Semantic vs Syntactic Properties of Graph Polynomials, I:

On the Location of Roots of Graph Polynomials

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Graph polynomial project:
http://www.cs.technion.ac.il/~janos/RESEARCH/gp-homepage.html

References

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  On the Location of Roots of Graph Polynomials

  Special issue of the Erdős Centennial
  Electronic Notes in Discrete Mathematics, Volume 43 (2013), Pages 201-206

- Johann A. Makowsky, Elena V. Ravve, Nicolas K. Blanchard

  On the location of roots of graph polynomials

  European Journal of Combinatorics, Volume 41 (2014), Pages 1-19

- Johann A. Makowsky, Elena V. Ravve

  Semantic Equivalence of Multivariate Graph
  Polynomials Definable in Second Order Logic

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Overview

Semantic Properties of Graph Polynomials

Graph polynomials

Let $\mathcal{R}$ be a (polynomial) ring.

A function $P : \mathcal{G} \rightarrow \mathcal{R}$ is a

graph parameter

if for any two isomorphic graphs $G_1, G_2 \in \mathcal{G}$ we have $P(G_1) = P(G_2)$.

It is a

graph polynomial

if for each $G \in \mathcal{G}$ it is a polynomial.

In this lecture we study univariate graph polynomials P with $\mathcal{R} = \mathbb{Z}[X]$ or $\mathbb{C}[X]$.

A complex number $z \in \mathbb{C}$ is a P-root if there is a graph $G \in \mathcal{G}$ such that $P(G, z) = 0$.

Similar graphs and similarity functions

Two graphs G_1, G_2 are similar if they have the same number of vertices, edges and connected components, i.e.,

- $|V(G_1)| = n(G_1) = n(G_2) = |V(G_2)|$,
- $|E(G_1)| = m(G_1) = m(G_2) = |E(G_2)|$, and
- $k(G_1) = k(G_2)$.

A graph parameter or graph polynomial is a similarity function if it is invariant and similarity.

(i) The nullity $\nu(G) = m(G) - n(G) + k(G)$ and the rank $\rho(G) = n(G) - k(G)$ of a graph G are similarity polynomials with integer coefficients.

(ii) Similarity polynomials can be formed inductively starting with similarity functions $f(G)$ not involving indeterminates, and monomials of the form $X^{g(G)}$ where X is an indeterminate and $g(G)$ is a similarity function not involving indeterminates. One then closes under pointwise addition, subtraction, multiplication and substitution of indeterminates X by similarity polynomials.

Distinctive power of graph polynomials, I

Two graph polynomials are usually compared via their distinctive power.

A graph polynomial $Q(G, X)$ is less distinctive than $P(G, Y)$, $Q \preceq P$, if for every two similar graphs G_1 and G_2

$$P(G_1, X) = P(G_2, X) \text{ implies } Q(G_1, Y) = Q(G_2, Y).$$

We also say the $P(G; X)$ determines $Q(G; X)$ if $Q \preceq P$.

Two graph polynomials $P(G, X)$ and $Q(G, Y)$ are equivalent in distinctive power (d.p-equivalent) if for every two similar graphs G_1 and G_2

$$P(G_1, X) = P(G_2, X) \text{ iff } Q(G_1, Y) = Q(G_2, Y).$$

The same definition also works for graph parameters and multivariate graph polynomials.

Distinctive power of graph polynomials, II

$\mathbb{C}^\infty$ denotes the set of finite sequences of complex numbers.
We denote by $cP(G) \in \mathbb{C}^\infty$ the sequence of coefficients of $P(G, X)$.

Proposition 1
Two graph polynomials $P(G, X_1, \ldots X_r)$ and $Q(G, Y_1, \ldots, Y_s)$ are equivalent in distinctive power (d.p-equivalent) *($P \sim_{d.p.} Q$) iff there are two functions $F_1, F_2 : \mathbb{C}^\infty \to \mathbb{C}^\infty$ such that for every graph G*

$$F_1(n(G), m(G), k(G), cP(G)) = cQ(G) \text{ and}$$
$$F_2(n(G), m(G), k(G), cQ(G)) = cP(G)$$

Proposition 1 shows that our definition of equivalence of graph polynomials is mathematically equivalent to the definition proposed by
C. Merino and S. Noble in 2009.

