Lecture 8-9
What did we do so far?

• Satisfiability, validity

• Logical equivalence

• Normal forms ($\text{NNF}$, $\text{CNF}$, $\text{DNF}$)

• Checking logical equivalence

To outline of Lecture 8-9
Outline of lecture 8 and 9

- Deduction systems
- Proof sequences
- Soundness and completeness of proof systems
- Deduction theorem
- Completeness of proof system
- Compactness
- Definability
Deduction Systems

We present a method of deduction:

Proof sequences

(Hilbert style deduction)

It is supposed to model human reasoning.

This is an exageration:
In the best case it models the way mathematicians and other educated people should write down their arguments, when pressed to do so.

In short, it models a stylized form of human reasoning.

In this sense it is quite user friendly.

Back to outline of Lecture 8-9
Deduction Systems: Soundness and completeness

Let $\text{WFF}_{myown} \subseteq \text{WFF}$ be such that for every $\phi \in \text{WFF}$ there is a $\psi \in \text{WFF}_{myown}$ with $\phi$ logically equivalent to $\psi$.

Recall: For $\Sigma \subseteq \text{WFF}_{myown}$ we set
\[ \text{Con}(\Sigma) = \{ \phi \in \text{WFF}_{myown} : \Sigma \models \phi \} \]

An inductive definition of $\text{Ded}(\Sigma)$ for $\text{Con}(\Sigma)$ is sound (for $\text{Con}(\Sigma)$) if
\[ \text{Ded}(\Sigma) \subseteq \text{Con}(\Sigma). \]
It is complete (for $\text{Con}(\Sigma)$) if
\[ \text{Con}(\Sigma) \subseteq \text{Ded}(\Sigma). \]
Many Deduction Systems

There are many sound and complete inductive definitions of $\text{Con}(\Sigma)$.

**Natural deduction, Gentzen style:**
- Trivial basis ($\Sigma \subseteq \text{Ded}(\Sigma)$)
- Many cases in the closure.
- Close to mathematical reasoning

**Hilbert style:**
- Many cases in the basis,
  only one case (Modus ponens) in the closure
- Not so close to mathematical reasoning

**minimal Hilbert style, our systems:**
- Few operators, small basis
  only one case (Modus ponens) in the closure
- Good for analysing the system.
David Hilbert and Gerhard Gentzen

P. Bernays was the real logician working for D. Hilbert and supervising G. Gentzen

Go directly to Proof sequences for $\text{WFF}_{\rightarrow, F}$
Deduction Systems: first try

For $\Sigma \subseteq \text{WFF}$ we could start to define $Ded(\Sigma)$ as follows:

**Basis:**
$T \in Ded(\Sigma)$, $\Sigma \subseteq Ded(\Sigma)$

**Closure:**
If $\phi, \psi \in Ded(\Sigma)$ so also $(\phi \land \psi)$ and $(\psi \land \phi)$.
If $(\phi \land \psi) \in Ded(\Sigma)$ so also $\phi$ and $\psi$.
If $\phi, \psi \in Ded(\Sigma)$ so also $(\phi \rightarrow \psi)$ and $(\psi \rightarrow \phi)$.
If $\phi, (\phi \rightarrow \psi) \in Ded(\Sigma)$ so also $\psi$.
If $\phi \in \text{WFF}, \psi \in Ded(\Sigma)$ so also $(\phi \lor \psi) \in Ded(\Sigma)$ and $(\psi \lor \phi) \in Ded(\Sigma)$.
If $F, (\phi \rightarrow F) \in Ded(\Sigma)$ so also $\phi \in Ded(\Sigma)$.
If $\phi \in \text{WFF}, F \in Ded(\Sigma)$ so also $\phi \in Ded(\Sigma)$.
If $\phi, \neg \phi \in Ded(\Sigma)$, so $F \in Ded(\Sigma)$.

*is this correct? is this enough?*

There will be many cases in the closure. We want to make our system complete but as simple as possible.

Go directly to Proof sequences for $\text{WFF}\{\rightarrow, F\}$
Deduction Systems: second try

For $\Sigma \subseteq \text{WFF}$ we could start to define $\text{Ded}(\Sigma)$ as follows:

**Basis:**
$T \in \text{Ded}(\Sigma), \Sigma \subseteq \text{Ded}(\Sigma)$

$(\phi \to (\psi \to \phi))$

$(((\phi \to (\psi \to \theta))) \to (((\phi \to \psi) \to (\phi \to \theta)))$

$((\neg \psi \to \neg \phi) \to ((\neg \psi \to \phi) \to \psi))$

$((\phi \land \psi) \to \phi) \in \text{Ded}(\Sigma)$ and $((\phi \land \psi) \to \psi) \in \text{Ded}(\Sigma)$

$(\phi \to (\phi \lor \psi)) \in \text{Ded}(\Sigma)$ and $(\psi \to (\phi \lor \psi)) \in \text{Ded}(\Sigma)$

**is this enough?**

**Closure:**

If $F, (\phi \to F) \in \text{Ded}(\Sigma)$ so also $\phi \in \text{Ded}(\Sigma)$.

**is this enough?**

There will be many cases in the basis.
We want to make our system complete but as simple as possible.

Go directly to Proof sequences for $\text{WFF}\{\to, F\}$
For $\Sigma \subseteq \text{WFF}$ let $Ded_1(\Sigma)$ be defined by

**Basis:** $\Sigma \subseteq Ded_1(\Sigma)$

**Closure:**
- If $\phi, \psi \in Ded_1(\Sigma)$ so also $(\phi \land \psi)$ and $(\psi \land \phi)$.
- If $(\phi \land \psi) \in Ded_1(\Sigma)$ so also $\phi$ and $\psi$.
- If $\phi, \psi \in Ded_1(\Sigma)$ so also $(\phi \rightarrow \psi)$ and $(\psi \rightarrow \phi)$.
- If $\phi, (\phi \rightarrow \psi) \in Ded_1(\Sigma)$ so also $\psi$.
- If $\phi \in \text{WFF}, \psi \in Ded_1(\Sigma)$ so also $(\phi \lor \psi) \in Ded_1(\Sigma)$ and $(\psi \lor \phi) \in Ded_1(\Sigma)$.
- If $F, (\phi \rightarrow F) \in Ded_1(\Sigma)$ so also $\phi \in Ded_1(\Sigma)$.
- If $\phi \in \text{WFF}, F \in Ded_1(\Sigma)$ so also $\phi \in Ded_1(\Sigma)$.
- If $\phi, \neg \phi \in Ded_1(\Sigma)$, so $F \in Ded_1(\Sigma)$.

