

# Linear Recurrences for Graph polynomials

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## Around 1960: Boaz and Monadic Predicate Calculus

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- B.A. Trakhtenbrot,  
Some constructions in the Monadic Predicate Calculus,  
Doklady AN SSR, 138.2 (1961) pp. 320-321
- B.A. Trakhtenbrot,  
Finite Automata and the Monadic Predicate Calculus,  
Doklady AN SSR, 140.2 (1961) pp. 326-329
- B.A. Trakhtenbrot,  
Finite Automata and the Monadic Predicate Calculus,  
Siberian Mathematical Journal, 3.1 (1962) pp. 103-131.

Today the Monadic Predicate Calculus is called

Monadic Second Order Logic (MSOL)

## Monadic Second Order Logic in the 1960ties

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- R. Büchi, C. Elgot, M. Rabin, D. Scott and B. Trakhtenbrot study Finite Automata and their generalizations.
- A. Ehrenfeucht, H. Läuchli study decidability questions.  
This line of work reaches its peak with papers by
  - M. Rabin on tree-automata.
  - S. Shelah on the decidability of MSOL on linear orders.

Many follow these lines of research.

## Monadic Second Order Logic in the 1980ties

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Now interest shifts to graph algorithms.

- B. Courcelle develops a systematic study of the Monadic Second Order Logic of graphs.
- B. Courcelle shows that all MSOL-properties of graphs can be decided in polynomial time on graphs of fixed tree-width  $k$ .  
In general the problem can be arbitrarily high in the polynomial hierarchy.

This is then extended by various authors to MSOL-optimization problems and beyond.

## Monadic Second Order Logic at the turn of the century

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- B. Courcelle, J.A.M. and U. Rotics extend this type of work to graph classes of bounded clique-width.
- J.A.M. extends this work to the computation of graph polynomials.

Today's talk deals with graph polynomials.

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For a more general view, see

J.A. Makowsky,  
From a Zoo to a Zoology:  
Descriptive Complexity for Graph Polynomials.

To appear in: Arnold Beckmann, Ulrich Berger, Benedikt Löwe, and John V Tucker (eds.): Logical Approaches to Computational Barriers, Second Conference on Computability in Europe, CiE 2006, Swansea, UK, July 2006, Proceedings, LNCS vol. 3988.

## Main theme

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We deal with a purely **graph theoretical** problem:

Given a **regularly constructed** indexed family of graphs  $G_n$  such as the paths  $P_n$ , the circles  $C_n$ , the wheels  $W_n$ , the cliques  $K_n$ , the grids  $Grid_{m,n}$

and a graph polynomial  $\mathfrak{P}$ , such as the matching, Tutte, clique, cover polynomial

compute all the values  $\mathfrak{P}(G_n)$ .

Often we have a (linear) recurrence relation, i.e. there is  $q \in \mathbb{N}$ , and polynomials  $p_1, \dots, p_q \in \mathbb{Z}[\bar{X}]$  such that for sufficiently large  $n$

$$\mathfrak{P}(G_{n+q+1}) = \sum_{i=1}^q p_i \cdot \mathfrak{P}(G_{n+i})$$

When is this the case?

We shall see that methods from LOGIC help clarifying the situation.

## Graph polynomials

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Let  $\mathcal{G}$  be the class of all finite graphs, and

$\mathbb{Z}[\bar{X}]$  be a polynomial ring over  $\mathbb{Z}$  with  $\bar{X} = (X_1, \dots, X_m)$ .

A graph polynomial is a map

$$\mathfrak{P} : \mathcal{G} \rightarrow \mathbb{Z}[\bar{X}]$$

which is invariant under graph isomorphisms.

There are obvious generalizations to

- vertex-labeled and edge-labeled (signed) graphs;
- hypergraphs; and to
- relational structures.
- Knot polynomials are defined on signed graphs, the shading graphs of knot diagrams, where invariance is additionally required under the Reidemeister moves.