Prefactor and subtsitution equivalence, I

- We say that $P(G;\bar{X})$ is prefactor reducible to $Q(G;\bar{X})$ and we write
$$P(G;\bar{Y}) \preceq_{prefactor} Q(G;\bar{X})$$
if there are similarity functions
$$f(G;\bar{X}), g_1(G;\bar{X}), \ldots, g_r(G;\bar{X})$$
such that
$$P(G;\bar{Y}) = f(G;\bar{X}) \cdot Q(G; g_1(G;\bar{Y}), \ldots, g(G;\bar{Y})).$$
- We say that $P(G;\bar{X})$ is substitutions reducible to $Q(G;\bar{X})$, and we write
$$P(G;\bar{Y}) \preceq_{subst} Q(G;\bar{X})$$
if $f(G;\bar{X}) = 1$ for all values of $\bar{X}$.
- $P(G;\bar{X})$ and $Q(G;\bar{X})$ are prefactor (substitution) equivalent if the relationship holds in both directions.

It follows that if $P(G;\bar{X})$ and $Q(G;\bar{X})$ are prefactor (substitution) equivalent then they are computably d.p.-equivalent.

Semantic properties of graph parameters

A semantic property is a class of graph parameters (polynomials) closed under d.p.-equivalence.

Let $p(G)$ be a graph parameter with values in $\mathbb{N}$, and $P(G;X)$ be a graph polynomial.

- The degree of $P(G;X)$ equals $p(G)$ is not a semantic property of $P(G;X)$.

  Using Proposition 1 we see that $P(G;X)$ and $P(G;X^2)$ are d.p.-equivalent, but they have different degrees.

- $P(G;X)$ determines $p(G)$ is a semantic property of $P(G;X)$.

Semantic vs syntactic properties of graph polynomials, I

Semantically meaningless properties:

(i) $P(G, X)$ is monic for each graph G, i.e., the leading coefficient of $P(G; X)$ equals 1.

Multiplying each coefficient by a fixed polynomial gives an equivalent graph polynomial.

(ii) The leading coefficient of $P(G, X)$ equals the number of vertices of G.

However, proving that two graphs G_1, G_2 with $P(G_1, X) = P(G_2, X)$ have the same number of vertices is semantically meaningful.

(iii) The graph polynomials $P(G; X)$ and $Q(G; X)$ coincide on a class $\mathcal{C}$ of graphs, i.e. for all $G \in \mathcal{C}$ we have $P(G; X) = Q(G; X)$.

The semantic content of this situation says that if we restrict our graphs to $\mathcal{C}$, then $P(G; X)$ and $Q(G; X)$ have the same distinguishing power.

The equality of $P(G; X)$ and $Q(G; a)X$ is a syntactic conincidence or reflects a clever choice in the definitions $P(G; X)$ and $Q(G; X)$.

Semantic vs syntactic properties of graph polynomials, II

Clever choices of can be often achieved.

Let $\mathcal{C}$ be class of finite graphs closed under graph isomorphisms.

Proposition 2
Assume that $P(G;X)$ and $Q(G;X)$ have the same distinguishing power on a class of graphs $\mathcal{C}$. Then there is $P' \sim_{d.p.} P$ such that the graph polynomials $P'(G;X)$ and $Q(G;X)$ coincide on a class $\mathcal{C}$ of graphs.

If, additionally, $\mathcal{C}, P(G;X)$ and $Q(G;X)$ are computable, then $P'(G;X)$ can be made computable, too.

Proposition 2 also holds when we replace computable by definable in SOL, as we shall see later.

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Prominent graph polynomials

Checking d.p.-equivalence

- The characteristic polynomial and the Laplacian polynomial are d.p.-incomparable.
  However, if restricted to k-regular graphs,
  they are d.p.-eqivalent.

- All versions of the matching polynomial are d.p.-equivalent.
  Bivariate matching, generating matching
  and defect matching polynomial.

- All versions of the Tutte polynomial are d.p.-equivalent
  Potts model, original definition, etc...