Show that $Ded_1(\Sigma) \subseteq Con(\Sigma)$ but $Con(\Sigma) \not\subseteq Ded_1(\Sigma)$.

Go directly to Proof sequences for $\text{WFF}\{\rightarrow, F\}$.
Homework

A sound and complete system

For $\Sigma \subseteq \text{WFF}$ let $\text{Ded}_2(\Sigma)$ be defined by

**Basis:**

$T \in \text{Ded}_2(\Sigma), \Sigma \subseteq \text{Ded}_2(\Sigma)$, and all of the following are in $\text{Ded}_2(\Sigma)$:

- $(\phi \rightarrow (\psi \rightarrow \phi))$
- $((\phi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \theta)))$
- $((\neg \psi \rightarrow \neg \phi) \rightarrow ((\neg \psi \rightarrow \phi) \rightarrow \psi))$
- $((\phi \land \psi) \rightarrow \phi)$ and $((\phi \land \psi) \rightarrow \psi)$
- $(\phi \rightarrow (\phi \lor \psi))$ and $(\psi \rightarrow (\phi \lor \psi))$

**Closure:**

If $\phi, (\phi \rightarrow \psi) \in \text{Ded}(\Sigma)$ so also $\psi \in \text{Ded}(\Sigma)$.

Show that $\text{Ded}_2(\Sigma) \subseteq \text{Con}(\Sigma)$.

$\text{Ded}_2(\Sigma)$ is also complete, i.e. $\text{Con}(\Sigma) \subseteq \text{Ded}_2(\Sigma)$.

You can show this later using the Completeness Theorem 25.

**Go directly to Proof sequences for** $\text{WFF}_{\{\rightarrow, \land, \lor, \neg\}}$
Deduction Systems: Resolution

In the booklet you will also find Resolution used in AI, Logic Programming and Datalog.

It is defined for formulas in **CNF** only, and tests satisfiability, rather than logical consequence.

Resolution is more fit for machine implementations, and in this sense it is more machine friendly.
Proof Sequences $\text{WFF}\{\rightarrow, F\}$

To keep things simple we restrict ourselves to the case of propositional formulas in $\text{WFF}\{\rightarrow, F\} \subseteq \text{WFF}$

Let $\Sigma \subseteq \text{WFF}\{\rightarrow, F\}$ be a set of formulas.

We first shall define inductively the set $Ded(\Sigma)$ of formulas deducible from $\Sigma$.

A generating sequence of formulas

$$\phi_1, \phi_2, \ldots \phi_n = \phi$$

which shows that $\phi \in Ded(\Sigma)$ will be called a


deduction of $\phi$ from $\Sigma$

or a proof sequence for $\phi$ using $\Sigma$. 

8s-2014
Why $\text{WFF}\{\rightarrow, F\}$?

The restriction to formulas in $\text{WFF}\{\rightarrow, F\}$ is not really a restriction.

**Proposition 1**

*Every formula $\phi \in \text{WFF}$ is logically equivalent to a formula $\psi$ such that*

(i) $\psi \in \text{WFF}\{\rightarrow, F\}$ and

(ii) $\phi$ and $\psi$ have the same variables.

**Proof:**

Use that $\neg \phi$ is equivalent to $(\phi \rightarrow F)$ and $(\phi \lor \psi)$ is equivalent to $(((\phi \rightarrow F) \rightarrow \psi)$

Q.E.D.

The same holds for $\text{WFF}\{\rightarrow, \neg\}$ or $\text{WFF}\{\land, \neg\}$ or $\text{WFF}\{\lor, \neg\}$. 

8s-2014
The Axioms for $F, \rightarrow$

(after Lyndon)

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**Definition 2 (3.3.1)**

For every $\phi, \psi, \theta \in \text{WFF}_{\{\rightarrow,F\}}$ the following formulas are axioms.

**A1:** $(\phi \rightarrow (\psi \rightarrow \phi))$

**A2:** $((\phi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \theta)))$

**A3:** $(((\phi \rightarrow F) \rightarrow F) \rightarrow \phi)$

We have seen already

**Proposition 3 (3.2.3)**

The axioms are logically valid (tautologies)
The Axioms for \( \neg, \rightarrow \)

(after Mendelson)

**Definition 4 (in other courses)**
For every \( \phi, \psi, \theta \in \text{WFF}\{\rightarrow, \neg}\) the following formulas are axioms.

**AA1:** \((\phi \rightarrow (\psi \rightarrow \phi))\)

**AA2:** \(((\phi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \theta)))\)

**AA3:** \(((\neg \psi \rightarrow \neg \phi) \rightarrow ((\neg \psi \rightarrow \phi) \rightarrow \psi))\)

It is easy to see (as A1=AA1, A2=AA2)

**Proposition 5**
The axioms are logically valid (tautologies)
**Definition 6 (3.3.2)**

*Deducible formulas*

Let $\Sigma \subseteq \text{WFF}\{\rightarrow, F\}$ be a set of formulas. We define inductively the set $Ded(\Sigma)$:

**Basis:**

(i) $\Sigma \subseteq Ded(\Sigma)$;

(ii) If $\phi \in \text{WFF}\{\rightarrow, F\}$ is an *axiom* then $\phi \in Ded(\Sigma)$.

**Closure:** (Modus ponens)

If $\phi \in Ded(\Sigma)$ and $(\phi \rightarrow \psi) \in Ded(\Sigma)$ then $\psi \in Ded(\Sigma)$. 
Definition 7 (3.3.5)

Proof Sequence

To show that a formula $\phi \in Ded(\Sigma)$ one has to unwind the inductive definition of $Ded(\Sigma)$. Such an unwinding (generating sequence) will be called a proof sequence.

Let $\Sigma \subseteq WFF_{\{\rightarrow, F\}}$ be a set of formulas and $\phi_1, \ldots, \phi_n$ be formulas in $WFF_{\{\rightarrow, F\}}$. We say that $\phi_1, \ldots, \phi_n$ is a proof sequence over $\Sigma$ if for each $i \leq n$ either

(i) $\phi_i$ is an axiom or $\phi_i \in \Sigma$ or

(ii) (Modus ponens) there are $k, l < i$ such that $\phi_l = (\phi_k \rightarrow \phi_i)$

We write $\Sigma \vdash \phi$ if there is a proof sequence $\phi_1, \ldots, \phi_n$ over $\Sigma$ such that $\phi_n = \phi$. 
Proposition-Exercise 8 (3.3.6)

Let $\Sigma \subseteq \text{WFF}\{\rightarrow, F\}$ be a set of formulas and $\phi \in \text{WFF}\{\rightarrow, F\}$.