## Examples of graph polynomials

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Well studied graph polynomials are:

- The chromatic polynomial; the characteristic polynomial.  
(G. Birkhoff, 1912; L. Collatz and U. Sinogowitz, 1957)
- The Tutte polynomial and its colored versions  
(W.T. Tutte 1954, B. Bollobas and O. Riordan, 1999);
- The various matching polynomials;  
(O.J. Heilman and E.J. Lieb, 1972)
- Various clique and independent set polynomials  
(I. Gutman and F. Harary 1983)
- The various cover polynomials for digraphs  
(F.R.K. Chung and R.L. Graham, 1995)
- The interlace-polynomials,  
(R. Arratia, B. Bollobás and G. Sorkin, 2000)

## The matching polynomial, I

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For a graph  $G$ , the **matching polynomial**  $\mu(G, X) \in \mathbb{Z}[X]$  is defined by

$$\mu(G, X) = \sum m_k(G) \cdot X^k$$

where  $m_k(G)$  is the number of  $k$ -matchings of  $G$ .

We compute  $\mu(P_n, X)$ :

We use auxiliary polynomials

$$\mu^+(P_n, X) = \sum m_k^+(P_n) \cdot X^k$$

and

$$\mu^-(P_n, X) = \sum m_k^-(P_n) \cdot X^k$$

where  $m_k^+(P_n)$  and  $m_k^-(P_n)$  is the number of  $k$ -matchings of  $P_n$  which **includes**, respectively **excludes** the last vertex.

Clearly we have  $m_k(P_n) = m_k^+(P_n) + m_k^-(P_n)$  hence

$$\mu(P_n, X) = \mu^+(P_n, X) + \mu^-(P_n, X)$$

## The matching polynomial, II

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It is easy to see that

$$\begin{aligned}\mu^-(P_{n+1}) &= \mu^-(P_n) + \mu^+(P_n) \\ \mu^+(P_{n+1}) &= X \cdot \mu^-(P_n)\end{aligned}$$

For  $\bar{\mu}_n = (\mu^-(P_n), \mu^+(P_n))^t$  we get

$$A\bar{\mu}_n = \bar{\mu}_{n+1}$$

with

$$a_{1,1} = 1, a_{1,2} = 1, a_{2,1} = X, a_{2,2} = 0$$

The characteristic polynomial of  $A$  is

$$\det(\lambda \mathbf{1} - A) = \lambda^2 - \lambda - X$$

so we get the linear recurrence relation (independent of  $n$ )

$$\mu(P_{n+2}) = \mu(P_{n+1}) + X \cdot \mu(P_n)$$

## The vertex-cover polynomial

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For a graph  $G$ , the **vertex-cover polynomial**  $vc(G, X) \in \mathbb{Z}[X]$  is defined by

$$vc(G, X) = \sum vc_k(G) \cdot X^k$$

where  $vc_k(G)$  is the number of  $k$ -vertex-covers of  $G$ .

- $vc(P_{n+1}, X) = X \cdot vc(P_n, X) + X \cdot vc(P_{n-1}, X)$
- $vc(C_{n+1}, X) = X \cdot vc(C_n, X) + X^2 \cdot vc(C_{n-2}, X)$
- Let  $L_n$  be the graph which consists of  $n$  isolated loops.  
 $vc(L_{n+1}, X) = X \cdot vc(L_n, X) = X^n$
- For the wheel graph  $W_n$  we have  
 $vc(W_{n+1}, X) = X \cdot vc(W_n, X) + X^n = X \cdot vc(W_n, X) + X \cdot vc(L_n, X)$   
 hence, using the characteristic polynomial of the matrix,  $A = (a_{i,j})$  with  
 $a_{1,1} = a_{1,2} = a_{2,2} = X$  and  $a_{2,1} = 0$   
 $vc(W_{n+1}, X) = 2X \cdot vc(W_n, X) - X^2 \cdot vc(W_{n-1}, X)$

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F.M. Dong, M.D. Hendy, K.L. Teo and C.H.C. Little, The vertex-cover polynomial of a graph, Discrete Mathematics 250 (2002), 71-78

## $\mathfrak{P}$ -recursive families of graphs, I

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Let  $\mathfrak{P}$  be a graph polynomial and  $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$  be a family of graphs.