Back to overview

Definability of Graph Polynomials

in Second Order Logic SOL

Why definability?

- There are too many d.p.-equivalent graph polynomials.
- We want to show that for each SOL-definable graph polynomial there is a d.p.-equivalent SOL-definable graph polynomial with a certain blue syntactic property.
- (Almost) all graph polynomials from the literature are SOL-definable.
- Every SOL-definable graph polynomial $P(G; X)$ with coefficients in a ring $\mathcal{R}$ is computable in exponential time in a model of computation suitable for $\mathcal{R}$.

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Roots of Graph Polynomials

P-roots

It is an established topic to study the locations of the roots of graph polynomials.

For a fixed graph polynomial $P(G,X)$ typical statements about roots are:

(i) For every G the roots of $P(G,X)$ are real.

(ii) For every G all real roots of $P(G,X)$ are positive (negative) or the only real root is 0.

(iii) For every G the roots of $P(G,X)$ are contained in a disk of radius $\rho(p(G))$ where $p(G)$ is some numeric graph parameter (degree, girth, clique number, etc).

(iv) For every G the roots of $P(G,X)$ are contained in a disk of constant radius.

(v) The roots of $P(G,X)$ are dense in the complex plane.

(vi) The roots of $P(G,X)$ are dense in some absolute region.

Studying P-roots

We now overview polynomials P for which P-roots have been studied.

- Spectra of graphs, chromatic polynomial, matching polynomial, independence polynomial.
  Studying the location of their roots is motivated by applications in chemistry, statistical mechanics.
- Edge cover polynomial and domination polynomial.
  Studying the location of their roots is motivated by analogy only.
- All these polynomials are SOL-definable.
- All are univariate.

Skip examples, Back to overview

Spectral graph theory

Let $G(V, E)$ be a simple undirected graph with $|V| = n$, and
Let A_G and L_G be the (symmetric) adjacency resp. Laplacian matrix of G.

The characteristic polynomial of G is defined as

$$P_A(G, \lambda) = \text{det}(\lambda \cdot 1 - A_G)$$

and the Laplacian polynomial of G is defined s

$$P_L(G, \lambda) = \text{det}(\lambda \cdot 1 - L_G)$$

Theorem 3
The roots of $P_A(G, \lambda)$ and $P_L(G, \lambda)$ are all real.

There is a rich literature.

A.E. Brouwer and W. H. Haemers, Spectra of Graphs, Springer 2010.

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The (vertex) chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A **λ-vertex-coloring** is a map

$$c : V(G) \to [\lambda]$$

such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$.

We define $\chi(G, \lambda)$ to be the number of λ-vertex-colorings

Theorem 4 (G. Birkhoff, 1912)
$\chi(G, \lambda)$ is a polynomial in $\mathbb{Z}[\lambda]$.

Proof:

(i) $\chi(E_n) = \lambda^n$ where E_n consists of n isolated vertices.

(ii) For any edge $e = E(G)$ we have $\chi(G - e, \lambda) = \chi(G, \lambda) + \chi(G/e, \lambda)$.

The Four Color Conjecture

Birkhoff wanted to prove the **Four Color Conjecture** using techniques from real or complex analysis.

Conjecture:(Birkhoff and Lewis, 1946)
If G is planar then $\chi(G, \lambda) \neq 0$ for $\lambda \in [4, +\infty) \subseteq \mathbb{R}$.

Theorem 5 (Birkhoff and Lewis, 1946)
For planar graphs G we have $\chi(G, \lambda) \neq 0$ for $\lambda \in [5, +\infty)$.

Still open: Are there planar graphs G such that

$\chi(G, \lambda) = 0$ for some $\lambda \in (4, 5)$?

More on chromatic roots, I

For real roots of χ we know:

Theorem 6 (Jackson, 1993, Thomassen, 1997)

For simple graphs G we have $\chi(G, \lambda) \neq 0$ for real $\lambda \in (-\infty, 0)$, $\lambda \in (0, 1)$ and $\lambda \in (1, \frac{32}{27})$.
The only real roots $\leq \frac{32}{27}$ are 0 and 1.

