5 Let $G = \langle V, E \rangle$ be a connected graph. $T = \langle V', E' \rangle$ is a **spanning tree** of G if

- T is a tree;
- $V' = V$ and $E' \subseteq E$.

If the graph G is not connected, and consists of connected components $G = G_1 \sqcup G_2 \dots G_k$ then a **spanning forest** S of G is defined as a disjoint union of the spanning trees of its components ²:

$$S = T_1 \sqcup T_2 \sqcup \dots \sqcup T_k$$

A total order \prec_ω over the edges of the graph uniquely defines a **minimum spanning tree** $T_{min} = \langle V, E' \rangle$ of a connected graph (or respectively a **minimum spanning forest** $S_{min} = \langle V, E' \rangle$ of a graph which is not connected), in the following manner: for every edge $e_{ext} \in E \setminus E'$, the graph $\langle V, E' \cup \{e_{ext}\} \rangle$ has a cycle $(e_1, e_2, \dots, e_k, e_{ext})$, such that $\forall 1 \leq i \leq k (e_i \prec_\omega e_{ext})$.

²We shall use this definition of a spanning forest, instead of the graph-theoretical “ S is a forest and its vertex set is V ”.

1.1.2 Vertex elimination

We define three basic vertex elimination operations on graphs³:

- *Deletion.* For a given vertex $v \in V(G)$, let $G-v$ be the graph obtained from G by removal of v and all edges that are incident to v . We call this operation *vertex deletion*:

$$G - v = \langle V \setminus \{v\}, E \setminus \{(v, x) : x \in V\} \rangle$$

- *Extraction.* Similarly, let $G - X$ be the graph obtained from G by removal of all vertices of the set $X \subseteq V$. Let $N(v)$ be the set of vertices that are adjacent to v in G (the neighborhood of v , not including v). We denote by $N[v]$ the *closed neighborhood* of a vertex v in G , i.e. the set of all vertices adjacent to v including v itself. The operation is denoted by $G \dagger v$ and called *vertex extraction*:

$$G \dagger v = G - N[v]$$

- *Contraction:* The graph G/v obtained from G by removal of v and insertion of edges between all pairs of non-adjacent neighbor vertices of v . Figure 1.1 shows an example graph and the graph obtained by vertex contraction.

$$G/v = \langle V \setminus v, E(G - v) \cup \{(x, y) : x, y \in N(v)\} \rangle$$

1.1.3 Edge elimination

We define three basic edge elimination operations on *multigraphs*:

- *Deletion.* For a given edge $e \in E(G)$, let $G-e$ be the graph obtained from G by removal of the edge e :

$$G - e = \langle V, E \setminus \{e\} \rangle$$

³The vertex elimination operations are defined on simple graphs. Extending this definition to multigraphs is straightforward; however, note that self-loops and multiple edges do not affect graph decomposition. Note also that all the vertex elimination operations preserve simplicity of the graph.

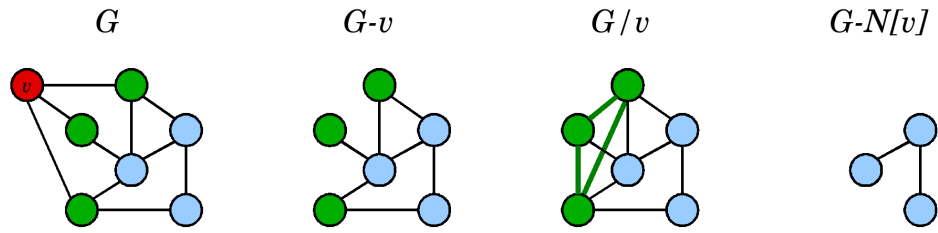


Figure 1.1: Vertex elimination operations

- *Extraction.* For a given edge $e \in E(G)$, let $G \dagger e$ be the subgraph of G induced by $V \setminus \{u, v\}$ provided $e = (u, v)$. Note that this operation removes also all the edges adjacent to e .

$$G \dagger e = G - \{u, v\}$$

- *Contraction.* For a given edge $e = (u, v) \in E(G)$, let G/e be the graph obtained from G by unifying the endpoints u and v of e into a new vertex w ; the edge set of G is preserved, except for the edge e itself. Note that this operation can produce multiple edges (if u and v have common neighbors), and self loops (when the edge e has some parallel edges).

$$V(G/e) = V \setminus \{u, v\} \cup \{w\},$$

$$E(G/e) = (E \setminus \{e\})[u \mapsto w, v \mapsto w].$$

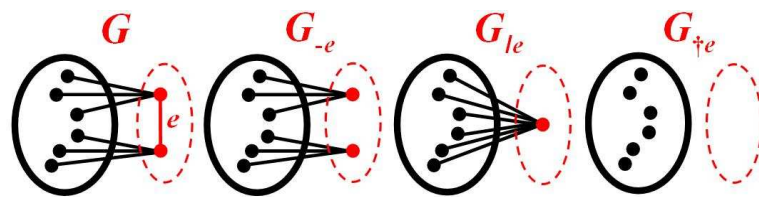


Figure 1.2: Edge elimination operations

1.1.4 k -sum, join and complement of a graph

Let $\Lambda = [k]$ be k distinct labels. Let $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ be subsets of size k , respectively, of V_1 and V_2 , and let the vertices of U_1 and vertices of U_2 be labeled by $1 \dots k$.

Then k -sum of two graphs $G_1 \sqcup_k G_2$ is obtained from $G_1 \sqcup G_2$ by unifying vertices from U_1 and U_2 having the same label. The edge sets are preserved (may produce multiple edges).

The *join* of two graphs denoted $G = G_1 \bowtie G_2$ is defined as a union of two graphs with disjoint vertex sets, and connecting every vertex of G_1 to every vertex of G_2 (the original edge sets are preserved):

$$G = \langle V_1 \sqcup V_2, E_1 \sqcup E_2 \sqcup (V_1 \times V_2) \rangle$$

Given a simple loop-free graph $G = \langle V, E \rangle$, its *complement* denoted by \overline{G} is defined as

$$\overline{G} = \langle V, (V \times V) \setminus E \setminus \{(v, v) : v \in V\} \rangle$$

Note that if G_1 and G_2 are two simple loop-free graphs, their join can be obtained by complementing of the disjoint union of their complements:

$$G_1 \bowtie G_2 = \overline{\overline{G_1} \sqcup \overline{G_2}}$$

Graph invariants and graph polynomials

Definition 6 A graph invariant is a function from the class of (finite) graphs \mathcal{G} into some range \mathcal{D} such that isomorphic graphs have the same picture:

$$f : \mathcal{G} \mapsto \mathcal{D}$$

such that

$$G_1 \simeq G_2 \rightarrow f(G_1) = f(G_2)$$

Definition 7 A graph polynomial is a graph invariant, which has a polynomial ring \mathbb{Z} , or, more generally, any commutative ring \mathcal{R} , over some (not necessarily finite) set of indeterminates \overline{X} , as its range:

$$p : \mathcal{G} \mapsto \mathcal{R}[\overline{X}]$$

While graph polynomials can in principle be defined over any polynomial ring, many of the definitions and proofs in the sequel assume the underlying ring to be a field such as \mathbb{R} . We henceforth use this assumption unapologetically.

Additive and multiplicative graph invariants

Definition 8 Let $G = G_1 \sqcup G_2$ denote the disjoint union of two graphs. A graph invariant p is **additive** if

$$p(G) = p(G_1) + p(G_2)$$

Observation: for every additive graph invariant p ,

$$p(\emptyset) = 0.$$

Definition 9 A graph invariant q is **multiplicative** if

$$q(G) = q(G_1) \cdot q(G_2)$$

Observation: for every multiplicative graph invariant q ,

$$q(\emptyset) = 1.$$

1.1.5 Line graph and source graph of a line graph

Given a graph $G = \langle V, E \rangle$, the *line graph* of G denoted as $L(G)$ is $L(G) = \langle E, F \rangle$, where

- The vertex set of $L(G)$ is the edge set of G ;
- Two vertices of $L(G)$ are connected by an edge if the corresponding edges of G share a vertex.

The graph G is called the *source graph* of $L(G)$. For every graph one can build its line graph, but not for any graph there is a source graph ⁴.

⁴Usually in the literature only simple source graphs are considered. However, a line graph can be built for a multigraph too, using the same rules.

Any two non-isomorphic connected simple loop-free graphs G_1 and G_2 have non-isomorphic line graphs, with exception of the pair S_3 and C_3 (a star and a cycle with 3 edges each), which share the same line graph C_3 .

There are efficient algorithms, for example, [38], that can output the simple source graph of an input graph if it is a line graph, or conclude that the input graph is not a line graph of any simple source graph.

1.2 Background

1.2.1 Comparing graph invariants

We shall use the following notions for comparing graph invariants:

Definition 10 *Distinctive power of graph invariants:*

Let $p : \mathcal{G} \rightarrow \mathcal{R}_1$ and $q : \mathcal{G} \rightarrow \mathcal{R}_2$ be two graph invariants. We say that the **distinctive power** of p does not exceed that of q , and write $p \preceq_{dp} q$ iff for every pair of graphs G_1 and G_2

$$q(G_1) = q(G_2) \quad \rightarrow \quad p(G_1) = p(G_2)$$

If $p \preceq_{dp} q$ and $q \preceq_{dp} p$ we say that the graph invariants p and q have the same distinctive power and write $p \simeq_{dp} q$. If neither $p \preceq_{dp} q$ nor $q \preceq_{dp} p$, we say that the graph invariants p and q have incomparable distinctive power.

Definition 11 *Substitution instance:*

Let $p : \mathcal{G} \rightarrow \mathcal{R}[\bar{X}]$ and $q : \mathcal{G} \rightarrow \mathcal{R}[\bar{Y}]$ be two graph polynomials with sets of indeterminates respectively \bar{X} and \bar{Y} . We say that p is a **substitution instance** of q and write $p \preceq_{subst} q$ iff there is a variable substitution $\sigma : \bar{Y} \rightarrow \mathcal{R}[\bar{X}]$ such that for all graphs G , $q(G, \bar{Y})$ under σ evaluates to $p(G; \bar{X})$:

$$p(G; \bar{X}) = \sigma(q(G; \bar{Y}))$$

1.2.2 DP-completeness and universality

Using the notions above, we can now define the “strongest” graph invariants in the following manner:

Definition 12 *Let S be a class of graph invariants.*

*We say that q is **dp-complete** for S iff $q \in S$ and for every $p \in S$, $p \preceq_{dp} q$.*

Definition 13 Let S be a class of graph invariants.

We say that q is **universal** for S iff $q \in S$ and for every $p \in S$, $p \preceq_{subst} q$.

As follows from the definitions of \preceq_{dp} and \preceq_{subst} , a universal invariant is also dp -complete. However, a dp -complete invariant is not necessary universal.

1.2.3 Edge elimination and vertex elimination

In this thesis we are interested in graph invariants which are multiplicative with respect to disjoint unions, and which satisfy certain linear recurrence relation with respect to some set of edge elimination or vertex elimination operations, defined respectively in Subsections 1.1.2 and 1.1.3.

Definition 14 Classes of multiplicative graph invariants with respect to their recurrence relations:

Let $p : \mathcal{G} \rightarrow \mathcal{R}$, where \mathcal{R} is either a field or a polynomial ring $\mathcal{F}[\bar{X}]$ with an underlying field \mathcal{F} , be a multiplicative graph invariant. Recall that for any multiplicative graph invariant the following boundary conditions are satisfied:

$$p(\emptyset) = 1 \text{ and } p(E_1) = \nu \text{ for some } \nu \in \mathcal{R} \quad (1.2.1)$$

We say that the graph invariant p is a

- **C-invariant** (chromatic invariant) and write $p \in C$ iff

1. There are elements $\alpha, \beta \in \mathcal{R}$ such that for every graph $G \in \mathcal{G}$ and every edge $e \in E(G)$,

$$p(G) = \alpha p(G - e) + \beta p(G/e)$$

2. The invariant p is uniquely defined by this recurrence relation and the initial conditions (1.2.1).

If $\alpha = 1$ we call p a **special chromatic invariant** and write $p \in sC$.

- **TG-invariant** (Tutte-Grothendieck invariant) and write $p \in TG$ iff

1. There are elements $x, y, \sigma, \tau \in \mathcal{R}$ such that for every graph $G \in \mathcal{G}$ and every edge $e \in E(G)$,

$$p(G) = \begin{cases} xp(G/e) & \text{if } e \text{ is a bridge,} \\ yp(G - e) & \text{if } e \text{ is a loop,} \\ \sigma p(G - e) + \tau p(G/e) & \text{otherwise.} \end{cases}$$

2. The invariant p is uniquely defined by this recurrence relation and the initial conditions (1.2.1).

- ***M-invariant*** (*Matching invariant*) and write $p \in M$ iff

1. There are elements $\alpha, \beta \in \mathcal{R}$ such that for every graph $G \in \mathcal{G}$ and for every edge $e \in E(G)$,

$$p(G) = \alpha p(G - e) + \beta p(G \dagger e);$$

2. The invariant p is uniquely defined by this recurrence relation and the initial conditions (1.2.1).

If $\alpha = 1$ we call p a ***special matching invariant*** and write $p \in sM$.

- ***EE-invariant*** (*Edge Elimination invariant*) and write $p \in EE$ iff

1. There are elements $\alpha, \beta, \gamma \in \mathcal{R}$ such that for every graph $G \in \mathcal{G}$ and for every edge $e \in E(G)$,

$$p(G) = \alpha p(G - e) + \beta p(G/e) + \gamma p(G \dagger e);$$

2. The invariant p is uniquely defined by this recurrence relation and the initial conditions (1.2.1).

If $\alpha = 1$ we call p a ***special EE invariant*** and write $p \in sEE$.

- ***VE-invariant*** (*Vertex Elimination invariant*) and write $p \in VE$ iff

1. There are elements $\alpha, \beta, \gamma \in \mathcal{R}$ such that for every graph $G \in \mathcal{G}$ and for every vertex $v \in V(G)$,

$$p(G) = \alpha p(G - v) + \beta p(G/v) + \gamma p(G \dagger v);$$

2. The invariant p is uniquely defined by this recurrence relation and the initial conditions (1.2.1).

Note that the set of EE-invariants trivially includes the entire sets of C-invariants and M-invariants. Additionally, we will show that the set of TG-invariants includes the entire set of C-invariants ⁵.

The main questions we study in this thesis:

Let X be the set of X -invariants ($X \in \{C, TG, M, EE, VE\}$).

1. Does X have some dp -complete element U_X ?
2. Does X have some universal element?
If not, for what largest subclass S of X can we define such a universal element? Can we characterize $X \setminus S$?
3. In general, what is the relationship between \preceq_{dp} and \preceq_{subst} on X ?
4. If U_X is dp -complete, how hard is it to describe? Can we obtain the coefficients of U_X without resorting to the recurrence relation ⁶?
5. What can we say about the complexity of the polynomials in X ?

In all the recurrence relations above, the coefficients are fixed for all graphs. However, if labeled graphs are considered, the coefficients may depend on the label of the edge or the vertex which is being eliminated. Such cases have been studied in the literature: for labeled TG invariants see [14, 49], for labeled M invariants see [34]. In chapter 4 we address similar questions with respect to the labeled graph invariants ⁷.

1.3 Results

The presented research was partially supported by the Graph Polynomials Project led by Prof. J.A. Makowsky. The project aims to conduct a wide

⁵ C -invariants are special cases of TG -invariants. In the book [1] by M.Aigner for both the term C -invariant is used. However, in most of the literature about Tutte polynomials the notion “Tutte-Grothendieck invariant“ is used.

⁶In the case of TG -invariants, U_{TG} has been determined and to be shown universal, and it has various explicit definitions.

⁷The authors of [14] use a term *edge-colorings* rather than *edge-labelings*. As we also discuss chromatic polynomials we prefer our terminology as it avoids confusions.

fundamental comparative study of graph polynomials. This thesis presents a part of results obtained in the framework of the project ⁸.

- We propose a general scheme that allows to compare graph invariants and in particular graph polynomials.
- We define notions of “dp-completeness” and “universality” of graph invariants within certain class.
- We use as a paradigm the universal Tutte polynomial and the class of Tutte-Grothendieck invariants. We introduce three more classes of graph invariants that include various existing graph polynomials. For every such class we introduce a new graph polynomial that appears to be *dp*-complete in this class. We also prove universality property of the new polynomials if it is possible. Otherwise we define the subclass, for which the new graph polynomial is universal, and describe the (relatively small) exception set.

DP-completeness and universality theorems:

Theorem 1.3.1 *The most distinctive matching polynomial*

There is a graph polynomial $U_M(G; x, y)$ which is *dp*-complete in the class of M -invariants. It has subset expansion as follows:

$$U_M(G; x, y) = \sum_{\substack{M \subseteq E, \\ M \text{ is a matching}}} y^{|M|} x^{|V|-2|M|} \quad (1.3.1)$$

Theorem 1.3.2 $U_M(G; x, y)$ is universal in the class of sM -invariants The exception subset of invariants $M \setminus sM$ consists of invariants of the kind $p(G) = x^{|V(G)|} y^{|E(G)|}$.

Theorem 1.3.3 *The most distinctive edge elimination polynomial*

There is a graph polynomial $\xi(G; x, y, z)$ which is *dp*-complete in the class of EE -invariants. It has subset expansion as follows:

$$\xi(G; x, y, z) = \sum_{(A \sqcup B) \subseteq E} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)}. \quad (1.3.2)$$

⁸Only structural theorems are stated here. Our results in the complexity theory are presented in Chapter 5.

where k and k_{cov} denote respectively the number of **spanning** and **covered** connected components with certain edge set. For a more precise notation please refer to Subsection 3.3.3.

Theorem 1.3.4 $\xi(G; x, y, z)$ is universal in the class of sEE -invariants. The exception subset $EE \setminus sEE$ consists of the **chromatic** invariants which are not special.

Theorem 1.3.5 *The most distinctive vertex elimination polynomial* There is a graph polynomial $Q_0(G; x, y)$ which is dp -complete in the class of VE -invariants. It has subset expansion as follows:

$$U_{VE}(G; x, y) = \sum_{U \subseteq V} x^{|U| - k(G[U])} (x + y)^{k(G[U])}, \quad (1.3.3)$$

where $G[U]$ denotes the induced subgraph of G with vertex set U , and $k(G[U])$ denotes number of spanning connected components in $G[U]$.

Theorem 1.3.6 $U_{VE}(G; x, y)$ is universal in the class of VE -invariants.

Chapter 2

A catalogue of graph polynomials and a survey of the literature

There is a variety of graph polynomials discussed in the literature. A very comprehensive (though, not exhaustive) survey of known graph polynomial by J. Ellis-Monaghan and C. Merino can be found at [23, 24]. We present here several graph polynomials and state to which class of graph invariants (from $\{C, TG, M, EE, VE\}$ defined in the Subsection 1.2.3) they belong.

2.1 Chromatic Invariants

2.1.1 Chromatic polynomial $\chi(G; x)$

Definition: Given a simple loop-free graph $G = \langle V, E \rangle$ and a natural k , the graph parameter $\chi(G; k)$ is defined as the number of proper vertex colorings of G by k colors. G. Birkhoff, who introduced it in [9], proved that $\chi(G, k)$ is a polynomial in k . A survey monograph is [22].

Subset expansion formula: Birkhoff proved that if $G = \langle V, E \rangle$ has m_{ij} spanning subgraphs of rank i and nullity j then

$$\chi(G; x) = \sum_{i,j} (-1)^{i+j} m_{ij} x^{|V|-i},$$

This definition can be rewritten in terms of edge subset expansion, using $r(G) = |V| - k(G)$, and $s(G) = |E| - |V| + k(G)$, where $k(G)$ is a number of spanning connected components:

$$\chi(G; x) = \sum_{A \subseteq E} (-1)^{r(A)+s(A)} x^{|V|-r(A)} = \sum_{A \subseteq E} (-1)^{|A|} x^{k(A)},$$

where by slight abuse of notation $k(A)$ denotes the number of connected components of the spanning subgraph of G with edge set A .

Recurrence relations. The chromatic polynomial is multiplicative with respect to disjoint union of graphs:

$$\chi(G_1 \sqcup G_2; x) = \chi(G_1; x) \cdot \chi(G_2; x)$$

It satisfies a linear recurrence relation with respect to edge deletion and edge contraction operations:

$$\chi(G; x) = \chi(G - e; x) - \chi(G/e; x),$$

with initial conditions $\chi(\emptyset) = 1$ and $\chi(E_1) = x$.

Based on this recurrence relation, we can conclude:

Proposition 2.1.1 *The chromatic polynomial is a special C -invariant.*

Additional combinatorial interpretations.

$(-1)^{|V|} \chi(G; -1)$ counts the number of acyclic orientations, [50].

Corollary 2.1.2 *The graph parameter “number of acyclic orientations” is a special C -invariant.*

Proof. Let $a(G)$ denote the number of acyclic orientations of G . By the results of [50], $a(G) = (-1)^{|V|} \chi(G; -1)$. Applying the recurrence relation of $\chi(G; x)$, we have

$$a(G) = (-1)^{|V|} (\chi(G - e; -1) - \chi(G/e; -1)) = a(G - e) + a(G/e).$$

■

2.1.2 Dichromatic polynomial $Z(G; q, v)$

Definition: Given a multigraph $G = \langle V, E \rangle$, $Z(G; q, v)$ is the unique polynomial in $\mathbb{Z}[q, v]$ defined recursively as follows ¹ :

$$Z(\emptyset; q, v) = 1$$

$$Z(E_n; q, v) = q^n \text{ for every } n \geq 1$$

$$Z(G_1 \sqcup G_2; q, v) = Z(G_1; q, v) \cdot Z(G_2; q, v)$$

$$Z(G; q, v) = Z(G - e; q, v) + v \cdot Z(G/e; q, v) \text{ for any edge } e \in E$$

By definition, it is a multiplicative graph invariant satisfying a recurrence relation with respect to edge deletion and edge contraction. We can conclude:

Proposition 2.1.3 *The dichromatic polynomial $Z(G; q, v)$ is a special C -invariant.*

Subset expansion formula: The dichromatic polynomial has edge subset expansion:

$$Z(G; q, v) = \sum_{A \subseteq E} q^{k(A)} v^{|A|}, \quad (2.1.1)$$

where $k(A)$ denotes the number of spanning connected components of $\langle V, A \rangle$.

Now we provide two models from Statistical Mechanics that are strongly related to the dichromatic polynomial [13]:

The partition function of the Potts ferromagnetic model with q states and inverse temperature β relates to the dichromatic polynomial as follows:

$$P(G; q, \beta) = e^{-\beta|E|} Z(G; q, e^\beta - 1)$$

Proposition 2.1.4 *$P(G; q, \beta)$ satisfies a linear recurrence relation as follows:*

$$P(G; q, \beta) = e^{-\beta} P(G - e; q, \beta) + (1 - e^{-\beta}) P(G/e; q, \beta)$$

¹The proof is available for example in [13].

Proof. We observe that both $G - e$ and G/e have exactly 1 edge less than G . Therefore, in order to obtain factor of $e^{-\beta|E|}$, we need to multiply every summand in the recurrence relation of $Z(G; q, e^\beta - 1)$ by $e^{-\beta}$. ■

Corollary 2.1.5 *The partition function of the Potts model $P(G; q, \beta)$ is a C -invariant (though, it is not special).*

Note that $P(G; q, \beta)$ is not a polynomial in β .

The partition function of the random cluster model with parameters p and q relates to the dichromatic polynomial as follows:

$$P(G; q, p) = (1 - p)^{|E|} Z(G; q, \frac{p}{1 - p})$$

Proposition 2.1.6 *$P(G; q, p)$ satisfies a linear recurrence relation as follows:*

$$P(G; q, p) = (1 - p)P(G - e; q, p) + pP(G/e; q, p)$$

Corollary 2.1.7 *The partition function of the random cluster model $P(G; q, p)$ is a C -invariant (though, it is not special).*

2.1.3 Bad Coloring Polynomial $B(G; x, y)$

The Bad Coloring Polynomial $B(G; x, y)$ [23] is the generating function

$$B(G; \lambda, t) = \sum_{j \geq 0} b_j(G; \lambda) t^j,$$

where $b_j(G; \lambda)$ is the number of λ -colorings of G with exactly j “bad” edges (i.e. the edges that connect vertices colored by the same color). It can be also written as

$$B(G; \lambda, t) = \sum_{f: V \rightarrow [\lambda]} t^{|b(f)|},$$

where $b(f)$ denotes the set of “bad” edges in the coloring f .

Proposition 2.1.8 *S. D. Noble (published in [23])*

$$B(G; \lambda, t) = \sum_{A \subseteq E} \lambda^{k(A)} (t - 1)^{|A|}$$

Proof.

$$\begin{aligned}
B(G; \lambda, t) &= \sum_{f:V \rightarrow [\lambda]} t^{|b(f)|} = \\
&= \sum_{f:V \rightarrow [\lambda]} \sum_{A \subseteq b(f)} (t-1)^{|A|} = \\
&= \sum_{A \subseteq E} \sum_{\substack{f:V \rightarrow [\lambda] \\ A \subseteq b(f)}} (t-1)^{|A|} = \\
&= \sum_{A \subseteq E} \lambda^{k(A)} (t-1)^{|A|} = \\
&= Z(G; \lambda, t-1).
\end{aligned}$$

■

Corollary 2.1.9 $B(G; \lambda, t)$ is a substitution instance of $Z(G; q, v)$ and it is a special C -invariant.

2.1.4 Universality and dp-completeness for C

Proposition 2.1.10 The following graph invariant is C -universal graph polynomial:

$$U_C(G; w, q, v) = w^{|E|} Z(G; q, \frac{v}{w}),$$

Proof.

- $U_C(G; w, q, v)$ is a C -polynomial:
It is multiplicative since both $w^{|E|}$ and $Z(G; q, v)$ are multiplicative. Moreover, it satisfies a linear recurrence relation

$$U_C(G; w, q, v) = wU_C(G - e; w, q, v) + vU_C(G/e; w, q, v)$$

by a similar argument as for Potts model partition function. Therefore it is a C -invariant, and it is a graph polynomial.

- For every C -invariant $p: \mathcal{G} \rightarrow \mathcal{R}$, $p(G) \preceq_{subst} U_C(G; w, q, v)$
Since p is multiplicative, there is some $\nu \in \mathcal{R}$ such that $p(E_n) = \nu^n$. Moreover, since p is a C -invariant, there are some $\alpha, \beta \in \mathcal{R}$ such that

$$p(G) = \alpha p(G - e) + \beta p(G/e)$$

Therefore, under the substitution $\sigma = \{q \mapsto \nu, w \mapsto \alpha, v \mapsto \beta\}$ $U_C(G; w, q, v)$ evaluates to p . ■

Corollary 2.1.11 $U_C(G; w, q, v)$ is dp-complete in the class of C-invariants.

Proposition 2.1.12 The dichromatic polynomial $Z(G; q, v)$ is dp-complete in the class of C-invariants.

Proof. We use the fact that $w^{|E|}$ is an evaluation of $Z(G; q, v)$ at $q = 1; v = (w - 1)$. Indeed,

$$Z(G; 1, w - 1) = \sum_{A \subseteq E} (w - 1)^{|A|} = w^{|E|}$$

Hence, $U_C(G; w, q, v) = Z(G; 1, w - 1) \cdot Z(G; q, \frac{v}{w})$, therefore $Z(G_1) = Z(G_2) \rightarrow U_C(G_1) = U_C(G_2)$. ■

2.2 Tutte-Grothendieck invariants

2.2.1 Tutte polynomial

The *dichromate*, named later the Tutte polynomial, has been introduced by W.T. Tutte [55] in 1954, building on H. Whitney's work on coefficients of the chromatic polynomial [56]. Monographs dealing with the Tutte polynomial are [13, 30].

Definition. The original definition of the Tutte polynomial uses *spanning tree expansion*. Let $G = (V, E)$ be a connected graph, and let $T = (V, F)$, $F \subseteq E$ be a spanning tree of G . For every edge $e \in E \setminus F$, let $cyc(T \cup e)$ denote the edges of the unique cycle in $T \cup e$. For each edge $f \in F$, let $cut(T - f)$ denote the set of edges of G that connect components of $T - f$. Finally, let $\phi : E \rightarrow 1, 2, 3 \dots |E|$ be an auxiliary total order over E .

An edge $e \in E \setminus F$ is called *externally active* with respect to T under order ϕ if it is the smallest edge (under ϕ) in the $cyc(T - e)$. Otherwise, this edge is called *externally inactive*.

An edge $f \in E$ is called *internally active* with respect to T under order ϕ if it is the smallest edge (under ϕ) in the $\text{cut}(T - f)$. Otherwise, this edge is called *internally inactive*.

The Tutte polynomial is defined by

$$T(G; x, y) = \sum_{i,j} t_{i,j} x^i y^j, \quad (2.2.1)$$

where $t_{i,j}$ denotes the number of spanning trees with exactly i internally active and j externally active edges. Tutte proved that $T(G; x, y)$ does not depend on the used auxiliary order ϕ .

This definition given for connected graphs, but it is naturally extended to all graphs by a multiplication rule:

$$T(G; x, y) = \prod_{i=1}^k T(G_i; x, y),$$

where $G_1 \dots G_k$ are the connected components of G .

Subset expansion formula. The Tutte polynomial can be represented by an edge subset expansion using notations of rank and nullity:

$$T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{r(G) - r(A)} (y - 1)^{s(A)}, \quad (2.2.2)$$

which can be simply rewritten in terms of spanning connected components:

$$T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{k(A) - k(E)} (y - 1)^{|A| + k(A) - |V|} \quad (2.2.3)$$

Recurrence relations. The Tutte polynomial is multiplicative with respect to the disjoint union of two graphs and with respect to *one-connection* of two graphs (in the latter case, G_1 and G_2 share exactly one vertex).

$$T(G_1 \sqcup G_2; x, y) = T(G_1 \sqcup_1 G_2; x, y) = T(G_1; x, y) \cdot T(G_2; x, y)$$

Additionally, it satisfies a linear recurrence relation with respect to edge deletion and edge contraction operations, with distinction between three kinds of edges: a bridge, a loop and neither bridge or loop.

$$T(G; x, y) = \begin{cases} xT(G/e; x, y) & \text{if } e \text{ is a bridge,} \\ yT(G - e; x, y) & \text{if } e \text{ is a loop,} \\ T(G/e; x, y) + T(G - e; x, y) & \text{otherwise} \end{cases}, \quad (2.2.4)$$

with initial conditions $T(\emptyset) = T(E_1) = 1$.

Based on this recurrence relation, we can conclude:

Proposition 2.2.1 *The Tutte polynomial is a TG-invariant.*

2.2.2 Whitney's rank generating polynomial, Whitney-Tutte dichromatic polynomial and the flow polynomial

In the next few paragraphs we provide several graph polynomials which are strongly related to the Tutte polynomial:

Whitney's Rank Generating Polynomial $R(G; x, y)$ [56] is defined as follows:

$$R(G; x, y) = \sum_{A \subseteq E} x^{r(G) - r(A)} y^{s(A)},$$

and it relates to the Tutte polynomial via $T(G; x, y) = R(G; x - 1, y - 1)$, and therefore it is a TG-invariant.

Whitney-Tutte Dichromatic Polynomial $Q(G; t, z)$ is defined by

$$Q(G; t, z) = \sum_{A \subseteq E} t^{k(A)} z^{s(A)}, \quad (2.2.5)$$

and it relates to the Tutte polynomial as

$$T(G; x, y) = (x - 1)^{-k(G)} Q(G; (x - 1), (y - 1))$$

. $Q(G; t, z)$ satisfies the following recurrence relation (cf. [14]):

$$Q(G; t, z) = \begin{cases} (z + 1)Q(G - e; t, z) & \text{if } e \text{ is a loop,} \\ Q(G - e; t, z) + Q(G/e; t, z) & \text{otherwise.} \end{cases}$$

One can rewrite this recurrence relation as

$$Q(G; t, z) = \begin{cases} (t+1)Q(G/e; t, z) & \text{if } e \text{ is a bridge,} \\ (z+1)Q(G-e; t, z) & \text{if } e \text{ is a loop,} \\ Q(G-e; t, z) + Q(G/e; t, z) & \text{otherwise.} \end{cases}$$

Indeed, if e is a bridge, according to the explicit definition 2.2.5, $Q(G-e; t, z) = tQ(G/e; t, z)$, as for any $A \subseteq (E \setminus \{e\})$, the rank $r(A)$ and the nullity $s(A)$ are preserved, whereas the number of connected components $k(A)$ is increased by 1. Therefore, we can state that $Q(G; t, z)$ is a TG-invariant.

Flow Polynomial $C^*(G; u)$ (cf. [36] for survey) can be obtained from the Tutte polynomial by

$$C^*(G; u) = (-1)^{|E|-r(G)}T(G; 0, 1-u)$$

Using the subset expansion (2.2.4) of $T(G; x, y)$ we have:

$$C^*(G; u) = \begin{cases} 0 & \text{if } e \text{ is a bridge,} \\ (u-1)C^*(G-e; u) & \text{if } e \text{ is a loop,} \\ C^*(G/e; u) - C^*(G-e; u) & \text{otherwise.} \end{cases}$$

We can conclude that $C^*(G; u)$ is a TG-invariant.

2.2.3 Connection to C-invariants

The universal chromatic invariant $U_C(G; w, q, v)$ discussed in Section 2.1, is defined as

$$U_C(G; w, q, v) = w^{|E|}Z(G; q, \frac{v}{w}),$$

where

$$Z(G; q, v) = \sum_{A \subseteq E} q^{k(A)} v^{|A|}$$

is a dichromatic polynomial. Recall that $U_C(G; w, q, v)$ is uniquely defined by its recurrence relation and initial conditions:

$$U_C(E_n; w, q, v) = q^n; \quad (2.2.6)$$

$$U_C(G; w, q, v) = wU_C(G - e; w, q, v) + vU_C(G/e; w, q, v) \quad (2.2.7)$$

One can rewrite this recurrence relation to the form of TG-invariants, using two facts:

1. If e is a loop, then $U_C(G/e; w, q, v) = U_C(G - e; w, q, v)$;
2. If e is a bridge, then $U_C(G - e; w, q, v) = q \cdot U_C(G/e; w, q, v)$.