$\mathcal{G}$  is said to be  **$\mathfrak{P}$ -recursive** if there is  $q \in \mathbb{N}$ , and polynomials  $p_1, \dots, p_q \in \mathbb{Z}[\bar{X}]$  such that for sufficiently large  $n$

$$\mathfrak{P}(G_{n+q+1}) = \sum_{i=1}^q p_i \cdot \mathfrak{P}(G_{n+i})$$

Let  $P_n$  be the path on  $n$  vertices. We get, for sufficiently large  $n$ ,

- for the chromatic polynomials:  $c(P_{n+1})(\lambda) = c(P_n, \lambda)(\lambda) \cdot (\lambda - 1)$ .
- for the clique polynomials:  
 $cl(P_{n+1})(X) = (1 + X) + cl(P_n)(X)$ ,  
 $cl(K_{n+1})(X) = \sum_k^{n+1} \binom{n}{k} X^k = (X + 1)^{n+1} = (X + 1) \cdot cl(K_n)(X)$
- for the matching polynomials:  $\mu(P_{n+1})(X) = X \cdot \mu(P_{n-1})(X) + \mu(P_n)(X)$ ,
- for the Tutte polynomials:  $T(P_{n+1})(X, Y) = Y \cdot T(P_n)(X, Y)$ .
- for the vertex-cover polynomials:  $vc(P_{n+1}, X) = X \cdot vc(P_n, X) + X \cdot vc(P_{n-1}, X)$

## Previous work, I

N.L. Biggs, R.M. Damerell and D.A. Sands, 1972

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In 1972 N.L. Biggs, R.M. Damerell and D.A. Sands introduced recursive families of graphs.

These are our ***T*-recursive families of graphs** where  $T$  is the Tutte polynomial.

They show that several families of graphs are recursive (in their sense). Among them there are:

cycles, ladders and wheels

All these families have in common that they can be constructed from an initial graph by the repeated application of a fixed graph operation

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N.L. Biggs, R.M. Damerell and D.A. Sands,  
Recursive families of graphs, J. Combin. Theory Ser. B 12 (1972), 123-131

## Previous work, II

M. Noy and A. Ribó, 2004

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In 2004 M. Noy and A. Ribó study which graph families  $G_n$ , constructed from an initial graph  $G_0$ , by the repeated application of a fixed graph operation  $F(G)$ , are ***T*-recursive families of graphs**.

They introduce a notion of **recursively constructible families of graphs**, and show that every such family is *T*-recursive.

Their notion is reminiscent of certain graph grammars.

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M. Noy and A. Ribó, Recursively constructible families of graphs, Advances in Applied Mathematics 32 (2004) 350-363.

## Our work

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We use the finite model theory of Monadic Second Order Logic (MSOL) to extend these results in several ways:

- We prove that for every  $\mathfrak{P}$  from a wide class of graph polynomials, the MSOL-definable graph polynomials, every recursively constructible family  $G_n$  is  $\mathfrak{P}$ -recursive.
- We extend the result to the class of iteration families of graphs which is proper extension of the class of recursively constructible families.
- We extend the result to signed graphs and knot diagrams and to various knot polynomials.
- We extend the result to hypergraphs and relational structures.

## Recursively constructible families, I

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In the absence of the formalisms of graph grammars Noy and Ribó give an adhoc definition of

repeated fixed succession  
of elementary operations

which can be applied to a graph with a *context*, i.e. a labeled graph.

Let  $F$  denote such an operation.

Given a graph (with context)  $G$ , we put

$$G_0 = G, G_{n+1} = F(G_n)$$

Then the family

$$\mathcal{G} = \{G_n : n \in \mathbb{N}\}$$

is called **recursively constructible** using  $F$ , or an  **$F$ -iteration family**.

## Recursively constructible families, II

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Given a graph polynomial  $\mathfrak{P}$ ,

the question now is to find

a characterization of those operations  $F$ ,

for which a linear recurrence for the polynomials  $\mathfrak{P}(G_n)$  holds.

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M. Noy and A. Ribó give only a **sufficient condition**  
in the case of the **Tutte polynomial**.

## General strategy

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We proceed as outlined in the case of the matching and the vertex-cover polynomial.