Then $\phi \in Ded(\Sigma)$ iff $\Sigma \vdash \phi$. 
Remark 9

To check whether $\phi \in Ded(\Sigma)$ or $\Sigma \vdash \phi$ may be hard:

We have to FIND the proof sequence (and we don’t know how long it is)

To check whether a sequence

$$\phi_1, \phi_2, \ldots, \phi_n$$

is a proof sequence over $\Sigma$ is simple.

Recognizing axioms is done best for formulas in tree form $WFTF$. 
Example 10

Let $\Sigma = \{((\phi \rightarrow F) \rightarrow F), (\phi \rightarrow \psi), (\psi \rightarrow \theta)\}$.  

We show $\Sigma \vdash \theta$.

\[ (((\phi \rightarrow F) \rightarrow F) \rightarrow \phi) \quad (1,A3) \]

\[ ((\phi \rightarrow F) \rightarrow F) \quad (2, \in \Sigma) \]

\[ \phi \quad (3, \text{MP}(1,2)) \]

\[ (\phi \rightarrow \psi) \quad (4, \in \Sigma) \]

\[ \psi \quad (5, \text{MP}(3,4)) \]

\[ (\psi \rightarrow \theta) \quad (6, \in \Sigma) \]

\[ \theta \quad (7, \text{MP}(5,6)) \]
Example 11

Let $\Sigma = \{\phi, (\phi \rightarrow F)\}$. We show for every $\theta$ $\Sigma \vdash \theta$.

\[
\begin{align*}
\phi & \\
(\phi \rightarrow F) & \quad (1, \epsilon \ Sigma) \\
F & \quad (2, \epsilon \ Sigma) \\
(\neg \phi \rightarrow \neg F) & \quad (3, \text{MP}(1,2)) \\
(\neg \phi \rightarrow F) & \quad (4, \text{A1})
\end{align*}
\]

with $\phi = F, \psi = (\theta \rightarrow F')$

\[
\begin{align*}
((\theta \rightarrow F) \rightarrow F) & \quad (5, \text{MP}(3,4)) \\
(((\theta \rightarrow F) \rightarrow F) \rightarrow \theta) & \quad (6, \text{A3})
\end{align*}
\]

with $\phi = \theta$

\[
\begin{align*}
\theta & \quad (7, \text{MP}(6,5))
\end{align*}
\]
Soundness and Completeness, once more

$\text{Ded}(\Sigma)$ turns out to be an inductive definition of the logical consequences of $\Sigma$.

**Soundness:**
If $\phi \in \text{Ded}(\Sigma)$ then $\phi \in \text{Con}(\Sigma)$
or, equivalently
If $\Sigma \vdash \phi$ then $\Sigma \models \phi$

**Completeness:**
If $\phi \in \text{Con}(\Sigma)$ then $\phi \in \text{Ded}(\Sigma)$
or, equivalently
If $\Sigma \models \phi$ then $\Sigma \vdash \phi$
Proposition-Exercise 12 (3.3.3 and 3.3.7)

Soundness of Ded(Σ)

Let \( \Sigma \subseteq \text{WFF}_{\rightarrow, F} \) be a set of formulas and \( \phi \in \text{WFF}_{\rightarrow, F} \).

If \( \phi \in \text{Ded}(\Sigma) \) then \( \Sigma \models \phi \).

Proof:

- Induction on the definition of \( \text{Ded}(\Sigma) \).
- The axioms are tautologies (3.2.3)
- For \( \Sigma \) this is trivial.
- For Modus ponens we seen this in Lecture 7, slide 16, Proposition 14 (3.2.10).

Q.E.D.
Examples 13 (3.3.4 and 3.3.8)

Prove the following statements:

(i) \( \text{Ded}(\emptyset) \) is a subset of the tautologies of \( \text{WFF}\{\rightarrow, F\} \).

\[ \ldots \]
If \( \emptyset \vdash \phi \) then \( \phi \) is a tautology.

(ii) \( \text{Ded}(\{F\}) = \text{WFF}\{\rightarrow, F\} \)

\[ \ldots \]
For every \( \phi \in \text{WFF}\{\rightarrow, F\} \) we have that \( F \vdash \phi \).

(iii) Let \( \Sigma \) be infinite and \( \phi \in \text{Ded}(\Sigma) \).
Then there is a finite subset \( \Sigma_0 \subseteq \Sigma \) such that \( \phi \in \text{Ded}(\Sigma_0) \).

\[ \ldots \]
Let \( \Sigma \) be infinite and \( \Sigma \vdash \phi \) there is a finite subset \( \Sigma_0 \subseteq \Sigma \) such that \( \Sigma_0 \vdash \phi \).
**Definition 14 (3.3.9)**

*Consistency*

We say that $\Sigma \subseteq \text{WFF}\{\to, F\}$ is *inconsistent* if $\Sigma \vdash F$.

If $\Sigma$ is *not* inconsistent, we say that $\Sigma$ is *consistent*.

$\Sigma$ is consistent iff $F \notin \text{Ded}(\Sigma)$ iff there is no proof sequence $\phi_1, \phi_2, \ldots, \phi_n$ over $\Sigma$ with $\phi_n \models F$.

**Remark 15 (3.3.10)**

*By the soundness of proof sequences,*

*if $\Sigma$ is inconsistent, then $\Sigma$ is not satisfiable.*
Proposition 16 (3.3.14)

Substitution in Proof Sequences

Proof sequences capture the essence of proofs and can be used for similar formulas in the following sense:

Let $\Sigma \vdash \phi$ and $s : \text{Var} \rightarrow \text{WFF}$ be a substitution. Then

$$\text{subst}(\Sigma, s) = \{ \text{subst}(\psi, s) : \psi \in \Sigma \} \vdash \text{subst}(\phi, s).$$

Proof:

If

$$\phi_1, \phi_2, \ldots \phi_n = \phi$$

is a proof sequence over $\Sigma$, so

$$\text{subst}(\phi_1, s), \text{subst}(\phi_2, s), \ldots \text{subst}(\phi_n, s) = \text{subst}(\phi, s)$$

is a proof sequence over $\text{subst}(\Sigma, s)$.

We have to verify this for axioms and Modus Ponens (exercise). Q.E.D.
Proposition 17 (3.3.11)  
Manipulations of Proof Sequences

The following are useful manipulations of proof sequences.

Let $\Sigma_0 \subseteq \Sigma \subseteq \text{WFF}\{\rightarrow, F\}$ and $\phi \in \text{WFF}\{\rightarrow, F\}$.

(i) If $\Sigma_0 \vdash \phi$ then $\Sigma \vdash \phi$;

(ii) If $\phi_1, \phi_2, \ldots, \phi_n$ is a proof sequence over $\Sigma$ then for each $i \leq n$ we have that $\Sigma \vdash \phi_i$.