These facts can be checked using the subset expansion of $Z(G; q, v)$ (2.1.1). Hence, we can rewrite the recurrence relation (2.2.7) as follows:

$$U_C(E_n) = q^n;$$

$$U_C(G) = \begin{cases} (wq + v)U_C(G/e) & \text{if } e \text{ is a bridge,} \\ (w + v)U_C(G - e) & \text{if } e \text{ is a loop,} \\ wU_C(G - e) + vU_C(G/e) & \text{otherwise.} \end{cases}$$

This allows us to state:

Proposition 2.2.2 *The universal chromatic invariant is also a TG-invariant.*

Corollary 2.2.3 *The entire class of C-invariants is included in the class of TG-invariants.*

2.2.4 Universality and dp-completeness for TG

The universal TG-invariant $U_{TG}(G; x, y, \alpha, \sigma, \tau)$

Theorem 2.2.4 *Brylawski [15]*

The universal TG-invariant is uniquely defined by

$$U_{TG}(E_n) = \alpha^n$$

$$U_{TG}(G) = \begin{cases} xU_{TG}(G/e) & \text{if } e \text{ is a bridge,} \\ yU_{TG}(G - e) & \text{if } e \text{ is a loop,} \\ \sigma U_{TG}(G - e) + \tau U_{TG}(G/e) & \text{otherwise.} \end{cases} \quad (2.2.8)$$

Furthermore, Oxley and Welsh showed in [47] that

$$U_{TG}(G) = \alpha^{k(G)} \sigma^{s(G)} \tau^{r(G)} T(G; \frac{x}{\tau}, \frac{y}{\tau}),$$

where $k(G)$, $r(G)$ and $s(G)$ are respectively the number of connected components, rank and nullity of G .

Corollary 2.2.5 $U_{TG}(G; x, y, \alpha, \sigma, \tau)$ is a dp-complete TG-invariant.

Note that the Tutte polynomial $T(G; x, y)$ is not a dp-complete TG-invariant:

$$T(E_n; x, y) = 1 \quad \text{whereas} \quad U_{TG}(E_n; x, y, \alpha, \sigma, \tau) = \alpha^n$$

However, the following can be shown:

Proposition 2.2.6 The dichromatic polynomial $Z(G; q, v)$ is dp-complete in the class of TG-invariants.

Proof. We use the known connections: the first is due to the results of [47], the others follow directly from the subset expansion of $Z(G; q, v)$:

- Between $T(G; x, y)$ and $U_{TG}(G; x, y)$

$$U_{TG}(G) = \alpha^{k(G)} \sigma^{s(G)} \tau^{r(G)} T(G; \frac{x}{\tau}, \frac{y}{\tau}),$$

- Between $Z(G; q, v)$ and $T(G; x, y)$:

$$T(G, x, y) = (x - 1)^{-k(E)} (y - 1)^{-|V|} Z(G, (x - 1)(y - 1), y - 1)$$

- Between $Z(G; q, v)$ and the number of vertices:

$$x^{|V|} = Z(G; x, 0)$$

- Between $Z(G; q, v)$ and the number of edges:

$$y^{|E|} = Z(G; 1, y - 1)$$

- Between $Z(G; q, v)$ and the number of connected components:

$$q^{k(G)} = \lim_{v \rightarrow \infty} \frac{Z(G; q, v)}{Z(G; 1, v - 1)}$$

For the last connection, $q^{k(G)}$ is the coefficient of the highest degree of v in $Z(G; q, v)$. The denominator equals $v^{|E|}$, the numerator has only one summand of the same degree of v . The equality follows by $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$.

Using the connections above, we get:

$$Z(G_1; q, v) = Z(G_2; q, v) \quad \longrightarrow \quad U_{TG}(G_1; x, y) = U_{TG}(G_2; x, y)$$

■

2.3 Matching invariants

2.3.1 Matching polynomial

The matching polynomial was introduced by C.J. Heilmann and E.H. Lieb in 1972 [34] motivated by a monomer-dimer problem in statistical physics. In combinatorics it was introduced by I. Gutman [32] and E.J. Farrell [25]. It is discussed in the monographs [40, 31]. It has appeared also in chemical literature (so-called topological resonance energy [53, 7, 8]).

Definition. Let $G(V, E)$ be a simple undirected loop-free graph with $|V| = n$ vertices. A k -*matching* in a graph G is a subset $M \subseteq E$ of k edges, no two of which have a vertex in common.

We denote by $m_k(G)$ the number of k -matchings of a graph G , when we define $m_0(G) = 1$ by convention. Of course, for $k > \frac{n}{2}$ always $m_k(G) = 0$. We denote by $V(M)$ the set of the vertices, which participate in the matching.

The most common form of matching polynomial (known in literature as *acyclic* or *matching defect* polynomial) was defined as:

$$\mu(G; x) = \sum_{k=0}^{\frac{n}{2}} (-1)^k m_k(G) x^{n-2k}$$

Another, may be a bit more natural form is called in the literature the *matching generating* polynomial. It is defined by a simple generating function:

$$g(G; x) = \sum_{k=0}^{\frac{n}{2}} m_k(G) x^k$$

The two forms above are related:

$$\mu(G, x) = x^n g(G, (-x^{-2}))$$

The matching polynomials are definable via subset expansion:

$$\begin{aligned} \mu(G; x) &= \sum_{\substack{M \subseteq E, \\ M \text{ is a matching}}} \prod_{e \in M} (-1) \prod_{v \notin V(M)} x \\ g(G; x) &= \sum_{\substack{M \subseteq E, \\ M \text{ is a matching}}} \prod_{e \in M} x \end{aligned}$$

Recurrence relations: Both $\mu(G; x)$ and $g(G; x)$ are multiplicative under disjoint union and satisfy recurrence relations with respect to edge deletion and edge extraction:

The matching defect polynomial:

$$\mu(G; x) = \mu(G - e; x) - \mu(G \dagger e; x)$$

The matching generating polynomial:

$$g(G; x) = g(G - e; x) + x \cdot g(G \dagger e; x)$$

Hence, we can conclude:

Proposition 2.3.1 *Both the matching defect polynomial $\mu(G; x)$ and the matching generating polynomial $g(G; x)$ are M -invariants.*

2.3.2 The Rook polynomial

The rook polynomial was introduced by J. Riordan in [48] as a generalization of the Rook problem.

Definition:

A generalized chessboard is a square matrix of $n \times n$ cells, where part of the cells are marked as “prohibited”, i.e. no figure can be placed there. Given a generalized chessboard B , the rook polynomial is defined as a generating

function:

$$R(B, x) = \sum_{k=0}^{|B|} r_k(G) x^k,$$

where r_k denotes the number of possibilities to place k non-attacking rooks on the chessboard B .

Recurrence relations:

First, the rook polynomial is multiplicative with respect to disjoint union: let B_1 and B_2 be two disjoint chessboards, then

$$R(B_1 \sqcup B_2; x) = R(B_1; x) \cdot R(B_2; x).$$

Let a denote some cell of the chessboard B , and let r_a and c_a denote respectively the row and the column of a . Then the rook polynomial is determined by the following recurrence relation with initial conditions:

$$\begin{aligned} R(B; x) &= R(B - a; x) + xR(B - c_a - r_a; x); \\ R(\emptyset; x) &= 1; \end{aligned}$$

where $B - a$ denotes the chessboard B with removed cell a , and $B - c_a - r_a$ denotes the chessboard B with removed both column and row of a (cf. [44]).

Connection to the matching polynomial:

Let $G_B = \langle V_r \sqcup V_c, E \rangle$ be a bipartite graph constructed from the chessboard B by the following way:

- Each vertex of V_r corresponds to a row and each vertex of V_c corresponds to a column;
- Two vertices $u \in V_r$ and $v \in V_c$ are connected by an edge iff there is a cell in B at the intersection of the corresponding row and column.

Proposition 2.3.2 (Farrell, [26]) *The rook polynomial $R(B; x)$ relates to the matching generating polynomial as follows:*

$$R(B; x) = g(G_B; x) = \sum_k m_k(G_B) x^k,$$

where m_k denotes the number of k -matchings in graph G_B .

The proof is based on an observation that every k -matching in G_B corresponds to a placement of k non-attacking rooks on the chessboard B .

The associated rook polynomial and the rook polynomial of a graph:

Let $r_k(B)$ denote the number of possibilities to place k non-attacking rooks on a chessboard B with n rows and n columns. Then the associated rook polynomial is defined by

$$r(B; x) = \sum_{k=0}^n r_k(B) (-1)^k x^{n-k} = x^n R(B; -\frac{1}{x})$$

In the book by L. Lovasz and M. D. Plummer [40] the authors use this version of the rook polynomial. Furthermore, they define the rook polynomial on graphs rather than on chessboards:

Definition 15 *The rook polynomial of a graph from [40]*

Let $G = \langle V_1 \sqcup V_2, E \rangle$ be a bipartite graph with $|V_1| = |V_2| = n$. Then the rook polynomial is defined by

$$\rho(G; x) = \sum_{k=0}^n m_k(G) (-1)^k x^{n-k},$$

where m_k is a number of k -matchings in G .

Proposition 2.3.3 *The rook polynomial of [40] is multiplicative under disjoint union and satisfies a linear recurrence relation as follows:*

$$\rho(G; x) = \rho(G - e; x) - \rho(G \uparrow e; x)$$

Moreover, it is determined by this recurrence relation with initial conditions $\rho(\emptyset; x) = 1$ and $\rho(E_1; x) = \sqrt{x}$.

For the proof one can analyze the matchings that do not include the edge e and therefore counted by $\rho(G - e; x)$, and those that include it and therefore counted by $\rho(G \uparrow e; x)$. Note that $\rho(G; x)$ is a graph invariant for every graph, though, it is a graph polynomial iff the number of vertices is even.

Corollary 2.3.4 *Both the original rook polynomial of [48] (via its connection to $g(G; x)$) and the rook polynomial of a graph $\rho(G; x)$ of [40] are M-invariants.*

2.3.3 Universality and dp-completeness for M

The *most distinctive matching polynomial* $U_M(G; x, y)$ subsumes both matching generating polynomial and matching defect polynomial as its substitution instances:

$$U_M(G; x, y) = \sum_{k=0}^{\frac{n}{2}} m_k(G) x^{n-2k} y^k$$

The most distinctive matching polynomial is multiplicative under disjoint union of graphs and satisfies a linear recurrence relation

$$U_M(G; x, y) = U_M(G; x, y) + y \cdot U_M(G; x, y)$$

Moreover, it is determined by this recurrence relation and the initial conditions $U_M(E_n; x, y) = x^n$.

In this thesis we prove that the most distinctive matching polynomial $U_M(G; x, y)$ is dp-complete in the class of M-invariants. Moreover, it is universal in the class of sM -invariants, when the exception subset of invariants $M \setminus sM$ consists of invariants of the kind $p(G) = x^{|V(G)|} y^{|E(G)|}$ (Theorems 1.3.1 and 1.3.2).

2.4 Edge Elimination invariants

In this section we introduce a new class of graph invariants. We start with the bivariate chromatic polynomial of [20], which turns out to satisfy a linear recurrence relation with respect to edge deletion, edge contraction and edge extraction operations defined in Subsection 1.1.3.

2.4.1 The Bivariate Chromatic polynomial

The bivariate chromatic polynomial was introduced by K.Dohmen, A.Pönitz and P.Tittmann in [20] as a natural extension of the chromatic polynomial. The authors of [20] noticed that the distinctive power of this polynomial

exceeds this of the chromatic polynomial, the matching polynomial and the vertex cover polynomial.

Definition: Let $G = \langle V, E \rangle$ be a graph, and let Y and Z be two disjoint sets of colors. We call X the union of the sets: $X = Y \sqcup Z$. Let $\varphi : V \mapsto X$ be a *valid* coloring of graph G if

$$\forall \{u, v\} \in E \ ((\varphi(u) \in Y \wedge \varphi(v) \in Y) \rightarrow \varphi(u) \neq \varphi(v))$$

In other words, two adjacent vertices can have the same color c only if $c \in Z$. We call the colors from the set Y “proper”, and the colors from the set Z “improper”.

The polynomial $P(G; x, y)$ is defined as the *number of valid colorings* of a graph G with total $|X| = x$ colors, when exactly $|Y| = y$ of them are proper.

Note that this combinatorial definition is valid only for positive integers $x \geq y$. Moreover, it is not trivial that it determines a polynomial in x and y . The following solves this issue:

Connection to the chromatic polynomial $\chi(G)$:

First, if all the colors are “proper”, the bivariate chromatic polynomial evaluates to the “classical” chromatic polynomial:

$$P(G; x, x) = \chi(G; x).$$

Moreover, Theorem 1 of [20] states that

$$P(G; x, y) = \sum_{U \subseteq V} (x - y)^{|U|} \chi(G - U; y),$$

where $G - U$ denotes the induced subgraph of G with vertex set $V \setminus U$. Therefore, $P(G; x, y)$ is a polynomial in x and y and its value can be extrapolated to any $x, y \in \mathbb{R}$.

Recurrence relations:

The authors of [20] state the multiplicativity of the bivariate chromatic polynomial with respect to disjoint union. We prove in this thesis the following theorem (the proof appears in Subsection 3.3.1):

Theorem 2.4.1 *The bivariate chromatic polynomial satisfies a linear recurrence relation with respect to edge deletion, edge contraction and edge extraction operations:*

$$P(G; x, y) = P(G - e; x, y) - P(G/e; x, y) + (x - y)P(G \dagger e; x, y),$$

and is determined by this recurrence relation together with initial conditions $P(E_n; x, y) = x^n$.

From the theorem above we can conclude that

Corollary 2.4.2 *The bivariate chromatic polynomial $P(G; x, y)$ is an sEE-invariant.*

2.4.2 The vertex cover polynomial

The vertex cover polynomial was introduced by F.M. Dong, M.D. Hendy, K.L. Teo and C.H.C. Little in [21]. motivated by biological systematics and theoretical chemistry.

Definition: Let $vc(G, k)$ denote the number of vertex-covers of G of size k , i.e. subsets $U \subseteq V$ s.t. $|U| = k$ and for every edge $(u, v) \in E$, either $u \in U$ or $v \in U$ or both.

$$\Psi(G, x) = \sum_{k=0}^{|V|} vc(G; k)x^k.$$

Proposition 2.4.3 *(Corollary 2 from [20])² :*

The vertex cover polynomial is a substitution instance of the bivariate chromatic polynomial $P(G; x, y)$:

$$\Psi(G, x) = P(G; x + 1; 1)$$

Proof. $P(G; x + 1; 1)$ counts the number of valid colorings of G with x improper colors and 1 proper color. Every vertex cover $U \subseteq V$ of G

²The authors of [20] speak about the independent set polynomial instead of the vertex cover polynomial. However, they use the definition of the independent set polynomial as $I(G; x) := \sum_{i=0}^n a_i(G)x^i$, where a_i is the number of independent sets of cardinality $n - i$, which is exactly the number of vertex covers of cardinality i .

induces a class of such colorings: the vertices of U are colored by improper colors, and the vertices of $V \setminus U$, which produce an independent set, are colored by the proper color. The size of the class is $x^{|U|}$. The classes are disjoint and they cover all the valid colorings of G . The proposition follows by summation over all the vertex covers of size k , for every k . ■

Corollary 2.4.4 *The vertex cover polynomial is a sEE -invariant; it is multiplicative under disjoint union and it satisfies a linear recurrence relation as follows:*

$$\Psi(G, x) = \Psi(G - e, x) - \Psi(G/e, x) + x \cdot \Psi(G \dagger e, x)$$

with initial conditions $\Psi(E_n, x) = (x + 1)^n$.

Proof. This follows from the Proposition 2.4.3 and the recurrence relation of the bivariate chromatic polynomial. ■

2.4.3 Universality and dp -completeness for EE

We introduce in this thesis a new graph polynomial: the most distinctive edge elimination polynomial $\xi(G; x, y, z)$ (A subset expansion of $\xi(G; x, y, z)$ appears in Subsection 3.3.3.)

We prove that $\xi(G; x, y, z)$ is dp -complete in the class of EE -invariants. Moreover, we prove that $\xi(G; x, y, z)$ is universal in the class of sEE -invariants, when the exception set is $C \setminus sC$ (Theorems 1.3.3 and 1.3.4).

2.4.4 The covered components polynomial

Inspired by the edge-subset expansion presentation of $\xi(G; x, y, z)$ published in [3], M. Trinks have introduced a new graph polynomial with remarkable combinatorial features [54]:

Definition:

Let $G = \langle V, E \rangle$ be a multigraph. The covered components polynomial $C(G; x, y, z)$ is defined as an ordinary generating function:

$$C(G; x, y, z) = \sum_{i,j,k} c_{i,j,k}(G) x^i y^j z^k,$$

where $c_{i,j,k}(G)$ denotes the number of edge subsets $A \subseteq E$ of size $|A| = j$, such that the graph $\langle V, A \rangle$ has exactly i components, from which k are covered by edges, (i.e. not singletons).

Recurrence relations:

M. Trinks proves that the covered components polynomial is multiplicative under disjoint union and that it satisfies a linear recurrence relation as follows:

$$C(G; x, y, z) = C(G - e; x, y, z) + y \cdot C(G/e; x, y, z) + xy(z - 1) \cdot C(G \uparrow e; x, y, z)$$

with initial condition $C(E_n; x, y, z) = x^n$.

The following proposition immediately follows from the universality of $\xi(G; x, y, z)$:

Proposition 2.4.5 *The covered components polynomial is an sEE invariant, and it is a variable substitution of $\xi(G; x, y, z)$:*

$$C(G; x, y, z) = \xi(G; x, y, xy(z - 1)).$$

M. Trinks provides a combinatorial proof of the proposition above in [54].

2.4.5 Connection to C-invariants, TG-invariants and M-invariants

It follows immediately from the definitions of C , M and EE that $C \subseteq EE$ and $M \subseteq EE$. Moreover, using our results for $\xi(G; x, y, z)$ and $U_M(G; x, y)$, and the dp-completeness of $Z(G; q, v)$, one can conclude that

Corollary 2.4.6 $\xi(G; x, y, z)$ determines any C , M , or TG invariant.

Proof. We use dp-completeness of $U_M(G; x, y)$ for M and $Z(G; q, v)$ for C -invariants. Those polynomials are evaluations of $\xi(G; x, y, z)$:

$$U_M(G; x, y) = \xi(G; x, 0, y) \text{ and } Z(G; q, v) = \xi(G; q, v, 0).$$

The conclusion about TG follows from dp-completeness for TG of the dichromatic polynomial $Z(G; q, v)$.³ ■

³We cannot state that $\xi(G; x, y, z)$ is dp-complete for the classes M , C or TG , because it does not belong to any of those classes.

2.5 Vertex Elimination Invariants

2.5.1 The Independent Set polynomial

Introduced by I. Gutman and F. Harary in [33]. as a generalization of the matching polynomial.

Definition: Let $in(G, k)$ denote the number of independent sets of size k which are induced subgraphs of G .

$$In(G, x) = \sum_{k=0}^{|V|} in(G, k) \cdot x^k$$

Recurrence relations: $In(G; x)$ satisfies a linear recurrence relation with respect to vertex deletion and vertex extraction (as defined in 1.1.2). Recall that $G - v$ denotes the induced subgraph of G with vertex set $V \setminus \{v\}$, and $G \dagger v$ denotes the graph obtained from G by removing of v together with its neighborhood.

Proposition 2.5.1 (*Gutman and Harary, [33]*)

The independent set polynomial is multiplicative with respect to disjoint union. Moreover, it satisfies a linear recurrence relation as follows:

$$In(G; x) = In(G - v; x) + x \cdot In(G \dagger v; x),$$

and it is determined uniquely by the recurrence relation above and the initial condition $In(\emptyset; x) = 1$.

Corollary 2.5.2 *The independent set polynomial $In(G; X)$ is a VE-invariant.*

2.5.2 The subgraph component polynomial

The subgraph component polynomial $Q(G; x, y)$ has been proposed by P. Tittmann and further developed in collaboration with J. A. Makowsky and the author in [51]. This graph polynomial arises from analyzing community structures in social networks.

Definition: Let $G = \langle V, E \rangle$ be a finite undirected graph with n vertices and let $k \leq n$ be a positive integer. For every vertex, decide with probability $q = 1 - p$ whether to remove that vertex or keep it. What is the probability that a subgraph of G with exactly k components survives? The solution of this problem leads to the enumeration of induced subgraphs of G with k components.

For a given vertex subset $U \subseteq V$, let $G[U]$ be the *induced subgraph* of G with vertex set U . We denote by $k(G)$ the number of components of G . Let $q_{ij}(G)$ be the number of vertex subsets $U \subseteq V$ with i vertices such that $G[U]$ has exactly j components:

$$q_{ij}(G) = |\{U \subseteq V : |U| = i \wedge k(G[U]) = j\}|$$

The subgraph component polynomial of G is an ordinary generating function for these numbers:

$$Q(G; x, y) = \sum_{i=0}^n \sum_{j=0}^n q_{ij}(G) x^i y^j.$$

Note: loops or parallel edges do not affect connectivity properties of a graph.

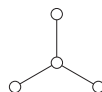


Figure 2.1: The star $Star_3 = K_{1,3}$

Example: The star $K_{1,3}$, presented in Figure 2.1, has the subgraph polynomial

$$Q(K_{1,3}; x, y) = 1 + 4xy + 3x^2y + 3x^3y + x^4y + 3x^2y^2 + x^3y^3.$$

The term $3x^2y^2$ tells us that there are 3 possibilities to select two vertices of G that are non-adjacent.

The empty set induces the null graph \emptyset that we consider as being connected by convention, which gives $q_{00}(G) = Q(G; 0, 0) = 1$ for any graph.

Substitution of 1 for y results in an univariate polynomial that is the ordinary generating function for all subsets of V , i.e. $Q(G; x, 1) = (1 + x)^n$.

Recurrence relations:

In this thesis we prove that the subgraph component polynomial $Q(G; x, y)$ is multiplicative with respect to disjoint union of graphs. Moreover, it satisfies a linear recurrence relation with respect to vertex deletion, contraction and extraction (as defined in 1.1.2). Recall that the vertex contraction is removing vertex and connecting its neighborhood to a clique.

$$Q(G; x, y) = Q(G - v; x, y) + xQ(G/v; x, y) + x(y - 1)Q(G \dagger v; x, y),$$

$Q(G; x, y)$ is uniquely determined by this recurrence relation and the initial conditions $Q(\emptyset; x, y) = 1$ and $Q(E_1; x, y) = 1 + xy$. The proof appears in Section 3.4, Theorems 3.4.1 and 3.4.3.

We can immediately conclude that

Proposition 2.5.3 *The subgraph component polynomial $Q(G; x, y)$ is a VE -invariant.*

2.5.3 Universality and dp-completeness for VE

We introduce in this thesis a new graph polynomial: the most distinctive vertex elimination polynomial $U_{VE}(G; x, y)$:

$$U_{VE}(G; x, y) = \sum_{U \subseteq V} x^{|U| - k(G[U])} (x + y)^{k(G[U])},$$

where $G[U]$ denotes the induced subgraph of G with vertex set U , and $k(G[U])$ denotes number of spanning connected components in $G[U]$.

We prove that $U_{VE}(G; x, y)$ is dp-complete in the class of VE -invariants. Moreover, we prove that $U_{VE}(G; x, y)$ is universal in the class of VE -invariants (Theorems 1.3.5 and 1.3.6).

Chapter 3

DP-complete and Universal Graph Polynomials for Unlabeled Graphs

In this chapter we discuss dp-complete graph polynomials for the classes of M , EE and VE invariants. Recall that the classes above are defined as the classes of graph invariants which are

1. Multiplicative with respect to disjoint union of the input graphs;
2. Satisfying a linear recurrence relation according to its class;
3. Uniquely determined by its recurrence relation and initial conditions defined for a null graph $\emptyset = \langle \emptyset, \emptyset \rangle$ and for a singleton $E_1 = \langle \{v\}, \emptyset \rangle$.

First, we refer to the problem of unique definition ¹.

3.1 Recurrence relations that uniquely define graph invariants.

From now on, we consider linear recurrence relations without case distinction. Let the recurrence relation $\mathcal{Y} = (\vartheta, \mathcal{W})$ be a set of graph transformations $\vartheta = \{T_1, \dots, T_k\}$, provided with their weight coefficients $\mathcal{W} :=$

¹The first paper to study general conditions under which linear recurrence relations define a graph invariant is D.N.Yetter [57]

$\{W_1, \dots, W_k\}$, which are elements of a target ring \mathcal{R} independent of the graph G . We say that a graph invariant p satisfies the recurrence relation \mathcal{T} , if

$$p(G) = \sum_{i=1}^k W_i p(T_i(G)). \quad (3.1.1)$$

The recurrence relation should satisfy two important conditions:

- **Well-foundedness:** The set of graph transformations should provide well-founded graph decomposition. In other words, any legal sequence of those transformations should lead, after a *finite* number of steps, to one of the initial conditions.
- **Order invariance:** Let Ω denote the set of all possible orders of decomposition of the graph G using transformations from ϑ . Given an input graph G , a well-founded linear recurrence relation $\mathcal{T} = (\vartheta, \mathcal{W})$, a set of initial conditions ϑ_0 and some auxiliary **order** $\omega \in \Omega$ of graph decomposition, one can obtain a map $p : \mathcal{G} \times \Omega \rightarrow \mathcal{R}$. The map p is a graph invariant if for every two orders $\omega_1, \omega_2 \in \Omega$ of graph decomposition,

$$p(G, \omega_1) = p(G, \omega_2)$$

We use and generalize the ideas of [47, 14] in order to define the most distinctive graph polynomial in the class $Y \in \{M, EE, VE\}$ of graph invariants defined with respect to some set of decomposition transformations ϑ and initial conditions ϑ_0 :

- First, we make sure that the set ϑ of graph transformations is well-founded, i.e. for any graph G and for any order $\omega \in \Omega$ of decomposition steps from ϑ , a graph satisfying some condition of ϑ_0 is obtained after applying a finite number of decomposition steps on G ;
- Then, we substitute the weight coefficients $W \in \mathcal{W}$ by indeterminates. At this point, we have defined a map

$$p : \mathcal{G} \times \Omega \rightarrow \mathbb{R}[\bar{X}],$$

where \mathcal{G} is a class of graphs, Ω is the set of all possible orders of graph decompositions using transductions from ϑ , and \bar{X} is the set of the

introduced indeterminates.

- We regard the introduced indeterminates as unresolved polynomials $q_1, q_2, \dots, q_k \in \mathbb{R}[\bar{X}]$. We look for the minimal set of equations $\varphi(q_1, q_2, \dots, q_k)$ that the polynomials $q_1(\bar{x}), q_2(\bar{x}), \dots, q_k(\bar{x})$ should satisfy to make $p : \mathcal{G} \times \Omega \rightarrow \mathbb{R}[\bar{X}]$ independent of the order, i.e. for any graph G and any two orders $\omega_1, \omega_2 \in \Omega$,

$$p|_{\varphi}(G, \omega_1; \bar{x}) = p|_{\varphi}(G, \omega_2; \bar{x}) = p|_{\varphi}(G, \bar{x})$$

It is possible that there are several solutions of $\varphi(q_1(\bar{x}), q_2(\bar{x}), \dots, q_k(\bar{x}))$ that make the resulting map order-independent. In this case, we look for a solution that makes the map $p|_{\varphi}(G, \bar{x})$ dp -complete in Y . This solution determines the subclass sY which we call “special Y -invariants” for which $p|_{\varphi}(G; \bar{x})$ is universal.

Now we provide our results regarding dp -completeness and universality of M , EE and VE invariants.

3.2 The most distinctive matching polynomial

$$U_M(G; x, y)$$

Our first example is the most distinctive matching polynomial. We look for the “most general” matching polynomial, which subsumes all the types of matching polynomials discussed above (as well as many others), as its particular cases.

3.2.1 The recurrence relation

Recall the definition of M -invariant:

$p(G) : \mathcal{G} \rightarrow \mathcal{R}$ is an M -invariant if

- $p(G)$ is multiplicative with respect to disjoint union of graphs: for all graphs G_1, G_2 ,

$$p(G_1 \sqcup G_2) = p(G_1) \cdot p(G_2);$$

- There are $\alpha, \beta \in \mathcal{R}$ such that for every graph $G = \langle V, E \rangle$ and every edge $e \in E$,

$$p(G) = \alpha p(G - e) + \beta p(G \dagger e), \quad (3.2.1)$$

where $G - e$ and $G \dagger e$ denote graphs obtained from G by respectively *deletion* and *extraction* (i.e. deletion together with endpoints) of the edge e .

- $p(G)$ is uniquely defined by the recurrence relation (3.2.1) together with the initial conditions

$$p(\emptyset) = 1; \quad p(E_1) = \nu, \quad (3.2.2)$$

where $\nu \in \mathcal{R}$ is some element ² of the ring \mathcal{R} that does not depend on the graph G .

The most general recursive definition of $P(G, \omega; \bar{X})$ is obtained by introducing indeterminates where possible. Any graph decomposition sequence consists of *edge removal* steps and non-empty disjoint subgraph *decomposition* steps. Let $\omega \in \Omega$ denote an auxiliary order of decomposition steps.

$$\begin{aligned} P(G) &= \begin{cases} yP(G - e) + zP(G \dagger e); & \text{for every edge removal step} \\ P(G_1) \cdot P(G_2); & \text{for every decomposition step} \end{cases} \\ P(E_1) &= x; \\ P(\emptyset) &= 1; \end{aligned} \quad (3.2.3)$$

Proposition 3.2.1 *The reduction of $P(G, \omega; \bar{X})$ is well-founded.*

Proof. Indeed, every step of the decomposition reduces either the number of edges or the number of vertices. Hence, after a final number of steps, only singletons and empty sets appear in the decomposition parse tree. ■

Currently, $P(G, \omega; \bar{X})$ is defined recursively for unlabeled graphs and set of indeterminates $\bar{X} = \{x, y, z\}$ onto the ring $\mathbb{R}[\bar{X}]$ and its result depends on the order ω of graph decomposition steps.

Theorem 3.2.2 *Let $x(\bar{x}), y(\bar{x}), z(\bar{x}) \in \mathbb{R}[\bar{x}]$ be unresolved polynomials, and let $\hat{P}(G, \omega; \bar{x})$ be obtained from $P(G, \omega; \bar{X})$ by substitution of respective in-*

²Recall that we assume that \mathcal{R} is a polynomial ring with underlying field \mathbb{R} or \mathbb{Q} .

determinates:

$$\hat{P}(G, \omega; \bar{x}) = P(G, \omega; \bar{X})[x \mapsto x(\bar{x}), y \mapsto y(\bar{x}), z \mapsto z(\bar{x})].$$

Then $\hat{P}(G, \omega_1; \bar{x}) = \hat{P}(G, \omega_2; \bar{x})$ for all graphs G and all pairs of orders ω_1 and ω_2 , if and only if

$$y(\bar{x})z(\bar{x}) = z(\bar{x}) \quad \text{or} \quad P(G) = x(\bar{x})^{|V|}$$

Proof. We are looking for conditions $\varphi(x(\bar{x}), y(\bar{x}), z(\bar{x}))$ such that if φ is satisfied, then $\hat{P}(G, \omega; \bar{x})$ is independent of the order.

We apply the order-invariance restriction to the family of graphs shown at Fig. 3.1: G consists of two disjoint graphs H_1 and H_2 connected by a bridge of two edges $e_1 = (u, v)$ and $e_2 = (v, w)$.

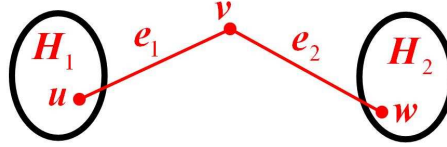


Figure 3.1: Graph G for applying order-invariance restriction

In order to be a unique graph invariant, $P(G)$ must return the same value when the edge reduction rule is applied first on the edge e_1 and then on the edge e_2 , as well as when it is applied first on the edge e_2 and then on the edge e_1 .

$$\begin{aligned} P(G, \{1 \rightarrow 2\}) &= y \cdot P(G - e_1) + z \cdot P(G \dagger e_1) \\ &= y \cdot (y \cdot P(G - e_1 - e_2) + z \cdot P(G \dagger e_2)) + \\ &\quad z \cdot P(G \dagger e_1) \\ &= xy^2 \cdot P(H_1)P(H_2) + \\ &\quad yz \cdot P(H_1)P(H_2 - w) + \\ &\quad zP(H_1 - u)P(H_2) \end{aligned}$$

On the other hand,

$$\begin{aligned}
P(G, \{2 \rightarrow 1\}) &= y \cdot P(G - e_2) + z \cdot P(G \dagger e_2) \\
&= y \cdot (y \cdot P(G - e_2 - e_1) + z \cdot P(G \dagger e_1)) + \\
&\quad z \cdot P(G \dagger e_2) \\
&= y^2 x \cdot P(H_1)P(H_2) + \\
&\quad yzP(H_1 - u)P(H_2) + \\
&\quad zP(H_1)P(H_2 - w)
\end{aligned}$$

Applying $P(G, \{2 \rightarrow 1\}) = P(G, \{2 \rightarrow 1\})$, we have:

$$\begin{aligned}
&yzP(H_1)P(H_2 - w) + zP(H_1 - u)P(H_2) = \\
&= yzP(H_1 - u)P(H_2) + zP(H_1)P(H_2 - w)
\end{aligned}$$

and hence

$$(yz - z)P(H_1)P(H_2 - w) = (yz - z)P(H_1 - u)P(H_2),$$

which leads to $yz = z$, or $P(H_1 - u)P(H_2) = P(H_1)P(H_2 - w)$ for all graphs H_1, H_2 and all vertices $u \in V(H_1)$ and $w \in V(H_2)$. In the latter case we get a trivial polynomial $P(H) = x^{|V(H)|}$. Indeed, if H_1 is a singleton, we get $1 \cdot P(H_2) = x \cdot P(H_2 - w)$ for any $w \in V(H_2)$, which leads to $P(H) = x^{|V(H)|}$.