To compute  $\mathfrak{P}(G_{n+1})$ , we try to find, depending on  $\mathfrak{P}$  and, possibly, on  $G_0$ , but **independently of  $n$**

- an  $m \in \mathbb{N}$ ,
- auxiliary polynomials  $\mathfrak{P}_i(G_{n+1}), i \leq m$ ,
- and a matrix  $A = (a_{i,j}) \in \mathbb{Z}[\bar{X}]^{m \times m}$

such that

$$\mathfrak{P}_j(G_{n+1})(\bar{X}) = \sum_i a_{i,j}(\bar{X}) \cdot \mathfrak{P}_i(G_n)(\bar{X})$$

Then we use the **characteristic polynomial of  $A$**  to convert this into a **linear recurrence** relation.

## Where logic enters for the polynomials?

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The polynomial is of the form

$$\mathfrak{P}(G) = \sum_{E' \subseteq E: (V, E') \in K_1} \left( \prod_{e \in E'} t_e(\bar{X}) \right)$$

or

$$\mathfrak{P}(G) = \sum_{(V' \subseteq V: (V', E|_{V'}) \in K_2} \left( \prod_{v \in V'} t_v(\bar{X}) \right)$$

where  $t_e(\bar{X})$  and  $T_v(\bar{X})$  are fixed terms in the indeterminates  $\bar{X}$  and  $K_1$  or  $K_2$  are definable in

### Monadic Second Order Logic (MSOL).

We call them *MSOL-definable graph polynomials*.

There are more general versions.

## *MSOL*-definable polynomials, I

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The **vertex-cover polynomials** are of the form

$$\mathfrak{P}(G) = \sum_{V':(V,EV') \in K_2} \left( \prod_{v \in V'} X \right)$$

because saying that  $V'$  is a vertex-cover of  $(V, E)$  is *MSOL*-definable.

Rearranging the terms we get

$$vc(G) = \sum_{V':(V,E,V') \in VC} \left( \prod_{v \in V'} X \right) = \sum_k vc_k(G) \cdot X^k$$

**Note:** The second order variable for  $V'$  is needed.

## *MSOL*-definable polynomials, II

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The **matching polynomial** is of the form

$$\mu(G) = \sum_{E': (V, E') \in \text{Matching}} \left( \prod_{e \in E'} X \right)$$

However, being a matching is

- NOT *MSOL*-definable if graphs are represented as  $G = (V, E)$ .
- but IS *MSOL*-definable, if the graph is represented by its incidence graph  $I(G) = (V \cup E, R)$ .

For the **Tutte polynomial**, we have to add a linear order on the edges, to make it *MSOL*-definable, and note, that the Tutte polynomial is then independent of the order on the edges.

## Where logic enters in the operation $F$ ?

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Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two  $\tau$ -structures. We write  $\mathfrak{A} \equiv_q^{MSOL} \mathfrak{B}$ , if  $\mathfrak{A}$  and  $\mathfrak{B}$  cannot be distinguished by  $MSOL(\tau)$ -formulas of quantifier rank  $q$ .

An operation  $F$  on  $\tau$ -structures is

### ***MSOL-smooth***

if whenever  $\mathfrak{A} \equiv_q^{MSOL} \mathfrak{B}$ , then also  $F(\mathfrak{A}) \equiv_q^{MSOL} F(\mathfrak{B})$ .

The operation  $F$  should be *MSOL-smooth* for the presentation of the graphs, for which the polynomial is *MSOL-definable*.

For forming the cliques  $K_n$  we need the operation of adding a vertex connected to all previous vertices.

This is *MSOL-smooth* for  $G = (V, E)$  but not for  $I(G) = (V \cup E, R)$ .

## $k$ -structures

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A vocabulary  $\tau$  is a set of relation symbols.

A  $\tau$ -structure  $\mathfrak{A}$   
is an interpretation of the vocabulary  $\tau$  over a non-empty universe  $A$ .

For  $k \in \mathbb{N}$ , a  $k$ - $\tau$ -structure is a  $\tau$ -structure with  $k$  additional unary relations  $C_1^A, \dots, C_k^A$ , called colours.

We denote by  $\tau_k$  the vocabulary  $\tau \cup \{C_1, \dots, C_k\}$ .