The real roots of all chromatic polynomials are dense in $[\frac{32}{27}, \infty)$

More on chromatic roots, II

For complex roots of χ we know:

Theorem 7 (Sokal, 2004)

The complex roots are dense in $\mathbb{C}$.

The complex roots are bounded by $7.963907 \cdot \Delta(G) \leq 8 \cdot \Delta(G)$ *where* $\Delta(G)$ *is the maximal degree of* G.

We shall see that this is not a semantic property of the chromatic polynomial.

However, we have an interpretation in physics:

The chromatic polynomial corresponds to the zero-temperature limit of the antiferromagnetic Potts model. In particular, theorems guaranteeing that a certain complex open domain is free of zeros are often known as Lee-Yang theorems.

The above theorem says that no such domain exists.

More on chromatic roots, III

Theorem 8 (C. Thomassen, 2000)

If the chromatic polynomial of a graph has a real noninteger root less than or equal to

$$t_0 = \frac{2}{3} + \frac{1}{3}\sqrt[3]{26 + 6\sqrt{33}} + \frac{1}{3}\sqrt[3]{26 - 6\sqrt{33}} = 1.29559\ldots$$

Then the graph has no Hamiltonian path.

This result is best possible in the sense that it becomes false if t_0 is replaced by any larger number.

This is not a semantic property of the chromatic polynomial.

A semantic version would be:

The chromatic polynomial determines the existence of Hamiltonian paths..

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The three matching polynomials

Let $m_i(G)$ be the number sets of independent edges of size i. We define

$$dm(G,x) = \sum_r (-1)^r m_r(G) x^{n-2r} \tag{1}$$

$$gm(G,x) = \sum_r m_r(G) x^r \tag{2}$$

$$M(G,x,y) = \sum_r m_r(G) x^r y^{n-2r} \tag{3}$$

We have $dm(G;x) = x^n gm(G;(-x)^{-2}) = M(G,-1,x)$ where $n = \mid V \mid$.

All three matching polynomials are d.p-equivalent.

Theorem 9 (Heilmann and Lieb 1972)
The roots of $dm(G,x)$ are real and symmetrically placed around zero, i.e., $dm(G,x) = 0$ iff $dm(G,-x) = 0$

The roots of $gm(G,x)$ are real and negative

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Independence polynomial

Let $in_i(G)$ be the number of independent sets of G of size i,
and the **independence polynomial**

$$I(G,X) = \sum_i in_i(G) X^i$$

Clearly there are no positive real independence roots.
For a survey see: V.E. Levit and E. Mandrescu,
The independence polynomial of a graph - a survey,
Proceedings of the 1st International Conference on Algebraic Informatics,
Thessaloniki, 2005, pp. 233-254.

J. Brown, C. Hickman and R. Nowakowski showed in Journal of Algebraic Combinatorics, 2004:

Theorem 10 (J. Brown, C. Hickman and R. Nowakowski, 2004)
The real roots are dense in $(-\infty, 0]$ *and the complex roots are dense in* $\mathbb{C}$.

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Edge cover polynomial

Let $e_i(G)$ be the number of edge coverings of G of size i, and the **edge cover polynomial**

$$E(G,X) = \sum_i e_i(G) X^i$$

Theorem 11 (P. Csikvári and M.R.Oboudi, 2011)
All roots of $E(G,X)$ are in the ball

$$\{z \in \mathbb{C} : |z| \leq \frac{(2+\sqrt{3})^2}{1+\sqrt{3}} = \frac{(1+\sqrt{3})^3}{4}\}.$$

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Domination polynomial

Inspired by the rich literature on dominating sets, **S. Alikhani** introduced in his Ph.D. thesis the **domination polynomial**;

Let $d_i(G)$ be the number of dominating sets of G of size i, and the **domination polynomial**

$$D(G,X) = \sum_i d_i(G)X^i$$

It is easy to see that 0 is a domination root, and that there are no real positive domination roots.

J. Brown and J. Tufts (Graphs and Combinatorics, , 2013) showed:

Theorem 12 (J. Brown and J. Tufts)
The domination roots are dense in $\mathbb{C}$.

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D.p.-Equivalence

and the

Location of the Roots

of SOL-Definable Graph Polynomials

From now on all graph polynomials
are supposed to be SOL-definable.