So far, we proved that the condition of the theorem is necessary. In order to prove that they are also sufficient, it is enough to show that *any two steps of graph decomposition are exchangeable*. Then we can use induction on the tree of graph deconstruction to prove the order-invariance.

First, we observe that any step of disjoint decomposition is interchangeable with any step of edge removal. This follows by distributivity of multiplication. Therefore, we can assume that we first perform all the possible edge removal steps, and then do the disjoint decomposition of remaining singletons.

Second, we observe that an order over edges uniquely determines the order of graph decomposition. We just skip the steps of removing edges that have been already removed by the preceding steps.

Therefore, it is enough to prove that any two edges can be removed in either order, and this does not affect the result. If the edges of interest do not have any common vertices, then we get:

$$\begin{aligned}
P(G, \{1 \rightarrow 2\}) &= y \cdot P(G - e_1) + z \cdot P(G \dagger e_1) \\
&= y \cdot (y \cdot P(G - e_1 - e_2) + z \cdot P(G - e_1 \dagger e_2)) + \\
&\quad z \cdot (y \cdot P(G \dagger e_1 - e_2) + z \cdot P(G \dagger e_1 \dagger e_2)) = \\
&= P(G, \{2 \rightarrow 1\})
\end{aligned}$$

otherwise, if the edges of interest have a vertex or two in common, we have

$$\begin{aligned}
P(G, \{1 \rightarrow 2\}) &= y \cdot P(G - e_1) + z \cdot P(G \dagger e_1) \\
&= y \cdot (y \cdot P(G - e_1 - e_2) + z \cdot P(G \dagger e_2)) + \\
&\quad z \cdot P(G \dagger e_1)
\end{aligned}$$

whereas

$$\begin{aligned}
P(G, \{2 \rightarrow 1\}) &= y \cdot P(G - e_2) + z \cdot P(G \dagger e_2) \\
&= y \cdot (y \cdot P(G - e_2 - e_1) + z \cdot P(G \dagger e_1)) + \\
&\quad z \cdot P(G \dagger e_2)
\end{aligned}$$

which are equal if $yz = z$.

Hence, the conditions of the theorem are necessary and sufficient. ■

Analyzing the conditions φ , we get:

Corollary 3.2.3 *Every M -polynomial $p : \mathcal{G} \rightarrow \mathbb{R}[\bar{X}]$ of a graph $G = \langle V, E \rangle$ is either of the form*

$$p(G) = x^{|V|}y^{|E|}, \tag{3.2.4}$$

or it satisfies a linear recurrence relation as follows:

$$p(G) = p(G - e) + yp(G \dagger e); \tag{3.2.5}$$

where x and y are elements of $\mathbb{R}[\bar{X}]$ independent of G .

Proof. The condition $yz = z$ has two solutions: $y = 1$ and $z = 0$. Under the latter root the polynomial (3.2.4) is obtained. This also includes

the exception case $p(G) = x^{|V|}$. The first root gives a rise to the recurrence relation (3.2.5). ■

Corollary 3.2.4 *There is a map $U_M : \mathcal{G} \rightarrow \mathbb{R}[x, y]$, which is uniquely defined by the recurrence relation and initial conditions:*

$$\begin{aligned}
U_M(G_1 \sqcup G_2; x, y) &= U_M(G_1; x, y) \cdot U_M(G_2; x, y); \\
U_M(G; x, y) &= U_M(G - e; x, y) + yU_M(G \dagger e; x, y); \\
U_M(E_1; x, y) &= x; \\
U_M(\emptyset; x, y) &= 1.
\end{aligned} \tag{3.2.6}$$

Corollary 3.2.5 *The map $U_M(G; x, y)$ from Corollary 3.2.4 is universal in the class of sM -invariants and dp -complete in the class of M -invariants.*

Proof. The universality property follows directly from the Corollary 3.2.4, for the dp -completeness we need also prove that $U_M(G; x, y)$ determines the number of edges $|E|$ and the number of vertices $|V|$. This can be shown for example by

$$U_M(G; 2, 0) = 2^{|V|} \quad \text{and} \quad \left. \frac{\partial U_M(G)}{\partial y} \right|_{x=1, y=0} = |E|,$$

The latter follows by counting paths in the graph decomposition tree with exactly one edge extraction step. ■

3.2.2 The subset expansion form of $U_M(G; x, y)$

We proved so far that there is a unique map $U_M(G; x, y)$ which is defined recursively by initial conditions and linear recurrence relation 3.2.6 with respect to edge deletion and edge extraction operations.

Additionally, by multiple application of the edge reduction rule, we can obtain the vertex reduction rule. Let $N(v) = \{u_1, u_2, \dots, u_d\}$ be the neigh-

borhood of v , and $e_i = u_i, v$ be the edges incident to v . Then,

$$\begin{aligned}
U_M(G) &= U_M(G - e_1) + yU_M(G \dagger e_1) = \\
&= U_M(G - e_1 - e_2) + yU_M(G \dagger e_2) + yU_M(G \dagger e_1) = \dots \\
&= U_M(G - v \sqcup E_1) + \sum_{i=1}^d yU_M(G \dagger e_i) = \\
&= x \cdot U_M(G - v) + y \sum_{i=1}^d U_M(G \dagger e_i) \tag{3.2.7}
\end{aligned}$$

Theorem 3.2.6 *The unique map $U_M : \mathcal{G} \rightarrow \mathbb{R}[x, y]$ satisfying the recursive definition 3.2.6 has an explicit subset expansion definition as follows:*

$$U_M(G; x, y) = \sum_{\substack{M \subseteq E, \\ M \text{ is a matching}}} y^{|M|} x^{|V| - 2|M|} \tag{3.2.8}$$

Proof. The proof is by induction on the size of the graph G .

Base: for graphs of size $n \leq 2$ we have:

- $U_M(\emptyset) = 1$
- $U_M(E_1) = x$
- $U_M(E_2) = x^2$
- $U_M(P_2) = U_M(G - e) + y \cdot U_M(\emptyset) = x^2 + y$

In all those cases the equation 3.2.8 holds.

Closure: we assume the equation 3.2.8 holds for all the graphs with at most n vertices, and proof that it holds also for the graphs with $n + 1$ vertices.

Let G be a graph with $n + 1$ vertices and v any its vertex. Then we apply the vertex reduction rule 3.2.7 and obtain:

$$U_M(G) = x \cdot U_M(G - v) + y \cdot \sum_{i=1}^d U_M(G \dagger e_i)$$

By the induction assumption we have:

$$\begin{aligned}
 U_M(G) &= x \cdot \sum_{\substack{M \subseteq E(G-v), \\ M \text{ is a matching}}} y^{|M|} x^{|V|-1-2|M|} + \\
 &+ y \cdot \sum_{i=1}^d \sum_{\substack{M \subseteq E(G \dagger e_i), \\ M \text{ is a matching}}} y^{|M|} x^{|V|-2-2|M|}
 \end{aligned}$$

The first part of the equation describes all the matchings which do not include the vertex v , the second part - all the matchings which include the vertex v . In the latter case the matching M can be extended to include the vertex v by adding exactly one of the edges e_1, \dots, e_d . By summation of the two parts, we get exactly the equation 3.2.8. ■

3.3 The most distinctive edge elimination polynomial $\xi(G; x, y, z)$

The class of EE invariants includes both classes of matching and chromatic invariants.

We looked for graph polynomial that subsumes both the Tutte polynomial and the matching polynomial. The Tutte polynomial satisfies linear recurrence relation with respect to edge deletion and edge contraction operations, the matching polynomial - with respect to edge deletion and edge extraction, i.e. deletion of the edge together with its endpoints. The formal definitions of the edge elimination operations appears in subsection (1.1.3).

We first proved that the bivariate chromatic polynomial of K.Dohmen, A.Pönitz and P.Tittmann [20] satisfies a linear recurrence relation with respect to the three operations above. Then a question arose: is there a most general multiplicative graph polynomial, which is most distinctive or even universal with respect to the three edge elimination operations?

3.3.1 The bivariate chromatic polynomial

Recall the definition of the bivariate chromatic polynomial $P(G; x, y)$ of K.Dohmen, A.Pönitz and P.Tittmann [20]:

Given two disjoint sets of colors Y and Z ; a generalized coloring of a graph $G = \langle V, E \rangle$ is a map $\phi : V \mapsto (Y \sqcup Z)$ such that for all $(u, v) \in E$, if $\phi(u) \in Y$ and $\phi(v) \in Y$, then $\phi(u) \neq \phi(v)$ (The set Y is called therefore "proper colors", the set Z - "improper colors", and the entire set of colors is $X = Y \sqcup Z$). For two positive integers $x > y$, the value of the polynomial is the number of generalized colorings by x colors, exactly y of which are proper. To make this definition meaningful for graphs with multiple edges and self-loops, we require that a vertex with a self-loop can be colored only by an "improper" color $c \in Z$ and that multiple edges do not affect colorings.

Proposition 3.3.1 *The polynomial $P(G, x, y)$ satisfies the initial conditions $P(E_1) = x$ and $P(\emptyset) = 1$, and the following recurrence relation:*

$$\begin{aligned}
P(G, x, y) &= P(G - e, x, y) - P(G/e, x, y) + \\
&\quad + (x - y) \cdot P(G \dagger e, x, y) \tag{3.3.1}
\end{aligned}$$

$$P(G_1 \sqcup G_2, x, y) = P(G_1, x, y) \cdot P(G_2, x, y) \tag{3.3.2}$$

Proof. The authors of [20] state that $P(G, x, y)$ is multiplicative, since each connected component can be colored independently. We prove here the recurrence relation (3.3.1). Let $G = \langle V, E \rangle$ be a graph, and $P(G; x, y)$ be the number of generalized colorings defined above. Let $v \in V$ be any vertex. We denote by $P^v(G; x, y)$ the number of generalized colorings of G , when v is colored by an improper color, i.e. $\phi(v) \in Z$.

Lemma 3.3.2 $P^v(G, x, y) = (x - y) \cdot P(G - v, x, y)$, where $G - v$ denotes the subgraph of G induced by $V \setminus \{v\}$.

Proof. By inspection: the vertex v can have any color in Z , and the coloring of the remainder does not depend on it. ■

Now we can prove the Proposition 3.3.1:

Let $e = (u, v) \in E$ be any edge of G , which is not a self-loop and not a multiple edge. Consider the number of colorings of $G - e$. Any such coloring is either a coloring of G , or a coloring of G/e , when the vertex w , which is produced by the contraction of u and v , is colored by a proper color. Together with Lemma 3.3.2, that raises:

$$\begin{aligned}
P(G - e, x, y) &= P(G, x, y) + [P(G/e, x, y) - P^w(G/e, x, y)] = \\
&= P(G, x, y) + [P(G/e, x, y) - (x - y) \cdot P(G \dagger e, x, y)],
\end{aligned}$$

and therefore

$$P(G, x, y) = P(G - e, x, y) - P(G/e, x, y) + (x - y) \cdot P(G \dagger e, x, y) \tag{3.3.3}$$

One can easily check that this equation is satisfied also for loops and multiple edges. Together with the multiplicativity and the fact that a singleton can be colored by any color, this proves Proposition 3.3.1. ■

3.3.2 The recurrence relation

Recall the definition of EE -invariant:

$p(G) : \mathcal{G} \rightarrow \mathcal{R}$ is an EE -invariant if

- $p(G)$ is multiplicative with respect to disjoint union of graphs: for all graphs G_1, G_2 ,

$$p(G_1 \sqcup G_2) = p(G_1) \cdot p(G_2);$$

- There are $\alpha, \beta, \gamma \in \mathcal{R}$ such that for every graph $G = \langle V, E \rangle$ and every edge $e \in E$,

$$p(G) = \alpha p(G - e) + \beta p(G/e) + \gamma p(G \dagger e), \quad (3.3.4)$$

where $G - e$, G/e and $G \dagger e$ denote graphs obtained from G by respectively *deletion*, *contraction* and *extraction* (i.e. deletion together with endpoints) of the edge e .

- $p(G)$ is uniquely defined by the recurrence relation (3.3.4) together with the initial conditions

$$p(\emptyset) = 1; \quad p(E_1) = \nu, \quad (3.3.5)$$

where $\nu \in \mathcal{R}$ is some element of the ring \mathcal{R} that does not depend on the graph G .

The most general recursive definition of $P(G, \omega; \bar{X})$ is obtained by introducing indeterminates where possible. Any graph decomposition sequence consists of *edge removal* steps and non-empty disjoint subgraph *decomposition* steps. Let $\omega \in \Omega$ denote an auxiliary order of decomposition steps.

$$\begin{aligned} P(G) &= \begin{cases} sP(G - e) + yP(G/e) + zP(G \dagger e); & \text{(edge removal step)} \\ P(G_1) \cdot P(G_2); & \text{(decomposition step)} \end{cases} \\ P(E_1) &= x; \\ P(\emptyset) &= 1; \end{aligned} \quad (3.3.6)$$

Proposition 3.3.3 *The reduction of $P(G, \omega; \bar{X})$ is well-founded.*

Proof. Indeed, every step of the decomposition reduces either the number of edges or the number of vertices. Hence, after a final number of steps, only singletons and empty sets appear in the decomposition parse tree. ■

Currently, $P(G, \omega; \bar{X})$ is defined recursively for unlabeled graphs and set of indeterminates $\bar{X} = \{s, x, y, z\}$ onto the ring $\mathbb{R}[\bar{X}]$ and its result depends on the order ω of graph decomposition steps.

Theorem 3.3.4 *Let $s(\bar{x}), x(\bar{x}), y(\bar{x}), z(\bar{x}) \in \mathbb{R}[\bar{x}]$ be unresolved polynomials, and let $\hat{P}(G, \omega; \bar{x})$ be obtained from $P(G, \omega; \bar{X})$ by substitution of respective indeterminates:*

$$\hat{P}(G, \omega; \bar{x}) = P(G, \omega; \bar{X})[s \mapsto s(\bar{x}), \quad x \mapsto x(\bar{x}), \quad y \mapsto y(\bar{x}), \quad z \mapsto z(\bar{x})].$$

Then $\hat{P}(G, \omega_1; \bar{x}) = \hat{P}(G, \omega_2; \bar{x})$ for all graphs G and all pairs of orders ω_1 and ω_2 , if and only if $s(\bar{x})z(\bar{x}) = z(\bar{x})$ or $P(G) = x(\bar{x})^{|V|}$

Proof. We are looking for conditions $\varphi(s(\bar{x}), x(\bar{x}), y(\bar{x}), z(\bar{x}))$ such that if φ is satisfied, then $\hat{P}(G, \omega; \bar{x})$ is independent of the order.

We apply the order-invariance restriction to the same family of graphs we have used for the universal matching polynomial (Fig. 3.1). In order to be a unique graph invariant, $P(G)$ must return the same result in both cases: when the edge reduction rule is applied first on the edge e_1 and then on the edge e_2 , and when it is applied first on the edge e_2 and then on the edge e_1 .

$$\begin{aligned} P(G, \{1 \rightarrow 2\}) = & \tag{3.3.7} \\ s \cdot P(G - e_1) + y \cdot P(G/e_1) + z \cdot P(G \dagger e_1) = & \\ s \cdot P(H_1) \cdot [x \cdot s \cdot P(H_2) + y \cdot P(H_2) + z \cdot P(H_2 - w)] + & \\ y \cdot [s \cdot P(H_1)P(H_2) + y \cdot P(G/e_1/e_2) + z \cdot P(H_1 - u)P(H_2 - w)] + & \\ z \cdot P(H_1 - u)P(H_2) & \end{aligned}$$

On the other hand,

$$\begin{aligned}
P(G, \{2 \rightarrow 1\}) &= \tag{3.3.8} \\
& s \cdot P(G - e_2) + y \cdot P(G/e_2) + z \cdot P(G \dagger e_2) = \\
& s \cdot P(H_2) \cdot [x \cdot s \cdot P(H_1) + y \cdot P(H_1) + z \cdot P(H_1 - u)] + \\
& y \cdot [s \cdot P(H_1)P(H_2) + y \cdot P(G/e_1/e_2) + z \cdot P(H_1 - u)P(H_2 - w)] + \\
& z \cdot P(H_2 - w)P(H_1)
\end{aligned}$$

Solving the equation $P(G, \{1 \rightarrow 2\}) = P(G, \{2 \rightarrow 1\})$, we get:

$$szP(H_1)P(H_2-w)+zP(H_1-u)P(H_2) = szP(H_1-u)P(H_2)+zP(H_1)P(H_2-w)$$

Hence, we have the following necessary conditions: either $sz = z$ or $P(H_1)P(H_2-w) = P(H_1-u)P(H_2)$ for any H_1 and H_2 . In the latter case we get a trivial polynomial $P_0(H) = x^{|V(H)|}$. Indeed, if H_1 is a singleton, we get $x \cdot P(H_2 - w) = 1 \cdot P(H_2)$ for any $w \in V(H_2)$, which leads to $P(G) = x^{|V(G)|}$.

So far, we proved that the condition the theorem is necessary. In order to prove that they are also sufficient, it is enough to prove that *any two steps of graph decomposition are exchangeable*. This includes two parts,

- Edge elimination and disjoint union.
- Decomposition of a graph by elimination of any two edges in different order;

The proof of the first part is simple. Let G be a disjoint union of two graphs: $G = H_1 \sqcup H_2$. Without loss of generality, assume that the edge e , which is being eliminated, is in $E(H_1)$. Then use the distributivity of multiplication to show that

$$\begin{aligned}
P(G) &= [sP(H_1 - e) + y \cdot P(H_1/e) + z \cdot P(H_1 \dagger e)] \cdot P(H_2) = \\
&= sP(H_1 - e) \cdot P(H_2) + y \cdot P(H_1/e) \cdot P(H_2) + z \cdot P(H_1 \dagger e) \cdot P(H_2)
\end{aligned}$$

We henceforth may assume, without loss of generality, that the disjoint decomposition steps are applied only on singletons.

For the second part, we shall consider order over edges rather than order over edge-removal steps: such an order uniquely determines the decompo-

sition process, if by convention, we just skip the steps of removing edges that have been already removed by the preceding steps. Since we now speak about a linear order over the edges, it is enough to show that any two consequent pieces can be swapped.

We analyze three possible cases (Fig. 3.2):

- Case 1: The two edges have no common vertices (graphs G_1, G_2, G_3);
- Case 2: The two edges have one common vertex, and at least one exclusive vertex (graphs G_4, G_5);
- Case 3: The two edges have no exclusive vertices (graphs G_6, G_7).

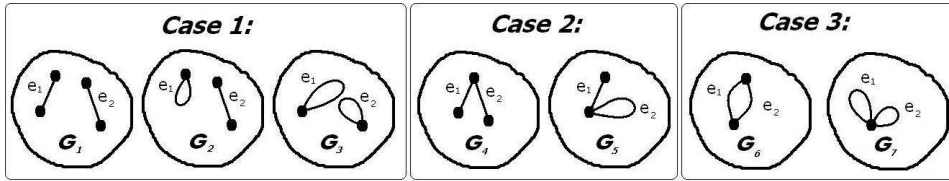


Figure 3.2: Different cases to check order invariance of edge elimination

In the first case, the edge elimination operations are independent and commutative, e.g. $G - e_1 / e_2 \cong G / e_2 - e_1$ and $G \dagger e_1 / e_2 \cong G / e_2 \dagger e_1$. Thus, if we first eliminate e_1 and then e_2 , we have:

$$\begin{aligned}
P(G, \{1 \rightarrow 2\}) &= sP(G - e_1) + yP(G / e_1) + zP(G \dagger e_1) = \\
&= s^2P(G - e_1 - e_2) + syP(G - e_1 / e_2) + szP(G - e_1 \dagger e_2) + \\
&+ syP(G / e_1 - e_2) + y^2P(G / e_1 / e_2) + yzP(G / e_1 \dagger e_2) + \\
&+ szP(G \dagger e_1 - e_2) + yzP(G \dagger e_1 / e_2) + z^2P(G \dagger e_1 \dagger e_2)
\end{aligned}$$

On the other hand, if we first eliminate e_2 and then e_1 , we have:

$$\begin{aligned}
P(G, \{2 \rightarrow 1\}) &= sP(G - e_2) + yP(G / e_2) + zP(G \dagger e_2) = \\
&= s^2P(G - e_2 - e_1) + syP(G - e_2 / e_1) + szP(G - e_2 \dagger e_1) + \\
&+ syP(G / e_2 - e_1) + y^2P(G / e_2 / e_1) + yzP(G / e_2 \dagger e_1) + \\
&+ szP(G \dagger e_2 - e_1) + yzP(G \dagger e_2 / e_1) + z^2P(G \dagger e_2 \dagger e_1)
\end{aligned}$$

It is simple to see that the two expressions are equal.

The second case is slightly more confusing, since the edge extraction operation is not commutative with others: indeed, if we have extracted for example the edge e_1 in G_4 , there is no more e_2 to eliminate, and vice versa. The other operations, deletion and contraction, are still commutative with each other. Therefore we should check only the following three sequences of non-commutative edge elimination operations:

- Extraction of the first edge eliminates also the second;
- Contraction of the first edge and then extraction of the second gives a graph with the two edges extracted;
- Deletion of the first edge and then extraction of the second is equivalent to simply extraction of the second edge.

Thus, if we first eliminate e_1 and then e_2 , we have:

$$\begin{aligned}
P(G, \{1 \rightarrow 2\}) &= sP(G-e_1) + yP(G/e_1) + zP(G\uparrow e_1) = \\
&= s^2P(G-e_1-e_2) + syP(G-e_1/e_2) + szP(G\uparrow e_2) + \\
&+ syP(G/e_1-e_2) + y^2P(G/e_1/e_2) + yzP(G\uparrow(e_1 \text{ and } e_2)) + \\
&+ zP(G\uparrow e_1)
\end{aligned}$$

On the other hand, if we first eliminate e_2 and then e_1 , we have:

$$\begin{aligned}
P(G, \{2 \rightarrow 1\}) &= sP(G-e_2) + yP(G/e_2) + zP(G\uparrow e_2) = \\
&= s^2P(G-e_2-e_1) + syP(G-e_2/e_1) + szP(G\uparrow e_1) + \\
&+ syP(G/e_2-e_1) + y^2P(G/e_2/e_1) + yzP(G\uparrow(e_1 \text{ and } e_2)) + \\
&+ zP(G\uparrow e_2)
\end{aligned}$$

These two expressions are equal provided $sz = z$.

In the third case, the edge elimination steps are symmetric in their transformations of G with respect to the order among e_1 and e_2 . ■

Analyzing the conditions φ , we get:

Corollary 3.3.5 *Every EE-polynomial $p : \mathcal{G} \rightarrow \mathbb{R}[\bar{X}]$ of a graph $G = \langle V, E \rangle$ satisfies either recurrence relation*

$$p(G) = yp(G-e) + zp(G/e) \tag{3.3.9}$$

or recurrence relation:

$$p(G) = p(G - e) + yp(G/e) + zp(G \dagger e); \quad (3.3.10)$$

where y and z are elements of $\mathbb{R}[\bar{X}]$ independent of G .

Proof. The condition $sz = z$ has two roots: $s = 1$ and $z = 0$. Under the latter root the recurrence relation (3.3.9) is obtained. Recall that this is the recurrence relation of C -invariants. This also includes the exception case $p(G) = x^{|V|}$. The first root gives a rise to the recurrence relation (3.3.10). ■

Corollary 3.3.6 *There is a map $\xi : \mathcal{G} \rightarrow \mathbb{R}[x, y, z]$, which is uniquely defined by the recurrence relation and initial conditions:*

$$\begin{aligned} \xi(G_1 \sqcup G_2; x, y, z) &= \xi(G_1; x, y, z) \cdot \xi(G_2; x, y, z); \\ \xi(G; x, y) &= \xi(G - e; x, y, z) + y\xi(G/e; x, y, z) + z\xi(G \dagger e; x, y, z); \\ \xi(E_1; x, y, z) &= x; \\ \xi(\emptyset; x, y, z) &= 1. \end{aligned} \quad (3.3.11)$$

Corollary 3.3.7 *The map $\xi(G; x, y, z)$ from Corollary 3.3.6 is universal in the class of sEE -invariants and dp -complete in the class of EE -invariants.*

Proof. The universality property follows directly from the Corollary 3.3.6, for the dp -completeness we need also prove that $\xi(G; x, y, z)$ determines every C -invariant. Recall that the dichromatic polynomial $Z(G; q, v)$ is dp -complete in C . On the other hand, $Z(G; q, v) \in sEE$ (by definition of sEE). ■

3.3.3 The subset expansion form of $\xi(G; x, y, z)$

We now give an explicit form of the polynomial $\xi(G, x, y, z)$ using 3-partition edge expansion³:

³A more precise name would be a “Pair of disjoint subsets expansion”. We chose the name 3-partition expansion, as any two disjoint subsets induce a partition into three sets.

Theorem 3.3.8 *Let $G = \langle V, E \rangle$ be a (multi)graph. Then the most distinctive edge elimination polynomial $\xi(G; x, y, z)$ can be calculated as*

$$\xi(G, x, y, z) = \sum_{(A \sqcup B) \subseteq E} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} \quad (3.3.12)$$

where by abuse of notation we use $(A \sqcup B) \subseteq E$ for summation over subsets $A, B \subseteq E$, such that the subsets of vertices $V(A)$ and $V(B)$, covered by respective subset of edges, are disjoint: $V(A) \cap V(B) = \emptyset$; $k(A)$ denotes the number of spanning connected components in $\langle V, A \rangle$, and $k_{cov}(B)$ denotes the number of covered connected components, i.e. the connected components of the graph $\langle V(B), B \rangle$.

Proof. We need to show that

- The expression (3.3.12) satisfies the initial conditions of (3.3.11);
- The expression (3.3.12) is multiplicative;
- The expression (3.3.12) satisfies the edge elimination rule of (3.3.11).

Then by induction on the number of edges in G the theorem holds. The first statement follows by inspection: indeed, a null graph $\emptyset = \langle \emptyset, \emptyset \rangle$ has no edges and no components, so its only summand is 1; a singleton has no edges and no covered components, so its only summand is x .

The second statement, multiplicativity, can be easily checked too: Indeed, the summation over subsets of edges of $\langle V, E \rangle = \langle V_1, E_1 \rangle \sqcup \langle V_2, E_2 \rangle$ can be regarded as a summation over the subsets of E_1 , multiplied by an independent summation over the subsets of E_2 .

Therefore, we just need to prove the third statement:

Let $G = \langle V, E \rangle$ be the (multi)graph of interest. Let $N(G)$ be defined as in (3.3.12):

$$N(G, x, y, z) = \sum_{(A \sqcup B) \subseteq E} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} \quad (3.3.13)$$

Let e be the edge we have chosen to reduce. Any particular choice of A and B can be regarded as a vertex-disjoint edge coloring in 2 colors A and B, when part of the edges remains uncolored. We separate all the possible colorings to three disjoint cases:

- Case 1: e is uncolored;
- Case 2: e is colored by B , and it is the only edge of a colored connected component;
- Case 3: All the rest. That means, e is colored by A , or e is colored by B but it is not the only edge of a colored connected component.

In case 1, we just sum over colorings of $G-e$:

$$N_1(G) = \sum_{(A \sqcup B) \models \text{Case 1}} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} = N(G-e) \quad (3.3.14)$$

In case 2, the edge e is a connected component of $\langle V(B), B \rangle$. Therefore, if we analyze now $N(G \dagger e)$, we will get

- The number of edges colored by A is the same;
- The number of edges colored by B is reduced by one;
- The total number of colored connected components is reduced by one;
- The number of covered connected components colored B is reduced by one;

This gives us

$$N_2(G) = \sum_{(A \sqcup B) \models \text{Case 2}} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} = z \cdot N(G \dagger e) \quad (3.3.15)$$

And finally, in case 3, e is a part of a bigger colored connected component, or it is alone a connected component colored by A . In this case, we analyze the colorings of G/e :

- Either $|A|$ or $|B|$ is reduced by 1, the other remained the same;
- The total number of colored connected components remained the same (in case when $e \in A$ is the only edge of a connected component, this component is contracted to one vertex);
- The number of covered connected components colored B remained the same.

According to the above,

$$N_3(G) = \sum_{(A \sqcup B) \models \text{Case 3}} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} = y \cdot N(G/e) \quad (3.3.16)$$

On the other hand, three the cases above are disjoint and cover all the possibilities, and therefore $N(G) = N_1(G) + N_2(G) + N_3(G)$. Hence, we have:

$$N(G; x, y, z) = \xi(G; x, y, z),$$

which completes the proof of the theorem. ■

3.4 The most distinctive vertex elimination polynomial $U_{VE}(G; x, y)$

The last class of graph invariants we discuss in this thesis is the class of the vertex elimination invariants. We start with the *subgraph component polynomial* $Q(G; x, y)$, which has been proposed by P. Tittmann and further developed in collaboration with J. A. Makowsky and the author in [51]. This graph polynomial arises from analyzing community structures in social networks.

It turned out that $Q(G; x, y)$ satisfies a linear recurrence relation with respect to three vertex elimination operations. We look for the most distinctive graph polynomial in the class of VE -invariants.

3.4.1 The subgraph component polynomial $Q(G; x, y)$

Recall the definition of $Q(G; x, y)$ given in Section 2.5: Let $G = \langle V, E \rangle$ be a finite undirected graph with $|V| = n$ vertices and let $i, j \leq n$ be positive integers. Let $q_{ij}(G)$ be the number of vertex subsets $X \subseteq V$ with i vertices such that $G[X]$ has exactly j components:

$$q_{ij}(G) = |\{X \subseteq V : |X| = i \wedge k(G[X]) = j\}|$$

Then the subgraph component polynomial is defined as

$$Q(G; x, y) = \sum_{i=0}^n \sum_{j=0}^n q_{ij}(G) x^i y^j.$$

We can rewrite this definition in a subset expansion form; instead of summation over i and j , we can sum over subsets of vertices:

$$Q(G; x, y) = \sum_{U \subseteq V} x^{|U|} y^{k(G[U])}$$

3.4.2 The recurrence relation of $Q(G; x, y)$

We first investigate the recurrence relation of the subgraph polynomial. The first statement concerns the multiplicativity with respect to disjoint union.

Theorem 3.4.1 (Multiplicativity) *Let $G = G_1 \sqcup G_2$ be the disjoint union of the graphs G_1 and G_2 . Then*

$$Q(G; x, y) = Q(G_1; x, y) \cdot Q(G_2; x, y).$$

Proof. Each subset $U \subseteq V$ is a disjoint union of two subsets $U_1 \subseteq V(G_1)$ that induces $k(G_1[U_1])$ connected components and $U_2 \subseteq V(G_2)$ that induces $k(G_2[U_2])$ connected components. Moreover, since the graphs G_1 and G_2 are disjoint, $k(G[U]) = k(G_1[U_1]) + k(G_2[U_2])$.

We obtain

$$Q(G; x, y) = \sum_{U \subseteq V} x^{|U|} y^{k(G[U])} = \sum_{U_1 \subseteq V_1} \sum_{U_2 \subseteq V_2} x^{|U_1|+|U_2|} y^{k(G_1[U_1])+k(G_2[U_2])}. \quad (3.4.1)$$

Thus for a disjoint union of two graphs, the subgraph polynomial satisfies $Q(G; x, y) = Q(G_1; x, y) Q(G_2; x, y)$ ■

Corollary 3.4.2 *If $G = (V, E)$ consists of c components G_1, \dots, G_c then the subgraph polynomial satisfies*

$$Q(G; x, y) = \prod_{j=1}^c Q(G_j; x, y).$$

Proof. We obtain the statement of the theorem by induction on the number of components. ■

Next, we prove that the subgraph component polynomial $Q(G; x, y)$ satisfies a linear recurrence relation with respect to three vertex elimination operations: vertex deletion, vertex contraction and vertex extraction.

Theorem 3.4.3 *Let $G = (V, E)$ be a graph and $v \in V$. Then the subgraph polynomial satisfies the decomposition formula*

$$Q(G; x, y) = Q(G - v; x, y) + xQ(G/v; x, y) + x(y - 1)Q(G \dagger v; x, y).$$

Proof. Let us first consider all vertex induced subgraphs of G that do not contain vertex v . These subgraphs are also vertex induced subgraphs of $G - v$. Consequently,

$$Q(G - v; x, y)$$

enumerates all induced subgraphs not including the vertex v .

In a second step we count all vertex induced subgraphs that contain vertex v but none of its neighbors in G . In this case, the vertex v forms a connected component consisting of v only. The rest of the induced subgraph is a subgraph of $G \dagger v$. All these subgraphs are enumerated by $Q(G \dagger v; x, y)$. However, the component built by v contributes one vertex and one component to the polynomial. Thus we obtain the contribution

$$xyQ(G \dagger v; x, y).$$

In our enumeration so far we missed exactly those vertex induced subgraphs that contain v and at least one of its neighbors together in one component. We include v in the corresponding candidate set, remove it from G , and multiply the generating function by x (not by xy because we do not increase the number of components). In order to trace the components, we have to simulate the paths using v in G . These paths are no longer present in $G - v$. This task is best performed by using G/v instead of $G - v$. Thus we obtain the contribution $xQ(G/v; x, y)$ to the generating function. Unfortunately, this polynomial enumerates induced subgraphs that do not contain any vertices from $N(v)$, too. We can fix this problem by subtraction of $xQ(G \dagger v; x, y)$, which gives

$$xQ(G/v; x, y) - xQ(G \dagger v; x, y)$$

as final contribution. ■

3.4.3 The Universal VE -invariant

Recall the definition of VE invariants:

$p(G) : \mathcal{G} \rightarrow \mathcal{R}$ is a VE -invariant if

- $p(G)$ is multiplicative with respect to disjoint union of graphs: for all graphs G_1, G_2 ,

$$p(G_1 \sqcup G_2) = p(G_1) \cdot p(G_2);$$

- There are $\alpha, \beta, \gamma \in \mathcal{R}$ such that for every graph $G = \langle V, E \rangle$ and every vertex $v \in V$,

$$p(G) = \alpha p(G - v) + \beta p(G/v) + \gamma p(G \dagger v), \quad (3.4.2)$$

where $G - v$, G/v and $G \dagger v$ denote graphs obtained from G by respectively *deletion*, *contraction* and *extraction* of the vertex v .