(iii) If $\Sigma \vdash \phi$ and $\Sigma \vdash (\phi \rightarrow \psi)$ then $\Sigma \vdash \psi$.

(iv) If $\Sigma \vdash (\phi \rightarrow (\theta \rightarrow \psi))$ and $\Sigma \vdash (\phi \rightarrow \theta)$ then $\Sigma \vdash (\phi \rightarrow \psi)$.
Proof of Proposition 17 (iv)

Proof: To show (iv) in proposition 17 it suffices to show
\[ \{ (\phi \to (\theta \to \psi)), (\phi \to \theta) \} \vdash (\phi \to \psi) \]

We proceed as follows:

\[ (((\phi \to (\theta \to \psi))) \to (((\phi \to \theta) \to (\phi \to \psi)))) \] (1,A2)
\[ (\phi \to (\theta \to \psi)) \] (2,\( \in \Sigma \))
\[ (((\phi \to \theta) \to (\phi \to \psi)) \] (3,MP(1,2))
\[ (\phi \to \theta) \] (4,\( \in \Sigma \))
\[ (\phi \to \psi) \] (5,MP(4,3))

Q.E.D.
Theorem 18 (3.3.12)

Deduction Theorem

Let \( \Sigma \subseteq \text{WFF}\{\rightarrow, F\} \) be a set of formulas and \( \phi, \psi \in \text{WFF}\{\rightarrow, F\} \).

\( \Sigma \vdash (\phi \rightarrow \psi) \) iff \( \Sigma \cup \{\phi\} \vdash \psi \).

The proof is part of the Tirgulim

Skip the proof of the Deduction Theorem
Proof of Theorem 18, I

(i) Assume $\Sigma \vdash (\phi \rightarrow \psi)$.

We have to prove that $\Sigma \cup \{\phi\} \vdash \psi$.

By proposition 17 (i) we have
$\Sigma \cup \{\phi\} \vdash (\phi \rightarrow \psi)$ and, using modus ponens,
$\Sigma \cup \{\phi\} \vdash \psi$. 
Proof of Theorem 18, II

(ii) Assume $\Sigma \cup \{\phi\} \vdash \psi$.

We have to show that $\Sigma \vdash (\phi \rightarrow \psi)$.

Equivalently, we can show that $(\phi \rightarrow \psi) \in Ded(\Sigma)$.

Let $K \subseteq Ded(\Sigma \cup \{\phi\})$ be the set of formulas $\psi$ such that $\Sigma \vdash (\phi \rightarrow \psi)$.

We show by induction that $K = Ded(\Sigma \cup \{\phi\})$. 
Proof of Theorem 18, III

**Basis:**
Σ ⊆ K and all the axioms are in K (Exercise).

In the case that ψ = φ we use that ∅ ⊢ (φ → φ).

**Closure:**
Assume θ and (θ → ψ) are in K.

We have to show that ψ ∈ K.

By assumption Σ ⊢ (φ → θ) and Σ ⊢ (φ → (θ → ψ)).

Therefore, by proposition 17 (iv), Σ ⊢ (φ → ψ). Q.E.D.
Theorem 19 (3.3.13)
Dychotomy Theorem

Let $\Sigma \subseteq \textsf{WFF}\{\rightarrow, F\}$ be a set of formulas and $\phi, \psi \in \textsf{WFF}\{\rightarrow, F\}$.

If both $\Sigma \cup \{\phi\} \vdash \psi$ and $\Sigma \cup \{(\phi \rightarrow F)\} \vdash \psi$.

Then $\Sigma \vdash \psi$. 

Proof of the Dychotomy Theorem \[19, I\]

Use the Deduction Theorem to prove

\[\{(\phi \to \psi), (\psi \to \theta)\} \vdash (\phi \to \theta)\]

Then use this to prove the following three tautologies:

\[(((\phi \to \psi) \to ((\psi \to F) \to (\phi \to F)))\]

\[(((\phi \to F) \to \psi) \to ((\psi \to F) \to ((\phi \to F) \to F)))\]

\[(((\psi \to F) \to ((\phi \to F) \to F)) \to (((\psi \to F) \to (\phi \to F)) \to \psi))\]

Then use these three tautologies and Modus Ponens to prove the Dychotomy Theorem.

Q.E.D.
Example 20

We show $\emptyset \vdash (\phi \rightarrow \phi)$:

With Deduction theorem it suffices to show $\phi \vdash \phi$ which is trivial.

******

Without Deduction Theorem we proceed as follows:

\[
(((\phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi)) \\
\rightarrow ((\phi \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi)))
\]

(1,A2)

with $\phi = \theta, \psi = (\theta \rightarrow \theta)$

\[
(\phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi))
\]

(2,A1)

with $\psi = (\phi \rightarrow \phi)$

\[
(((\phi \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi))
\]

(3,MP(1,2))

\[
((\phi \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi)
\]

(4,A1)

\[
(\phi \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi)
\]

(5,MP(3,4))

Q.E.D.

Back to outline of Lecture 8-9
Example 21 (Double Negation)

We show $\phi \vdash ((\phi \rightarrow F) \rightarrow F)$.

With Deduction Theorem it suffices to show

$$\{\phi, (\phi \rightarrow F)\} \vdash F$$

This is achieved using once Modus Ponens.

*****

Exercise:

Prove it without the Deduction Theorem.
Example 22 (Contrapositive)

We show \((\phi \rightarrow \psi) \vdash ((\psi \rightarrow \text{F}) \rightarrow (\phi \rightarrow \text{F}))\).

With Deduction Theorem it suffices to show

\[
\{(\phi \rightarrow \psi), (\psi \rightarrow \text{F}), \phi\} \vdash \text{F}
\]

But this is achieved using twice Modus Ponens.

*****

Exercise:

Prove it without the Deduction Theorem.
Example 23 (Disjunctions)

We want to show \( \phi \vdash (\psi \lor \phi) \)

We replace \( \lor \) using \( F \) and \( \rightarrow \):

\[
\phi \vdash ((\psi \rightarrow F) \rightarrow \phi)
\]

\[
(\phi \rightarrow ((\psi \rightarrow F) \rightarrow \phi)) \quad (1,A1)
\]

\[
\phi
\]

\[
((\psi \rightarrow F) \rightarrow \phi) \quad (2, \in \Sigma)
\]

\[
(\psi \rightarrow F) \rightarrow \phi) \quad (3, MP(1,2))
\]

Q.E.D.