Roots vs distinctive power, I

Let $s(G)$ be a similarity function.

Theorem 13 (MRB)

For every univariate graph polynomial $P(G;X) = \sum_{i=0}^{s(G)} h_i(G)X^i$

where $s(G)$ and $h_i(G), i = 0, \dots s(G)$ are graph parameters with values in $\mathbb{N}$,

there exists a univariate graph polynomials $Q_1(G;X)$,

prefactor equivalent to $P(G;X)$ such that for every G

all real roots of $Q_1(G;X)$ are

positive (negative) or the only real root is 0.

Show proof, Skip remaining theorems

Roots vs distinctive power, II

Let $s(G)$ be a similarity function.

Theorem 14 (MRB)

For every univariate graph polynomial

$$P(G;X) = \sum_{i=0}^{i=s(G)} h_i(G)X^i \in \mathbb{Z}[X], \text{ resp. } \mathbb{R}[X]$$

there is a d.p.-equivalent graph polynomial

$$Q_2(G;X) = \sum_{i=0}^{i=s(G)} H_i(G)X^i \in \mathbb{Z}[X], \text{ resp. } \mathbb{R}[X]$$

such that all the roots of $Q(G;X)$ are real.

Show proof, Skip remaining theorems

Roots vs distinctive power, III

Let $P(G; X)$ as before.

Theorem 15 (MRB)

For every univariate graph polynomial $P(G; X)$

there exist univariate graph polynomials $Q_3(G; X)$

substitution equivalent to $P(G; X)$ such that

for every G the roots of $Q_3(G; X)$ are contained in a disk of constant radius.

If we want to have all roots real and bounded in $\mathbb{R}$,

we have to require d.p.-equivalence.

Show proof Skip remaining theorems

Roots vs distinctive power, IV

Let $P(G; X)$ as before.

Theorem 16 (MRB)

For every univariate graph polynomial $P(G; X)$

there exists a univariate graph polynomial $Q_4(G; X)$

prefactor equivalent to $P(G; X)$ such that

$Q_4(G; X)$ has only countably many roots,

and its roots are **dense in the complex plane***.*

If we want to have all roots **real and dense in** $\mathbb{R}$*,*

we have to require d.p.-equivalence.

Show proof

The proofs use various tricks!

Skip proofs
Back to overview

Proofs: Theorem 13

Let $P(G,X) = \sum_i c_i(G)X^i = \sum_{A \subset V(G)^r} X^{|A|}$ be SOL-definable. We want to show:

For every G all real roots of $P(G,X)$ are negative.

This is true, because all coefficients of $P(G,X)$ are non-negative integers, due to SOL-definability.

If we want to find $Q_1(G;X)$ d.p.-equivalent to $P(G;X)$ such that

for every G all real roots of $Q_1(G,X)$ are positive,

we put $Q_1(G,X) = P(G,-X) = \sum_i c_i(G)(-X)^i = \sum_i (-1)^i c_i(G)(X)^i$.

If we want to find $Q_1'(G;X)$ d.p.-equivalent to $P(G;X)$ such that

for every G the only real root of $Q_1(G,X)$ is 0,

we put $Q_1'(G,X) = P(G,X^2) = \sum_i c_i(G)(X)^{2i}$.

Q.E.D.

Go to next theorem, Skip remaining proofs

Proofs: Theorem 14

Let $P(G, X)$ as before be SOL-definable.

We want to find $Q_3(G; X)$ d.p.-equivalent to $P(G; X)$ such that all roots of $Q_2(G; X)$ are real.

We define $Q_2(G; X) = \prod_{i=0}^{s(G)} (X - h_i(G))$.

Q.E.D.

Go to next theorem, Skip remaining proofs

Proofs: Theorem 15

Let $P(G, X)$ be SOL-definable.

We want to show:

For every G the roots of $Q_3(G, X)$ are contained in a disk of constant radius.

To relocate the roots of $P(G, X)$ we use Rouché's Theorem in the following form:

Lemma 17
Let $P(X) = \sum_{i=0}^{d} h_i X^i$ and $P'(X) = A \cdot X^{2d}$ with $A \geq \max_i\{h_i : 0 \leq i \leq d-1\}$.