- $p(G)$ is uniquely defined by the recurrence relation (3.4.2) together with the initial conditions

$$p(\emptyset) = 1; \quad p(E_1) = \nu, \quad (3.4.3)$$

where $\nu \in \mathcal{R}$ is some element of the ring \mathcal{R} that does not depend on the graph G .

The most general recursive definition of $P(G, \omega; \bar{X})$ is obtained by introducing indeterminates where possible. Any graph decomposition sequence consists of *vertex removal* steps and non-empty disjoint subgraph *decomposition* steps. Let $\omega \in \Omega$ denote an auxiliary order of decomposition steps.

$$\begin{aligned} P(G) &= \begin{cases} sP(G - v) + yP(G/v) + zP(G \dagger v); & (\text{vertex removal step}) \\ P(G_1) \cdot P(G_2); & (\text{decomposition step}) \end{cases} \\ P(E_1) &= x; \\ P(\emptyset) &= 1; \end{aligned} \quad (3.4.4)$$

Proposition 3.4.4 *The reduction of $P(G, \omega; \bar{X})$ is well-founded.*

Proof. Indeed, every step of the decomposition reduces the number of vertices. Hence, after a final number of steps, only singletons and empty sets appear in the decomposition parse tree. ■

Currently, $P(G, \omega; \bar{X})$ is defined recursively for unlabeled graphs and set of indeterminates $\bar{X} = \{s, x, y, z\}$ onto the ring $\mathbb{R}[\bar{X}]$ and its result depends on the order ω of graph decomposition steps.

Theorem 3.4.5 *Let $s(\bar{x}), x(\bar{x}), y(\bar{x}), z(\bar{x}) \in \mathbb{R}[\bar{x}]$ be unresolved polynomials, and let $\hat{P}(G, \omega; \bar{x})$ be obtained from $P(G, \omega; \bar{X})$ by substitution of respective indeterminates:*

$$\hat{P}(G, \omega; \bar{x}) = P(G, \omega; \bar{X})[s \mapsto s(\bar{x}), x \mapsto x(\bar{x}), y \mapsto y(\bar{x}), z \mapsto z(\bar{x})].$$

Then $\hat{P}(G, \omega_1; \bar{x}) = \hat{P}(G, \omega_2; \bar{x})$ for all graphs G and all pairs of orders ω_1 and ω_2 , if and only if

$$(s = 1 \text{ and } x = s + y + z) \text{ or } P(G; \bar{X}) = x^{|V(G)|} \text{ or } P(G; \bar{X}) = 1.$$

Proof. We are looking for conditions $\varphi(s(\bar{x}), x(\bar{x}), y(\bar{x}), z(\bar{x}))$ such that if φ is satisfied, then $\hat{P}(G, \omega; \bar{x})$ is independent of the order.

First, from $E_1 - v = E_1 \dagger v = E_1/v = \emptyset$ we obtain $x = (s + y + z)$.

Second, we apply the recurrence relation (3.4.4) to compute $P(P_3, \omega_1)$ and $P(P_3, \omega_2)$, where under ω_1 we first eliminate a vertex of degree 1 from P_3 , whereas under ω_2 we first eliminate vertex of degree 2.

$$\begin{aligned} P(P_3, \omega_1) &= sP(P_2) + yP(P_2) + zP(E_1) = \\ &= (s + y)[(s + y)(s + y + z) + z] + z(s + y + z) = \\ &= (s + y)^2(s + y + z) + (s + y)z + z(s + y + z) \end{aligned}$$

$$\begin{aligned} P(P_3, \omega_2) &= sP(E_2) + yP(P_2) + z = \\ &= s(s + y + z)^2 + y[(s + y)(s + y + z) + z] + z = \\ &= s(s + y + z)^2 + y(s + y)(s + y + z) + yz + z \end{aligned}$$

Solving $P(P_3, \omega_1) = P(P_3, \omega_2)$ we get:

$$z = 0 \quad \text{or} \quad s = 1 \quad \text{or} \quad s + y + z = 1$$

In case of $z = 0$, we get the polynomial $P = (s + y)^{|V|} = x^{|V|}$. In case of $w + y + z = 1$, the result is $P = 1$. The only remaining choice, $s = 1$, completes the conditions of the theorem. \blacksquare

Theorem 3.4.6 *VE-universality:*

1. Every *VE*-invariant $p : \mathcal{G} \rightarrow \mathcal{R}$ of a graph $G = \langle V, E \rangle$ satisfies the recurrence relation

$$p(G) = p(G - v) + ap(G/v) + bp(G \dagger E), \quad (3.4.5)$$

where a and b are elements of the target ring \mathcal{R} that do not depend on the graph G .

2. There is a map $U_{VE} : \mathcal{G} \rightarrow \mathbb{R}[x, y]$, which is uniquely defined by this recurrence relation and initial conditions:

$$\begin{aligned} U_{VE}(G_1 \sqcup G_2) &= U_{VE}(G_1) \cdot U_{VE}(G_2); \\ U_{VE}(G) &= U_{VE}(G - v) + xU_{VE}(G/v) + yU_{VE}(G \dagger v); \\ U_{VE}(E_1) &= 1 + x + y; \\ U_{VE}(\emptyset) &= 1. \end{aligned} \quad (3.4.6)$$

Proof. Analyzing the conditions φ , we obtain that all *VE* invariants satisfy the recurrence relation (3.4.5), including the exceptional polynomials $P(G) = x^{|V|}$ and $P(G) = 1$ (with $a \mapsto x - 1, b \mapsto 0$ and $a \mapsto 0, b \mapsto 0$ respectively).

In order to prove that the map $U_{VE}(G; x, y)$ is well-defined, we will use the connection to the subgraph component polynomial and to the independent set polynomial:

Assuming $x \neq 0$, $U_{VE}(G)$ is a substitution instance of $Q(G; x, y)$:

$$U_{VE}(G; x, y) = Q(G; x, \frac{y}{x} + 1) = \sum_{U \subseteq V} x^{|U|} \left(\frac{y}{x} + 1\right)^{k(G[U])}$$

Otherwise, it turns a substitution instance of the independent set polynomial

$$U_{VE}(G; 0, y) = In(G; y)$$

Both those statements can be proven by induction using the recurrence relations satisfied by the respective graph polynomials. Both $Q(G; x, y)$ and $In(G; y)$ are well-defined graph polynomials. Therefore, the map $U_{VE}(G; x, y)$ is well-defined. \blacksquare

Corollary 3.4.7 *$U_{VE}(G; x, y)$ is a VE -universal and dp -complete graph invariant.*

Corollary 3.4.8 *The subgraph component polynomial $Q(G; x, y)$ is a dp -complete VE -invariant.*

Proof. For $x \neq 0$, the universal VE -invariant $U_{VE}(G; x, y)$ can be obtained as a substitution instance of $Q(G; x, y)$:

$$U_{VE}(G; x, y) = Q(G; x, \frac{y}{x} + 1)$$

The only exception is the independent set polynomial, which, in turn, can be read out of $Q(G; x, y)$ as a subset of its coefficients:

$$in(G; i) = q_{ii} = [x^i y^i] Q(G; x, y).$$

Therefore, for any two graphs G_1 and G_2 ,

$$Q(G_1; x, y) = Q(G_2; x, y) \rightarrow U_{VE}(G_1; x, y) = U_{VE}(G_2; x, y),$$

which completes the proof. \blacksquare

Corollary 3.4.9 *The subset-expansion definition of $U_{VE}(G; x, y)$ is as follows:*

$$U_{VE}(G; x, y) = \sum_{U \subseteq V} x^{|U| - k(G[U])} (x + y)^{k(G[U])}, \quad (3.4.7)$$

where $G[U]$ denotes the induced subgraph of G with vertex set U , and $k(G[U])$ denotes number of spanning connected components in $G[U]$.

Proof. Like the fact that $U_{VE}(G; x, y)$ is well-defined, its subset expansion form can be derived from its connection to the subgraph component polynomial $Q(G; x, y)$ and to the independent set polynomial $In(G; y)$. Indeed, if $x \neq 0$, then the equation (3.4.7) can be rewritten as

$$U_{VE}|_{x \neq 0}(G; x, y) = \sum_{U \subseteq V} x^{|U|} \left(\frac{x+y}{x} \right)^{k(G[U])} = Q(G; x, \frac{y}{x} + 1).$$

Otherwise, if $x = 0$, the only summands that contribute to the result are those with U independent set, and therefore,

$$U_{VE}|_{x=0}(G; x, y) = \sum_{\substack{U \subseteq V \\ U \text{ is indep. set}}} y^{|U|} = In(G; y).$$

■

3.4.4 Vertex Eliminations vs Edge Elimination

In this section we will show the connection of $Q(G; x, y)$ to the most distinctive edge elimination polynomial $\xi(G; x, y, z)$.

We rewrite the decomposition of $Q(G; x, y)$ using Theorem 3.4.3:

$$\begin{aligned} Q(G; x, y) &= Q(G - v; x, y) + xQ(G/v; x, y) + x(y-1)Q(G \dagger v; x, y) \\ Q(G_1 \sqcup G_2; x, y) &= Q(G_1; x, y)Q(G_2; x, y) \\ Q(E_1; x, y) &= xy + 1 \\ Q(\emptyset) &= 1 \end{aligned} \tag{3.4.8}$$

Theorem 3.4.10 *Let $G = \langle V, E \rangle$ be a graph. Let $L(G) = \langle V_e, E_e \rangle$ denote the line graph of G . Then the following equation holds:*

$$\xi(G; 1, x, x(y-1)) = Q(L(G); x, y)$$

Proof. First, let us analyze the correspondence of the edge elimination operations in a graph to the vertex elimination operations in its line graph.

Let $v_e \in V_e$ be the vertex in the line graph that corresponds to the edge $e \in E$ of the original graph. By the definition of the edge and vertex elimination operations:

$$L(G - e) = L(G) - v_e \quad (3.4.9)$$

$$L(G/e) = L(G)/v_e \quad (3.4.10)$$

$$L(G \dagger e) = L(G) \dagger v_e \quad (3.4.11)$$

First let us check the connected graphs with up to one edge:

If $G \in \{\emptyset, E_1\}$, $L(G) = \emptyset$,

The equivalence $\xi(G; 1, x, x(y-1)) = 1 = Q(\emptyset)$ holds.

If G is a single point with a loop, or $G = P_2$, $L(G)$ is a singleton, The equivalence $\xi(G; 1, x, x(y-1)) = 1 + x + x(y-1) = 1 + xy = Q(E_1)$ holds.

Next, we note that $L(G_1 \sqcup G_2) = L(G_1) \sqcup L(G_2)$. Therefore, if the theorem holds for graphs G_1 and G_2 , then it holds also for $G_1 \sqcup G_2$. Finally, the theorem follows by induction on the number of edges, using the decomposition formulae (3.4.8) and (3.3.11) and the correspondence of edge and vertex elimination operations. ■

Chapter 4

DP-complete and Universal Graph Polynomials for Labeled Graphs

In this chapter we extend our results presented so far to labeled graph invariants. First, we define classes of labeled graph invariants, with respect to edge elimination and vertex elimination, in parallel to the classes C, TG, M, EE and VE discussed above.

4.1 Extending definitions to labeled graphs

Let \mathcal{LG} be the class of labeled graphs. Recall that labeled graph $G = \langle V, E, lab_V, lab_E \rangle$ is provided with the labeling functions $lab_V : V \mapsto \Lambda_V$ and $lab_E : E \mapsto \Lambda_E$, which map the vertices and the edges, respectively, to sets of vertex and edge labels Λ_V and Λ_E . Two labeled graphs are isomorphic if there is an isomorphism that preserves both vertex and edge labelings. We define also subclasses of \mathcal{LG} : we denote by \mathcal{LG}_E the class of edge-labeled graphs and by \mathcal{LG}_V – the class of vertex-labeled graphs.

Labeled graph invariant is a function $f : \mathcal{LG} \rightarrow \mathcal{R}$ which maps labeled graphs into a ring \mathcal{R} such that isomorphic labeled graphs $G_1 \simeq G_2$ get the same image: $f(G_1) = f(G_2)$.

Labeled graph polynomials have as their domains polynomial ring $\mathcal{R}[\bar{X}]$,

where the set of indeterminates \bar{X} is labeled:

$$\bar{X} = \{X^V \times \Lambda_V, X^E \times \Lambda_E\},$$

where $X^V = \{x_0^V, x_1^V, \dots\}$ is the set of vertex-labeled indeterminates, and $X^E = \{x_0^E, x_1^E, \dots\}$ is the set of edge-labeled indeterminates.

4.1.1 Comparing labeled graph invariants

Like the unlabeled graph invariants, we use the following notions for comparing labeled graph invariants:

Definition 16 *Distinctive power of labeled graph invariants:*

Let $p : \mathcal{LG} \rightarrow \mathcal{R}_1$ and $q : \mathcal{LG} \rightarrow \mathcal{R}_2$ be two labeled graph invariants. We say that the **distinctive power** of p does not exceed that of q , and write $p \preceq_{dp} q$ iff for every pair of labeled graphs G_1 and G_2

$$q(G_1) = q(G_2) \quad \rightarrow \quad p(G_1) = p(G_2)$$

If $p \preceq_{dp} q$ and $q \preceq_{dp} p$ we say that the graph invariants p and q have the same distinctive power and write $p \simeq_{dp} q$. If neither $p \preceq_{dp} q$ nor $q \preceq_{dp} p$, we say that the graph invariants p and q have incomparable distinctive power.

Definition 17 *Substitution instance:*

Let $p : \mathcal{G} \rightarrow \mathcal{R}[\bar{X}]$ and $q : \mathcal{G} \rightarrow \mathcal{R}[\bar{Y}]$ be two labeled graph polynomials with sets of indeterminates respectively \bar{X} and \bar{Y} . We say that p is a **substitution instance** of q and write $p \preceq_{subst} q$ iff there is a variable substitution $\sigma : \bar{Y} \rightarrow \mathcal{R}[\bar{X}]$ such that for all labeled graphs $G \in \mathcal{LG}$, $q(G, \bar{Y})$ under σ evaluates to $p(G; \bar{X})$:

$$p(G; \bar{X}) = \sigma(q(G; \bar{Y}))$$

Based on the notions above, we can use the same definitions of dp -completeness and universality as for unlabeled graph invariants.

4.1.2 Classes of labeled graph invariants with respect to edge elimination and vertex elimination

Definition 18 Let $p : \mathcal{G} \rightarrow \mathcal{R}$, where \mathcal{R} is either a field or a polynomial ring $\mathcal{F}[\bar{X}]$ with an underlying field \mathcal{F} , be a multiplicative labeled graph invariant

that satisfies boundary conditions as follows:

$$p(\emptyset) = 1 \text{ and } p(\langle \{v\}, \emptyset, \{v \mapsto c\}, \emptyset \rangle) = \nu_c \text{ for some } \nu_c \in \mathcal{R} \quad (4.1.1)$$

Note that if vertex-labeling exists, the value of $p(\{v\})$ depends on the label c . If there is no vertex labeling, the initial conditions are like the unlabeled case:

$$p(\emptyset) = 1 \text{ and } p(\{v\}) = \nu \text{ for some } \nu \in \mathcal{R} \quad (4.1.2)$$

We say that a graph invariant p is a

- **labeled C-invariant** and write $p \in LC$ iff

1. It is defined on the class of edge-labeled¹ graphs: $p : \mathcal{LG}_E \rightarrow \mathcal{R}$;
2. For every edge-label $d \in \Lambda_E$ there are elements $\alpha_d, \beta_d \in \mathcal{R}$ such that for every graph $G \in \mathcal{LG}$ and every edge $e \in E(G)$ with $\text{lab}_E(e) = d$ the following equation holds:

$$p(G) = \alpha_d p(G - e) + \beta_d p(G/e)$$

3. The invariant p is uniquely defined by this recurrence relation and the initial conditions (4.1.2).

If $\alpha_d = 1$ for all $d \in \Lambda_E$, we call p a **special LC-invariant** and write $p \in sLC$.

- **labeled TG-invariant** and write $p \in LTG$ iff

1. It is defined on the class of edge-labeled graphs: $p : \mathcal{LG}_E \rightarrow \mathcal{R}$;
2. For every edge-label $d \in \Lambda_E$ there are elements $x_d, y_d, \sigma_d, \tau_d \in \mathcal{R}$ such that for every graph $G \in \mathcal{LG}$ and every edge $e \in E(G)$ with

¹Allowing vertex labeling would make the definition of the edge contraction operation unclear: it is not clear which label should be given to a vertex produced by a contraction of an edge (u, v) with different labels.

$lab_E(e) = d$ the following holds:

$$p(G) = \begin{cases} x_d p(G - e) & \text{if } e \text{ is a bridge,} \\ y_d p(G - e) & \text{if } e \text{ is a loop,} \\ \sigma_d p(G - e) + \tau_d p(G/e) & \text{otherwise.} \end{cases}$$

3. The invariant p is uniquely defined by this recurrence relation and the initial conditions (4.1.2).

• **labeled M-invariant** and write $p \in LM$ iff

1. It is defined on the entire class of labeled graphs: $p : \mathcal{LG} \rightarrow \mathcal{R}$;
2. For every edge-label $d \in \Lambda_E$ there are elements $\alpha_d, \beta_d \in \mathcal{R}$ such that for every graph $G \in \mathcal{LG}$ and for every edge $e \in E(G)$ with $lab_E(e) = d$ the following holds:

$$p(G) = \alpha_d p(G - e) + \beta_d p(G \dagger e);$$

3. The invariant p is uniquely defined by this recurrence relation and the initial conditions (4.1.1).

If $\alpha_d = 1$ for all $d \in \Lambda_E$, we call p a **special LM-invariant** and write $p \in sLM$.

• **labeled EE-invariant** and write $p \in LEE$ iff

1. It is defined on the class of edge-labeled graphs: $p : \mathcal{LG}_E \rightarrow \mathcal{R}$;
2. For every edge-label $d \in \Lambda_E$ there are elements $\alpha_d, \beta_d, \gamma_d \in \mathcal{R}$ such that for every graph $G \in \mathcal{LG}$ and for every edge $e \in E(G)$ with $lab_E(e) = d$ the following holds:

$$p(G) = \alpha_d p(G - e) + \beta_d p(G/e) + \gamma_d p(G \dagger e);$$

3. The invariant p is uniquely defined by this recurrence relation and the initial conditions (4.1.2).

If $\alpha_d = 1$ for all $d \in \Lambda_E$, we call p a **special LEE invariant** and write $p \in sLEE$.

• **labeled VE-invariant** and write $p \in LVE$ iff

1. It is defined on the class of vertex-labeled² graphs: $p : \mathcal{LG}_V \rightarrow \mathcal{R}$;
2. For every vertex-label $c \in \Lambda_V$ there are elements $\alpha_c, \beta_c, \gamma_c \in \mathcal{R}$ such that for every graph $G \in \mathcal{LG}$ and for every vertex $v \in V(G)$ with $\text{lab}_V(v) = c$ the following holds:

$$p(G) = \alpha_c p(G - v) + \beta_c p(G/v) + \gamma_c p(G \dagger v);$$

3. The invariant p is uniquely defined by this recurrence relation and the initial conditions (4.1.1).

Note that the set of LEE-invariants trivially includes the entire set of LC-invariants. However, in contrast to the unlabeled case, it does not include the class of LM-invariants, because it is not defined for vertex-labeled graphs.

In the remainder of this chapter we present some examples of known labeled graph polynomials, and then provide our results regarding dp-completeness and universality of LM , LEE and LVE invariants.

4.2 Examples of known labeled graph polynomials

In this section we provide some examples of labeled graph polynomials from the literature.

4.2.1 The Sokal polynomial

A. Sokal defines in [49] a weighted generalization of the dichromatic polynomial:

$$Z(G; q, \bar{v}) = \sum_{A \in \mathcal{E}} q^{k(A)} \prod_{e \in A} v_e,$$

where v_e denotes a weight defined per edge, and $k(A)$ denotes the number of connected components in $\langle V, A \rangle$.

²Similarly, the vertex contraction operation would become unclear if edge labeling were allowed, because it is not clear what label should be given to the new edges produced by this operation.

This graph polynomial satisfies a linear recurrence relation:

$$\begin{aligned} Z(G_1 \sqcup G_2) &= Z(G_1) \cdot Z(G_2), \\ Z(G) &= Z(G - e) + v_e \cdot Z(G/e); \end{aligned} \quad (4.2.1)$$

The Sokal polynomial is uniquely defined by the recurrence relation (4.2.1) and initial conditions $Z(\emptyset) = 1$ and $Z(\{v\}) = q$. From here we can conclude:

Proposition 4.2.1 *The Sokal polynomial $Z(G; q, \bar{v})$ is an LC-invariant³. Moreover, it is dp-complete in the class of LC graph invariants and universal in the class of sLC graph invariants.*

Proof. Let $U_{LC}(G; q, \bar{w}, \bar{v})$ be defined as the most general map satisfying the conditions of LC-invariants:

$$\begin{aligned} U_{LC}(G_1 \sqcup G_2) &= U_{LC}(G_1) \cdot U_{LC}(G_2), \\ U_{LC}(G) &= w_e U_{LC}(G - e) + v_e U_{LC}(G/e); \\ U_{LC}(\{v\}) &= q; \\ U_{LC}(\emptyset) &= 1. \end{aligned} \quad (4.2.2)$$

Analyzing the paths of graph decomposition, one can conclude that, independently of the decomposition order, each path contains either contraction or deletion of each edge. Hence, the recurrence relation (4.2.2) can be rewritten as

$$U_{LC}(G) = w_e \cdot (U_{LC}(G - e) + \frac{v_e}{w_e} U_{LC}(G/e)),$$

and therefore, by induction on the number of edges,

$$U_{LC}(G; q; \bar{w}, \bar{v}) = \left(\prod_{e \in E} w_e \right) Z \left(G; q, \overline{\left(\frac{v}{w} \right)} \right).$$

The factor $(\prod_{e \in E} w_e)$ can be derived, in turn, as the highest-degree term of $Z(G; 1, \bar{w})$. ■

³Here every edge gets its own label. This fits our definition of labeled graphs provided Λ_E can be infinite.

4.2.2 The Bollobas-Riordan polynomial

The labeled Tutte polynomial, widely known as the *Bollobas-Riordan* polynomial [14] is defined as follows:

Let $G = (V, E)$ be a connected graph, and let $c : E \rightarrow \Lambda$ be an edge labeling function, with some set of labels Λ . Let $T = (V, F)$, $F \subseteq E$ be a spanning tree of G . For every edge $e \in E \setminus F$, let $\text{cyc}(T \cup e)$ denote the edges of the unique cycle in $T \cup e$. For each edge $f \in F$, let $\text{cut}(T - f)$ denote the set of edges of G that connect components of $T - f$. Finally, let $\phi : E \rightarrow 1, 2, 3 \dots |E|$ be an auxiliary total order over E .

An edge $e \in E \setminus F$ is called *externally active* with respect to T under order ϕ if it is the smallest edge (under ϕ) in the $\text{cyc}(T \cup e)$. Otherwise, this edge is called *externally inactive*.

An edge $f \in F$ is called *internally active* with respect to T under order ϕ if it is the smallest edge (under ϕ) in the $\text{cut}(T - f)$. Otherwise, this edge is called *internally inactive*.

The *weight* of an edge e with respect to labeling c , order ϕ and spanning tree T is denoted by $w(G, c, \phi, T, e)$ and defined by

$$w(G, c, \phi, T, e) = \begin{cases} X_{c(e)} & \text{if } e \text{ is internally active,} \\ Y_{c(e)} & \text{if } e \text{ is externally active,} \\ x_{c(e)} & \text{if } e \text{ is internally inactive,} \\ y_{c(e)} & \text{if } e \text{ is externally inactive,} \end{cases}$$

The *weight* of the spanning tree T is

$$w(G, c, \phi, T) = \prod_{e \in E} w(G, c, \phi, T, e)$$

The *labeled Tutte polynomial* is defined as

$$W_0(G, c, \phi) = \sum_{T \in \mathcal{T}_G} w(G, c, \phi, T), \quad (4.2.3)$$

where \mathcal{T}_G is a set of all the spanning trees of G . For the disconnected graphs the Bollobas-Riordan polynomial is defined as a product over the connected

components of the graph, with prefactor that depends on their number:

$$W_0(G, c, \phi) = \alpha_{k(G)} \prod_{i=1}^k W_o(G_i, c, \phi)$$

Note: the above does not actually define a graph polynomial, since its final value may depend on the order ϕ . To become a well defined graph polynomial, it must take its values from the polynomial ring modulo the ideal generated by the following identities:

$$X_\lambda y_\mu - y_\lambda X_\mu = x_\lambda Y_\mu - Y_\lambda x_\mu = x_\lambda y_\mu - y_\lambda x_\mu, \quad (4.2.4)$$

holds for any labels λ and μ , or

$$X_\lambda = Y_\lambda = 0 \quad (4.2.5)$$

holds for any label λ (see [14] for the proof). For the multiplicativity, one must assume $\alpha_{k(G)} = \alpha^{k(G)}$ for some α .

Recurrence relations. The Bollobas-Riordan polynomial, when it is well-defined, satisfies the following recurrence relation:

$$W(G; c) = \begin{cases} X_\lambda W(G/e; c) & \text{if } e \text{ is a bridge,} \\ Y_\lambda W(G - e; c) & \text{if } e \text{ is a loop,} \\ x_\lambda W(G/e; c) + y_\lambda W(G - e; c) & \text{otherwise;} \end{cases}$$

with initial condition $W(E_n; c, \phi) = \alpha_n$.

The recurrence relation above leads to the conclusion:

Proposition 4.2.2 *The (multiplicative) Bollobas-Riordan polynomial is an LTG-invariant.*

Moreover, according to the results of [14], this polynomial subsumes all the other known labeled versions of the Tutte polynomial, cf. [14, 52, 27, 49], which allows us to conclude:

Proposition 4.2.3 *The Bollobas-Riordan polynomial is a dp-complete LTG-invariant.*

4.3 The most distinctive labeled matching polynomial $U_{LM}(G; \bar{x}, y)$

Like in the unlabeled case, we first approach the most distinctive labeled matching polynomial.

4.3.1 The recurrence relation

Recall the definition of LM -invariant:

$p(G) : \mathcal{LG} \rightarrow \mathcal{R}$ is an LM -invariant if

- $p(G)$ is multiplicative with respect to disjoint union: for all labeled graphs G_1, G_2 ,

$$p(G_1 \sqcup G_2) = p(G_1) \cdot p(G_2);$$

- For every edge-label $d \in \Lambda_E$ there are $\alpha_d, \beta_d \in \mathcal{R}$ such that for every labeled graph $G = \langle V, E, lab_V, lab_E \rangle$ and every edge $e \in E$ with $lab_E(e) = d$,

$$p(G) = \alpha_d p(G - e) + \beta_d p(G \dagger e), \quad (4.3.1)$$

where $G - e$ and $G \dagger e$ denote graphs obtained from G by respectively *deletion* and *extraction* (i.e. deletion together with endpoints) of the edge e .

- $p(G)$ is uniquely defined by the recurrence relation (4.3.1) together with the initial conditions

$$p(\emptyset) = 1; \quad p(\{v\}) = \nu_c, \quad (4.3.2)$$

where $\nu_c \in \mathcal{R}$ is some element of the ring \mathcal{R} that depends on the label c of the singleton $\{v\}$, but does not depend on the graph G .

The most general recursive definition of $P(G, \omega; \bar{X})$ is obtained by introducing indeterminates where possible. Any graph decomposition sequence consists of *edge removal* steps and non-empty disjoint subgraph *decomposition* steps. Let $\omega \in \Omega$ denote an auxiliary order of decomposition

steps.

$$\begin{aligned}
P(G) &= \begin{cases} y_\mu P(G - e) + z_\mu P(G \dagger e); & \text{edge removal step} \\ & \text{(provided } \text{lab}_E(e) = \mu) \\ P(G_1) \cdot P(G_2); & \text{decomposition step} \end{cases} \\
P(\{v\}) &= x_\gamma \quad \text{provided } \text{lab}_V(v) = \gamma; \\
P(\emptyset) &= 1;
\end{aligned} \tag{4.3.3}$$

Proposition 4.3.1 *The reduction of $P(G, \omega; \bar{X})$ is well-founded.*

Proof. Indeed, every step of the decomposition reduces either the number of edges or the number of vertices. Hence, after a final number of steps, only singletons and empty sets appear in the decomposition parse tree. ■

$P(G, \omega; \bar{X})$ is defined recursively for labeled graphs and set of indeterminates $\bar{X} = \{x_\gamma, y_\lambda, z_\lambda : \gamma \in \Lambda_V, \lambda \in \Lambda_E\}$ onto the ring $\mathbb{Z}[\bar{X}]$ and its result depends on the order ω of graph decomposition steps.

Theorem 4.3.2 *Let $\hat{P}(G, \omega; \bar{x})$ be obtained from $P(G, \omega; \bar{X})$ by substitution of every respective indeterminate by an unresolved polynomial. Then $\hat{P}(G, \omega_1; \bar{x}) = \hat{P}(G, \omega_2; \bar{x})$ for all graphs G and all pairs of orders ω_1 and ω_2 , if and only if*

$$y_\lambda z_\mu = z_\mu \text{ for all } \lambda, \mu \in \Lambda_E, \text{ or } P(G) = x^{|V(G)|}.$$

Proof. We are looking for a minimal set of conditions φ over the introduced unresolved polynomials, such that if φ is satisfied, then $\hat{P}(G, \omega; \bar{x})$ is independent of the order.

We apply the order-invariance restriction to the family of graphs shown at Fig. 4.1: G consists of two disjoint graphs H_1 and H_2 connected by a bridge of two edges $e_1 = (u, v)$ and $e_2 = (v, w)$. Let the edges e_1 and e_2 be labeled respectively by $\text{lab}_E(e_1) = \lambda$ and $\text{lab}_E(e_2) = \mu$, and the vertex v be labeled $\text{lab}_V(v) = \gamma$. In order to be a unique graph invariant, $P(G)$ must return the same result when the edge reduction rule is applied first on the edge e_1 and then on the edge e_2 , as well as when it is applied first on the

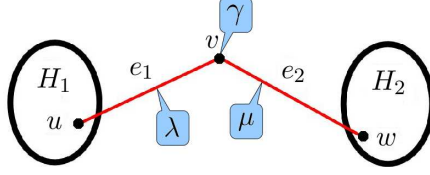


Figure 4.1: Graph G for applying order-invariance of edge elimination

edge e_2 and then on the edge e_1 .

$$\begin{aligned}
P(G, \{1 \rightarrow 2\}) &= y_\lambda \cdot P(G - e_1) + z_\lambda \cdot P(G \dagger e_1) \\
&= y_\lambda \cdot (y_\mu \cdot P(G - e_1 - e_2) + \\
&\quad z_\mu \cdot P(G \dagger e_2)) + \\
&\quad z_\lambda \cdot P(G \dagger e_1) \\
&= y_\lambda y_\mu x_\gamma P(H_1) P(H_2) + \\
&\quad y_\lambda z_\mu P(H_1) P(H_2 - w) + \\
&\quad z_\lambda P(H_1 - u) P(H_2)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
P(G, \{2 \rightarrow 1\}) &= y_\mu \cdot P(G - e_2) + z_\mu \cdot P(G \dagger e_2) \\
&= y_\mu \cdot (y_\lambda \cdot P(G - e_2 - e_1) + \\
&\quad z_\lambda \cdot P(G \dagger e_1)) + \\
&\quad z_\mu \cdot P(G \dagger e_2) \\
&= y_\mu y_\lambda x_\gamma P(H_1) P(H_2) + \\
&\quad y_\mu z_\lambda P(H_1 - u) P(H_2) + \\
&\quad z_\mu P(H_1) P(H_2 - w)
\end{aligned}$$

Applying $P(G, \{2 \rightarrow 1\}) = P(G, \{2 \rightarrow 1\})$, we obtain:

$$\begin{aligned}
&y_\lambda z_\mu P(H_1) P(H_2 - w) + z_\lambda P(H_1 - u) P(H_2) = \\
&= y_\mu z_\lambda P(H_1 - u) P(H_2) + z_\mu P(H_1) P(H_2 - w)
\end{aligned}$$

Analyzing the case $lab_E(e_1) = lab_E(e_2) = \lambda$ leads to $y_\lambda z_\lambda = z_\lambda$ for all $\lambda \in \Lambda_E$, or $P(H_1 - u)P(H_2) = P(H_1)P(H_2 - w)$ for all graphs H_1, H_2 and all vertices $u \in V(H_1)$ and $w \in V(H_2)$. In the latter case we get a trivial polynomial $P(H) = x^{|V(H)|}$. Indeed, if H_1 is a singleton, we get $1 \cdot P(H_2) = x_{lab_V(u)} \cdot P(H_2 - w)$ for any $w \in V(H_2)$, which leads to $P(H) = x^{|V(H)|}$, where $x_{lab_V(u)} = x$ for any $w \in V(H)$.

On the other hand, if $\mu \neq \lambda$, we get either $y_\lambda z_\mu = z_\mu$ for all $\lambda, \mu \in \Lambda_E$, or $P(H_1 - u)P(H_2) = 0$ for all graphs H_1 and H_2 , which requires $P(H) = 0$.

So far, we proved that the condition of the theorem is necessary. In order to prove that they are also sufficient, it is enough to prove that *any two steps of graph decomposition are exchangeable*. Then we use induction on the depth of graph deconstruction tree to prove the uniqueness of $P(G)$.

First, we observe that any step of disjoint decomposition is interchangeable with any step of edge removal. This follows by distributivity of multiplication. Therefore, we can assume that we first perform all the possible edge removal steps, and then do the disjoint decomposition of remaining singletons.