We do not use the Deduction Theorem here.
Example 24 (Conjunction)

We want to show \( \theta = (\phi \land \psi) \vdash \psi \)

We replace \( \land \) using \( F \) and \( \rightarrow \):

\[
\theta = \neg(\neg \phi \lor \neg \psi) = \neg(\neg \phi \rightarrow \neg \psi) = (((((\phi \rightarrow F) \rightarrow F) \rightarrow (\psi \rightarrow F)) \rightarrow F)
\]

Using the Deduction Theorem it suffices to prove

\( \vdash (\theta \rightarrow \psi) \)

Using the Contrapositive it suffices to prove

\( \vdash ((\psi \rightarrow F) \rightarrow (\theta \rightarrow F)) \)

Using the Contrapositive and the Deduction Theorem once more we need

\( \{ (\psi \rightarrow F), \theta \} \vdash F \)

For this we look at A2 in the form

\( ((\psi \rightarrow F) \rightarrow (((\phi \rightarrow F) \rightarrow F) \rightarrow (\psi \rightarrow F))) \)

and use Modus Ponens twice. Q.E.D.
Homework

Prove all of the following tautologies using proof sequences:

\[
\begin{align*}
T  \\
& ((\phi \to (\psi \to \phi)) \\
& (((\phi \to (\psi \to \theta)) \to ((\phi \to \psi) \to (\phi \to \theta))) \\
& (((\neg \psi \to \neg \phi) \to ((\neg \psi \to \phi) \to \psi)) \\
& ((\phi \land \psi) \to \phi) \in Ded(\Sigma) \\
& ((\phi \land \psi) \to \psi) \in Ded(\Sigma) \\
& (\phi \to (\phi \lor \psi)) \in Ded(\Sigma) \\
& (\psi \to (\phi \lor \psi)) \in Ded(\Sigma)
\end{align*}
\]

Is this enough to prove completeness?
To pass between $\text{WFF}_{\{\neg, \rightarrow\}}$ and $\text{WFF}_{\{F, \rightarrow\}}$ we use the definition of $\neg$ and $F$:

- $\neg \phi$ is defined by $(\phi \rightarrow F)$ and
- $F$ is defined by $\neg (p_0 \rightarrow p_0)$

(i) Prove the following tautologies using proof sequences in $\text{WFF}_{\{F, \rightarrow\}}$:
   
   $(F \rightarrow ((p_0 \rightarrow p_0) \rightarrow F'))$ and $(((p_0 \rightarrow p_0) \rightarrow F') \rightarrow F')$

(ii) Prove the following tautologies using proof sequences in $\text{WFF}_{\{\neg, \rightarrow\}}$:

   $(\neg \phi \rightarrow (\phi \rightarrow \neg (p_0 \rightarrow p_0)))$ and $((\phi \rightarrow \neg (p_0 \rightarrow p_0)) \rightarrow \neg \phi)$
Completeness Theorem for Deductions

The following shows that the method of proof sequences is sufficiently powerful to obtain all tautologies, or, more generally, all logical consequences of a given set of formulas.

**Theorem 25 (3.3.15)**

Let $\Sigma \subseteq \text{WFF}_{\{\rightarrow, F\}}$ and $\phi \in \text{WFF}_{\{\rightarrow, F\}}$.

$$\Sigma \models \phi \iff \Sigma \vdash \phi.$$  

**Proof:** We have seen the soundness: If $\Sigma \vdash \phi$ then $\Sigma \models \phi$.

The proof that if $\Sigma \models \phi$ then $\Sigma \vdash \phi$ will be given in the next lecture.  

Q.E.D.
Corollary 26 (3.3.21)

Compactness Theorem

Let $\Sigma \subseteq \text{WFF}_{\{\to, F\}}$ be an infinite set of formulas.

$\Sigma$ is satisfiable iff every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable.
Proof of the Compactness Theorem

Proof:
By the completeness theorem above Σ is satisfiable iff Σ is consistent.

Clearly, if Σ is consistent, so is every finite subset Σ₀ ⊆ Σ.

Conversely, assume Σ is inconsistent. Therefore there is a proof sequence

$$\psi_1, \ldots, \psi_k, \theta_1, \ldots \theta_m, \text{F}$$

with exactly the $$\psi_i \in \Sigma$$, which shows the inconsistency.

Therefore,

$$\{\psi_1, \ldots, \psi_k\} \subseteq \Sigma$$

is a finite inconsistent subset. Q.E.D.
Example 27

Colorability of graphs

Let \( \text{red}_i, \text{green}_i, \text{blue}_i, \text{white}_i \), and \( e_{i,j} \) be propositional variables for \( i, j \in \mathbb{N} \).

Let \( G \subseteq \mathbb{N}^2 \).

- \( \text{GRAPH} = \{e_{i,j} : (i, j) \in G\} \) says there is an edge between \( i \) and \( j \).

- \( \text{COLOR}_i = (\text{red}_i \lor \text{green}_i \lor \text{blue}_i \lor \text{white}_i) \) says each \( i \) has one of the four colors.

- \( \text{CR}_i = (\text{red}_i \rightarrow (\neg\text{blue}_i \land \neg\text{green}_i \land \neg\text{white}_i)) \) says if \( i \) is red, then it has no other color. Similarly for \( \text{CG}_i, \text{CB}_i \) and \( \text{CW}_i \).

- \( \text{NR}_{i,j} = ((\text{red}_i \land e_{i,j}) \rightarrow \neg\text{red}_j) \) says if \( i \) is red and \( j \) is a neighbor of \( i \), then \( j \) is not red. Similarly for \( \text{NG}_i, \text{NB}_i \) and \( \text{NW}_i \).
Example 28

Colorability of graphs, contd

Now we put

- \( COLOR = \{COLOR_i : i \in \mathbb{N}\} \)
- \( C = \{(CR_i \land CB_i \land CW_i \land CG_i) : i \in \mathbb{N}\} \)
- \( N = \{(NR_i \land NB_i \land NW_i \land NG_i) : i \in \mathbb{N}\} \)
- \( \Sigma_G = GRAPH \cup COLOR \cup C \cup N \)

For a fixed \( G \)  \( \Sigma_G \) is satisfiable iff
\( GRAPH \) can be colored with four colors.

Compactness now says:
If every finite subgraph of \( GRAPH \) can be colored with four colors, so can \( GRAPH \).
Proof of the Completeness Theorem

To prove the Completeness theorem we observe first that it suffices to prove

$$\Sigma \text{ is satisfiable iff } \Sigma \text{ is consistent.}$$

Proof: Assume $$\Sigma \models \phi.$$ This is equivalent to $$\Sigma_1 = \Sigma \cup \{\neg \phi\} \models F$$ i.e. $$\Sigma_1$$ is not satisfiable. But then $$\Sigma_1$$ is inconsistent, hence $$\Sigma_1 = \Sigma \cup \{((\phi \rightarrow F)) \vdash F$$ and, using the Deduction Theorem, $$\Sigma \vdash ((\phi \rightarrow F) \rightarrow F)$$ and, using A3, $$\Sigma \vdash \phi.$$ 

The other direction is soundness. Q.E.D.
How we prove the Completeness Theorem

We reduce it to showing that
if a set $\Sigma \subseteq WFF_{\{\rightarrow, F\}}$ is consistent,
then it has a satisfiable assignment $z$.