Let $Q_3(X) = P(X) + P'(X)$.

Then all complex roots ξ of $Q_3(X)$ satisfy $|\xi| \leq 2$.

Clearly, $P'(G, X)$ is SOL-definable and d.p. equivalent to $P(G, X)$. Q.E.D.

Reference: P. Henrici, Applied and Computational Complex Analysis, volume 1,
Wiley Classics Library, John Wiley, 1988.
Section 4.10, Theorem 4.10c

Go to next theorem, Skip remaining proofs

Proofs: Theorem 16

Lemma 18
There exist univariate similarity polynomials $D^i_{\mathbb{C}}(G;X), i = 1,2,3,4$ of degree 12 such that all its roots of $D^i_{\mathbb{C}}(G;X)$ are dense in the ith quadrant of $\mathbb{C}$.

We use this lemma and put

$$Q_4(G;X) = \left(\prod_{i=1}^{i=4} D^i(G;X)\right) \cdot P(G;X).$$

To get the real roots to be dense we proceed similarily.

Q.E.D.

The multivariate case: Stability and the Half-plane property

We study now the location of zeros of multivariate polynomials.

Let $P(G; \bar{X})$ be multivariate graph polynomial, and G_0 be a graph.

- $P(G_0; \bar{X})$ is stable if the imaginary part of its zeros is negative.
- $P(G_0; \bar{X})$ is Hurwitz-stable (H-stable) if the real part of its zeros is negative.
  We also say that $P(G_0; \bar{X})$ has the half-plane property.
- $P(G_0; \bar{X})$ is Schur-stable (S-stable) if all its roots are in the open unit ball.
- $P(G; \bar{X})$ is stable if it is stable for every G.
  Same for H-stable and S-stable.

The origins of stability theory in engineering

In engineering and stability theory, a square matrix A is called stable matrix (or sometimes Hurwitz matrix) if every eigenvalue of A has strictly negative real part. These matrices were first studied in the landmark paper by A. Hurwitz in 1895. The Hurwitz stability matrix plays a crucial part in control theory. A system is stable if its control matrix is a Hurwitz matrix. The negative real components of the eigenvalues of the matrix represent negative feedback. Similarly, a system is inherently unstable if any of the eigenvalues have positive real components, representing positive feedback.

In the engineering literature, one also considers Schur-stable univariate polynomials, which are polynomials such that all their roots are in the open unit disk.

Recently, stable and Hurwitz-stable polynomials have attracted the attention of combinatorial research, mainly in the study of graph and matroid invariants and knot theory.

File:halfplane

Example of stable graph polynomials

(folklore) Univariate polynomials are stable iff they have only real roots. In particular, the characteristic polynomial P_{cc} and its Laplacian version P_L are stable because they have only real roots.

[COSW] Let Tree$(G;X) = \sum_{T \subseteq E(G)} \prod_{e \in T} X$, where T ranges over all trees of $G = (V(G), E(G))$, be the tree polynomial. Tree$(G;X)$ is Hurwitz-stable.

[COSW] Let $G = (V(G), E(G))$ be a graph and let $\bar{X}_E = (X_e : e \in E(G))$ be commutative indeterminates. Let S be a family of subsets of $E(G)$, i.e., $S \subset \wp(E(G))$ and let $\mathsf{P}_S(G;\bar{X}_E) = \sum_{A \in S} \prod_{e \in A} X_e$. If S is the family of trees of $E(G)$ then $\mathsf{P}_S(G;\bar{X}_E)$ is a multivariate version of the tree polynomial, which is also Hurwitz-stable.

[COSW] It is open for which S is the polynomial $\mathsf{P}_S(G;\bar{X}_E)$ Hurwitz-stable. Actually they ask the corresponding question for matroids $M = (E(M), S(M))$.

[HiMu] M. Hirasawa and K. Murasugi (2013) study the stability of multivariate knot polynomials .

References for stability in combinatorics

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A criterion for the half-plane property.
Discrete Mathematics, 309(6):1385–1390, 2009.

[HiMu] M. Hirasawa and K. Murasugi.
Various stabilities of the Alexander polynomials of knots and links.
arXiv preprint arXiv:1307.1578, 2013.