Second, we observe that an order over edges uniquely determines the order of graph decomposition. We just skip the steps of removing edges that have been already removed by the preceding steps. Therefore, it is enough to prove that any two edges can be removed in either order, and this does not affect the result. Let $G = \langle V, E, lab_V, lab_E \rangle$ be any labeled graph, and let e_1 and e_2 be any two edges, labeled respectively $lab_E(e_1) = \lambda$ and $lab_E(e_2) = \mu$. If the edges of interest do not have any common vertices, then we get:

$$\begin{aligned} P(G, \{1 \rightarrow 2\}) &= y_\lambda \cdot P(G - e_1) + z_\lambda \cdot P(G \dagger e_1) \\ &= y_\lambda \cdot (y_\mu \cdot P(G - e_1 - e_2) + z_\mu \cdot P(G - e_1 \dagger e_2)) + \\ &\quad z_\lambda \cdot (y_\mu \cdot P(G \dagger e_1 - e_2) + z_\mu \cdot P(G \dagger e_1 \dagger e_2)) = \\ &= P(G, \{2 \rightarrow 1\}) \end{aligned}$$

otherwise, if the edges of interest have a vertex or two in common, we have

$$\begin{aligned} P(G, \{1 \rightarrow 2\}) &= y_\lambda \cdot P(G - e_1) + z_\lambda \cdot P(G \dagger e_1) \\ &= y_\lambda \cdot (y_\mu \cdot P(G - e_1 - e_2) + z_\mu \cdot P(G \dagger e_2)) + \\ &\quad z_\lambda \cdot P(G \dagger e_1) \end{aligned}$$

whereas

$$\begin{aligned}
P(G, \{2 \rightarrow 1\}) &= y_\mu \cdot P(G - e_2) + z_\mu \cdot P(G \dagger e_2) \\
&= y_\mu \cdot (y_\lambda \cdot P(G - e_2 - e_1) + z_\lambda \cdot P(G \dagger e_1)) + \\
&\quad z_\mu \cdot P(G \dagger e_2)
\end{aligned}$$

which are equivalent provided $y_\lambda z_\mu = z_\mu$ for all $\lambda, \mu \in \Lambda_E$.

Hence, the conditions of the theorem are necessary and sufficient. \blacksquare

Analyzing the conditions φ , we get:

Corollary 4.3.3 *Every LM-polynomial $p : \mathcal{LG} \rightarrow \mathbb{R}[\bar{X}]$ of a graph $G = \langle V, E, \text{lab}_V, \text{lab}_E \rangle$ is either of the form*

$$p(G) = \left(\prod_{v \in V} x_{\text{lab}_V(v)} \right) \left(\prod_{e \in E} y_{\text{lab}_E(e)} \right), \quad (4.3.4)$$

or it satisfies a linear recurrence relation as follows:

$$p(G) = p(G - e) + z_{\text{lab}_E(e)} p(G \dagger e); \quad (4.3.5)$$

where for every $\gamma \in \Lambda_V$ and $\lambda \in \Lambda_E$, x_γ and y_λ are elements of $\mathbb{R}[\bar{X}]$ independent of G .

Proof. The condition $\{y_\lambda z_\mu - z_\mu : \lambda, \mu \in \Lambda_E\}$ has two solutions: $y_\lambda = 1$ for any $\lambda \in \Lambda_E$, and $z_\mu = 0$ for any $\mu \in \Lambda_E$. Under the latter solution the polynomial (4.3.4) is obtained. This also includes the exception case $p(G) = x^{|V|}$. The first solution gives a rise to the recurrence relation (4.3.5). \blacksquare

Corollary 4.3.4 *There is a map $U_{LM} : \mathcal{LG} \rightarrow \mathbb{R}[\bar{X}]$, with $\bar{X} = (x_\gamma, y_\lambda : \gamma \in \Lambda_V, \lambda \in \Lambda_E)$, which is uniquely defined by the recurrence relation and initial conditions:*

$$\begin{aligned}
U_{LM}(G_1 \sqcup G_2) &= U_{LM}(G_1) \cdot U_{LM}(G_2); \\
U_{LM}(G) &= U_{LM}(G - e) + y_\lambda U_{LM}(G \dagger e), \quad \text{provided } \text{lab}_E(e) = \lambda; \\
U_{LM}(\{v\}) &= x_\gamma, \quad \text{provided } \text{lab}_V(v) = \gamma; \\
U_{LM}(\emptyset) &= 1.
\end{aligned} \quad (4.3.6)$$

Corollary 4.3.5 *The map $U_{LM}(G)$ from Corollary 4.3.4 is universal in the class of sLM -invariants and dp -complete in the class of LM -invariants.*

Proof. The universality property follows directly from the Corollary 4.3.4, for the dp -completeness we need also prove that $U_{LM}(G)$ determines the following invariants:

- $|E_\lambda(G)|$ = number of edges labeled λ for every $\lambda \in \Lambda_E$;
- The product $p_V(G; \bar{X}) = \prod_{v \in V} x_{lab_V(v)}$.

The latter can be obtained as a substitution instance of $U_{LM}(G)$ as follows:

$$p_V(G; \bar{X}) = U_{LM}(G; \bar{X})|_{y_\lambda=0 \text{ for each } \lambda \in \Lambda_E}$$

For the number of edges labeled λ we need to determine the coefficient of y_λ in U_{LM} . The paths in the graph decomposition tree that contribute to this summand contain exactly one step of extraction (of some edge labeled λ). Number of such paths is exactly E_λ . In order to determine this coefficient, we will use partial derivatives:

$$|E_\lambda(G)| = \left. \frac{\partial U_{LM}(G)}{\partial y_\lambda} \right|_{x_\gamma=1, y_\mu=0 \text{ for each } \gamma \in \Lambda_V, \mu \in \Lambda_E}$$

■

4.3.2 The subset expansion form of $U_{LM}(G; \bar{x}, \bar{y})$

We proved so far that there is a unique map $U_{LM}(G; x, y)$ which is defined recursively by initial conditions and linear recurrence relation 4.3.6 with respect to edge deletion and edge extraction operations.

Additionally, by multiple application of the edge reduction rule, we can obtain a vertex reduction rule. Let $N(v) = \{u_1, u_2, \dots, u_d\}$ be the neigh-

borhood of v , and $e_i = \{u_i, v\}$ be the edges incident to v . Then,

$$\begin{aligned}
U_{LM}(G) &= U_{LM}(G - e_1) + y_{lab_E(e_1)} U_{LM}(G \dagger e_1) = \\
&= U_{LM}(G - e_1 - e_2) + y_{lab_E(e_2)} U_{LM}(G \dagger e_2) + \\
&\quad + y_{lab_E(e_1)} U_{LM}(G \dagger e_1) = \dots \tag{4.3.7} \\
&= U_{LM}(G - v \sqcup \{v\}) + \sum_{i=1}^d y_{lab_E(e_i)} U_{LM}(G \dagger e_i) = \\
&= x_{lab_V(v)} \cdot U_{LM}(G - v) + \sum_{i=1}^d y_{lab_E(e_i)} U_{LM}(G \dagger e_i) \tag{4.3.8}
\end{aligned}$$

Theorem 4.3.6 *The unique map $U_{LM} : \mathcal{G} \rightarrow \mathbb{R}[\bar{X}]$, with $\bar{X} = (x_\gamma, y_\lambda : \gamma \in \Lambda_V, \lambda \in \Lambda_E)$, satisfying the recursive definition 4.3.6 has an explicit subset expansion definition as follows:*

$$U_{LM}(G; \bar{X}) = \sum_{\substack{M \subseteq E, \\ M \text{ is a matching}}} \left[\left(\prod_{e \in M} y_{lab_E(e)} \right) \left(\prod_{v \in V \setminus V(M)} x_{lab_V(v)} \right) \right] \tag{4.3.9}$$

Proof. The proof is by induction on the size of the graph G .

Base: for graphs of size $n \leq 2$ we have:

- $U_{LM}(\emptyset) = 1$
- $U_{LM}(E_1) = x_{lab_V(v)}$
- $U_{LM}(E_2) = x_{lab_V(u)} x_{lab_V(v)}$ (vertices u and v)
- $U_{LM}(P_2) = U_{LM}(G - e) + y_{lab_E(e)} \cdot U_{LM}(\emptyset) = x_{lab_V(u)} x_{lab_V(v)} + y_{lab_E(e)}$ (vertices u and v and edge e)

In all those cases the equation 4.3.9 holds.

Closure: we assume that the equation 4.3.9 holds for all the graphs with at most n vertices, and prove that it holds also for the graphs with $n + 1$ vertices.

Let G be a graph with $n + 1$ vertices and let v be any its vertex. Then we apply the vertex reduction rule 4.3.7 and obtain:

$$U_{LM}(G) = x_{lab_V(v)} \cdot U_{LM}(G - v) + \sum_{i=1}^d y_{lab_E(e_i)} \cdot U_{LM}(G \uparrow e_i)$$

By the induction assumption we have:

$$P(G) = x_{lab_V(v)} \cdot \sum_{\substack{M \subseteq E(G-v), \\ M \text{ is a matching}}} \left[\left(\prod_{e \in M} y_{lab_E(e)} \right) \left(\prod_{v \in V \setminus V(M)} x_{lab_V(v)} \right) \right] + \\ + \sum_{i=1}^d y_{lab_E(e_i)} \cdot \sum_{\substack{M \subseteq E(G \uparrow e_i), \\ M \text{ is a matching}}} \left[\left(\prod_{e \in M} y_{lab_E(e)} \right) \left(\prod_{v \in V \setminus (V(M) \cup \{u_i, v\})} x_{lab_V(v)} \right) \right]$$

The first part of the equation describes all the matchings which do not include the vertex v , hence, $x_{lab_V(v)}$ can be inserted into its second product. The second part of the equation describes all the matchings which include the vertex v . Those are the matchings, which include exactly one of the edges e_1, \dots, e_d , so the corresponding variable $y_{lab_E(e_i)}$ can be inserted into the first product. By summation of the two parts, we get exactly the equation 4.3.9. ■

4.4 The most distinctive labeled edge elimination polynomial $\xi_{lab}(G; x, y, z, \bar{t})$

In this section we extend the most distinctive EE graph polynomial described in Section 3.3 to edge-labeled graphs.

4.4.1 The recurrence relation

Recall the definition of LEE -invariant:

$p(G) : \mathcal{LG}_E \rightarrow \mathcal{R}$ is an LEE -invariant if

- $p(G)$ is multiplicative with respect to disjoint union of graphs: for all labeled graphs G_1, G_2 ,

$$p(G_1 \sqcup G_2) = p(G_1) \cdot p(G_2);$$

- For every edge-label $d \in \Lambda_E$ there are $\alpha_d, \beta_d, \gamma_d \in \mathcal{R}$ such that for every edge-labeled graph $G = \langle V, E, lab_E \rangle$ and every edge $e \in E$,

$$p(G) = \alpha_d p(G - e) + \beta_d p(G/e) + \gamma_d p(G \dagger e), \quad (4.4.1)$$

where $G - e$, G/e and $G \dagger e$ denote graphs obtained from G by respectively *deletion*, *contraction* and *extraction* (i.e. deletion together with endpoints) of the edge e .

- $p(G)$ is uniquely defined by the recurrence relation (4.4.1) together with the initial conditions

$$p(\emptyset) = 1; \quad p(E_1) = \nu, \quad (4.4.2)$$

where $\nu \in \mathcal{R}$ is some element of the ring \mathcal{R} that does not depend on the graph G .

The most general recursive definition of $P(G, \omega; \bar{X})$ is obtained by introducing indeterminates where possible. Any graph deconstruction sequence consists of *edge removal* steps and non-empty disjoint subgraph *decomposition* steps. Let $\omega \in \Omega$ denote an auxiliary order of deconstruction steps.

$$\begin{aligned} P(G) &= \begin{cases} s_\mu P(G - e) + y_\mu P(G/e) + z_\mu P(G \dagger e); & \text{(edge removal step)} \\ & \text{(provided } lab_E(e) = \mu) \\ P(G_1) \cdot P(G_2); & \text{(decomposition step)} \end{cases} \\ P(E_1) &= x; \\ P(\emptyset) &= 1; \end{aligned} \quad (4.4.3)$$

Proposition 4.4.1 *The reduction of $P(G, \omega; \bar{X})$ is well-founded.*

Proof. Indeed, every step of the deconstruction reduces either the number of edges or the number of vertices. Hence, after a final number of steps, only singletons and empty sets appear in the deconstruction parse tree. ■

Currently, $P(G, \omega; \bar{X})$ is defined recursively for edge-labeled graphs and set of indeterminates $\bar{X} = \{x, s_\mu, y_\mu, z_\mu : \mu \in \Lambda_E\}$ onto the ring $\mathbb{R}_{\Lambda_E} = \mathbb{R}[\bar{X}]$ and its result depends on the order ω of graph deconstruction steps.

Theorem 4.4.2 *Let $\hat{P}(G, \omega; \bar{x})$ be obtained from $P(G, \omega; \bar{X})$ by substitution of every respective indeterminate by an unresolved polynomial.*

Then $\hat{P}(G, \omega_1; \bar{x}) = \hat{P}(G, \omega_2; \bar{x})$ for all graphs G and all pairs of orders ω_1 and ω_2 , if and only if

$$s_\lambda z_\mu = z_\mu \quad \text{and} \quad y_\lambda z_\mu = y_\mu z_\lambda$$

for all $\lambda, \mu \in \Lambda_E$, or if

$$P(G) = P_0(G) = x^{|V(G)|}$$

.

Proof. We are looking for a minimal set of conditions φ over the introduced unresolved polynomials, such that if φ is satisfied, then $\hat{P}(G, \omega; \bar{x})$ is independent of the order.

We use the same family of graphs as for LM-invariants (Fig. 4.1): G consists of two disjoint graphs H_1 and H_2 connected by a bridge of two edges $e_1 = (u, v)$ and $e_2 = (v, w)$, labeled respectively by $lab_E(e_1) = \lambda$ and $lab_E(e_2) = \mu$ (unlike the LM case, vertices are not labeled). In order to be a unique graph invariant, $P(G)$ must give the same result when the edge reduction rule is applied first on the edge e_1 and then on the edge e_2 , as well as when it is applied first on the edge e_2 and then on the edge e_1 .

$$\begin{aligned} P(G, \{1 \rightarrow 2\}) &= s_\lambda \cdot P(G - e_1) + y_\lambda \cdot P(G/e_1) + z_\lambda \cdot P(G \dagger e_1) \\ &= s_\lambda \cdot (s_\mu \cdot P(G - e_1 - e_2) + \\ &\quad y_\mu \cdot P(G - e_1/e_2) + \\ &\quad z_\mu \cdot P(G - e_1 \dagger e_2)) + \\ &\quad y_\lambda \cdot (s_\mu \cdot P(G/e_1 - e_2) + \\ &\quad y_\mu \cdot P(G/e_1/e_2) + \\ &\quad z_\mu \cdot P(G/e_1 \dagger e_2)) + \\ &\quad z_\lambda \cdot P(G \dagger e_1) \end{aligned}$$

$$\begin{aligned}
&= s_\lambda \cdot (s_\mu \cdot x \cdot P(H_1)P(H_2) + \\
&\quad y_\mu \cdot P(H_1)P(H_2) + \\
&\quad z_\mu \cdot P(H_1)P(H_2 - w)) + \\
& y_\lambda \cdot (s_\mu \cdot P(H_1)P(H_2) + \\
&\quad y_\mu \cdot P(H_1 \sqcup_1 H_2) + \\
&\quad z_\mu \cdot P(H_1 - u)P(H_2 - w)) + \\
& z_\lambda \cdot P(H_1 - u)P(H_2)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
P(G, \{1 \rightarrow 2\}) &= s_\mu \cdot P(G - e_2) + y_\mu \cdot P(G/e_2) + z_\mu \cdot P(G \dagger e_2) \\
&= s_\mu \cdot (s_\lambda \cdot P(G - e_2 - e_1) + \\
&\quad y_\lambda \cdot P(G - e_2/e_1) + \\
&\quad z_\lambda \cdot P(G - e_2 \dagger e_1)) + \\
& y_\mu \cdot (s_\lambda \cdot P(G/e_2 - e_1) + \\
&\quad y_\lambda \cdot P(G/e_2/e_1) + \\
&\quad z_\lambda \cdot P(G/e_2 \dagger e_1)) + \\
& z_\mu \cdot P(G \dagger e_2)
\end{aligned}$$

$$\begin{aligned}
&= s_\mu \cdot (s_\lambda \cdot x \cdot P(H_1)P(H_2) + \\
&\quad y_\lambda \cdot P(H_1)P(H_2) + \\
&\quad z_\lambda \cdot P(H_1 - u)P(H_2)) + \\
& y_\mu \cdot (s_\lambda \cdot P(H_1)P(H_2) + \\
&\quad y_\lambda \cdot P(H_1 \sqcup_1 H_2) + \\
&\quad z_\lambda \cdot P(H_1 - u)P(H_2 - w)) + \\
& z_\mu \cdot P(H_1)P(H_2 - w)
\end{aligned}$$

Applying $P(G, \{2 \rightarrow 1\}) = P(G, \{2 \rightarrow 1\})$, we obtain:

$$\begin{aligned}
&P(H_1 - u)P(H_2 - w) \cdot [y_\lambda z_\mu - y_\mu z_\lambda] + \\
&P(H_1)P(H_2 - w) \cdot [s_\lambda z_\mu - z_\mu] + \\
&P(H_1 - u)P(H_2) \cdot [z_\lambda - s_\mu z_\lambda] = 0
\end{aligned} \tag{4.4.4}$$

Analyzing the case $lab_E(e_1) = lab_E(e_2) = \lambda$ leads to

$$w_\lambda z_\lambda = z_\lambda \text{ for all } \lambda \in \Lambda_E, \quad (4.4.5)$$

despite $P(H_1 - u)P(H_2) = P(H_1)P(H_2 - w)$ for all graphs H_1, H_2 and all vertices $u \in V(H_1)$ and $w \in V(H_2)$. In the latter case we get a trivial polynomial $P_0(H) = x^{|V(H)|}$, by the same argument as for the universal matching polynomial.

Assume there are two labels λ and μ such that $s_\lambda z_\mu \neq z_\mu$, then by (4.4.5) $s_\mu = 1$ and $z_\lambda = 0$, and by (4.4.4) we get

$$(1 - s_\lambda)P(H) = y_\lambda P(H - u)$$

for all labeled graphs H and all vertices u , and therefore we get a trivial polynomial $P_0(H; \frac{y_\lambda}{1-s_\lambda})$. Hence, for $P(G) \neq P_0(G)$, $s_\lambda z_\mu = z_\mu$ for all $\lambda, \mu \in \Lambda_E$ is a necessary condition. Then, from 4.4.4 we get that also $y_\lambda z_\mu = y_\mu z_\lambda$ for all $\lambda, \mu \in \Lambda_E$.

So far, we proved that the condition the theorem is necessary. In order to prove that they are also sufficient, it is enough to prove that *any two steps of graph deconstruction are exchangeable*. Then we use induction on the depth of graph deconstruction tree to prove uniqueness of $P(G)$ if φ is satisfied.

The proof of the sufficiency is identical to the unlabeled case (Theorem 3.3.4). ■

Analyzing the conditions φ , we get:

Corollary 4.4.3 *Every LEE-polynomial $p : \mathcal{LG}_E \rightarrow \mathbb{R}[\bar{X}]$ of a graph $G = \langle V, E, lab_E \rangle$ satisfies one of the following linear recurrence relations:*

$$p(G) = s_{lab_E(e)}p(G - e) + y_{lab_E(e)}p(G/e); \quad (4.4.6)$$

$$p(G) = p(G - e) + y \cdot t_{lab_E(e)}p(G/e) + z \cdot t_{lab_E(e)}p(G \dagger e); \quad (4.4.7)$$

where, for every $\lambda \in \Lambda_E$, in the former case, s_λ and y_λ are elements of $\mathbb{R}[\bar{X}]$ independent of G , and in the latter case, t_λ is an element of $\mathbb{R}[\bar{X}]$ independent of G , and $y, z \in \mathbb{R}[\bar{X}]$ do not depend neither on G nor on λ .

Proof. The conditions φ obtained in Theorem 4.4.2, have two cases:

$$(i) \quad z_\lambda = 0 \text{ for any } \lambda \in \Lambda_E \quad (4.4.8)$$

$$(ii) \quad s_\lambda = 1, \quad y_\lambda = y \cdot t_\lambda, \text{ and } z_\lambda = z \cdot t_\lambda \text{ for any } \lambda \in \Lambda_E, \\ \text{where } y, z \in \mathbb{R}[\bar{X}] \text{ do not depend on the label } \lambda. \quad (4.4.9)$$

In the first case, we obtain the recurrence relation (4.4.6), which we called previously “the universal LC -invariant $U_{LC}(G; q, \bar{s}, \bar{y})$ ” (4.2.2); it also includes the exceptional case $P_0(G) = x^{|V|} = U_{LC}(G; x, \bar{1}, \bar{0})$. The second case gives a rise to the recurrence relation (4.4.7). ■

From Theorem 4.4.2 we conclude:

Corollary 4.4.4 *There is a map $\xi_{lab} : \mathcal{LG}_E \rightarrow \mathbb{R}[x, y, z, t]$, which is uniquely defined by the recurrence relation and initial conditions:*

$$\begin{aligned} \xi_{lab}(G_1 \sqcup G_2) &= \xi_{lab}(G_1) \cdot \xi_{lab}(G_2); \\ \xi_{lab}(G) &= \xi_{lab}(G - e) + y \cdot t_\lambda \xi_{lab}(G/e) + z \cdot t_\lambda \xi_{lab}(G \dagger e), \\ &\quad \text{provided } lab_E(e) = \lambda; \\ \xi_{lab}(E_1) &= x; \\ \xi_{lab}(\emptyset) &= 1. \end{aligned} \quad (4.4.10)$$

Corollary 4.4.5 *The map $\xi_{lab}(G)$ from Corollary 4.4.4 is universal in the class of $sLEE$ -invariants and dp -complete in the class of $L EE$ -invariants.*

Proof. The universality property follows directly from the Corollary 4.3.4, for the dp -completeness we observe that

- Non-special labeled EE invariants are $L EE \setminus sL EE = LC \setminus sLC$, and
- The Sokal polynomial $Z(G; q, \bar{v})$, which is dp -complete for the class of LC invariants, is determined by ξ_{lab} :

$$Z(G; q, \bar{v}) = \xi_{lab}(G; q, 1, 0, \bar{v}).$$

The latter fact follows by induction on the depth of graph decomposition tree, and the first one follows directly from the Corollary 4.4.3. ■

4.4.2 Subset expansion form of $\xi_{lab}(G; x, y, z, \bar{t})$

We are looking for an explicit subset expansion form $\xi_{lab}(G; x, y, z, \bar{v})$.

Theorem 4.4.6 *The following expression defines the same graph polynomial as the recurrence relation (4.4.10):*

$$\xi_{lab}(G; x, y, z, \bar{t}) = \sum_{(A \sqcup B) \subseteq E} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} \cdot \left(\prod_{e \in A \sqcup B} t_{lab_E(e)} \right), \quad (4.4.11)$$

where by abuse of notation we use $(A \sqcup B) \subseteq E$ for summation over subsets $A, B \subseteq E$, such that the subsets of vertices $V(A)$ and $V(B)$, covered by respective subset of edges, are disjoint: $V(A) \cap V(B) = \emptyset$; $k(A)$ denotes the number of spanning connected components in $\langle V, A, \Lambda_E \rangle$, and $k_{cov}(B)$ denotes the number of covered connected components, i.e. the connected components of the graph $\langle V(B), B, \Lambda_E \rangle$.

Proof. The proof is similar to the unlabeled version. We need to show that

- The expression (4.4.11) satisfies the initial conditions of (4.4.10);
- The expression (4.4.11) is multiplicative;
- The expression (4.4.11) satisfies the edge elimination rule of (4.4.10).

Then by induction on the number of edges in G the theorem holds. The first statement follows by inspection: indeed, a null graph $\emptyset = \langle \emptyset, \emptyset \rangle$ has no edges and no components, so its only summand is 1, whereas a singleton has no edges and just one component, so its contribution is x .

The second statement, multiplicativity, can be easily checked too: Indeed, the summation over subsets of edges of $G = G_1 \sqcup G_2$ can be regarded as a summation over the subsets of G_1 , multiplied by an independent summation over the subsets of G_2 .

Therefore, we just need to prove the third statement:

Let $G = \langle V, E, \Lambda_E \rangle$ be the labeled (multi)graph of interest. Let $N(G)$ be

defined as in (4.4.11):

$$N(G; x, y, z, \bar{t}) = \sum_{(A \sqcup B) \subseteq E} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} \cdot \left(\prod_{e \in A \sqcup B} t_{lab_E(e)} \right),$$

Let e be the edge we have chosen to reduce. Any particular choice of A and B can be regarded as a vertex-disjoint edge coloring in 2 colors A and B, when part of the edges remains uncolored. We separate all the possible colorings to three disjoint cases:

- Case 1: e is uncolored;
- Case 2: e is colored by B , and it is the only edge of a colored connected component;
- Case 3: All the rest. That means, e is colored by A , or e is colored by B but it is not the only edge of a colored connected component.

In case 1, we just sum over colorings of $G - e$: $N_1(G) = N(G - e)$;

In case 2, the edge e is a connected component of $\langle V(B), B, \Lambda_E \rangle$. Therefore, if we analyze now $N(G \uparrow e)$, we get $N_2(G) = z \cdot t_{lab_E(e)} N(G \uparrow e)$;

Finally, in case 3, if we analyze $N(G/e)$, we get $N_3(G) = y \cdot t_{lab_E(e)} N(G/e)$.

The cases above are disjoint and cover all the possibilities, and therefore $N(G) = N_1(G) + N_2(G) + N_3(G)$. Hence, we have:

$$N(G; x, y, z, \bar{t}) = \xi(G; x, y, z, \bar{t}),$$

which completes the proof. ■

4.5 The most distinctive labeled vertex elimination polynomial $U_{LVE}(G; y, z, \bar{t})$

In this section we extend the most distinctive VE graph polynomial described in Section 3.4 to vertex-labeled graphs.

4.5.1 The recurrence relation

Recall the definition of *LVE*-invariant:

$p(G) : \mathcal{LG}_V \rightarrow \mathcal{R}$ is an *LVE*-invariant if

- $p(G)$ is multiplicative with respect to disjoint union of graphs: for all labeled graphs G_1, G_2 ,

$$p(G_1 \sqcup G_2) = p(G_1) \cdot p(G_2);$$

- For every vertex-label $d \in \Lambda_V$ there are $\alpha_d, \beta_d, \gamma_d \in \mathcal{R}$ such that for every vertex-labeled graph $G = \langle V, E, lab_V \rangle$ and every vertex $v \in V$,

$$p(G) = \alpha_d p(G - v) + \beta_d p(G/v) + \gamma_d p(G \dagger v), \quad (4.5.1)$$

where $G - v$, G/v and $G \dagger v$ denote graphs obtained from G by respectively *deletion*, *contraction* and *extraction* of the vertex v (defined in Section 1.1.2).

- $p(G)$ is uniquely defined by the recurrence relation (4.5.1) together with the initial conditions

$$p(\emptyset) = 1; \quad p(\{v\}) = \nu_d, \text{ provided } lab_V(v) = d \quad (4.5.2)$$

where $\nu_d \in \mathcal{R}$ is some element of the ring \mathcal{R} that does not depend on the graph G .

The most general recursive definition of $P(G, \omega; \bar{X})$ is obtained by introducing indeterminates where possible. Any graph deconstruction sequence consists of *vertex removal* steps and non-empty disjoint subgraph *decomposition* steps. Let $\omega \in \Omega$ denote an auxiliary order of deconstruction steps.

$$\begin{aligned} P(G) &= \begin{cases} w_\mu P(G - v) + y_\mu P(G/v) + z_\mu P(G \dagger v); & \text{(vertex removal step)} \\ & \text{(provided } lab_V(v) = \mu) \\ P(G_1) \cdot P(G_2); & \text{(decomposition step)} \end{cases} \\ P(E_1) &= x_\mu; \\ P(\emptyset) &= 1; \end{aligned} \quad (4.5.3)$$

Proposition 4.5.1 *The reduction of $P(G, \omega; \bar{X})$ is well-founded.*

Proof. Indeed, every step of the deconstruction reduces the number of vertices. Hence, after a final number of steps, only singletons and empty sets appear in the deconstruction parse tree. ■

Currently, $P(G, \omega; \bar{X})$ is defined recursively on vertex-labeled graphs and set of indeterminates $\bar{X} = \{x_\mu, s_\mu, y_\mu, z_\mu : \mu \in \Lambda_V\}$ onto the ring $\mathbb{R}_{\Lambda_V} = \mathbb{R}[\bar{X}]$ and its result depends on the order ω of graph deconstruction steps.

Theorem 4.5.2 *Let $\hat{P}(G, \omega; \bar{x})$ be obtained from $P(G, \omega; \bar{X})$ by substitution of every respective indeterminate by an unresolved polynomial.*

Then $\hat{P}(G, \omega_1; \bar{x}) = \hat{P}(G, \omega_2; \bar{x})$ for all graphs G and all pairs of orders ω_1 and ω_2 , if and only if

$$x_\lambda = s_\lambda + y_\lambda + z_\lambda \quad \text{and} \quad s_\lambda z_\mu = z_\mu \quad \text{and} \quad y_\lambda z_\mu = y_\mu z_\lambda$$

for all $\lambda, \mu \in \Lambda_V$, or if

$$P(G) = P_0(G) = x^{|V|}$$

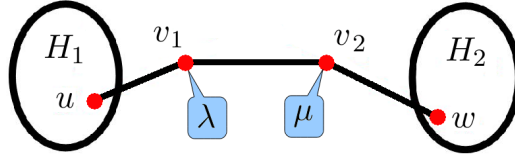


Figure 4.2: Graph G for applying order-invariance of vertex elimination

Proof. We are looking for a minimal set of conditions φ over the introduced unresolved polynomials, such that if φ is satisfied, then $\hat{P}(G, \omega; \bar{x})$ is independent of the order.

First, from $E_1 - v = E_1 \uparrow v = E_1 / v = \emptyset$ we obtain

$$x_\lambda = (s_\lambda + y_\lambda + z_\lambda) \quad \text{for every } \lambda \in \Lambda_V.$$

Second, we use the family of graphs shown on Fig. 4.2: G consists of two disjoint graphs H_1 and H_2 connected by a bridge of three edges $e_1 = (u, v_1)$, $e_2 = (v_1, v_2)$ and $e_3 = (v_2, w)$. The vertices v_1 and v_2 labeled respectively by $lab_V(v_1) = \lambda$ and $lab_V(v_2) = \mu$. In order to be a unique graph invariant, $P(G)$ must give the same result when the vertex reduction rule is applied first on the vertex v_1 and then on the vertex v_2 , as well as when it is applied first on the vertex v_2 and then on the vertex v_1 . We look on the case $lab_V(v_1) = lab_V(v_2) = lab_V(u) = \lambda$, $H_1 = \{u\}$ and $H_2 = \emptyset$. By applying the same argument as in the proof of Theorem 3.4.5, we get:

$$z_\lambda = 0 \text{ or } s_\lambda = 1 \text{ or } s_\lambda + y_\lambda + z_\lambda = 1 \quad (4.5.4)$$

Back to the general case:

$$\begin{aligned}
P(G, \{1 \rightarrow 2\}) &= s_\lambda \cdot P(G-v_1) + y_\lambda \cdot P(G/v_1) + z_\lambda \cdot P(G \dagger v_1) \\
&= s_\lambda \cdot (s_\mu \cdot P(G-v_1-v_2) + \\
&\quad y_\mu \cdot P(G-v_1/v_2) + \\
&\quad z_\mu \cdot P(G-v_1 \dagger v_2)) + \\
&\quad y_\lambda \cdot (s_\mu \cdot P(G/v_1-v_2) + \\
&\quad y_\mu \cdot P(G/v_1/v_2) + \\
&\quad z_\mu \cdot P(G/v_1 \dagger v_2)) + \\
&\quad z_\lambda \cdot P(G \dagger v_1) \\
&= s_\lambda \cdot (s_\mu \cdot P(H_1)P(H_2) + \\
&\quad y_\mu \cdot P(H_1)P(H_2) + \\
&\quad z_\mu \cdot P(H_1)P(H_2-w)) + \\
&\quad y_\lambda \cdot (s_\mu \cdot P(H_1)P(H_2) + \\
&\quad y_\mu \cdot P(H_1 \sqcup_{\{uw\}} H_2) + \\
&\quad z_\mu \cdot P(H_1-u)P(H_2-w)) + \\
&\quad z_\lambda \cdot P(H_1)P(H_2-w) \quad (4.5.5)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
P(G, \{1 \rightarrow 2\}) &= s_\mu \cdot P(G-v_2) + y_\mu \cdot P(G/v_2) + z_\mu \cdot P(G \uparrow v_2) \\
&= s_\mu \cdot (s_\lambda \cdot P(G - v_2 - v_1) + \\
&\quad y_\lambda \cdot P(G - v_2/v_1) + \\
&\quad z_\lambda \cdot P(G - v_2 \uparrow v_1)) + \\
&\quad y_\mu \cdot (s_\lambda \cdot P(G/v_2 - v_1) + \\
&\quad y_\lambda \cdot P(G/v_2/v_1) + \\
&\quad z_\lambda \cdot P(G/v_2 \uparrow v_1)) + \\
&\quad z_\mu \cdot P(G \uparrow v_2) \\
&= s_\mu \cdot (s_\lambda \cdot P(H_1)P(H_2) + \\
&\quad y_\lambda \cdot P(H_1)P(H_2) + \\
&\quad z_\lambda \cdot P(H_1 - u)P(H_2)) + \\
&\quad y_\mu \cdot (s_\lambda \cdot P(H_1)P(H_2) + \\
&\quad y_\lambda \cdot P(H_1 \sqcup_{\{uw\}} H_2) + \\
&\quad z_\lambda \cdot P(H_1 - u)P(H_2 - w)) + \\
&\quad z_\mu \cdot P(H_1 - u)P(H_2)
\end{aligned} \tag{4.5.6}$$

Applying $P(G, \{2 \rightarrow 1\}) = P(G, \{2 \rightarrow 1\})$, we obtain:

$$\begin{aligned}
&P(H_1 - u)P(H_2 - w) \cdot [y_\lambda z_\mu - y_\mu z_\lambda] + \\
&P(H_1)P(H_2 - w) \cdot [s_\lambda z_\mu - z_\mu] + \\
&P(H_1 - u)P(H_2) \cdot [z_\lambda - s_\mu z_\lambda] = 0
\end{aligned} \tag{4.5.7}$$

If there is a label λ such that $s_\lambda z_\lambda \neq z_\lambda$, then, using $lab_V(v_1) = lab_V(v_2) = \lambda$, we have

$$P(H_1)P(H_2 - w) = P(H_1 - u)P(H_2)$$

for all graphs H_1 and H_2 , and all vertices $u \in V(H_1)$ and $w \in V(H_2)$. Let H_2 be a singleton labeled λ . Then, by (4.5.4), we have $P(H_2) = P(H_2 - w) = 1$, and therefore $P(H_1) = 1$ for any graph H_1 .