$\Sigma$ may have many assignments,
and they may be hard to describe.

So we reduce the choices:
Finding a consistent $\Sigma' \supseteq \Sigma$ will have fewer satisfying assignments
(if any at all).

We will construct $\Sigma_\omega \supseteq \Sigma$ with at most one satisfying assignment.

Then we show that indeed it has
**exactly one** satisfying assignment:

$$z(p_i) = 1 \text{ iff } p_i \in \Sigma_\omega$$
How to find proof sequences

The above proof of the completeness theorem is mathematically very elegant, but hides the construction of proof sequence, whose very existence we show.

This proof method was independently suggested by L. Henkin, G. Hasenjaeger and J. Hintikka (all in 1949).

The actual construction of proof sequences was shown by P. Bernays (1918), E. Post (1921) and L. Kalmar (1934).

It is the topic of courses in Automated Theorem Proving or in Textbooks on Logic prior to 1960.

Our proof adapts easily to First Order Logic.
Maximally consistent sets

We use the notions consistent and inconsistent also for sets $\Sigma \subseteq \text{WFF}\{F, \rightarrow\}$.

**Definition 29 (3.3.16)**

A set $\Sigma$ of $\text{WFF}\{\rightarrow, F\}$ is maximally consistent if it is consistent, and for every $\phi \in \text{WFF}\{\rightarrow, F\}$ either $\Sigma \vdash \phi$ or $\Sigma \vdash (\phi \rightarrow F)$. 

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Maximally consistent extensions

Lemma 30 (3.3.17)
Let $\Sigma \subseteq \text{WFF}\{\rightarrow, F\}$ be consistent.

Then there is a maximally consistent $\Sigma^*$ such that $\Sigma \subseteq \Sigma^*$.

The proof is in four stages.
Proof: [Stage (i)]

Assume now, that $\Sigma$ is consistent.

Let $\{\phi_i : i \in \mathbb{N}\}$ be an enumeration of the formulas in $WFF\{\rightarrow, F\}$.

We define in stages a set $\Sigma_\omega \subseteq WFF\{\rightarrow,F\}$ in the following way:

- $\Sigma_0 = \Sigma$;
- $\Sigma_{n+1} = \begin{cases} 
\Sigma_n \cup \{\phi_n\} & \text{if } \Sigma_n \cup \{\phi_n\} \text{ is consistent, and} \\
\Sigma_n \cup \{(\phi_n \rightarrow F)\} & \text{otherwise.} 
\end{cases}$
- $\Sigma_\omega = \bigcup \{ \Sigma_i : i \in \mathbb{N} \}$.

Q.E.D.
**Proof: [Stage (ii)]**

For every $i \in \mathbb{N}$ we use the Dychotomy Theorem to show that $\Sigma_{i+1}$ is consistent.

Assume $\Sigma_i$ is consistent, but

$\Sigma_i \cup \{\phi_i\} \vdash \bot$

and

$\Sigma_i \cup \{(\phi_i \rightarrow \bot')\} \vdash \bot$.

Using the Dychotomy Theorem we get that $\Sigma_i \vdash \bot$ hence, $\Sigma_i$ is inconsistent, a contradiction. \(\text{Q.E.D.}\)
\textbf{Proof: [Stage (iii)]}

\[ \Sigma_\omega \text{ is consistent.} \]

For otherwise, there are formulas \( \{\psi_1, \ldots, \psi_k\} \) such that \( \{\psi_1, \ldots, \psi_k\} \vdash F \).

So there is \( m \in \mathbb{N} \) such that

\[ \psi_1, \ldots, \psi_k, F \in \Sigma_m \]

and therefore, \( \Sigma_m \) is inconsistent, contrary to Stage (ii).  \textbf{Q.E.D.}
Proof: [Stage (iv)]

Σω is a maximally consistent extensions of Σ, i.e. for every ψ ∈ WFF{→,F} either Σω ⊨ ψ or Σω ⊨ (ψ → F).

As {φi : i ∈ N} is an enumeration of the formulas in WFF{→,F}, ψ = φn for some n ∈ N.

Therefore ψ ∈ Σn or (ψ → F) ∈ Σn.

Now we put Σ* = Σω, which completes the proof. Q.E.D.
Remark 31 (3.3.18)

Note that different enumerations of $\text{WFF}_{\{\rightarrow,F\}}$ give different sets $\Sigma_\omega$ for the same $\Sigma$.

In general, for countable $\Sigma$, there are $2^\mathbb{N}$ maximally consistent extensions.

Exercise:
Prove this.
Assignment for maximally consistent set

Lemma 32 (3.3.19)
Let $\Sigma \subseteq WFF\{\rightarrow, F\}$ be maximally consistent. Then there is an assignment $z : Var \rightarrow \{0, 1\}$ such that $M_{PL}(\Sigma, z) = 1$.

In other words, $\Sigma$ is satisfiable.
Proof: [Stage (i)]

We first define a propositional assignment

\[ z : Var \rightarrow \{0, 1\} \]

for \( \Sigma \) in the following way:

\[ z(p_i) = 1 \text{ iff } p_i \in \Sigma \]

As \( \Sigma \) is maximally consistent, this is well defined. Q.E.D.
Proof: [Stage (ii)]

All we need to show now, is that
$M_{PL}(\psi, z) = 1$ iff $\Sigma \vdash \psi$.

This we do by induction for every $\psi \in WFF\{\rightarrow, F\}$.

**Basis:**
For $\psi$ atomic this follows from the definition of $z$.

**Closure:** Assume we have shown it for $\psi_1$, and that $\psi_2$, and $\psi = (\psi_1 \rightarrow \psi_2)$.

From left to right:

$M_{PL}(\psi, z) = 1$ iff
$M_{PL}((\psi_1 \rightarrow \psi_2), z) = TT_\rightarrow(M_{PL}(\psi_1, z), M_{PL}(\psi_2, z)) = 1$.

By the induction hypothesis $M_{PL}(\psi_1, z) = 1$ iff $\Sigma \vdash \psi_1$.

So assume, for contradiction, that
$M_{PL}(\psi_1, z) = 1$, $M_{PL}(\psi_2, z) = 0$ and $\Sigma \vdash \psi$.