## Basic operations on $k$ - $\tau$ -structures

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$Add_{C_i}(\mathfrak{A})$ : For  $i \leq k$ , add a new element to  $A$  of colour  $C_i$ .

$\rho_{i,j}(\mathfrak{A})$ : For  $i, j \leq k$ , recolour all elements of  $A$  of colour  $i$  with colour  $j$ .

$\eta_{R,i_1,\dots,i_m}(\mathfrak{A})$ : For  $R \in \tau$  an  $m$ -ary relation symbol  
and for each  $a_1 \in C_{i_1}^A, \dots, a_m \in C_{i_m}^A$   
add the tuple  $(a_1, \dots, a_m)$  to  $R^A$ .

$\delta_{R,i_1,\dots,i_m}(\mathfrak{A})$ : For  $R \in \tau$  an  $m$ -ary relation symbol  
and for each  $a_1 \in C_{i_1}^A, \dots, a_m \in C_{i_m}^A$   
delete the tuple  $(a_1, \dots, a_m)$  from  $R^A$ .

### Quantifierfree transductions:

For each  $R \in \tau_k$  of arity  $\alpha(R)$

let  $\phi_R(x_1, \dots, x_{\alpha(R)})$  be

a quantifierfree  $\tau_k$  formula with free variables as indicated.

A quantifier free transduction redefines all the predicates  $R^A$  in  $\mathfrak{A}$  by  $\phi_R^A$ .

## *MSOL*-elementary and *MSOL*-smooth operations

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An operation  $F$  on  $\tau_k$ -structures is *MSOL*-elementary if  $F$  is a finite composition of basic operations on  $\tau_k$ -structures.

### **Proposition:**

Let  $F$  be *MSOL*-elementary and  $\mathfrak{A}$  and  $\mathfrak{B}$  two  $\tau_k$  structures with  $\mathfrak{A} \equiv_q^{MSOL} \mathfrak{B}$ , then  $F(\mathfrak{A}) \equiv_q^{MSOL} F(\mathfrak{B})$ ,  
Hence,  $F$  is a *MSOL*-smooth.

### **Proposition:**

Let  $F$  be *MSOL*-elementary and  $\mathcal{G}$  be an  $F$ -iteration family. Then  $\mathcal{G}$  is of bounded clique-width.

### **Corollary:**

$I(K_n)$ ,  $Grid_{n,n}$  are not  $F$ -iteration families for any  $F$  which is *MSOL*-elementary.

## Main Result

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**THEOREM:** Let

- $F$  be an *MSOL*-smooth operation on  $\tau_k$ -structures.
- $\mathfrak{P}$  be a  $\tau$ -polynomial which is *MSOL*( $\tau$ )-definable.
- $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$  be an  $F$ -iteration family of  $\tau$ -structures.

Then  $\mathcal{A}$  is  $\mathfrak{P}$ -recursive, i.e. there is  $q \in \mathbb{N}$ , and polynomials  $p_1, \dots, p_q \in \mathbb{Z}[\bar{X}]$  such that for sufficiently large  $n$

$$\mathfrak{P}(G_{n+q+1}) = \sum_{i=1}^q p_i \cdot \mathfrak{P}(G_{n+i})$$

## Proof ingredients

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- For fixed  $q$  and a fixed number of free variables, there are, up to logical equivalence, only finitely many  $MSOL(\tau)$ -formulas of quantifier rank  $q$ .  
Let  $\bar{\mathfrak{P}} = (\bar{\mathfrak{P}}_1, \dots, \bar{\mathfrak{P}}_\alpha)$  be the vector of all  $MSOL(\tau)^q$ -definable polynomials.

- Feferman-Vaught Theorem for  $MSOL$ -definable graph polynomials

J.A. Makowsky, Algorithmic uses of the Feferman-Vaught Theorem, *Annals of Pure and Applied Logic*, 126 (2004), 159-213

- Bilinear version of the Feferman-Vaught Theorem for graph polynomials.  
With an  $MSOL$ -elementary operation  $F$  and a fixed  $q$  there is a matrix  $M_F$  such that

$$\bar{\mathfrak{P}}(F(G)) = M_F \cdot \bar{\mathfrak{P}}(G)$$

- Use the characteristic polynomial of  $M_F$ .