Now, assume that there are labels $\lambda, \mu \in \Lambda_V$, such that $s_\lambda z_\mu \neq z_\mu$. Then, if $P(G)$ is not constant 0 or 1, then by the previous analysis $s_\mu = 1$ and

$z_\lambda = 0$. By equation (4.5.7) we have

$$y_\lambda P(H_1 - u) = (1 - s_\lambda)P(H_1),$$

and therefore,

$$P(H_1) = \left(\frac{y_\lambda}{1 - s_\lambda} \right)^{|V(H_1)|} \quad (4.5.8)$$

The last possible case is that for all $\lambda, \mu \in \Lambda_V$, $s_\lambda z_\mu = z_\mu$. In this case, by (4.5.7) we have:

$$y_\lambda z_\mu = y_\mu z_\lambda.$$

Hence, with exception of a trivial polynomial $P_0(G) = x^{|V|}$, the unspecified polynomials must satisfy

$$s_\lambda z_\mu = z_\mu \text{ and } y_\lambda z_\mu = y_\mu z_\lambda \text{ for all } \lambda, \mu \in \Lambda_V$$

So far, we proved that the condition the theorem is necessary. In order to prove that they are also sufficient, it is enough to prove that *any two steps of graph deconstruction are exchangeable*. Then we use induction on the depth of graph deconstruction tree to prove uniqueness of $P(G)$.

Any disjoint subgraph decomposition step is exchangeable with any other step of graph deconstruction by commutativity of multiplication. Therefore, we can assume without loss of generality that they are applied only on singletons. In such case a linear order over the edges uniquely determines the decomposition tree. Hence, we only have to show that any two consequent vertex removal steps are exchangeable. Let v_1 and v_2 be two vertices of interest, labeled respectively $lab_V(v_1) = \lambda$ and $lab_V(v_2) = \mu$. We observe that if v_1 and v_2 are not adjacent, then any two vertex elimination operation applied consequently on v_1 and v_2 in any order produces the same graph, so the result of recursive computation of $P_I(G)$ is preserved. The case of adjacent v_1 and v_2 is a bit more complicated, because the contraction and extraction operations applied to one vertex affect another.

In order to show that $P(G)$ is still preserved, we'll observe:

- Two vertex deletion operations are commutative;
- Two vertex contraction operations are commutative;
- Vertex deletion and vertex contraction operations are commutative;

- Deletion of v_1 and then extraction of v_2 is equivalent to extraction of v_2 ;
- Contraction of v_1 and then extraction of v_2 is equivalent to contraction of v_2 and then extraction of v_1 .

If we analyze result of the two possible sequences of vertex elimination, we get exactly the equations (4.5.5) and (4.5.6). Combining those equations with the conditions φ , we get

$$\begin{aligned}
& P|_I(G, \{1 \rightarrow 2\}) - P|_I(G, \{2 \rightarrow 1\}) = \\
& w_\lambda w_\mu P(G - v_1 - v_2) + w_\lambda y_\mu P(G - v_1/v_2) + w_\lambda z_\mu P(G - v_1 \dagger v_2) + \\
& y_\lambda w_\mu P(G/v_1 - v_2) + y_\lambda y_\mu P(G/v_1/v_2) + y_\lambda z_\mu P(G/v_1 \dagger v_2) + \\
& z_\lambda P(G \dagger v_1) - \\
& w_\mu w_\lambda P(G - v_2 - v_1) - w_\mu y_\lambda P(G - v_2/v_1) - w_\mu z_\lambda P(G - v_2 \dagger v_1) - \\
& y_\mu w_\lambda P(G/v_2 - v_1) - y_\mu y_\lambda P(G/v_2/v_1) - y_\mu z_\lambda P(G/v_2 \dagger v_1) - \\
& z_\mu P(G \dagger v_2) = \\
& (w_\lambda z_\mu - z_\mu)P(G \dagger v_2) + (z_\lambda - w_\mu z_\lambda)P(G \dagger v_1) + \\
& (y_\lambda z_\mu - y_\mu z_\lambda)P(G/v_2 \dagger v_1) = 0
\end{aligned}$$

■

Analyzing the conditions φ obtained in the theorem above, we get:

Corollary 4.5.3 *Every LVE-polynomial $p : \mathcal{LG}_V \rightarrow \mathbb{R}[\bar{X}]$ of a graph $G = \langle V, E, lab_V \rangle$ is either of the form*

$$p(G) = \prod_{v \in V} (s_{lab_V(v)} + y_{lab_V(v)}) = \prod_{v \in V} x_{lab_V(v)}, \quad (4.5.9)$$

or it satisfies the following linear recurrence relation:

$$p(G) = p(G - v) + y \cdot t_{lab_V(v)} p(G/v) + z \cdot t_{lab_V(v)} p(G \dagger v) \quad (4.5.10)$$

where, for every $\lambda \in \Lambda_V$, in the former case, s_λ and y_λ are elements of $\mathbb{R}[\bar{X}]$ independent of G , and in the latter case, t_λ is an element of $\mathbb{R}[\bar{X}]$ independent of G , and $y, z \in \mathbb{R}[\bar{X}]$ do not depend on λ either.

Proof. The conditions φ obtained in the Theorem 4.5.2 include two cases:

$$(i) \quad z_\lambda = 0 \text{ for any } \lambda \in \Lambda_V \quad (4.5.11)$$

$$(ii) \quad s_\lambda = 1, \quad y_\lambda = y \cdot t_\lambda, \text{ and } z_\lambda = z \cdot t_\lambda \text{ for any } \lambda \in \Lambda_V, \\ \text{where } y, z \in \mathbb{R}[\bar{X}] \text{ do not depend on the label } \lambda. \quad (4.5.12)$$

In the first case, we obtain the exceptional case (4.5.9). The second case gives a rise to the recurrence relation (4.5.10). \blacksquare

From Theorem 4.5.2 we conclude:

Corollary 4.5.4 *There is a map $U_{LVE} : \mathcal{LG}_V \rightarrow \mathbb{R}[y, z, \bar{t}]$, which is uniquely defined by the recurrence relation and initial conditions:*

$$\begin{aligned} U_{LVE}(G_1 \sqcup G_2) &= U_{LVE}(G_1) \cdot U_{LVE}(G_2); \\ U_{LVE}(G) &= U_{LVE}(G - v) + y \cdot t_\lambda U_{LVE}(G/v) + z \cdot t_\lambda U_{LVE}(G \dagger v), \\ &\quad \text{provided } \text{lab}_V(v) = \lambda; \\ U_{LVE}(\{v\}) &= 1 + y \cdot t_\lambda + z \cdot t_\lambda; \\ &\quad \text{provided } \text{lab}_V(v) = \lambda; \\ U_{LVE}(\emptyset) &= 1. \end{aligned} \quad (4.5.13)$$

Corollary 4.5.5 *The map $U_{LVE}(G)$ from Corollary 4.5.4 is universal in the class of LVE-invariants.*

Proof. We observe that the exceptional case (4.5.9) is a substitution instance of $U_{LVE}(G)$:

$$U_{LVE}(G; 1, 0, \bar{w} + \bar{y} - \bar{1}) = \prod_{v \in V} (w_{\text{lab}_V(v)} + y_{\text{lab}_V(v)}).$$

This follows by induction on the depth of graph decomposition tree. The rest follows directly from the Corollary 4.5.3. \blacksquare

4.5.2 Subset expansion form of $U_{LVE}(G; y, z, \bar{t})$

We are looking for an explicit subset expansion form $U_{LVE}(G; y, z, \bar{t})$.

Theorem 4.5.6 *The following expression defines the same graph polynomial as the recurrence relation (4.5.13):*

$$U_{LVE}(G; y, z, \bar{t}) = \sum_{U \subseteq V} y^{|U| - k(G[U])} \cdot (y + z)^{k(G[U])} \cdot \left(\prod_{v \in U} t_{lab_V(v)} \right),$$

where $k(G[U])$ denotes the number of connected component in the induced subgraph of G with vertex set U .

Proof. The proof is similar to the most distinctive EE polynomial. We need to show that

- The expression (4.5.14) satisfies the initial conditions of (4.5.13);
- The expression (4.5.14) is multiplicative;
- The expression (4.5.14) satisfies the vertex elimination rule of (4.5.14).

Then by induction on the number of vertices in G the theorem holds. The first statement follows by inspection: indeed, a null graph $\emptyset = \langle \emptyset, \emptyset \rangle$ has no vertices and no components, so its only summand is 1, whereas a singleton has one vertex and just one component, so its contribution is $1 + (y + z)t_\lambda$.

The second statement, multiplicativity, can be easily checked too: Indeed, the summation over subsets of vertices of $G = G_1 \sqcup G_2$ can be regarded as a summation over the subsets of G_1 , multiplied by an independent summation over the subsets of G_2 .

Therefore, we just need to prove the third statement:

Let $G = \langle V, E, \Lambda_V \rangle$ be the labeled (multi)graph of interest. Let $N(G)$ be defined as in (4.5.14):

$$N(G; y, z, \bar{t}) = \sum_{U \subseteq V} y^{|U| - k(G[U])} \cdot (y + z)^{k(G[U])} \cdot \left(\prod_{v \in U} t_{lab_V(v)} \right),$$

Let v be the vertex we have chosen to reduce. There are three disjoint options to choose U :

- Case 1: $v \notin U$;
- Case 2: $v \in U$ but it is not adjacent to any other vertex in U ;

- Case 3: $v \in U$ and it is adjacent to some other vertex in U .

In case 1, we just sum over subsets of $G-v$: $N_1(G) = N(G-v)$;

In case 2, the neighborhood of v does not participate in any component of $G[U]$, so if we analyze $N(G \uparrow v)$, we get $N_2(G) = (y+z) \cdot t_{lab_V(v)} N(G \uparrow v)$;

In case 3, the neighborhood of v participates in the components of $G[U]$ as if it was a clique. Therefore, we need $N(G/v)$. However, this summation includes the cases that we have already counted, the subsets U that do not include any of neighbors of v . Hence, we will reduce those cases by subtracting $N(G \uparrow v)$. Finally, we get $N_3(G) = y \cdot t_{lab_V(v)} N(G/v) - N(G \uparrow v)$.

The cases above are disjoint and cover all the possibilities, and therefore $N(G) = N_1(G) + N_2(G) + N_3(G)$. Hence, we have:

$$N(G; y, z, \bar{t}) = U_{LVE}(G; y, z, \bar{t}),$$

which completes the proof. ■

Chapter 5

Some remarks on complexity

5.1 *MSOL*-definability of graph polynomials

Recall that simple graphs have two presentations: as a pair $\langle V, E \rangle$, where V is a set of vertices, and E is a binary relation $E \subseteq V \times V$ representing the edges of graph, or as a *two-sorted structure* $G = \langle V, E, R \rangle$, where V is a vertex set, E is an edge set, and R is a binary relation $R \subseteq V \times E$ that defines the incidence of vertices and edges.

The two presentations of finite graphs discussed here can be regarded as relational structures. In the first case $G = \langle V, E \rangle$, the universe of the relational structure is the vertex set V , and its vocabulary, denoted by τ_1 , consists of a binary edge relation E and possibly (for vertex-labeled graphs) unary predicates $U_1 \dots U_k$ indicating the available labels $\Lambda_V = [k]$. For edge-labeled graphs, several binary relations $E_1 \dots E_\ell$ are defined for $\Lambda_E = [\ell]$ available labels.

In the second case $G = \langle V, E, R \rangle$, the universe of the relational structure is $V \cup E$, and its vocabulary denoted by τ_2 , consists of a binary relation R , two disjoint unary predicates P_V and P_E (distinguishing between vertices and edges), and possibly two sets of unary predicates $U_1^V \dots U_k^V$ for vertex labeling and $U_1^E \dots U_\ell^E$ for edge labeling respectively.

Monadic Second Order Logic (MSOL) is an extension of *FOL*, while in addition to the first-order operators, quantification over sets is allowed. In the case of τ_1 , quantification is allowed over sets of vertices (*MSOL*₁), and in the case of τ_2 quantification is allowed over both sets of vertices and sets of edges (*MSOL*₂). An *auxiliary order* over the vertices or over the edges

may expand the expressive power of certain logic. In this work we allow an auxiliary order if the result is order-invariant.

***MSOL*-definable graph polynomials.**

We say that graph polynomial p is *MSOL*-definable, if its subset-expansion presentation can be written as

$$p(G; \bar{X}) = \sum_{(G, A_1, \dots, A_m) \models \varphi} \left[\left(\prod_{a \in A_1} f_1 \right) \cdots \left(\prod_{a \in A_m} f_m \right) \right],$$

where the summation ranges over tuples A_1, \dots, A_m of subsets of the universe (in particular, vertex subsets in case of *MSOL*₁, or vertex and edge subsets in case of *MSOL*₂), $\varphi(G, A_1, \dots, A_m)$ is in Monadic Second Order Logic, and $f_i(\bar{X})$ are polynomials in $\mathcal{R}[\bar{X}]$ that do not depend on the graph G .

5.1.1 Examples of *MSOL* definitions of graph polynomials

The chromatic polynomial is definable in *MSOL*₂ with an auxiliary order over vertices:

$$\chi(G; x) = \sum_{A \subseteq E} \left(\prod_{e \in A} (-1) \right) \left(\prod_{v \in F_A} x \right),$$

where F_A denotes the set of minimal (under the used auxiliary order) vertices of each connected component of $\langle V, A \rangle$:

$$F_A = \{v \in V : \forall_{u \neq v \in V} (\text{conn}(u, v) \rightarrow v \prec_{\omega} u)\},$$

$$\text{conn}(u, v) = \forall_{U \subseteq V} [(u \in U \wedge v \in V \setminus U) \rightarrow \exists_{u' \in U, v' \in V \setminus U} (\{u', v'\} \in E)]$$

The most distinctive matching polynomial $U_M(G; x, y)$ is definable in *MSOL*₂:

$$U_M(G; \bar{X}) = \sum_{\substack{M \subseteq E, \\ \text{matching}(M)}} \left[\left(\prod_{e \in M} y \right) \left(\prod_{v \in V \setminus V(M)} x \right) \right],$$

where $matching(M)$ is satisfied if M is a matching:

$$matching(M) = \forall_{e_1, e_2 \in M} \forall_{v \in V} (v \notin e_1 \vee v \notin e_2),$$

and

$$V(M) = \{v \in V : \exists_{e \in M} (v \in e)\}.$$

The Tutte polynomial is definable in $MSOL_2$ with an auxiliary order over vertices and edges:

$$T(G; x, y) = \sum_{A \subseteq E} \left(\prod_{e \in F_A \setminus F_E} (x - 1) \right) \left(\prod_{e \in A \setminus S_A} (y - 1) \right),$$

where F_A denotes the set of minimal (under the used auxiliary order) vertices of each connected component of $\langle V, A \rangle$, and S_A denotes the minimal (under the used auxiliary order) spanning forest of $\langle V, A \rangle$.

The subgraph component polynomial is definable in $MSOL_1$ with an auxiliary order over vertices:

$$Q(G; x, y) = \sum_{A \subseteq V} \left(\prod_{v \in A} x \right) \left(\prod_{u \in F_A} y \right),$$

where F_A denotes the set of minimal (under the used auxiliary order) vertices of each connected component of $\langle V, A \rangle$.

Note that in all the examples above the result does not depend on the used auxiliary order.

5.2 Computational complexity of graph polynomials

Generally, most of the known graph polynomials are hard to compute. When we speak about computing of graph polynomials, it can have different meanings: it can mean computing of the integer *coefficients* of the polynomial, or *evaluation* of the polynomial value in some particular point $\overline{X_0}$. Note that considering the second case, Turing computational model is not always

applicable, since the polynomial can have arbitrary values in the polynomial ring. For complexity issues, in this case, the BSS unit-cost model for Real computations can be used (cf. L.Blum, F.Cucker, M.Shub and S.Smale [11]).

We will further denote the first problem by *the computation*, and the second - by *the evaluation* of graph polynomial.

Most graph polynomials have "easy" points such that it is possible to efficiently evaluate the graph polynomial in these points for arbitrary input graph. We say that the polynomial has "*Difficult Point Property*" if it is easy (\mathbf{P} -time) to evaluate at any point, or it is $\#\mathbf{P}$ -hard at almost all the points, with possible semi-algebraic exception set of lower dimension.

It is proven that the chromatic polynomial [39], the Tutte polynomial [37], the interlace polynomial [10], and many others have the DPP. It is conjectured in [42] that all the *MSOL*-definable polynomials have DPP.

5.2.1 Parameterized complexity classes

In this section we consider the problem of computation of the integer coefficients of graph polynomial. It happens that for certain classes of graphs, restricted by some parameter k , the coefficients are computable in polynomial time. Two such parametric classes are widely studied: graph classes of *bounded tree-width* and of *bounded clique-width*. Precise definitions of those graph classes appear in the appendix.

We distinguish between the following upper bounds for the running time of algorithms with input graphs of size n , restricted by some parameter k :

- **Fixed parameter exponential time (FPEXP)**. Runtime less than $2^{n^{c_1(k)}}$ with $c_1(k) \geq 1$;
- **Fixed parameter sub-exponential time (FPSUBEXP)**. Runtime less than $2^{c_2(k) \cdot n^{1-\epsilon(k)}}$;
- **Fixed parameter polynomial time (FPPT)**. Runtime less than $n^{c_3(k)}$;
- **Fixed parameter tractable (FPT)**. Runtime less than $c_4(k) \cdot n^d$;
- **Polynomial time (P)**. Runtime less than $O(n^d)$.

Here d is independent of k , whereas the other constants may depend on k .

General theorems.

We cite here several general theorems regarding complexity of computation of graph properties and graph parameters on graph classes of bounded tree width or bounded clique width:

Theorem 5.2.1 Courcelle, 1990, [19]

Let G be a graph of tree-width k . Then any $MSOL_2$ -definable graph property can be tested efficiently on G .

Theorem 5.2.2 Bodlaender, 1996, [12]

For any graph G and constant k , it is possible to efficiently find k -tree-decomposition of G , if it exists, or provide a negative answer if it does not exist.

Theorem 5.2.3 Courcelle-Makowsky-Rotics, 2000, [41, 17]

Let G be a graph and let t be the corresponding parse term of constant clique-width k , determining G . Then any $MSOL_1$ -definable graph property can be tested efficiently on G .

Theorem 5.2.4 Oum, 2005, [45]

For any graph G and constant k , it is possible to efficiently distinguish between graphs of clique-width at most k and graphs of clique width $> 2^k$. Moreover, if the graph has clique-width at most k , a clique parse term of width 2^k can be efficiently found.

5.3 Complexity aspects of the new graph polynomials

We discuss here complexity aspects of three graph polynomials: The most distinctive matching polynomial $U_M(G; x, y)$, the most distinctive edge elimination polynomial $\xi(G; x, y, z)$, and the subgraph component polynomial $Q(G; x, y)$.

For each of these graph polynomials we discuss the following questions:

1. Can we use the parameterized complexity results of the general theorems [19, 12, 41, 16, 45], based on $MSOL$ definability of certain graph polynomial?

2. The algorithms provided by the general theorems suffer from huge hidden constants. Is there an explicit algorithm for computing the graph polynomial of interest that improves the constants?
3. And finally, does the graph polynomial have Difficult Point Property?

The following table presents the results in short. The DPP-related results are taken from C.Hoffmann's thesis [35]. The precise *MSOL* definitions and the explicit algorithms appear in Appendix A.

Complexity aspect	$U_M(G; x, y)$	$\xi(G; x, y, z)$	$Q(G; x, y)$
<i>MSOL</i> definable	<i>MSOL</i> ₂	<i>MSOL</i> ₂ (with order)	<i>MSOL</i> ₁ (with order)
Bounded tree-width (General theorems)	FPT	FPT	FPT
Bounded clique-width (General theorems)	No promise	<i>W</i> [1]-hard	FPT
Explicit algorithm for bounded tree-width	Exists	Exists	Exists
Explicit algorithm for bounded clique-width	Exists (FPPT)	No	still open
DPP (results by C. Hoffmann)	Yes, if multiple edges allowed	Yes	Yes, by reduction to $\xi(G; x, y, z)$

Table 5.1: Complexity aspects of the new graph polynomials.

Chapter 6

Conclusions and open questions

6.1 Conclusions

Inspired by the work of Brylawski [15], Oxley and Welsh [46], Bollobas and Riordan [14], on the Tutte polynomial, we studied classes of graph invariants satisfying linear recurrence relations with respect to edge and vertex elimination operations. We defined notions of *dp-complete* (most distinctive) and *universal* (most general) graph invariants in the classes above.

The Tutte polynomial satisfies recurrence relation with respect to edge deletion and edge contraction. In the case of edge elimination, we added an operation of edge extraction, which is deletion of an edge together with its end points. In the case of vertex elimination, we used vertex deletion, vertex contraction (deletion of a vertex and connecting its neighborhood to a clique), and vertex extraction (deletion of a vertex together with its neighborhood). The choice of these three operations was suggested by P. Tittmann.

We proposed five classes of graph invariants, according to operations used by the recurrence relations:

- *C*-invariants: edge deletion and contraction, no case distinction,
- *TG*-invariants: edge deletion and contraction, with distinction between loops, bridges and ordinary edges,

- M -invariants: edge deletion and extraction,
- EE -invariants: edge deletion, contraction and extraction,
- VE -invariants: vertex deletion, contraction and extraction.

For each of these classes, we proved a universality theorem, identifying a dp-complete, and if possible, universal graph polynomial. We used previous work related to C and TG graph invariants as a paradigm, and provided such a graph polynomial for M , EE and VE classes, in both recursive and explicit subset-expansion forms (Theorems 3.2.2, 3.2.6, 3.3.4, 3.3.8, 3.4.5, and 3.4.6).

We extended the results above to labeled graphs. Using the approach of Bollobas and Riordan [14], we identified dp-complete and, if possibly, universal graph polynomials for the following classes

- LM -invariants: edge deletion and extraction, edge- and vertex-labeled graphs,
- LEE -invariants: edge deletion, contraction and extraction, edge-labeled graphs,
- LVE -invariants: vertex deletion, contraction and extraction, vertex-labeled graphs,

and provided such graph polynomials in both recursive and subset-expansion forms.

The explicit subset expansion form of the new graph polynomials (from Theorems 3.2.6, 3.3.8 and 3.4.6) shows that they the most distinctive M , EE and VE -invariants, respectively $U_M(G; x, y)$, $\xi(G; x, y, z)$, and $Q(G; x, y)$, are definable in $MSOL$. In case of $U_M(G; x, y)$ and $\xi(G; x, y, z)$, allowing quantification over sets of edges, and in case of $Q(G; x, y)$, quantification is over sets of vertices. Based on fundamental theorems of [19, 12, 41, 16, 45], this allows us to conclude that $U_M(G; x, y)$ and $\xi(G; x, y, z)$ are fixed-parameter tractable on graph classes of bounded tree-width, and $Q(G; x, y)$ is fixed-parameter tractable on graph classes of bounded clique-width. In Appendix A we provide explicit algorithms for computing $U_M(G; x, y)$ and $\xi(G; x, y, z)$ on graphs of tree-width at most k , and a fixed-parameter P -time algorithm for computing $U_M(G; x, y)$ on graphs of clique-width at most k .

The results that appear in this thesis have been presented in the following publications:

***EE* graph invariants:**

- [3] I. Averbouch, B. Godlin, and J.A. Makowsky.
An extension of the bivariate chromatic polynomial.
European Journal of Combinatorics, 31.1:1–17, 2010.
- [4] Ilya Averbouch, Benny Godlin, and Johann A. Makowsky.
A most general edge elimination polynomial.
In Hajo Broersma, Thomas Erlebach, Tom Friedetzky, and Daniël Paulusma, editors, *WG*, volume 5344 of
Lecture Notes in Computer Science, pages 31–42, 2008.

***VE* graph invariants:**

- [51] P. Tittmann, I. Averbouch, and J.A. Makowsky.
The enumeration of vertex induced subgraphs with respect to number of components.
European Journal of Combinatorics, 3x.x:xx–yy, 2010.
- [5] Ilya Averbouch, Johann A. Makowsky, and Peter Tittmann.
A graph polynomial arising from community structure (extended abstract).
In Christophe Paul and Michel Habib, editors, *WG*, volume 5911 of
Lecture Notes in Computer Science, pages 33–43, 2009.

Complexity of Matching polynomial:

- [43] Johann A. Makowsky, Udi Rotics, Ilya Averbouch, and Benny Godlin.
Computing graph polynomials on graphs of bounded clique-width.
In Fedor V. Fomin, editor, *WG*, volume 4271 of
Lecture Notes in Computer Science, pages 191–204. Springer, 2006.

6.2 Further research

B. Godlin, E. Katz and J. A. Makowsky introduce in [28] a general framework of recurrence relations of graph polynomials, which generalizes edge elimination and vertex elimination to other graph deconstructions. They show that, under suitable conditions, each graph polynomial which is well-defined using a recurrence over such deconstruction, has a *SOL* subset expansion.

The classes of graph invariants C , M , TG , EE , and VE , introduced in this thesis, as well as their labeled counterparts, fit into this framework. It is natural to ask, what conditions are required to prove in the general framework the following properties of a graph polynomial:

- (i) Universality property;
- (ii) dp-completeness property;
- (iii) *MSOL* definability of the subset expansion.

Appendix A

Explicit efficient algorithms

In this chapter we provide explicit algorithms developed for computing graph polynomials on graphs of tree-width or clique-width at most k . Although the existence of such algorithms is promised by general theorems, the algorithms provided by the general methods suffer from huge hidden constants.

The appendix is organized as follows: We first provide the definitions of tree-width and clique-width. Then, for each particular case, we show the partitioning of specific graph polynomial to auxiliary polynomials, describe the dynamic algorithm and provide a brief proof of correctness and complexity analysis.

A.1 Tree-width and Clique-width of graphs

A.1.1 Tree-width

A *tree of decomposition* of the graph $G = \langle V, E \rangle$ is defined as a pair (S, T) , where $T = \langle W, F \rangle$ is a rooted tree and $S = \{S_w \mid w \in W\}$ is a collection of subsets of V such that

- $\bigcup_{w \in W} S_w = V$
- For all $e = (u, v) \in E$, there exists $w \in W$ with $u, v \in S_w$.
- For every $v \in V$, $W_v = \{w \in W \mid v \in S_w\}$ induces a connected subtree of T .

The *width* of a tree decomposition $(S = \{S_w \mid w \in W\}, T = \langle W, F \rangle)$ is $\max_{w \in W} |S_w| - 1$. The *tree-width* of a graph G , denoted as $tw(G)$ is the minimum width over all the tree decompositions of G .

Proposition A.1.1 *For every graph $G = \langle V, E \rangle$ with $|V| = n$ vertices and $tw(G) = k$, there exists a tree of decomposition (S, T) of width k with $|W| \leq n$.*

Proof. Let (S, T) be any tree of decomposition of G of width k . Every child subset S_w includes at least one vertex $v \in V$ that does not belong to its father subset (otherwise it could be unified with its father subset without increasing of the decomposition width). Let us call such a vertex "exclusive". According to the decomposition rules, it cannot happen that two subsets S_{w_1} and S_{w_2} , have the same "exclusive" vertex. ■

Proposition A.1.2 *For every tree of decomposition $(S = \{S_w \mid w \in W\}, T = \langle W, F \rangle)$ of width k , such that $|W| = M$, there exists tree of decomposition $(S' = \{S'_w \mid w \in W'\}, T' = \langle W', F' \rangle)$ of the same width k , such that $|W'| \leq 2M$ and T' is binary.*

Proof. It can be obtained, for example, by replacing each node with $m > 2$ children by a path of m nodes, each one with one additional child. The number of added nodes is not more than the number of edges in the original tree of decomposition. ■

The above propositions allow us to assume in our algorithms that for an input graph $G \langle V, E \rangle$ with n vertices and tree-width at most k we are given a *binary tree of decomposition* (S, T) defining it, where $T = \langle W, F \rangle$ has at most $|W| \leq 2|V|$ nodes.

Subgraph defined by subtree of T .

Let $w \in W$ be a tree node with children l and r , and father f . We denote by $T|_w = \langle W', F' \rangle$ the induced subtree of T with root w , including all the descendants of w . The subgraph defined by $T|_w$, is denoted by $G|_w = \langle V|_w, E|_w \rangle$ and defined inductively:

- if w is a leaf:
 $V|_w = S_w$, and
 $E|_w = E_w = \{\{u, v\} \in E : ((u, v \in S_w) \wedge \neg(u, v \in S_f))\}$

- if w is an internal vertex:
 $V|_w = V|_l \cup V|_r \cup S_w$, and
 $E|_w = E|_l \cup E|_r \cup E_w$, where
 $E_w = \{\{u, v\} \in E : ((u, v \in S_w) \wedge \neg(u, v \in S_f))\}$

By definition of tree decomposition, we have

Proposition A.1.3 *if w is a root of T , then $G|_w = \langle V|_w, E|_w \rangle$ is isomorphic to $G = \langle V, E \rangle$.*

A.1.2 Clique-width

For a *vertex labeled* graph $H = \langle V, E, c \rangle$ with label set $\Lambda = \{1, 2, \dots, k\}$ we denote by P_α the set of vertices labeled by α , when $1 \leq \alpha \leq k$.

$$P_\alpha = \{v \in V : c(v) = \alpha\}$$

The *clique-width* of a graph G , denoted by $cw(G)$, is defined as the minimum number of labels needed to construct G , using the four graph operations:

1. $i(v)$ – creation of a new vertex v with label i ;
 $H = \langle V, E \rangle = i(v)$ means $V = \{v\}$, $E = \emptyset$ and v is labeled by i ;
2. \sqcup – disjoint union of graphs;
 $H = \langle V, E \rangle = H_1 \sqcup H_2 = \langle V_1, E_1 \rangle \sqcup \langle V_2, E_2 \rangle$ means
 $V = V_1 \sqcup V_2$, $E = E_1 \sqcup E_2$, and labels on the vertices are preserved;
3. $\eta_{i,j}$ – only for $i \neq j$, connecting all vertices with label i to all vertices with label j ;
 $H = \langle V, E \rangle = \eta_{i,j}(H_1) = \eta_{i,j}(\langle V_1, E_1 \rangle)$ means
 $V = V_1$, $E = E_1 \cup \{\{u, v\} | u, v \in V, u \in P_i, v \in P_j\}$, labels are preserved;
4. $\rho_{i \rightarrow j}$ – only for $i \neq j$, renaming labels i as j .
 $H = \langle V, E \rangle = \rho_{i \rightarrow j}(H_1) = \rho_{i \rightarrow j}(\langle V_1, E_1 \rangle)$ means
 $V = V_1$, $E = E_1$, $P_j = P_j \cup P_i$, $P_i = \emptyset$.

A parse term built from the above four operations using k labels is called *k-construction*. For the set of labels C we denote by $T(C)$ all the possible constructions over this set of labels. In our notation the labels

(colors) are integers $1, \dots, k$. Each k -construction $t \in T(C)$ uniquely defines a labeled graph $val(t)$ where the labels are integers $1, \dots, k$ associated with the vertices and each vertex has exactly one label. We say that k -expression t defines a graph G if G is isomorphic to the graph obtained from the labeled graph $val(t)$ after removing the labels. The clique-width of a graph G is equal to the minimum k such that there exists a k -construction defining G .

Irredundant k -construction: We say that k -construction $t \in T(C)$ is *irredundant* if and only if for every sub-expression $\eta_{i,j}(t')$ of t no vertex labeled i in $val(t')$ is adjacent to a vertex labeled j in $val(t')$ (i.e. all the edges that could be added by $\eta_{i,j}(t')$ are actually added by this operation). Courcelle and Olariu have introduced this definition in [18]. They also have proven that if there exists a k -construction $t \in T(C)$ defining the graph G , then there is also $t' \in T(C)$ equivalent to t , which is irredundant.

In our algorithms we will assume that for input graph $G = \langle V, E \rangle$ of clique-width at most k we are given an irredundant k -construction.

A.1.3 General remarks

The algorithms presented in this chapter elaborate the ideas of L. Traldi [52], that work for both graphs of bounded tree-width and bounded clique-width:

- The input graph is presented by a tree of construction, which is a tree of decomposition for graph classes of bounded tree-width, or a parse-tree of a k -construction for graph classes of bounded clique-width.
- The graph polynomial in question is presented as a sum of auxiliary polynomials that carry all the information needed for further graph construction.
- A dynamic algorithm traverses the tree of construction of the input graph bottom-up, computing at every node the set of auxiliary polynomials based on those of its children nodes.
- The final result is obtained by summation of the auxiliary polynomials at the root.

A.2 Computing of the most distinctive matching polynomial U_M on graph classes of bounded tree-width

We introduce an explicit algorithm for computation of the most distinctive matching polynomial $U_M(G; x, y)$ on graph classes of bounded tree-width. Although the general theorem of [19] provides an *FPT* algorithm for this class of graphs with runtime $c(k) \cdot n^d$, our algorithm significantly improves the parameter $c(k)$ compared to the general one.

Compared to the Transfer Matrix method, introduced by D. Babic et al. for polygraphs in [6], our method seems to be more powerful, since it works on bigger class of graphs.

A.2.1 The algorithm

The graphs we consider in this section are simple, undirected and loop-free. We assume in our algorithm that we are given an input graph $G = \langle V, E \rangle$ together with a binary tree of decomposition (S, T) of width k defining it, where $T = \langle W, F \rangle$ has at most $|W| \leq 2|V|$ nodes.