Then $\Sigma \vdash \psi_1$, $\Sigma \vdash (\psi_2 \rightarrow F)$, and also $\Sigma \vdash \psi$,
which contradicts the consistency of $\Sigma$. 

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Proof: [Stage (ii)] continued

From right to left:

\[ M_{PL}(\psi, z) = 0 \text{ iff } M_{PL}(\psi_1 \rightarrow \psi_2, z) = M_{\rightarrow}(M_{PL}(\psi_1, z), M_{PL}(\psi_2, z)) = 0. \]

So assume, for contradiction, that \( M_{PL}(\psi_2, z) = 1 \) and \( \Sigma \not\vdash \psi \).

Then, by maximal consistency, \( \Sigma \vdash (\psi \rightarrow F) \), hence \( \Sigma \vdash ((\psi_1 \rightarrow \psi_2) \rightarrow F) \).

By induction hypothesis, \( \Sigma \vdash \psi_2 \) iff \( M_{PL}(\psi_2, z) = 1 \) hence \( \Sigma \vdash \psi_2 \).

Using Axiom \( \textbf{A1} \), and the Deduction Theorem, we get that

\[ ((\psi_1 \rightarrow \psi_2) \rightarrow F), \psi_2 \vdash F \]

hence \( \Sigma \) is inconsistent.

Conclusion:

\( M_{PL}(\Sigma, z) = 1 \), and as \( \Sigma \subseteq \Sigma \), \( M_{PL}(\Sigma, z) = 1 \) which shows that \( \Sigma \) is satisfiable.

Q.E.D.
End of the proof of the Completeness Theorem

**Proof:** The proof is in several stages.

(i) We have observed already that it suffices to prove that

If $\Sigma$ is consistent then $\Sigma$ is satisfiable.

For this we used the Deduction and the Dychotomy Theorems.

(ii) By the soundness of proof sequences,

if $\Sigma$ is satisfiable, then $\Sigma$ is consistent.

(iii) Assume now, that $\Sigma$ is consistent.

So there is $\Sigma^*$ maximally consistent with $\Sigma \subseteq \Sigma^*$ by lemma \[30\].
But $\Sigma^*$ is satisfiable by lemma \[32\].

**Q.E.D.**
Applications of Compactness: Definability

**Definition 33 (3.4.4)**
We look now at sets \( K \) of assignments which are defined by sets of formulas \( \Sigma \subseteq \text{WFF}\{\to, F\} \).

(i) Let \( \Sigma \) be a set of formulas in \( \text{WFF}\{\to, F\} \). We define

\[
\text{Ass}(\Sigma) = \{z \in \text{Ass} : M_{PL}(\Sigma, z) = 1\}.
\]

(ii) Let \( K \) be a subset of \( \text{Ass} \). We say that \( K \) is **(set)-definable in \( \text{WFF}\{\to, F\} \)** if there is a set \( \Sigma \subseteq \text{WFF}\{\to, F\} \) such that \( K = \text{Ass}(\Sigma) \).

(iii) A set \( K \) of truth assignments is **finitely definable** if there is a **finite** \( \Sigma \) such that \( K = \text{Ass}(\Sigma) \).
Examples 34 (3.4.5)

(o) $\text{Ass}(T) = \text{Ass}$, the set of all truth assignments.

(i) Let $K_1$ be the set of truth assignments $z$ such that $z(p_0) = 1$.
Then $K_1 = \text{Ass}({p_0})$, and therefore is finitely definable.

(ii) Let $K_2$ be the set of truth assignments $z$ such that $z(p_i) = z(p_{2i})$ for every $i \in \mathbb{N}$.
Then $K_2 = \text{Ass}(\Sigma_2)$, for

$$\Sigma_2 = \{((p_i \rightarrow p_{2i}) \land (p_{2i} \rightarrow p_i)) : i \in \mathbb{N}\},$$
and therefore is set-definable.
Examples 35 (3.4.5, contd)

(iii) If $K = \text{Ass}(\Sigma)$ and $K' = \text{Ass}(\Sigma')$, then $K \cap K'$ is definable with
$$K \cap K' = \text{Ass}(\Sigma \cup \Sigma').$$

(iv) If $K = \text{Ass}(\Sigma)$ and $K' = \text{Ass}(\Sigma')$, then $K \cup K'$ is definable with
$$K \cup K' = \text{Ass}(\{\phi \lor \psi : \phi \in \Sigma, \psi \in \Sigma'\}).$$

(v) If $\Sigma \subseteq \Sigma'$ then $\text{Ass}(\Sigma') \subseteq \text{Ass}(\Sigma)$. 
Examples 36 (3.4.5, contd)  

Exercises

(vi) Let $K = \text{Ass}(\Sigma)$ and $K' = \text{Ass}(\Sigma')$, what can you say about $\text{Ass}(\Sigma \cap \Sigma')$?  
    
    **Exercise:** Study several special cases.

(vii) Let

$$K_T = \{ z \in \text{Ass} : z(p_i) = 1, i \in \mathbb{N}\}$$

Note that $K_T$ has exactly one element and $K_T$ is definable by $\Sigma_T = \{p_i : i \in \mathbb{N}\}$.

**Exercise:**
Show that $K_T$ is not definable by any finite subset $\Sigma \subseteq \Sigma_T$, hence is not finitely definable.

**Hint:** In such a $\Sigma$ there are only finitely many variables.
Examples 37 (3.4.5, contd)

Exercises

(viii) If $K = \text{Ass}(\Sigma)$ for some finite $\Sigma$.

Then the complement
$K' = \text{Ass} - K$

is also definable.

Exercise:
Describe $\Sigma'$ which defines $K'$. 

Main Theorem on Definability in Propositional Logic

Theorem 38 (proposition 3.4.6)

(i) Let \( K \subseteq \text{Ass} \) be definable and \( \text{Ass} - K \) be also definable, i.e. \( K = \text{Ass}(\Sigma) \) and \( \text{Ass} - K = \text{Ass}(\Sigma') \).

Then there is a finite subset \( \Sigma_0 \subseteq \Sigma \) such that \( K = \text{Ass}(\Sigma_0) \).

(ii) Let \( K \) be a set of truth assignments.

\( K \) is finitely definable iff both \( K \) and its complement are definable.
Proof of Theorem 38

(i) By assumption and example 34 (iii) above

\[ K \cap K' = \emptyset = \text{Ass}(\Sigma \cup \Sigma') \]

Therefore \( \Sigma \cup \Sigma' \) is not satisfiable.

By the Compactness Theorem there is a finite subset

\[ \Sigma_1 \subseteq \Sigma \cup \Sigma' \]

which is not satisfiable.