Stability vs distinctive power, I

Hypothesis:

Let $P(G;\bar{X})$ be a graph polynomial with integer coefficients and with SOL-definition

$$P(G;\bar{X}) = \sum_{\phi} \prod_{\psi_1} X_1 \cdot \ldots \cdot \prod_{\psi_m} X_m,$$

with coefficients $(c_{i_1,\ldots,i_m} : i_j \leq d(G), j \in [m])$

$$P(G;\bar{X}) = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots,i_m} X_1^{i_1} X^{i_2} \ldots X^{i_m} \in \mathbb{N}[\bar{X}],$$

such that in each indeterminate the degree of $P(G,\bar{X})$ is less than $d(G)$. We put $M(G) = d(G)^m$ which serves as a bound on the number of relevant coefficients, some of which can be 0.

Stability vs distinctive power, II

Theorem 1:

There is a stable graph polynomial $Q^s(G; Y, \bar{X})$ with integer coefficients such that

(i) the coefficients of $Q^s(G)$ can be computed uniformly in polynomial time from the coefficients of $P(G)$;

(ii) there is $a_0 \in \mathbb{N}$ such that $Q^s(G; a_0, \bar{X})$ is d.p.-equivalent to $P(G; \bar{X})$;

(iii) $Q^s(G; Y, \bar{X})$ is SOL-definable and its SOL-definition can be computed uniformly in polynomial time from $\phi, \psi_1, \ldots, \psi_m$.

Stability vs distinctive power, III

Theorem 2:

If additionally, $P(G; \bar{X})$ has only non-negative coefficients, there is a Hurwitz-stable graph polynomial $Q^h(G; Y, \bar{X})$ with non-negative integer coefficients and one more indeterminate Y such that

(i) The coefficients of $Q^h(G)$ can be computed uniformly in polynomial time from the coefficients of $P(G)$;

(ii) there is $\bar{a} \in \mathbb{N}^{M-n}$ such that $Q^h(G; \bar{a}, \bar{X})$ is d.p.-equivalent to $P(G; \bar{X})$;

(iii) $Q^h(G; Y, \bar{X})$ is SOL-definable and its SOL-definition can be computed uniformly in polynomial time from $\phi, \psi_1, \ldots, \psi_m$.

Go to conclusions, Back to overview

Criteria for stability

Let $\bar{X} = (X_1, \ldots, X_m)$ be indeterminates, and $\mathcal{X}$ be the diagonal matrix of n indeterminates with $(\mathcal{X})_{i,i} = X_i$.

[BoBra] For $i \in [m]$ let each A_i be a positive semi-definite Hermitian $(n \times n)$-matrix and let B be Hermitian. Then

$$f(\bar{X}) = \det(X_1 A_1 + \ldots + X_m A_m + B) \in \mathbb{R}[\bar{X}]$$

is stable.

[HeVi] For $m = 2$ and $f(X_1, X_2) \in \mathbb{R}[X_1, X_2]$ we have $f(X_1, X_2)$ is stable iff there are Hermitian matrices A_1, A_2, B with A_1, A_2 positive semi-definite such that

$$f(X_1, X_2) = \det(X_1 A_1 + X_2 A_2 + B).$$

[Br] If A is a Hermitian $(m \times m)$ matrix then the polynomials $\det(\mathcal{X} + A)$ and $\det(\mathrm{I} + A \cdot \mathcal{X})$ are real stable.

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Criteria for Hurwitz-stability

[WaWe] If $f(\bar{X}) \in \mathbb{R}[\bar{X}]$ is a real homogeneous then
$f(\bar{X})$ is stable iff $f(\bar{X})$ is Hurwitz-stable.

[COSW] Let A be a complex $(r \times m)$-matrix, A^* be its Hermitian conjugate, then the polynomial in m-indeterminates $Q(\bar{X}) = \det(A \mathcal{X} A^*)$ is multiaffine, homogeneous and Hurwitz-stable.

[Br] If B is a skew-Hermitian $(n \times n)$ matrix
then $\det(\mathcal{X} + B)$ and $\det(\mathrm{I} + B \cdot \mathcal{X})$ are Hurwitz-stable.