The algorithm traverses the tree of decomposition (S, T) bottom-up, while constructing the subgraphs $G|_w$ defined by subtrees $T|_w$, and keeping the auxiliary polynomials corresponding to $G|_w$ denoted by $R(G|_w)$. When the scan of (S, T) is finished, we will obtain the graph G , and the corresponding set of auxiliary polynomials $R(G)$. The final graph polynomial is obtained from $R(G)$.

The set of auxiliary polynomials.

Given a graph $G = \langle V, E \rangle$, $A \subseteq V$ and $B \subseteq V$, such that $A \cap B = \emptyset$. Let $M \subseteq E$ be a matching in G and let $V(M)$ denote the set of vertices covered by this matching. We say that M is a fit for A and B on G if the following predicate is satisfied:

$$\psi_{A,B}(M) = (A \subseteq V(M)) \wedge (B \cap V(M) = \emptyset).$$

In the other words, we enforce the matchings to use vertices of A , and not to use the vertices of B .

Recall that we aim to compute the most distinctive matching polynomial $U_M(G; x, y)$, which is defined as

$$U_M(G; x, y) = \sum_{\substack{M \subseteq E \\ M \text{ is a matching}}} x^{|V|-|V(M)|} y^{|M|}$$

We define the following constrained polynomials:

$$U_M|_{A,B}(G; x, y) = \sum_{\substack{M \subseteq E \\ M \text{ is a matching} \\ M \models \psi_F}} x^{|V|-|V(M)|} y^{|M|}.$$

We observe that $U_M(G; x, y)$ is a special case of such constrained polynomial:

$$U_M(G; x, y) = U_M|_{\emptyset, \emptyset}(G; x, y)$$

Let $G|_w$ be a subgraph defined by subtree $T|_w = \langle W', F' \rangle$, when w is not the root of T . Let f be the father of w in the tree of decomposition. We denote by S_{fw} the vertices shared by S_f and S_w : $S_{fw} = S_f \cap S_w$. Let A_{fw} run over all the subsets of S_{fw} . We define the set of auxiliary polynomials as follows:

$$R(G|_w) = \{Q_{A_{fw}}^w = U_M|_{A_{fw}, S_{fw} \setminus A_{fw}}(G|_w; x, y) : A_{fw} \subseteq S_{fw}\}$$

If w is a root, we will compute only one auxiliary polynomial:

$$R(G|_w) = \{Q_{\emptyset}^w = U_M|_{\emptyset, \emptyset}(G|_w; x, y)\},$$

which equals to $U_M(G; x, y)$.

Note that at every node of T we need at most 2^{k+1} auxiliary polynomials.

Base: auxiliary polynomials of leaves Let w be a leaf of T , and f be its father. Let $G|_w$ be a subgraph defined by w . This graph has at most $k+1$ vertices. Let M_w be the set of all the matchings of this subgraph. For every $A_{fw} \subseteq S_{fw}$,

$$Q_{A_{fw}}^w = U_M|_{A_{fw}, S_{fw} \setminus A_{fw}}(G|_w; x, y) = \sum_{\substack{p \in M_w, \\ V(p) \cap S_{fw} = A_{fw}}} x^{|V_w|-|V(p)|} y^{|p|}$$

Step: auxiliary polynomials of an internal node Let w be an internal node of T , l and r - its children, and f - its father. We need to compute the auxiliary polynomials

$$R(G|_w) = \{Q_{A_{fw}}^w : A_{fw} \subseteq S_{fw}\}$$

given the auxiliary polynomials of the subgraphs defined by w 's children:

$$R(G|_l) = \{Q_{A_{wl}}^l : A_{wl} \subseteq S_{wl}\}$$

$$R(G|_r) = \{Q_{A_{wr}}^r : A_{wr} \subseteq S_{wr}\}$$

Let M_w denote the set of all the matchings, including only edges added by w , i.e. $M_w = \{p \subseteq E_w : p \text{ is a matching}\}$.

Proposition A.2.1 *Auxiliary polynomials for subgraph $G|_w$ can be computed as follows:*

$$Q_{A_{fw}}^w = x^{|S_w \setminus (S_l \cup S_r)| - |S_l \cap S_r|} \sum_{(A_{wl}, A_{wr}, p \in M_w) \models \varphi} Q_{A_{wl}}^l \cdot Q_{A_{wr}}^r \cdot \left(\frac{y}{x^2}\right)^{|p|} \quad (\text{A.2.1})$$

where

$$\varphi = \begin{cases} A_{wl} \cap A_{wr} = \emptyset \wedge \\ V(p) \subseteq (S_w \setminus (A_{wl} \cup A_{wr})) \wedge \\ A_{fw} = (A_{wl} \cup A_{wr} \cup V(p)) \cap S_f \end{cases}$$

A.2.2 Proof of correctness

We use induction on the structure of the tree decomposition. Since the base equations are computed by definition, we should only prove that the proposition A.2.1 is correct.

First, we observe that the degree of x in every auxiliary polynomial $Q_{A_{fw}}^w$ equals to the number of vertices that are not covered by matchings. This holds in the leaves; if we assume that it holds in the auxiliary polynomials produced by the children nodes, $Q_{A_{wl}}^l$ and $Q_{A_{wr}}^r$, then the equation A.2.1 just adds an x for every new vertex, and reduces an x for every vertex that appears in both S_l and S_r , and therefore, added twice by $Q_{A_{wl}}^l$ and $Q_{A_{wr}}^r$.

Therefore, it is enough to prove that the number of matchings is reflected correctly in the degree of y . We shall find 1 - 1 and *onto* mapping between

the matchings of $G|_w$ and the summands at the right side of the equation A.2.1, i.e. for every matching of $G|_w$, we should find the corresponding summand, and show that for every two different matchings there are different summands. The second direction is the opposite: for every summand on the right side of the equation, we should find the corresponding matching on the left side, and show that different summands correspond to different matchings.

Let $M|_w$ be set of all the matchings of $G|_w$, $M|_l$ be the set of all the matchings of $G|_l$ and $M|_r$ be the set of all the matchings of $G|_r$.

Direction 1

Let $p|_w$ be a matching of $G|_w$, which is counted by $Q_{A_{fw}}^w$. Every matching of $G|_w$ consists of disjoint parts: a matching of $G|_l$, $p|_l \in M|_l$, a matching of $G|_r$, $p|_r \in M|_r$, and a matching $p_w \in M_w$ that consists only of the edges from E_w . The matching $p|_l$ uses certain subset A_{wl} of S_{wl} , and it is counted in $Q_{A_{wl}}^l$. The matching $p|_r$ uses certain subset A_{wr} of S_{wr} , and it is counted in $Q_{A_{wr}}^r$. The matching p_w uses only the vertices of S_w and adds the $y^{|p_w|}$. Moreover, since $p|_l$, $p|_r$ and p_w are disjoint, and $p|_w$ uses exactly the subset A_{fw} of S_{fw} , the components A_{wl} , A_{wr} and p_w satisfy the formula φ . If two matchings $p|_w$ and $p'|_w$ of $G|_w$ are different, without loss of generality, $p|_w$ includes some edge $p'|_w$ doesn't include. Since every edge is added only once during the graph construction, it can belong to exactly one of the components: $G|_l$, $G|_r$ or E_w , so the corresponding components of the summand of the right side of equation are different too.

Direction 2

For every triple of matchings $p|_l \in M|_l$, $p|_r \in M|_r$ and $p_w \in M_w$, if they are disjoint, and include together some subset A_{fw} of S_{fw} , they can be combined into a matching of $G|_w$, which would be counted by $Q_{A_{fw}}^x$. Otherwise, if they are not disjoint, the only vertices that can be common are the vertices of w . In that case, the formula φ will not be satisfied. Since $E|_l$, $E|_r$ and E_w are disjoint, two combinations, in which at least one component is different, will correspond to different matchings of $G|_w$. \square

A.2.3 Complexity analysis

In our algorithm we need calculate the number of matchings of small graphs (with up to $k + 1$ vertices). This procedure requires generally the time of

$O(2^{k^2})$. But we can instead *pre-calculate* the number of matchings for *every* graph with up to $k + 1$ vertices, and the algorithm will only find in the table the required number, which takes only $O(k^2)$ (searching in the table of 2^{k^2} entries).

In the most naïve way, the operation on the polynomials of degree up to $\frac{n}{2}$ have the time cost $O(n^2)$.

The tree of decomposition includes up to $2n = O(n)$ vertices.

Each step of algorithm requires:

- Compute up to $2^{k+1} = O(2^k)$ auxiliary polynomials
- For each polynomial, run over all the partitions of S_w to four parts ($A_{wl}, A_{wr}, V(p)$ and the rest of S_w). This takes up to $4^{k+1} = O(2^{2k})$.
- Find the number of possible matchings $V(p)$ provides, and check that the combination of A_{wl}, A_{wr} and $V(p)$ satisfies the formula φ . This takes up to $(O(k^2))$.

Total time complexity: $O(n^3) \cdot O(k^2 \cdot 2^{3k})$

A.3 Computing of the most distinctive matching polynomial U_M on graph classes of bounded clique-width

We introduce an explicit algorithm for computation of the most distinctive matching polynomial $U_M(G; x, y)$ on graph classes of bounded clique-width. This result is new, since the theorems of [41, 17] do not hold for the case of $MSOL_2$ -definable polynomials, such as the matching polynomial. This algorithm was developed in joint work with B.Godlin. It develops further an idea of B.Godlin for computing the permanent of adjacency matrices of graphs of fixed clique-width, cf. [29].

A.3.1 The algorithm

The graphs we consider in this section are simple, undirected and loop-free. For a *vertex-labeled* graph $H = \langle V, E, c \rangle$ with labels $\{1, 2, \dots, k\}$ we denote by P_α the set of vertices labeled by α , when $1 \leq \alpha \leq k$.

We assume in our algorithm that we are given an input graph $G = \langle V, E \rangle$ together with an irredundant k -construction t defining it. We denote by $tree(t)$ the parse tree of t . The leaves of this tree are the vertices of G with their initial labels, and the internal nodes correspond to the operations of t , which can be either binary (corresponding to \sqcup), or unary (corresponding to η or ρ).

The algorithm traverses $tree(t)$ from bottom to top while constructing at each step the labeled graph H corresponding to a subtree of $tree(t)$, scanned so far, and keeping the set of auxiliary polynomials corresponding to H denoted by $R(H)$. When the scan of $tree(t)$ is finished, we will obtain a labeled copy of the graph G , and the corresponding set of auxiliary polynomials $R(G)$. The final graph polynomial is obtained by summation of $R(G)$.

The set of auxiliary polynomials Let $H \langle V, E, c \rangle$ be a labeled graph with labels $1, \dots, k$. Let $F = (f_1, f_2, \dots, f_k)$ be a vector of k integers. Let $|F|$ denote the sum of fields of F .

Let $M \subseteq E$ be a matching in H and let $V(M)$ denote the set of vertices covered by this matching. We say that M is a fit for F on H if the following predicate is satisfied:

$$\psi_F(M) = \forall \alpha \in [k] (|P_\alpha \setminus V(M)| = f_\alpha).$$

We observe that the set of vectors F such that ψ_F can be satisfied is

$$\mathcal{F} = \{F = (f_1, f_2, \dots, f_k) : \forall \alpha : 1 \leq \alpha \leq k (0 \leq f_\alpha \leq n)\}.$$

Recall that we aim to compute the most distinctive matching polynomial $U_M(G; x, y)$, which is defined as

$$U_M(H; x, y) = \sum_{\substack{M \subseteq E \\ M \text{ is a matching}}} x^{|V| - |V(M)|} y^{|M|}$$

Every matching leaves certain number of vertices of each color unused.

Therefore, we define the following set of auxiliary polynomials

$$R(H) = \left\{ U_{M|F}(H; x, y) = \sum_{\substack{M \subseteq E \\ M \text{ is a matching} \\ M \models \psi_F}} x^{|V|-|V(M)|} y^{|M|} \right\},$$

such that $U_M(H; x, y) = \sum_{F \in \mathcal{F}} U_{M|F}(H; x, y)$. Note that it has total of n^k auxiliary polynomials to keep.

Additionally, we observe that

$$U_{M|F}(H; x, y) = x^{|F|} y^{|V|-|F|} \left| \left\{ M \subseteq E : \begin{array}{l} M \text{ is a matching and} \\ M \models \psi_F \end{array} \right\} \right|. \quad (\text{A.3.1})$$

Base: auxiliary polynomials of leaves.

Every leaf of $tree(t)$ is of type " $i(v)$ " and corresponds to a single vertex v colored by i . This single vertex has only one matching (empty), leaving exactly one vertex of color i unused. So, for the leaf $H = i(v)$ we have:

$$U_{M|F}(H; x, y) = \begin{cases} x & \text{when } F = (0, \dots, 0, 1, 0, \dots, 0) \text{ with 1 at } i\text{-th position} \\ 0 & \text{otherwise.} \end{cases}$$

Handling of the \sqcup operator: $H = H_1 \sqcup H_2$.

We need to compute $R(H)$ given $R(H_1)$ and $R(H_2)$. Since the graphs H_1 and H_2 are disjoint, the number of free vertices of each color in H is a sum of those of H_1 and H_2 :

$$U_{M|F}(H; x, y) = \sum_{F_1 \leq F} U_{M|F_1}(H_1; x, y) \cdot U_{M|(F-F_1)}(H_2; x, y) \quad (\text{A.3.2})$$

Handling of the ρ operator: $H = \rho_{i,j}(H_1), i \neq j$.

We need to compute $R(H)$ given $R(H_1)$. The ρ operator only changes the labels of the vertices previously labeled by i to j . Therefore,

$$U_{M|F}(H; x, y) = \begin{cases} 0 & \text{if } f_i \neq 0 \\ \sum_{F' \models \varphi} U_{M|F'}(H_1; x, y) & \text{otherwise} \end{cases} \quad (\text{A.3.3})$$

where

$$\varphi = \left(\forall \ell: 1 \leq \ell \leq k : f_\ell = \begin{cases} f'_i + f'_j & \text{if } \ell = j \\ 0 & \text{if } \ell = i \\ f'_\ell & \text{otherwise} \end{cases} \right) \quad (\text{A.3.4})$$

Handling of the η operator: $H = \eta_{i,j}(H_1), i \neq j$.

We need to compute $R(H)$ given $R(H_1)$. The η operator adds edges between all the vertices labeled i to all the vertices labeled j . Moreover since t is an irredundant k -construction, all these edges are new, they did not appear in H_1 . Any matching of H consists of a matching of H_1 and some new edges. If there are f'_i free vertices labeled i and f'_j free vertices labeled j , and the matching includes q new edges, it chooses:

1. q of f'_i free vertices colored i
2. q of f'_j free vertices colored j
3. 1-1 mapping between two previous sets of vertices

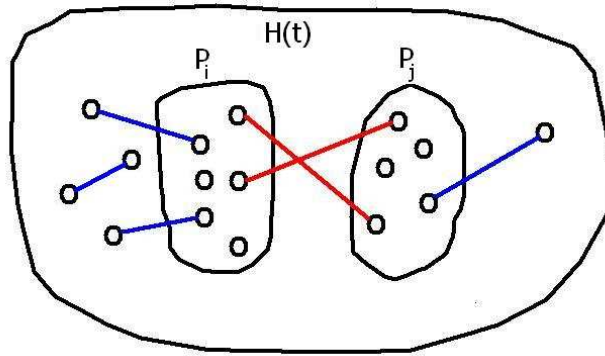


Figure A.1: Possible matching after $\eta_{i,j}$ operation

According to the mentioned above,

$$U_M|_F(H; x, y) = \sum_{q, F'=\varphi} \left(\frac{y}{x^2} \right)^q \cdot U_M|_{F'}(H_1; x, y) \cdot \binom{f'_i}{q} \cdot \binom{f'_j}{q} \cdot q! \quad (\text{A.3.5})$$

where

$$\varphi = \left(\forall \ell: 1 \leq \ell \leq k : f_\ell = \begin{cases} f'_i - q & \text{if } \ell = i \\ f'_j - q & \text{if } \ell = j \\ f'_\ell & \text{otherwise} \end{cases} \right) \quad (\text{A.3.6})$$

Computation of $U_M(G; x, y)$ out of $R(\text{val}(t))$:

When the scan of $\text{tree}(t)$ is finished, we obtain the set of auxiliary polynomials $R(H)$, where $H = \text{val}(t)$ is a colored copy of G . We need now to derive the matching polynomial of the graph G . As we already observed,

$$U_M(H; x, y) = \sum_{F \in \mathcal{F}} U_M|_F(H; x, y) \quad (\text{A.3.7})$$

A.3.2 Proof of correctness

We use induction on the structure of the clique-width k -construction. We should prove that the equations A.3.2, A.3.3, A.3.5 and A.3.7 are correct. For every equation we will show that the set of summands on both sides is equal.

Given a matching w of size k , we will say that its *weight* is $(\frac{y}{x^2})^k$.

Equation A.3.2 - handling of \sqcup operation Each matching w on the left side of the equation corresponds to union of two matchings w_1 in H_1 and w_2 in H_2 . The sum of the numbers of unused vertices of each color by w_1 (which is F_1) and w_2 (which is $F - F_1$) is F . Hence, w_1 is counted by $U_M|_{F_1}(H_1; x, y)$, and w_2 is counted by $U_M|_{F-F_1}(H_2; x, y)$. The product of their weights appears as a summand on the right side of the equation.

In the other direction, any two matchings w_1 (in $U_M|_{F_1}(H_1; x, y)$) and w_2 (in $U_M|_{F-F_1}(H_2; x, y)$) can be combined to a matching w in $U_M|_F(H; x, y)$.

Equation A.3.3 - handling of ρ operation Any matching of the right side is also the matching of the left side, with F and F' count the same unused vertices under the colorings of H and H_1 .

Equation A.3.5 - handling of η operation We can rewrite the equation A.3.5 to the following form:

$$\begin{aligned}
U_M|_F(H; x, y) &= \\
&= \sum_{q, F' \models \varphi} \left(\frac{y}{x^2}\right)^q \cdot U_M|_{F'}(H_1; x, y) \cdot \binom{f'_i}{q} \cdot \binom{f'_j}{q} \cdot q! \\
&= \sum_{q, F' \models \varphi} \binom{f'_i}{q} \cdot \binom{f'_j}{q} \cdot q! \left(\frac{y}{x^2}\right)^q \cdot U_M|_{F'}(H_1; x, y) \\
&= \sum_{q, F' \models \varphi} \sum_{\substack{S \subseteq P_i \times P_j, \\ S \text{ is } q\text{-matching}}} \left(\frac{y}{x^2}\right)^{|S|} \cdot U_M|_{F'}(H_1; x, y)
\end{aligned}$$

Any matching w on the left side contains some q edges between P_i and P_j . The rest of the edges of w form a matching in H_1 when the numbers of unused vertices of each color are F' , which satisfies A.3.6.

In the other direction, if F' and q satisfy A.3.6, any matching w_1 in $U_M|_{F'}(H_1; x, y)$ can be extended by any combination of q edges between the f'_i free vertices labeled i and f'_j free vertices labeled j to form matching w with unused vertices as in F .

A.3.3 Complexity analysis

First of all, we will evaluate the complexity of every step of our algorithm. As we have already shown in A.3.1, every auxiliary polynomial we keep is actually monomial. Thus, we can consider every operation on those monomials as cost of 1.

1. Creation of the vertex.

We should initialize up to (n^k) auxiliary monomials by 0 or 1. Time complexity is $O(n^k)$

2. \sqcup operation.

The calculation of single monomial $U_M|_F(H; x, y)$ requires summation over all possible F_1 , which is $O(n^k)$. This calculation should be done for all the (n^k) auxiliary monomials. Total complexity is $O(n^{2k})$.

3. ρ operation.

Although the summation is over F_1 , there are only up to n vectors which can fit the formula φ . The only free value is the j 'th field of the vector. Thus, the calculation of a single monomial $U_M|_F(H; x, y)$ has the time cost of $O(n)$. Total complexity is $O(n^{k+1})$

4. η operation.

The summation is over q , which uniquely defines F' . All the arithmetical operations, such as $\binom{f'_i}{q}$, $\binom{f'_j}{q}$ and $q!$, are considered as time cost of 1. The calculation of a single monomial $U_M|_F(H; x, y)$ has the time cost of $O(n)$. Total complexity is $O(n^{k+1})$

5. Summation of all the auxiliary monomials. Time cost of $O(n^k)$.

The number of times each operation can be applied depends on the term t . But we can see, that the most expensive operation (\sqcup) can appear at most n times in t , because every time the number of vertices in $val(t)$ grows at least by 1. Also, η operation cannot be applied more than n^2 times, because it adds edges every time. ρ operation can appear at most $n \cdot k$ times. Finally, the summation of all the monomials is performed once.

Total time complexity of the algorithm (for $k > 1$):

$$O(n \cdot (n^k) + n \cdot (n^{2k}) + n^2 \cdot (n^{k+1}) + n \cdot k \cdot (n^{k+1}) + (n^k)) = O(n^{2k+1})$$

A.4 Computing of the most distinctive edge elimination polynomial $\xi(G; x, y, z)$ on graph classes of bounded tree-width

In this section we provide an explicit algorithm which computes the edge elimination polynomial on graphs of tree-width at most k in polynomial time. The graphs we consider in this section are unlabeled and undirected. Self-loops and multiple edges are allowed.

Constrained edge elimination polynomial

Recall that the edge elimination polynomial of the graph $G = \langle V, E \rangle$ is

defined by

$$\xi(G; x, y, z) = \sum_{A \sqcup B} x^{k(A \sqcup B) - k_{cov}(B)} y^{|A| + |B| - k_{cov}(B)} z^{k_{cov}(B)},$$

where A and B are subsets of edges, such that $V_A = \{v : \exists u|(u, v) \in A\}$ and $V_B = \{v : \exists u|(u, v) \in B\}$ are disjoint, and $k_{cov}(A)$ denotes the number of connected components in (V_A, A) .

We use variable substitution $p = z/y$. Additionally, we denote by $I(A \cup B) = V \setminus (V_A \cup V_B)$ is the set of vertices that are not adjacent to any of edges of A or B . We get after the substitution:

$$\xi(G; x, y, p) = \sum_{A \sqcup B} x^{k_{cov}(A) + |I(A \cup B)|} y^{|A| + |B|} p^{k_{cov}(B)}. \quad (\text{A.4.1})$$

Let $C_A \subseteq V$, $C_B \subseteq V$ and $C_0 \subseteq V$ be disjoint subsets of vertices, and $M \subseteq (C_A \cup C_B)^2$ be some subset of pairs of vertices among C_A and C_B . We denote by $\varphi(A, B, C_A, C_B, C_0, M)$ the following predicate:

$$\begin{aligned} \varphi(A, B, C_A, C_B, C_0, M) = & (C_A \subseteq V_A) \wedge \\ & (C_B \subseteq V_B) \wedge \\ & (C_0 \subseteq I) \wedge \\ & (M = TCl(A \cup B) \cap (C_A \cup C_B)^2) \end{aligned} \quad (\text{A.4.2})$$

Informally, $\varphi(A, B, C_A, C_B, C_0, M)$ is satisfied if all the vertices of C_A are incident to one of the edges of A , all the vertices of C_B are incident to one of the edges of B , all the vertices of C_0 are not incident to any of the edges of A or B , and a pair of vertices $(u, v) \in M$ if and only if u and v are connected through the edges of A or B . Using this notation, we define the *constrained* edge elimination polynomial:

$$\xi_{C_A, C_B, C_0, M}(G; x, y, p) = \sum_{\substack{(A \sqcup B) \subseteq E, \\ \varphi(A, B, C_A, C_B, C_0, M)}} x^{k_{cov}(A) + |I(A \cup B)|} y^{|A| + |B|} p^{k_{cov}(B)} \quad (\text{A.4.3})$$

Note that the edge elimination polynomial is a special case of the constrained

edge elimination polynomial:

$$\xi_{\emptyset, \emptyset, \emptyset, \emptyset}(G; x, y, p) = \xi(G; x, y, p) \quad (\text{A.4.4})$$

A.4.1 The algorithm

The algorithm traverses the tree of decomposition T from bottom to top, while constructing the subgraphs $G|_w$ defined by subtrees $T|_w$, and keeping the auxiliary polynomials $R(G|_w)$. The final polynomial is derived from the set $R(G)$.

Let $G = \langle V, E \rangle$ be a graph with $|V| = n$ vertices and bounded tree-width. We suppose we are given a binary tree decomposition $T = \langle W, F \rangle$, defining the graph G . We need to compute its edge elimination polynomial $\xi(G, x, y, p)$.

The input graph can have multiple edges. Therefore, the edge set $|E|$ can be larger than $|V|^2$, which makes the time complexity of counting edge subsets unclear. To prevent this, we will use a notion of "thickness" of an edge, i.e. for the edge $e = (u, v)$, t_e denotes the number of edges connecting the vertices u and v . In other words, we will compute the polynomial over a *simple* graph $\tilde{G} = (V, \tilde{E})$, where $\tilde{E} = \{(u, v) \in V^2 : (u, v) \in E\}$. In the sequel of this section, \tilde{G} and \tilde{E} denote correspondingly such a simplified graph and a simplified edge set.

Set of auxiliary polynomials to keep.

Let $G|_w$ be a subgraph defined by subtree $T|_w = \langle W', F' \rangle$, when w is not the root of T . Let f be the father of w in the tree decomposition. We denote by S_{fw} the vertices shared between S_f and S_w : $S_{fw} = S_f \cap S_w$. Let $C_{A_{fw}}$, $C_{B_{fw}}$ and $C_{0_{fw}}$ run over all the tri-partitions of S_{fw} , and M_{fw} run over all the subsets of $(S_{fw})^2$. We define the set of auxiliary polynomials as follows:

$$\begin{aligned}
R(G|_w) = \{ & \\
& Q_{C_{A_{fw}}, C_{B_{fw}}, C_{0_{fw}}, M_{fw}}^w = \xi_{C_{A_{fw}}, C_{B_{fw}}, C_{0_{fw}}, M_{fw}}(G|_w; x, y, p) : \\
& C_{A_{fw}} \sqcup C_{B_{fw}} \sqcup C_{0_{fw}} = S_{fw}, \\
& M_{fw} \subseteq (S_{fw})^2 \\
& \} \tag{A.4.5}
\end{aligned}$$

If w is a root, we will compute only one auxiliary polynomial:

$$R(G|_w) = \{Q_{\emptyset, \emptyset, \emptyset, \emptyset}^w = \xi_{\emptyset, \emptyset, \emptyset, \emptyset}(G; x, y, p)\},$$

which equals to the edge elimination polynomial. Note that there are at most $3^{k+1} \times 2^{(k+1)^2}$ auxiliary polynomials. For sake of compactness, we shall use the index \mathcal{I}_{fw} instead of $\{C_{A_{fw}}, C_{B_{fw}}, C_{0_{fw}}, M_{fw}\}$.

Base: auxiliary polynomials of leaves

Let w be a leaf of T , and f be its father. Let $G|_w$ be a subgraph defined by w . This graph has at most $k+1$ vertices. For every tripartition $C_{A_{fw}} \sqcup C_{B_{fw}} \sqcup C_{0_{fw}} = S_{fw}$, and every subset $M_{fw} \subseteq (S_{fw})^2$, by the definition of constrained edge elimination polynomial:

$$\begin{aligned}
Q_{\mathcal{I}_{fw}}^w &= \xi_{\mathcal{I}_{fw}}(G|_w; x, y, p) = \\
&= \sum_{\substack{(A \sqcup B) \subseteq E|_w, \\ (A, B, \mathcal{I}_{fw}) \models \varphi}} x^{k_{cov}(A) + |I(A \cup B)|} \cdot \left(\prod_{e \in A \cup B} y \right) \cdot p^{k_{cov}(B)} = \\
&= \sum_{\substack{(A \sqcup B) \subseteq \bar{E}|_w, \\ (A, B, \mathcal{I}_{fw}) \models \varphi}} x^{k_{cov}(A) + |I(A \cup B)|} \cdot \left(\prod_{e \in A \cup B} ((y+1)^{t_e} - 1) \right) \cdot p^{k_{cov}(B)}
\end{aligned}$$

Step: auxiliary polynomials of an internal node

Let w be an internal node of T , l and r - its children nodes, and f - its father. We need to compute the auxiliary polynomials

$$R(G|_w) = \{ Q_{\mathcal{I}_{fw}}^w : C_{A_{fw}} \sqcup C_{B_{fw}} \sqcup C_{0_{fw}} = S_{fw}, M_{fw} \subseteq (S_{fw})^2 \}$$

given the auxiliary polynomials of the subgraphs defined by w 's children:

$$R(G|_l) = \{ Q_{\mathcal{I}_{wl}}^l : C_{A_{wl}} \sqcup C_{B_{wl}} \sqcup C_{0_{wl}} = S_{wl}, M_{wl} \subseteq (S_{wl})^2 \}$$

$$R(G|_r) = \{ Q_{\mathcal{I}_{wr}}^r : C_{A_{wr}} \sqcup C_{B_{wr}} \sqcup C_{0_{wr}} = S_{wr}, M_{wr} \subseteq (S_{wr})^2 \}$$

Recall that E_w is a set of edges added by w , and \tilde{E}_w is its simplified copy. Let $A_w \subseteq \tilde{E}_w$ and $B_w \subseteq \tilde{E}_w$ be vertex-disjoint subsets of \tilde{E}_w , covering correspondingly vertex sets V_{A_w} and V_{B_w} . We define auxiliary graphs over the vertex set S_w with edge sets defined as follows:

$$\begin{aligned} H_{A_{wl}} &= (S_w, F_{A_{wl}}), \text{ where } F_{A_{wl}} = M_{wl} \cap C_{A_{wl}}^2, \\ H_{A_{wr}} &= (S_w, F_{A_{wr}}), \text{ where } F_{A_{wr}} = M_{wr} \cap C_{A_{wr}}^2, \\ H_{B_{wl}} &= (S_w, F_{B_{wl}}), \text{ where } F_{B_{wl}} = M_{wl} \cap C_{B_{wl}}^2, \\ H_{B_{wr}} &= (S_w, F_{B_{wr}}), \text{ where } F_{B_{wr}} = M_{wr} \cap C_{B_{wr}}^2, \\ H_{A_w} &= (S_w, F_{A_w}), \text{ where } F_{A_w} = F_{A_{wl}} \cup F_{A_{wr}}, \\ H_{B_w} &= (S_w, F_{B_w}), \text{ where } F_{B_w} = F_{B_{wl}} \cup F_{B_{wr}}, \\ H'_{A_w} &= (S_w, F'_{A_w}), \text{ where } F'_{A_w} = F_{A_w} \cup A_w, \\ H'_{B_w} &= (S_w, F'_{B_w}), \text{ where } F'_{B_w} = F_{B_w} \cup B_w, \\ H'_w &= (S_w, F'_w), \text{ where } F'_w = F'_{A_w} \cup F'_{B_w}. \end{aligned}$$

Finally, the contribution function of A_w and B_w with child indexes \mathcal{I}_{wr} and \mathcal{I}_{wl} is defined as follows:

$$\begin{aligned} \mathcal{C}(A_w, B_w, \mathcal{I}_{wr}, \mathcal{I}_{wl}) &= \\ &= x^{(k_{cov}(H'_{A_w}) + |I(H'_w)|) - (k_{cov}(H_{A_{wl}}) + |C_{0_{wl}}| + k_{cov}(H_{A_{wr}}) + |C_{0_{wr}}|)} \\ &\quad \cdot \left(\prod_{e \in A_w \cup B_w} (y+1)^{t_e} - 1 \right) \\ &\quad \cdot p^{k_{cov}(H'_{B_w}) - (k_{cov}(H_{B_{wl}}) + k_{cov}(H_{B_{wr}}))} \end{aligned} \tag{A.4.6}$$

Here $k_{cov}(H)$ and $I(H)$ denotes correspondingly the number of covered connected components and the number of isolated nodes in certain auxiliary graph.

Proposition A.4.1 *Auxiliary polynomials for subgraph $G|_w$ can be computed by the following formula:*

$$Q_{\mathcal{I}_{fw}}^w = \sum_{(\mathcal{I}_{fw}, \mathcal{I}_{wl}, \mathcal{I}_{wr}, A_w B_w) \models \psi} Q_{\mathcal{I}_{wl}}^l \cdot Q_{\mathcal{I}_{wr}}^r \cdot \mathcal{C}(A_w, B_w, \mathcal{I}_{wr}, \mathcal{I}_{wl}), \quad (\text{A.4.7})$$

where

$$\psi = \begin{cases} (C_{A_{wl}} \cup C_{A_{wr}} \cup V_{Aw}) \cap (C_{B_{wl}} \cup C_{B_{wr}} \cup V_{Bw}) = \emptyset, \\ C_{A_{fw}} = (C_{A_{wl}} \cup C_{A_{wr}} \cup V_{Aw}) \cap S_{fw} \\ C_{B_{fw}} = (C_{B_{wl}} \cup C_{B_{wr}} \cup V_{Bw}) \cap S_{fw} \\ C_{0_{fw}} = S_{fw} \setminus (C_{A_{fw}} \cup C_{B_{fw}}) \\ M_{fw} = (C_{A_{fw}}^2 \cup C_{B_{fw}}^2) \cap TCl[M_{wl} \cup M_{wr} \cup A_w \cup B_w] \end{cases}$$

A.4.2 Proof of correctness

We use induction on the structure of the tree decomposition. Since the base equations are computed by definition, we should only prove that the proposition A.4.1 is correct.

We shall find 1 – 1 and *onto* mapping between the consistent choices of A and B in $G|_w$ and the summands at the right side of the equation A.4.7, i.e. for every consistent choice of A and B in $G|_w$, we should find the corresponding summand, and show that for every two different choices there are different summands. The second direction is the opposite: for every summand on the right side of the equation, we should find the corresponding choice of A and B in $G|_w$ on the left side, and show that different summands correspond to different choices.

Let $Ch|_w$ be set of all the consistent choices of A and B in $G|_w$, $Ch|_l$ be the same in $G|_l$, and $Ch|_r$ be the same in $G|_r$.

Direction 1

Let $ch|_w$ be a particular choice of A and B in $G|_w$, which is counted by $Q_{\mathcal{I}_{fw}}^w$.