Let \( \Sigma_0 = \Sigma \cap \Sigma_1 \).

By example 35 (vi) above, we have \( K \subseteq \text{Ass}(\Sigma_0) \).

To show that \( \text{Ass}(\Sigma_0) \subseteq K \) it suffices to show, that \( \text{Ass}(\Sigma_0) \cap (\text{Ass} - K) = \emptyset \).

But, as \( \Sigma_1 \) is not satisfiable, neither is \( \Sigma_0 \cup \Sigma' \), because \( \Sigma_1 \subseteq \Sigma_0 \cup \Sigma' \).

(ii) follows from (i) and example 35 (viii) above. Q.E.D.
(i) Show that \( \text{Ass} - K_T \) is not definable.

(ii) Let \( K_{\text{even}} \) be the set of truth assignments \( z \) such that \( z(p_{2i}) = 1 \) for every \( i \in \mathbb{N} \).

Show that \( K_{\text{even}} \) is not finitely definable.

(iii) Let \( K_{\text{fin-true}} \) be the set of assignments \( z \) with finitely many \( i \)'s such that \( z(p_i) = 1 \).

Show that \( K_{\text{fin-true}} \) is not (set)-definable.
Homework

(i) Let $K \subseteq \text{Ass}$ and $z \notin K$.
Assume $\Sigma$ defines $K$.
Let
\[ \Sigma_1 = \{ p_i : z(p_i) = 1 \} \cup \{ \neg p_i : z(p_i) = 0 \} \]
Then $\Sigma \cup \Sigma_1$ is not satisfiable.

(ii) Let $K$ be the set of assignments $z$ such that for exactly one $i \in \mathbb{N}$ $z(p_i) = 1$.
Use (i) and compactness to show that neither $K$ nor its complement $\text{Ass} \setminus K$ are definable.

Skip **Finite Support** and go to **What Else?**
Definition 40 (3.4.8)

Dependence on variables

Let $K$ be a set of truth assignments.
Let $D$ be a set of propositional variables.

We say that $K$ does depend on the variables $p_i \in D$ if there are truth assignments $z$ and $z'$ which only differ for $p_i \in D$, i.e. with $z(p_j) = z'(p_j)$ for every $p_j \notin D$, such that $z \in K$ and $z' \notin K$. 
Definition 41 (3.4.8) 
Finite Support

Let $K$ be a set of truth assignments. 
Let $S$ be a set of propositional variables.

(i) $S$ is a support for $K$ 
if for any two assignments $z$ and $z'$ which coincide on $S$, 
i.e. for every $p_i \in S$ $z(p_i) = z'(p_i)$, 
$z \in K$ iff $z' \in K$.

(ii) We say that $K$ has finite support 
if there is a finite $S$ 
which is a support for $K$. 

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Example 42
Two critical examples

Let $K_1$ be all the assignments $z$ with $z(p_i) = 1$ for finitely many variables.

Let $K_2$ be all the assignments $z$ with $z(p_{2i}) = 1$ for all $i \in \mathbb{N}$ and $z(p_{2i+1}) = 1$ finitely many $i \in \mathbb{N}$.

(i) The set of all variables is a support for $K_1$ and $K_2$.

(ii) A proper subset $S \subset \mathbb{N}$ is a support for $K_1$ iff $\mathbb{N} \setminus S$ is finite.

(iii) **Exercise:** Which proper subsets $S \subset \mathbb{N}$ are a support for $K_2$?

(iv) $K_1$ does not depend on any proper subset $D \subset \mathbb{N}$.

(v) $K_2$ does depend on each single variable $p_{2i}$ (does depend on $\{p_{2i}\}$).
Homework

Let $S$ and $S'$ be supports for $K$ and $D$ a set of variables.

Prove or disprove the following:

(i) If $D \subseteq Var \setminus S$ then $K$ does not depend on $D$.
(ii) If $K$ depends on $D$ then $D \subseteq S$.
(iii) If $D \subseteq S$ then $K$ depends on $D$.
(iv) $S \cap S'$ is a support for $K$.
(v) $S \cup S'$ is a support for $K$.
(vi) $S = \{p_i \in Var : K \text{ depends on } \{p_i\}\}$
(vii) $S = \bigcup\{D \subseteq Var : K \text{ depends on } D\}$
Examples 43 (3.4.9)

(i) If $K$ is finitely definable,
    then $K$ has finite support.

(ii) $K_{even}$ does not have finite support.

(iii) If $K$ is itself finite,
    it does not have finite support.
**Theorem 44 (proposition 3.4.10)**

*Finite Support Theorem*

Let $K$ be a set of truth assignments. $K$ is finitely definable iff $K$ has finite support.

**Proof:** By (i) in the example above, if $K$ is finitely definable, then $K$ has finite support.

So assume, $K$ has finite support.

Then it suffices to describe the truth table describing $K$ on the variables of $\text{Support}(K)$.

Q.E.D.
**SAT**: The most important open problem in Propositional Logic

**Input**: Given a finite set $\Sigma \in CNF$.

**Question**: Is $\Sigma$ satisfiable.

Find efficient algorithms for **SAT**.
Propositional Modal Logic

Propositional Logic analyses the words

**and, or, not, if ... then**

Modal Propositional Logic analyses additionally the words

**possibly, necessarily**

We study this in the course **Logic II**.
Propositional Temporal Logic

Propositional Logic analyses the words

and, or, not, if ... then

Temporal Propositional Logic analyses additionally the words

next time, previous time
some time in the future,
some time in the past,
always in the future,
always in the past,
until, since

We study this in the course Logic II.
Where do we use Propositional Logics

Propositional Logic (and its Modal and Temporal versions) are used in

- **Hardware** specification and verification.
- **Software** specification and verification.
- **Automated reasoning** and **Artificial Intelligence (AI)**.
- **Databases**.
Now we are ready for

Predicate Logic aka First Order Logic \textbf{FOL}

- Predicate Logic for Arithmetics
- Semantics of \textbf{FOL}
- Syntax of \textbf{FOL}
- Learning to read and understand formulas of \textbf{FOL}
- Handling quantifiers
- Expressive power of \textbf{FOL}
- Stating Gödel’s Completeness Theorem
Outlook: Predicate aka First Order Logic FOL

FOL for the natural numbers with their arithmetic operations

- Inductive definition of the syntax
- Inductive definition of the meaning function
- Reading, writing and understanding FOL-formulas

FOL-structures as interpretation of vocabularies $\tau$

- What is a FOL-vocabulary?
- Inductive definition of the syntax and semantics for FOL($\tau$)
- Proof systems for FOL.
- Definability and non-definability in FOL