[COSW] Let A be a real $(r \times m)$-matrix with non-negative entries. Then the polynomial in m-indeterminates

$$Q(\bar{X}) = \mathrm{per}(AX) = \sum_{S \subseteq [m], |S| = r} \mathrm{per}(A\,|_S) \prod_{i \in S} X_i$$

is Hurwitz-stable.

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More references for multivariate stable polynomials

[HeVi] J.W. Helton and V. Vinnikov.
Linear matrix inequality representation of sets.
Communications on pure and applied mathematics, 60(5):654–674, 2007.

[Wa] D.G. Wagner.
Multivariate stable polynomials: theory and applications.
Bulletin of the American Mathematical Society, 48(1):53–84, 2011.

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Proof of Theorem 1 (sketch)

We use

> For $i \in [m]$ let each A_i be a positive semi-definite Hermitian $(n \times n)$-matrix and let B be Hermitian. Then $f(\bar{X}) = \det(X_1 A_1 + \ldots + X_m A_m + B) \in \mathbb{R}[\bar{X}]$ is stable.

Let $\alpha : \mathbb{N}^m \to \mathbb{N}$ which maps $(i_1, \ldots i_m) \in \mathbb{N}^m$ into its position in the lexicographic order of $\mathbb{N}^m$. We relabel the coefficients of $P(G; \bar{X})$ such that $d_i = c_{i_1,\ldots,i_m}$ with $\alpha(i_1, \ldots, i_m) = i, i \in [M]$ and $M = m^{d(G)}$.

We put B to be the $(M \times M)$ diagonal matrix with $B_{i,i} = d_i \cdot Y_i$ and $A_1 = A_2 = \ldots = A_m$ to be the $(M \times M)$ identity matrix. The identity matrix is both Hermitian and positive semi-definite. Furthermore, $B \mid_{Y=a} = B(a)$ being a diagonal ma trix, is Hermitian for every $a \in \mathbb{C}$. Hence,

$$Q_a^s(G; a, \bar{X}) = \det(B(a) + \sum_{i=1}^{M} X_i \cdot A_i) = \prod_{i=1}^{M} (d_i + \sum_{i=1}^{M} X_i)$$

is stable for every $a \in \mathbb{C}$.

Proof of Theorem 1 (sketch, contd)

We have to show

(i) The coeffficents can be computed fast.

(ii) How to find a.

(iii) SOL-definability.

The main difficulty is showing that $Q^s(G; Y, \bar{X})$ is SOL-definable. Back to overview

Conclusions

Are the locations of P-roots semantically meaningfull?

Our results seems to suggest:

- The location of P-roots depends strongly on the syntactic presentation of P.
- We still don't understand the particular rôle syntactic presentation of graph polynomials have to play.
- d.p. equivalence garantees that the information conveyed by coefficients or roots is inherent in every presentation.
  The choice of presentation only serves in making it more or less visible.
- Although the location of chromatic roots is easily interpretable, the same is not true for edge cover or domination roots.
- The study of P-roots needs better justifications besides mere mathematical curiosity.

The rôle of recurrence relations

The chromatic polynomial, Tutte polynomial and the matching polynomial satisfy **recurrence relations** of the type

$$P(G, X) = \alpha \cdot P(G_{-e}, X) + \beta \cdot P(G_{/e} X) + \gamma \cdot P(G_{\dagger e}, X)$$

where G_{-e} is deletion of the edge e,
$G_{/e}$ is contraction of the edge e, and
$G_{\dagger e}$ is extraction of the edge e, and
$\alpha, \beta, \gamma \in \mathbb{Z}[X]$ are suitable polynomials.

It is conceivable, and the proofs use these relations, that the location of the corresponding P-roots are intrinsically related to these recurrence relations.

Note: It is not clear how recurrence relations **behave** under d.p. equivalence.

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Note: Ilia Averbouch, PhD Thesis, Haifa, February 2011

"Completeness and Universality Properties of Graph Invariants and Graph Polynomials",

http://www.cs.technion.ac.il/ janos/RESEARCH/averbouch-PhD.pdf

Thank you for your attention!

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