Every choice of A and B in $G|_w$ consists of six disjoint parts:

Subset A consists of disjoint subset A_l of $G|_l$, subset A_r of $G|_r$ and a subset of \tilde{E}_w , which we call A_w ;

Subset B consists of disjoint subset B_l of $G|_l$, subset B_r of $G|_r$ and a subset of \tilde{E}_w , which we call B_w ;

The choices A_l and B_l are determined by $ch|_l$, and A_r and B_r are determined

by $ch|_r$.

The choice $ch|_l$ uses some subsets $C_{A_{wl}}$ and $C_{B_{wl}}$ of S_{wl} , and it is counted by $Q_{\mathcal{I}_{wl}}^l$ (induction hypothesis). Moreover, under this choice, the vertices of $C_{A_{wl}}$ and $C_{B_{wl}}$ are connected or not connected (via chosen edges) according to the matrix M_{wl} . The same holds for $ch|_r$.

The contribution of the node w to $Q_{\mathcal{I}_{fw}}^w$ is counted as follows:

- The degree of x is affected by the number of covered connected components marked A , and number of unmarked vertices. The previously counted contributions of $Q_{\mathcal{I}_{wl}}^l$ and $Q_{\mathcal{I}_{wr}}^r$, which are respectively $x^{k_{cov}(H_{A_{wl}})+|C_{0_{wl}}|}$ and $x^{k_{cov}(H_{A_{wr}})+|C_{0_{wr}}|}$, are subtracted from the current contribution $x^{k_{cov}(H'_{A_w})+|I(H'_w)|}$.
- The degree of y is affected by the number of edges added by the node w .
- The degree of p is affected by the number of covered connected components marked B . The previously counted contributions of $Q_{\mathcal{I}_{wl}}^l$ and $Q_{\mathcal{I}_{wr}}^r$, which are respectively $x^{k_{cov}(H_{B_{wl}})}$ and $x^{k_{cov}(H_{B_{wr}})}$, are subtracted from the current contribution $x^{k_{cov}(H'_{B_w})}$.

Moreover, if $ch|_l$, $ch|_r$, A_w and B_w are consistent, and $ch|_w$ fits the index \mathcal{I}_{fw} , the predicate ψ is satisfied.

If two choices $ch|_w$ and $ch'|_w$ of A and B in $G|_w$ are different, without loss of generality, A or B under $ch|_w$ includes some edge $ch'|_w$ doesn't include. Since every edge is added only once during the graph construction, it can belong to exactly one of the components: $ch|_l$, $ch|_r$, A_w or B_w , so the corresponding components of the summand of the right side of equation are different too.

Direction 2

For every set of choices $ch|_l \in Ch|_l$, $ch|_r \in Ch|_r$, and $A_w, B_w \subseteq \tilde{E}_w$, if they are consistent and fit together some index \mathcal{I}_{fw} , they can be combined into a choice of A and B in $G|_w$, which would be counted by $Q_{\mathcal{I}_{fw}}^x$. Otherwise, if they are not consistent, the formula ψ will not be satisfied. Since $ch|_l \in Ch|_l$, $ch|_r \in Ch|_r$, and $A_w, B_w \subseteq \tilde{E}_w$ are disjoint, two combinations, in which at least one component is different, will correspond to different choices of A and B in $G|_w$. \square

A.4.3 Complexity analysis

Each step of algorithm requires:

- Compute up to $3^{k+1} \times 2^{(k+1)^2} = O(2^{(1+o(1))k^2})$ auxiliary polynomials
- For each polynomial, run over all the pairs of possible child indexes. This takes up to $O(2^{2k^2})$.
- Check that ψ is satisfied. This takes up to $O(k^2)$.

Total time complexity: $O(n^3) \cdot O(k^2 \cdot 2^{(3+o(1))k^2})$

A.5 Computing of the subgraph component polynomial $Q(G; x, y)$ on graph classes of bounded tree-width

In this section we provide an explicit algorithm which computes the subgraph component polynomial on graphs of tree-width at most k in polynomial time. The graphs we consider in this section are unlabeled and undirected, without multiple edges and self-loops.

Constrained subgraph component polynomial

Recall that the subgraph component polynomial of the graph $G = \langle V, E \rangle$ is defined by

$$Q(G; x, y) = \sum_{U \subseteq V} x^{|U|} y^{k(G[U])},$$

where U runs over the subsets of vertices of G , and $k(G[U])$ denotes the number of connected components in the subgraph of G induced by U .

Let $A, B \subseteq V$ be subsets of vertices of G and let $M \subseteq A^2$ be a symmetric binary relation on A . We denote by $\varphi(U, A, B, M)$ the following predicate:

$$\begin{aligned} \varphi(U, A, B, M) = & (M = (TCU(E \cap U^2)) \cap A^2) \wedge \\ & (A \subseteq U) \wedge (B \cap U = \emptyset) \end{aligned} \quad (\text{A.5.1})$$

Informally, $\varphi(U, A, B, M)$ is satisfied if all the vertices of A are in U , none of the vertices of B are in U , and a pair $\{u, v\} \in M$ iff u and v are connected in

$G[U]$. Using this notation, we define the *constrained* subgraph component polynomial:

$$Q_{A,B,M}(G; x, y) = \sum_{\substack{U \subseteq V, \\ \varphi(U, A, B, M)}} x^{|U|} y^{k(G[U])}. \quad (\text{A.5.2})$$

Note that the subgraph component polynomial is a special case of the constrained polynomial:

$$Q_{\emptyset, \emptyset, \emptyset}(G; x, y) = Q(G; x, y) \quad (\text{A.5.3})$$

A.5.1 The algorithm

The algorithm traverses the tree of decomposition T from bottom to top, while constructing the subgraphs $G|_w$ defined by subtrees $T|_w$, and keeping the auxiliary polynomials $R(G|_w)$. The final polynomial is derived from the set $R(G)$.

Let $G = \langle V, E \rangle$ be a graph with $|V| = n$ vertices and bounded tree-width. We suppose we are given a binary tree decomposition $T = \langle W, F \rangle$, defining the graph G . We need to compute its subgraph component polynomial $Q(G; x, y)$.

Set of auxiliary polynomials to keep.

Let $G|_w$ be a subgraph defined by subtree $T|_w = \langle W', F' \rangle$, when w is not the root of T . Let f be the father of w in the tree decomposition. We denote by S_{fw} the vertices shared between S_f and S_w : $S_{fw} = S_f \cap S_w$. Let A_{fw} run over all subsets of S_{fw} , and M_{fw} run over all the subsets of $(S_{fw})^2$. We define the set of auxiliary polynomials as follows:

$$R(G|_w) = \{Q_{A_{fw}, M_{fw}}^w = Q_{A_{fw}, S_{fw} \setminus A_{fw}, M_{fw}}(G|_w; x, y) : A_{fw} \subseteq S_{fw}, M_{fw} \subseteq (S_{fw})^2\}$$

If w is a root, we will compute only one auxiliary polynomial:

$$R(G|_w) = \{Q_{\emptyset, \emptyset, \emptyset}^w = Q_{\emptyset, \emptyset, \emptyset}(G; x, y)\},$$

which equals to the subgraph component polynomial. Note that there are at most $2^{k+1} \times 2^{(k+1)^2}$ auxiliary polynomials.

Base: auxiliary polynomials of leaves

Let w be a leaf of T , and f be its father. Let $G|_w$ be a subgraph defined by w . This graph has at most $k + 1$ vertices. For every subset $A_{fw} \subseteq S_{fw}$, and every subset $M_{fw} \subseteq (S_{fw})^2$, by the definition of constrained subgraph component polynomial:

$$\begin{aligned} \mathcal{Q}_{A_{fw}, M_{fw}}^w &= \mathcal{Q}_{A_{fw}, M_{fw}}(G|_w; x, y) = \\ &= \sum_{\substack{U \subseteq V|_w, \\ (U, A_{fw}, S_{fw} \setminus A_{fw}, M_{fw}) \models \varphi}} x^{|U|} y^{k(G[U])}. \end{aligned}$$

Step: auxiliary polynomials of an internal node

Let w be an internal node of T , l and r - its children nodes, and f - its father. We need to compute the auxiliary polynomials

$$R(G|_w) = \{ \mathcal{Q}_{A_{fw}, M_{fw}}^w : A_{fw} \subseteq S_{fw}, M_{fw} \subseteq (S_{fw})^2 \}$$

given the auxiliary polynomials of the subgraphs defined by w 's children:

$$R(G|_l) = \{ \mathcal{Q}_{A_{wl}, M_{wl}}^l : A_{wl} \subseteq S_{wl}, M_{wl} \subseteq (S_{wl})^2 \}$$

$$R(G|_r) = \{ \mathcal{Q}_{A_{wr}, M_{wr}}^w : A_{wr} \subseteq S_{wr}, M_{wr} \subseteq (S_{wr})^2 \}$$

Recall that E_w is a set of edges added by w . Let $A_w \subseteq S_w$ be a subset of S_w . We denote by E_{Aw} the edges between vertices of A_w : $E_{Aw} = E_w \cap A_w^2$.

We define auxiliary graphs over the vertex sets and edge sets defined as follows:

$$\begin{aligned} H_{wl} &= (S_{wl}, F_{wl}), \text{ where } F_{wl} = M_{wl}, \\ H_{wr} &= (S_{wr}, F_{wr}), \text{ where } F_{wr} = M_{wr}, \\ H_w &= (S_w, F_w), \text{ where } F_w = TCl(F_{wl} \cup F_{wr} \cup E_{Aw}). \end{aligned}$$

The contribution function of $A_w \subseteq S_w$ with children indexes A_{wl}, M_{wl} and

A_{wr}, M_{wr} is defined as follows:

$$\mathcal{C}(A_w, A_{wl}, M_{wl}, A_{wr}, M_{wr}) = x^{|A_w| - (|A_{wl}| + |A_{wr}|)} y^{k(H_w) - (k(H_{wl}) + k(H_{wr}))} \quad (\text{A.5.4})$$

Proposition A.5.1 *Auxiliary polynomials for subgraph $G|_w$ can be computed as follows:*

$$\mathcal{Q}_{A_{fw}, M_{fw}}^w = \sum_{\left(\begin{array}{c} A_{fw}, M_{fw}, S_{fw} \\ A_{wl}, M_{wl}, S_{wl} \\ A_{wr}, M_{wr}, S_{wr} \\ A_w \end{array} \right) \models \psi} \mathcal{Q}_{A_{wl}, M_{wl}}^l \cdot \mathcal{Q}_{A_{wr}, M_{wr}}^r \cdot \mathcal{C}(A_w, A_{wl}, M_{wl}, A_{wr}, M_{wr}), \quad (\text{A.5.5})$$

where

$$\psi = \begin{cases} A_w \cap S_{wl} = A_{wl}, \\ A_w \cap S_{wr} = A_{wr}, \\ A_{fw} = A_w \cap S_{fw}, \\ M_{fw} = A_{fw}^2 \cap TCl(M_{wl} \cup M_{wr} \cup E_{A_w}) \end{cases}$$

A.5.2 Proof of correctness

We use induction on the structure of the tree decomposition. Since the base equations are computed by definition, we should only prove that the proposition A.5.1 is correct.

We shall find 1 – 1 and *onto* mapping between the consistent choices of U in $G|_w$ and the summands at the right side of the equation A.5.5, i.e. for every consistent choice of U in $G|_w$, we should find the corresponding summand, and show that for every two different choices there are different summands. The second direction is the opposite: for every summand on the right side of the equation, we should find the corresponding choice of U in $G|_w$ on the left side, and show that different summands correspond to different choices.

Let $Ch|_w$ be set of all the consistent choices of U in $G|_w$, $Ch|_l$ be the same in $G|_l$, and $Ch|_r$ be the same in $G|_r$.

Direction 1

Let $U|_w$ be a particular choice of U in $G|_w$, which is counted by $\mathcal{Q}_{A_{fw}, M_{fw}}^w$. Every choice of U in $G|_w$ consists of three parts that coincide on the shared vertices:

subset $U|_l$ of $G|_l$, subset $U|_r$ of $G|_r$ and A_w that consists of vertices of S_w . By the induction hypothesis, $\mathcal{Q}_{A_{wl}, M_{wl}}^l$ and $\mathcal{Q}_{A_{wr}, M_{wr}}^r$ count the choices of $U|_l$, and $U|_r$. Moreover, connectedness of vertices of A_{wl} and A_{wr} are according to the matrices M_{wl} and M_{wr} .

The contribution of the node w to $\mathcal{Q}_{A_{fw}, M_{fw}}^w$ is counted as follows:

- The degree of x is affected by the number of vertices added by A_w . The previously counted contributions of $\mathcal{Q}_{A_{wl}, M_{wl}}^l$ and $\mathcal{Q}_{A_{wr}, M_{wr}}^r$, which are respectively $|A_{wl}|$ and $|A_{wr}|$, are subtracted from the current contribution $|A_w|$.
- The degree of y is affected by the number of connected components added by A_w . The previously counted contributions of $\mathcal{Q}_{A_{wl}, M_{wl}}^l$ and $\mathcal{Q}_{A_{wr}, M_{wr}}^r$, which are respectively $k(H_{wl})$ and $k(H_{wr})$ are subtracted from the current contribution $k(H_w)$.

Moreover, if $U|_l$, $U|_r$, and A_w are consistent, and $U|_w$ is a fit for the index A_{fw}, M_{fw} , the predicate ψ is satisfied.

If two choices $U|_w$ and $U'|_w$ of U in $G|_w$ are different, without loss of generality, U under $U|_w$ includes some vertex $U'|_w$ doesn't include. Since all the components $U|_l$, $U|_r$, and A_w coincide on the shared vertices, the corresponding components of the summand of the right side of equation are different too.

Direction 2

For every set of choices $U|_l \in Ch|_l$, $U|_r \in Ch|_r$, and $A_w \subseteq S_w$, if they are consistent, they can be combined into a choice of U in $G|_w$, which would be counted by $\mathcal{Q}_{A_{fw}, M_{fw}}^w$. Otherwise, if they are not consistent, the predicate ψ will not be satisfied. Since $U|_l \in Ch|_l$, $U|_r \in Ch|_r$, and $A_w \subseteq S_w$ must coincide on shared vertices, two combinations, in which at least one component is different, will correspond to different choices of U in $G|_w$. \square

A.5.3 Complexity analysis

Each step of algorithm requires:

- Compute up to $2^{k+1} \times 2^{(k+1)^2} = O(2^{(1+o(1))k^2})$ auxiliary polynomials
- For each polynomial, run over all the pairs of possible child indexes. This takes up to $O(2^{2k^2})$.

- Check that ψ is satisfied. This takes up to $(O(k^2))$.

Total time complexity: $O(n^3) \cdot O(k^2 \cdot 2^{(3+o(1))k^2})$.

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תכונות שלמות ואוניברסליות של
שמורות-גרפים ופולינומי-גרפים.

איליה אברבוך

תכונות שלמות ואוניברסליות של שמורות-גרפים ופולינומי-גרפים.

חיבור על מחקר

לשם מילוי חלקי של הדרישות לקבלת התואר
דוקטור לפילוסופיה

איליה אברבוך

הוגש לסנט הטכניון – מכון טכנולוגי לישראל
אדר א תשע"א חיפה פברואר 2011

המחקר נעשה בהנחיית פרופ' יוהן מקובסקי בפקולטה למדעי המחשב.

ברצוני להודות לפרופ' יוהן מקובסקי על ההנחיה, התמיכה והעזרה שקיבלתי בכל שלבי המחקר. אני רוצה להודות גם לכל אחד ממחברי המאמרים שפרסמנו ביחד: פרופ' יוהן מקובסקי, בני גודלין, דר' אודי רוטיץ', ופרופ' פטר טיטמן. תודה גם לכל משתתפי הסמינרים בנושא: אמיליה כץ, אריה מצליח, יניב אלטשולר, תומר קוטק, אייל רוזנברג ואחרים, על דיונים פוריים ומעניינים. תודה לפרופ' ברוננו קורסל, שלראשונה הפנה את תשומת ליבנו לפולינום ההתאמות הכי כללי. תודה לקריסטיאן הופמן על גילוי פער בהוכחה שלנו ל"משפט הנקודה הקשה" של פולינום ההתאמות. תודה לחברי ועדת הבוחנים, פרופ' רון אהרוני, פרופ' תופיק מנסור ופרופ' אלדר פישר, על הערות והצעות חשובות רבות. תודה לדר' מריה ילנבסקיה על עזרתה בניסוחים באנגלית. תודה מיוחדת לעמיתיי וחבריי אריאל בירנבאום, בני גודלין, ודר' ילנה רווה, על קריאה קפדנית של החיבור הזה ועל עזרתם הרבה.

ולבסוף, תודה לכל המשפחה שלי על סבלנותם ותמיכתם במשך כל תקופת ההשתלמות שלי.

אני מודה לטכניון, לקרן הלאומית למדע, ולקרן הל ואינג' מרקוס על תמיכתם הכספית הנדיבה.

תקציר

פולינומי-גרפים הם כלים חזקים ומפותחים לתיאור תכונות של גרפים. בדרך כלל פולינומי-גרפים מושווים אחד לשני באמצעים אד-הוקיים המאפשרים להחליט האם פולינום-הגרפים החדש מכליל (או מוכלל בתוך) פולינום-גרפים אחר. אנו בוחנים את כוח ההפרדה שלהם ומגדירים מושגים של שלמות יכולת ההפרדה (dp -שלמות) ואוניברסליות של פולינומי-הגרפים על-מנת להגדיר באופן פורמלי את הקשרים ביניהם.

לפולינומי-גרפים ידועים רבים יש תכונות רקורסיביות לינאריות ביחס לקבוצה מסוימת של פעולות הורדת-צומת או הורדת-קשת. בהשפעת עבודתם של Oxley ו-Brylawski על הפולינום של Tutte, אנו מגדירים כמה מחלקות של פולינומי-גרפים בהתאם לתכונות הרקורסיביות שלהם, ומוכיחים משפטים אודות dp -שלמות ואוניברסליות במחלקות אלה. אנו מרחיבים את התוצאות הנ"ל לגרפים עם סימונים על הקשתות או הצמתים.

אנו מספקים כמה אבחנות לגבי הסיבוכיות החישובית של פולינומי-הגרפים ה- dp -שלמים הנידונים לעיל. בייחוד, אנו משתמשים בגדירות של פולינומי-הגרפים האלה כדי להראות שהם ניתנים לחישוב בצורה יעילה כאשר גרף המקור מוגבל בפרמטר מסוים, ומספקים אלגוריתמים מפורטים.

מבוא

תהי \mathcal{G} המחלקה של כל הגרפים. שמורת-גרפים היא פונקציה $f : \mathcal{G} \rightarrow \mathcal{R}$ שמ-מפה גרפים אל טווח כלשהו \mathcal{R} כך שעבור גרפים איזומורפיים $G_1 \simeq G_2$ הפונ-קציה f מקבלת אותו ערך: $f(G_1) = f(G_2)$. לשמורות-גרפים טווחים שונים. אם הטווח \mathcal{R} הוא שדה בינרי $\{0, 1\}$, מדובר בתכונת הגרף, למשל היותו קשיר, מישורי, אוילרי, המילטוני, וכו'. כאשר \mathcal{R} הוא המספרים השלמים \mathbb{Z} או המספר-

ים הממשיים \mathbb{R} , שמורות-גרפים כזו נקראת פרמטר הגרף, או שמורת-גרפים מספרית. דוגמאות של שמורות-גרפים מספריות: מספר הצמתים, מספר הקשתות, מספר הרכיבים הקשירים, מספר 3-צביעות, וכו'. כאשר \mathcal{R} הוא חוג הפולינומים, $\mathbb{Z}[\bar{X}]$ או $\mathbb{R}[\bar{X}]$, מעל קבוצת משתנים \bar{X} כלשהי, שמורת-גרפים כזו נקראת פולינום-גרפים. פולינומי-גרפים מקודדים אינסוף שמורות-גרפים, בתוך השערוכים, המקדמים, הדרגות והשורשים שלהם. בין היתר, פולינומי-גרפים גם יכולים להיות מקודדים בתוך פולינומי-גרפים אחרים, למשל על-ידי הצבת משתנים, או על-ידי טרנספורם-ציות על קבוצות המקדמים. במקרים כאלה אנו מדברים על פולינום-גרפים שמכליל פולינום-גרפים אחר.

היסטורית, פולינומי-גרפים הופיעו אחד-אחד, הושוו אחד לשני בשיטות פרטניות, בעיקר כדי לברר, האם פולינום-הגרפים החדש מכליל, או מוכלל, בפולינום-גרפים אחר כלשהו שהוגדר קודם לכן בספרות.

כדי להראות כי פולינום-גרפים מסוים p אינו מכליל פולינום-גרפים אחר q , החוקר-ים הציגו, בדרך כלל, שני גרפים G_1 ו- G_2 , כך ש- $p(G_1) = p(G_2)$, אבל $p(G_1) \neq q(G_2)$. כדי להראות כי p אכן מכליל את q , השתמשו בדרך כלל בטכניקה שונה: החוקרים הראו טרנספורמציה פשוטה למדי, שייצרה q מ- p בצורה אחידה. טרנספורמציות שונות הופיעו במאמרים שונים, חלק השתמשו רק בהצבת משתנים, בחלק אחר אפשרו פעולות אלגבריות פשוטות, או טרנספורמציות פשוטות מעל גרפים.

פולינומי-גרפים רבים שהופיעו בספרות מספקים יחסי-רקורסיה לינאריים, ביחס לפעולות הורדת-קשת והורדת-צומת שונות. קטלוג של פולינומי-הגרפים הנידונים בחיבור זה וסקר ספרות בנושא מופיע בפרק 2.

תהי S מחלקה של שמורות-גרפים שמספקות יחס רקורסיבי מסוג מסוים. האם יש ב- S פולינום-גרפים שמכליל את כל האחרים? על-מנת לענות לשאלה זו, אנו משתמשים ביחסי סדר חלקי של \leq_{dp} ו- \leq_{subst} בין פולינומי-גרפים, שמתאר-ים, בהתאמה, את היכולת של הפולינום להפריד בין הגרפים, ואת היכולת שלו לקודד שמורות-גרפים אחרות בהצבת משתנים. בהתבסס על היחסים האלה, אנו מגדירים מונחים של אוניברסליות ו- dp -שלמות של פולינומי-גרפים ביחס למחלקת שמורות-גרפים מסוימת. אנו מגדירים מספר מחלקות כאלה, בהתבסס על יחסי-הרקורסיה המסופקים, וקובעים תוצאות אודות אוניברסליות ו- dp -שלמות עבור מחלקות אלה.

החלק העיקרי (פרקים 4 - 2) של החיבור מוקדש לתיאוריה המבנית הזו. אנו מסווג גם מספר פולינומי-גרפים ידועים למחלקות שונות וקובעים את תכונות האוני-

ברסליות וה- dp -שלמות שלהם אם קיימות. אזי, בפרק 3 אנו מגדירים פולינומי-גרפים חדשים, ומראים את ה- dp -שלמות שלהם, ואם ניתן, גם את האוניברסליות, ביחס למחלקות שלהם. בפרק 4 אנו מרחיבים את המסגרת שהגדרנו קודם לגרפים עם סימונים על הקשתות או הצמתים, ומוכיחים תוצאות דומות לאלה של פרק 3.

התוצאות שלנו בתיאורית הסיבוכיות מופיעות בפרק 5. אנו משתמשים בתכונות הגדירות של פולינומי-גרפים שהגדרנו בפרקים הקודמים, על-מנת להראות כי תוצאות המשפטים הכלליים של [91, 21, 14, 61, 54] מתאימות גם למקרה שלנו. בנוסף, בנספח א אנו מספקים אלגוריתמים מפורטים יעילים לחישוב של פולינומי-גרפים החדשים על מחלקות גרפים עם רוחב-עץ מוגבל, שמשפרים בצורה משמעותית את זמן הריצה המובטח על-ידי המשפטים הכלליים. יתרה מכך, אנו מספקים אלגוריתם מפורט לחישוב פולינום-ההתאמות בשני משתנים על מחלקות גרפים עם רוחב-קליק מוגבל. התוצאה הזו היא חדשה כי היא לא נובעת מהמשפטים הכלליים.

השוואה של שמורות-גרפים

כוח הפרדה של שמורות-גרפים: יהיו $p : \mathcal{G} \rightarrow \mathcal{R}$ ו- $q : \mathcal{G} \rightarrow \mathcal{R}$ שתי שמורות-גרפים. נגיד שכוח ההפרדה של p לא עולה על זה של q , ונכתוב $p \preceq_{dp} q$ אם"ם לכל זוג גרפים G_1 ו- G_2

$$q(G_1) = q(G_2) \rightarrow p(G_1) = p(G_2)$$

אם מתקיים $p \preceq_{dp} q$ וגם $q \preceq_{dp} p$ אזי נגיד שכוח ההפרדה של שתי שמורות-הגרפים זהה. אם לא מתקיים לא $p \preceq_{dp} q$ ולא $q \preceq_{dp} p$, נגיד שכוח ההפרדה של שתי שמורות-הגרפים אינו ניתן להשוואה.

הצבת משתנים: יהיו $p : \mathcal{G} \rightarrow \mathcal{R}[\bar{X}]$ ו- $q : \mathcal{G} \rightarrow \mathcal{R}[\bar{Y}]$ שתי פולינומי-גרפים מעל קבוצות המשתנים \bar{X} ו- \bar{Y} בהתאמה. נגיד ש- p הוא הצבת משתנים של q , ונכתוב $p \preceq_{subst} q$ אם"ם ישנה הצבת משתנים $\sigma : \bar{Y} \rightarrow \bar{X}$ כך שלכל גרף G

$$p(G; \bar{X}) = \sigma(q(G; \bar{Y}))$$

dp-שלמות:

תהי S מחלקה של שמורות-גרפים. נגיד כי שמורת-הגרפים q היא dp-שלמה עבור S אם"ם $q \in S$ וכן לכל $p \in S$ מתקיים $p \preceq_{dp} q$.

אוניברסליות:

תהי S מחלקה של שמורות-גרפים. נגיד כי פולינום-הגרפים q הוא אוניברסלי עבור S אם"ם $q \in S$ וכן לכל $p \in S$ מתקיים $p \preceq_{subst} q$.

מחלקות של שמורות-גרפים מבוססות על יחס-רקורסיה

אנו מעוניינים בשמורות הגרפים שהן מולטיפליקטיביות (כפליות) ביחס לאיחוד זר של גרפים, ומספקות יחס-רקורסיה לינארי עם פעולות הבאות על גרפים:

פעולות הורדת קשת:

- $G - e$ מסמן את הגרף המתקבל מגרף G על-ידי מחיקת קשת e
- G/e מסמן את הגרף המתקבל מגרף G על-ידי צמצום קשת e ואיחוד של הקצוות שלה לצומת יחיד
- $G \dagger e$ מסמן את הגרף המתקבל מגרף G על-ידי עקירת קשת e ומחיקת הקצוות שלה

פעולות הורדת צומת:

- $G - v$ מסמן את הגרף המתקבל מגרף G על-ידי מחיקת צומת v
- G/v מסמן את הגרף המתקבל מגרף G על-ידי "צמצום" צומת v : מחיקת v וחיבור של כל השכנים שלו לקליק
- $G \dagger v$ מסמן את הגרף המתקבל מגרף G על-ידי עקירת צומת v : מחיקת v יחד עם כל השכנים שלו

בטבלה הבאה אנו מגדירים מחלקות של שמורות-גרפים מבוססות על יחס-רקורסיה:

שם המחלקה	יחס-רקורסיה
שמורות C (צביעה)	$p(G) = \alpha p(G - e) + \beta p(G/e)$
שמורות TG (Tutte – Grothendieck)	$p(G) = \begin{matrix} xp(G - e) & (e \text{ היא גשר}) \\ yp(G - e) & (e \text{ היא לולאה}) \\ \sigma p(G - e) + \tau p(G/e) & (\text{קשת אחרת}) \end{matrix}$
שמורות M (התאמות)	$p(G) = \alpha p(G - e) + \beta p(G \dagger e)$
שמורות EE (הורדת קשת)	$p(G) = \alpha p(G - e) + \beta p(G/e) + \gamma p(G \dagger e)$
שמורות VE (הורדת צומת)	$p(G) = \alpha p(G - v) + \beta p(G/v) + \gamma p(G \dagger v)$

כאן המקדמים $\alpha, \beta, \gamma, \sigma, \tau, x, y$ הם איברים בחוג מעליו מוגדרת שמורת-הגרפים $p(G)$ שאינם תלויים בגרף G .

בנוסף, למחלקות C, M, EE הגדרנו תת-מחלקות מנורמלות sC, sM, sEE , בהתאמה, המתקבלות מהמחלקות המקוריות ע"י הצבת $\alpha = 1$.

השאלות העיקריות שאנו חוקרים

תהי X מחלקת שמורות-הגרפים $(X \in C, TG, M, EE, VE)$.

1. האם קיים $U_X \in X$ פולינום-גרפים dp -שלם עבור X ?
2. האם U_X אוניברסלי עבור X ? אם לא, עבור איזו תת-מחלקה $S \subseteq X$ הפולינום U_X אוניברסלי? האם ניתן לתאר את $X \setminus S$?
3. באופן כללי, מהו היחס בין \preceq_{dp} ו- \preceq_{subst} ב- X ?
4. אם U_X הוא dp -שלם עבור X , כמה קשה לתאר אותו? האם וכיצד ניתן להגדירו ללא שימוש ברקורסיה?
5. מה ניתן להגיד על סיבוכיות החישוב של U_X ?

תוצאות

המחקר המוצג כאן התבצע במסגרת הפרויקט "פולינומי-הגרפים" בראשותו של פרופ' י.מקובסקי. פרויקט נועד לבצע מחקר השוואתי רחב היסוד של פולינומי-הגרפים. בתזה זו מופיע חלק מהתוצאות שהושגו במסגרת הפרויקט.

- אנו מציעים גישה כללית המאפשרת להשוות שמורות-גרפים בכלל ופולינומי-גרפים בפרט.
- אנו מגדירים את המושגים של אוניברסליות ו- dp -שלמות של פולינומי-גרפים ביחס למחלקת שמורות-גרפים מסוימת.
- כפרדיגמה, אנו משתמשים בפולינום האוניברסלי של $Tutte$ ומחלקת שמורות-הגרפים המופיעה בספרות כ- $Tutte - Grothendieck$ invariants. אנחנו מגדירים עוד שלוש מחלקות של שמורות-הגרפים שכוללות פולינומי-גרפים רבים קיימים. לכל מחלקה כזו אנחנו מציגים את פולינום-גרפים חדש שהוא dp -שלם במחלקה זו. אנחנו גם מוכיחים תכונת האוניברסליות של הפולינום-ים החדשים אם זה אפשרי. אחרת אנחנו צריכים להפחית מהמחלקה איזה חלק קטן
- בנוסף, פיתחנו אלגוריתמים מפורשים עבור חישוב של כמה פולינומי-הגרפים כאשר לגרף הקלט רוחב-עץ ($Tree - width$) לכל היותר k , או רוחב-הקליקה ($Clique - width$) לכל היותר k .

משפטי ה- dp -שלמות:

משפט 1.3.1: פולינום ההתאמות הכללי ביותר. קיים פולינום-הגרפים dp -שלם במ-חלקת שמורות M , והגדרתו

$$U_M(G; x, y) = \sum_{\substack{M \subseteq E \\ M \text{ is a matching}}} y^{|M|} x^{|V|-2|M|}$$

משפט 1.3.2: $U_M(G; x, y)$ הוא אוניברסלי למחלקת שמורות sM , כאשר כל יוצאי-הדופן $M \setminus sM$ הם מסוג $p(G) = x^{|V(G)|} y^{|E(G)|}$

משפט 1.3.3: פולינום הורדת-הקשת הכללי ביותר. קיים פולינום-הגרפים dp -שלם במחלקת שמורות EE , והגדרתו

$$\xi(G; x, y, z) = \sum_{(A \cup B) \subseteq E} x^{k(A \cup B) - k_{cov}(B)} y^{|A| + |B| - k_{cov}(B)} z^{k_{cov}(B)},$$

כאן k ו- k_{cov} מהווים, בהתאם, מספר הרכיבים הקשירים של תת-גרף פורש עם קבוצת קשתות מתאימה, ושל תת-גרף מושרה ע"י קבוצת קשתות מתאימה.

משפט 1.3.4: $\xi(G; x, y, z)$ הוא אוניברסלי למחלקת שמורות sEE , כאשר כל יוצאי-הדופן הם שמורות C לא מנורמלות: $EE \setminus sEE = C \setminus sC$.

משפט 1.3.5: פולינום הורדת-הצומת הכללי ביותר. קיים פולינום-הגרפים dp -שלם במחלקת שמורות VE , והגדרתו

$$U_{VE}(G; x, y) = \sum_{U \subseteq V} x^{|U| - k(G[U])} (x + y)^{k(G[U])},$$

כאן $G[U]$ הוא תת-גרף של G המושרה ע"י קבוצת הצמתים U , ו- $k(G[U])$ הוא מספר הרכיבים הקשירים של $G[U]$.

משפט 1.3.6: $U_{VE}(G; x, y)$ הוא אוניברסלי למחלקת שמורות VE .

תוצאות בתורת הסיבוכיות:

אנו דנים בתכונות הסיבוכיות החישובית של שלושה פולינומי-הגרפים החדש-ים: פולינום ההתאמות הכללי ביותר $U_M(G; x, y)$, פולינום הורדת-הקשת הכללי ביותר $\xi(G; x, y, z)$, ופולינום הורדת-הצומת הכללי ביותר $U_{VE}(G; x, y)$. על כל אחד מהפולינומים אנו קובעים:

- האם תוצאות המשפטים הכלליים של [91, 21, 14, 61, 54] חלים על הפולינום הנידון על-סמך גדירותו?
- האלגוריתמים הנובעים מהמשפטים הכלליים סובלים מקבועים מוסווים ענ-קיים. האם ניתן למצוא אלגוריתם יעיל יותר עבור הפולינום הספציפי?

התוצאות שלנו מפורטות בטבלה 5.1.